Chapter 5

Conformal supergravity

As we saw in the previous chapters, Scherk-Schwarz dimensional reduction can be used as a tool to obtain scalar potentials in lower dimensional gauged supergravity theories. There are unfortunately many different ways in which the compactification process can be performed and therefore it is a priori not clear how to obtain the most general vacuum solutions.

Another approach to this problem is the construction of matter coupled Poincaré supergravity in lower dimensions. One possibility is the explicit coupling of matter multiplets of the super Poincaré algebra to the supergravity multiplet. The method we will use in the following chapters however, is that of conformal supergravity for reasons explained below.

Conformal supergravities have been constructed in various dimensions (for a review, see [123]) but not yet in five dimensions. By using conformal tensor calculus, conformal supergravities form an elegant way to construct general couplings of Poincaré-supergravities to matter [124]. It also provides a method to find the auxiliary fields of off-shell Poincaré supergravities, like e.g. for $\mathcal{N} = 1, D = 4$ supergravity [125]. The reason for using a theory based on the superconformal group instead of the Poincaré group is the presence of more symmetries, generally resulting in more structure and therefore simplifying the calculations. Furthermore, the conformal group is the largest possible group of space-time symmetries and turns out to be gauge equivalent to the Poincaré group. In the five-dimensional case these matter coupled supergravities have recently attracted renewed attention for reasons motivated in chapter 2.

Although quite some progress has been made in these areas, it is clear that it is important to have an independent derivation of the most general matter couplings derived in [76] where most of the current results are based on. Especially since past experience has shown that superconformal constructions lead to new insights in the structure of matter couplings. A recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal invariant matter couplings with hypermultiplets [126]. For these reasons a superconformal construction of general matter couplings in $\mathcal{N} = 2, D = 5$ is desirable.

The superconformal program

The procedure will be as follows.

- In this chapter we take the first step in this investigation by constructing the $\mathcal{N} = 2, D = 5$

conformal supergravity theory. In our construction we use the methods developed first for $\mathcal{N} = 1, D = 4$ [127, 128], which were inspired by the geometrical methods of [129]. They are based on gauging the conformal superalgebra [130] which in our case is $F^2(4)$. The superconformal multiplet one obtains this way contains all the (independent) gauge fields of the superconformal algebra and is called the Weyl multiplet. In general one needs to include matter fields to have an equal number of bosons and fermions. Although there are two sets of auxiliary fields one can use, in this chapter we will restrict to the one leading to the so-called Standard Weyl multiplet.

- The second step will be performed in chapter 6, where we construct the actions for matter multiplets in the background of the Standard Weyl multiplet. This step already produces a nice geometrical framework on the scalar manifolds resulting from this construction.
- Finally, in chapter 7 we will gaugefix the symmetries not present in the Poincaré algebra and construct $\mathcal{N} = 2, D = 5$ matter coupled Poincaré supergravity.

This chapter is based on the work published in [131]. Note that many details have been left out for reasons of brevity and clarity; we refer the reader to [131, 132] for more details. Note that shortly after our publication interesting results have been obtained on conformal supergravity in five dimensions [133] that have some overlap with our work.

For more information on the conformal supergravity approach, see [134–136].

5.1 Definition of rigid conformal (super-)symmetry

We start this chapter by giving a short review of rigid conformal supersymmetry; for a more extended discussion, see e.g. [137]. We first introduce conformal symmetry and in a second step extend this to conformal supersymmetry. Given a space-time with a metric tensor $g_{\mu\nu}(x)$, the conformal transformations are defined as the general coordinate transformations that leave "angles" invariant. The parameters of these special coordinate transformations define a conformal Killing vector $k^\mu(x)$. The defining equation for this conformal Killing vector is given by

$$\delta_{\text{g.c.t.}}(k) g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x) g_{\mu\nu}(x)\,, \tag{5.1}$$

where $\omega(x)$ is an arbitrary function, $k_\mu = g_{\mu\nu}k^\nu$ and the covariant derivative is given by $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu} k_\rho$. In flat D-dimensional Minkowski space-time, (5.1) implies

$$\partial_{(\mu} k_{\nu)}(x) - \tfrac{1}{D}\eta_{\mu\nu}\partial_\rho k^\rho(x) = 0\,. \tag{5.2}$$

In dimensions $D > 2$, the conformal algebra is finite-dimensional. The solutions of (5.2) are given by

$$k^\mu(x) = \xi^\mu + \lambda_M^{\mu\nu} x_\nu + \lambda_D x^\mu + \left(x^2 \Lambda_K^\mu - 2x^\mu x \cdot \Lambda_K\right). \tag{5.3}$$

Corresponding to the parameters ξ^μ are the translations P_μ, the parameters $\lambda_M^{\mu\nu}$ correspond to Lorentz rotations $M_{\mu\nu}$, to λ_D are associated the dilatations D, and Λ_K^μ are the parameters of 'special conformal transformations' K_μ. Thus, the full set of conformal transformations δ_C can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{5.4}$$

The commutators between different generators define the conformal algebra which is isomorphic to the algebra of $SO(D,2)$.

We wish to consider representations of the conformal algebra on fields $\phi^\alpha(x)$ where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = 0$. From the expression (5.3) for the conformal Killing vector, we deduce that this algebra is isomorphic to the algebra generated by $M_{\mu\nu}$, D and K_μ. We denote the generators of this stability subalgebra by $\Sigma_{\mu\nu}$, Δ and κ_μ. Applying the theory of induced representations, it follows that any representation (Σ, Δ, κ) of the stability subalgebra induces a representation of the full conformal algebra with the following transformation rules [135] (we suppress any internal indices):

$$\begin{aligned}
\delta_P\phi(x) &= \xi^\mu\partial_\mu\phi(x)\,,\\
\delta_M\phi(x) &= \tfrac{1}{2}\lambda_M^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \delta_\Sigma(\lambda_M)\phi(x)\,,\\
\delta_D\phi(x) &= \lambda_D x^\lambda\partial_\lambda\phi(x) + \delta_\Delta(\lambda_D)\phi(x)\,,\\
\delta_K\phi(x) &= \lambda_K^\mu(x^2\partial_\mu - 2x_\mu x^\lambda\partial_\lambda)\phi(x)\\
&\quad + \Big(\delta_\Delta(-2x\cdot\Lambda_K) + \delta_\Sigma(-4x_{[\mu}\lambda_{K\nu]}) + \delta_\kappa(\lambda_K)\Big)\phi(x)\,.
\end{aligned} \tag{5.5}$$

We now look at the non-trivial representation (Σ, Δ, κ) that we use in this chapter.

- Firstly, concerning the Lorentz representations, in this chapter we will encounter anti-symmetric tensors $\phi_{a_1\cdots a_n}(x)$ ($n = 0, 1, 2, \ldots$) and spinors $\psi_\alpha(x)$:

$$\begin{aligned}
\delta_\Sigma(\lambda_M)\phi_{a_1\cdots a_n}(x) &= -n(\lambda_M)_{[a_1}{}^b\phi_{|b|a_2\cdots a_n]}(x)\,,\\
\delta_\Sigma(\lambda_M)\psi(x) &= -\tfrac{1}{4}\lambda_M^{ab}\gamma_{ab}\psi(x)\,.
\end{aligned} \tag{5.6}$$

- Secondly, we consider the dilatations. For most fields, the Δ transformation is just determined by a number w, which is called the Weyl weight of ϕ^α:

$$\delta_\Delta(\lambda_D)\phi^\alpha(x) = w\lambda_D\phi^\alpha(x)\,. \tag{5.7}$$

An exception is given in the next chapter for the scalars of the hypermultiplet, on which dilatation transformations are realized nonlinearly. Namely, for scalar fields it is often convenient to consider the set of fields ϕ^α as the coordinates of a scalar manifold with affine connection $\Gamma_{\alpha\beta}{}^\gamma$. With this understanding, the transformation of ϕ^α under dilatations can be characterized by

$$\delta_\Delta(\lambda_D)\phi^\alpha = \lambda_D k^\alpha(\phi)\,. \tag{5.8}$$

Requiring dilatational invariance of kinetic terms determined by a metric $g_{\alpha\beta}$, leads to the interpretation of the vector k^α as a homothetic Killing vector, i.e. it should satisfy the conformal Killing equation (5.1) for *constant* $\omega(x)$:

$$\mathfrak{D}_\alpha k_\beta + \mathfrak{D}_\beta k_\alpha = (D-2)g_{\alpha\beta}\,, \tag{5.9}$$

where D denotes the space-time dimension and $\mathfrak{D}_\alpha k_\beta = \partial_\alpha k_\beta - \Gamma_{\alpha\beta}{}^\gamma k_\gamma$. However, (5.5) shows that the Δ-transformation also enters in the special conformal transformation. It turns out that invariance of the kinetic terms under these special conformal transformations restricts $k^\alpha(\phi)$ further to a so-called *exact* homothetic Killing vector, i.e.

$$k_\alpha = \partial_\alpha \mathcal{K}\,, \tag{5.10}$$

for some function $\mathcal{K}(\phi)$. One can show that the restrictions (5.9) and (5.10) are equivalent to

$$\mathfrak{D}_\alpha k^\beta \equiv \partial_\alpha k^\beta + \Gamma_{\alpha\gamma}{}^\beta k^\gamma = w\delta_\alpha{}^\beta\,. \tag{5.11}$$

The constant w is identified with the Weyl weight of ϕ^α and is given by $w = (D-2)/2$, i.e. 3/2 in five dimensions. The proof of the necessity of (5.11) can be extracted from [138], see also [139, 140]. In these papers the conditions for conformal invariance of a sigma model with either gravity or supersymmetry are investigated. Note that the condition (5.11) can be formulated *independent* of a metric. Only an affine connection is necessary.

For the special case of a zero affine connection, the homothetic Killing vector is given by $k^\alpha = w\phi^\alpha$ and the transformation rule (5.8) reduces to $\delta_\Delta(\lambda_D)\phi^\alpha = w\lambda_D\phi^\alpha$. Note that the homothetic Killing vector $k^\alpha = w\phi^\alpha$ is indeed exact with $\mathcal{K}$ given by

$$\mathcal{K} = \tfrac{1}{(D-2)} k^\alpha g_{\alpha\beta} k^\beta\,. \tag{5.12}$$

- Finally, all (non-gauge)fields that we will discuss in this thesis are invariant under the internal special conformal transformations, i.e. $\delta_\kappa\phi^\alpha = 0$.

We next consider the extension to conformal supersymmetry. The parameters of these supersymmetries define a conformal Killing spinor $\epsilon^i(x)$ whose defining equation is given by

$$\nabla_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\gamma^\nu\nabla_\nu\epsilon^i(x) = 0\,. \tag{5.13}$$

In D-dimensional Minkowski space-time this equation implies

$$\partial_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\partial\!\!\!/\epsilon^i(x) = 0\,. \tag{5.14}$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + \mathrm{i}\, x^\mu\gamma_\mu\eta^i\,, \tag{5.15}$$

where the (constant) parameters ϵ^i correspond to "ordinary" supersymmetry transformations Q^i_α and the parameters η^i define special conformal supersymmetries generated by $S^{\ i}_{\alpha}$. The conformal transformation (5.3) and the supersymmetries (5.15) do not form a closed algebra. To obtain closure, one must introduce additional R-symmetry generators. In particular, in the case of 8 supercharges Q^i_α in $D = 5$, there is an additional SU(2) R-symmetry with generators $U_{ij} = U_{ji}$ $(i = 1, 2)$. Thus, the full set of superconformal transformations δ_C is given by:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu + \Lambda^{ij} U_{ij} + \mathrm{i}\,\bar\epsilon Q + \mathrm{i}\,\bar\eta S\,. \tag{5.16}$$

The factors of i in the last two terms appear due to the reality properties, as explained in appendix A. The full superconformal algebra $F^2(4)$ formed by (anti-)commutators between the (bosonic and fermionic) generators will be given in section 5.2.1.

To construct field representations of the superconformal algebra, one can again apply the method of induced representations. In this case one must use superfields $\Phi^a(x^\mu, \theta^i_\alpha)$, where a stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = \theta^i_\alpha = 0$. This algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D, K_\mu, U_{ij}$ and $S^{\;i}_\alpha$.

An additional complication, not encountered in the bosonic case, is that the representation one obtains is reducible. To obtain an irreducible representation, one must impose constraints on the superfield. It is at this point that the transformation rules become nonlinear in the fields. In this chapter we will follow a different approach; instead of working with superfields we will work with the component "ordinary" fields. The different nonlinear transformation rules are obtained by imposing the superconformal algebra.

In the supersymmetric case, we must specify the SU(2)-properties of the different fields as well as the behavior under S-supersymmetry. Concerning the SU(2), we will only encounter scalars ϕ, doublets ψ^i and triplets $\phi^{(ij)}$ whose transformations are given by

$$\begin{aligned} \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\phi &= 0\,, \\ \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\psi^i(x) &= -\Lambda^i{}_j\psi^j(x)\,, \\ \delta_{\mathrm{SU(2)}}(\Lambda^{ij})\phi^{ij}(x) &= -2\Lambda^{(i}{}_k\phi^{j)k}(x)\,. \end{aligned} \tag{5.17}$$

Note that the scalars of the hypermultiplet will also have an SU(2) transformation despite the absence of an i index, as we will see in the following chapter in section 6.3.2.

This leaves us with specifying how a given field transforms under the special supersymmetries generated by $S^{\;i}_\alpha$. In superfield language the full S-transformation is given by a combination of an x-dependent translation in superspace, with parameter $\epsilon^i(x) = \mathrm{i}\, x^\mu\gamma_\mu\eta^i$, and an internal S-transformation. This is a perfect analogy to the bosonic case. In terms of component fields, the same is true. The x-dependent contribution is obtained by making the substitution

$$\epsilon^i \to \mathrm{i}\, \not{x}\eta^i \tag{5.18}$$

in the Q-supersymmetry rules. The internal S-transformations can be deduced by imposing the superconformal algebra.

5.2 Gauging the Superconformal Algebra

In this section we will construct the Standard Weyl multiplet by using the methods developed first for $\mathcal{N} = 1$ in four dimensions [128]. They are based on gauging the conformal superalgebra [130] which, in our case, is $F^2(4)$. We start by giving the commutation relations defining the $F^2(4)$ algebra. Next we discuss the general method, and then apply this to construct the full nonlinear Standard Weyl multiplet. For clarity, we have collected the final results in section 5.3.

5.2.1 The $D = 5$ superconformal algebra $F^2(4)$

Our starting point is the five dimensional superconformal algebra. There exist many varieties of superconformal algebras when one allows for central charges [141, 142]. However, so far a suitable superconformal Weyl multiplet has only been constructed from those superconformal algebras[1] that appear in the Nahm's classification [144]. In that classification there appears one exceptional algebra, which is $F(4)$. The particular real form that we need here is denoted by $F^2(4)$, see tables 5 and 6 in [137].

As we saw in section 5.1, the algebra consists of the bosonic generators M_{ab}, P_a, K_a, D and the fermionic generators $Q_{i\alpha}$ and $S_{i\alpha}$, where $a, b, \ldots$ are Lorentz indices, α is a spinor index and $i = 1, 2$ is an SU(2) index. M_{ab} is the Lorentz generator, P_a are the conformal transformations, K_a is the special conformal transformation, D the dilatation, $Q_{i\alpha}$ and $S_{i\alpha}$ are the supersymmetry and the special supersymmetry generators, respectively, which are symplectic Majorana spinors, 8 real components in total. Finally, $U^{ij} = U^{ji}$ are the generators of the SU(2) R-symmetry group. For more details on the $F^2(4)$ algebra and the rigid superconformal transformations, see [137]. The non-trivial (anti)commutation relations of the generators defining the $F^2(4)$ algebra are given by

$$
\begin{array}{rclrcl}
[P_a, M_{bc}] &=& \eta_{a[b}P_{c]}\,, & [K_a, M_{bc}] &=& \eta_{a[b}K_{c]}\,,\\
[D, P_a] &=& P_a\,, & [D, K_a] &=& -K_a\,,\\
\left[M_{ab}, M^{cd}\right] &=& -2\delta_{[a}{}^{[c}M_{b]}{}^{d]}\,, & [P_a, K_b] &=& 2(\eta_{ab}D + 2M_{ab})\,,\\
&&&&&\\
[M_{ab}, Q_{i\alpha}] &=& -\tfrac{1}{4}(\gamma_{ab}Q_i)_\alpha\,, & [M_{ab}, S_{i\alpha}] &=& -\tfrac{1}{4}(\gamma_{ab}S_i)_\alpha\,,\\
[D, Q_{i\alpha}] &=& \tfrac{1}{2}Q_{i\alpha}\,, & [D, S_{i\alpha}] &=& -\tfrac{1}{2}S_{i\alpha}\,,\\
[K_a, Q_{i\alpha}] &=& \mathrm{i}\,(\gamma_a S_i)_\alpha\,, & [P_a, S_{i\alpha}] &=& -\mathrm{i}\,(\gamma_a Q_i)_\alpha\,,\\
&&&&&\\
\left\{Q_{i\alpha}, Q_{j\beta}\right\} &=& -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta}P_a\,, & \left\{S_{i\alpha}, S_{j\beta}\right\} &=& -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta}K_a\,,\\
\left\{Q_{i\alpha}, S_{j\beta}\right\} &=& \multicolumn{4}{l}{-\tfrac{1}{2}\,\mathrm{i}\left(\varepsilon_{ij}C_{\alpha\beta}D + \varepsilon_{ij}(\gamma^{ab})_{\alpha\beta}M_{ab} + 3C_{\alpha\beta}U_{ij}\right),}\\
&&&&&\\
[Q_{i\alpha}, U_{kl}] &=& \varepsilon_{i(k}Q_{l)\alpha}\,, & [S_{i\alpha}, U_{kl}] &=& \varepsilon_{i(k}S_{l)\alpha}\,,\\
\left[U_{ij}, U^{kl}\right] &=& 2\delta_{(i}{}^{(k}U_{j)}{}^{l)}\,, &&&
\end{array}
\tag{5.19}
$$

where $C_{\alpha\beta}$ is the charge conjungation matrix, see appendix A. The first six commutation relations define the bosonic conformal subgroup SO(5, 2).

We give below some of the commutators of the (rigid) superconformal algebra expressed in terms of commutators of variations of the fields. The commutators between Q- and S-supersymmetry are given by

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\right), \tag{5.20}$$

$$[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_D\left(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\eta\right) + \delta_M\left(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\eta\right) + \delta_U\left(-\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\eta^{j)}\right), \tag{5.21}$$

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K\left(\tfrac{1}{2}\bar\eta_2\gamma^a\eta_1\right). \tag{5.22}$$

Note that to verify these commutators one should use not only the internal but the *full* superconformal transformation rules including the x-dependent translations (5.5) and Q-supersymmetries (5.18).

[1] One exception is the ten dimensional Weyl multiplet [143], which is not based on a known algebra.

Generators	P_a	M_{ab}	D	K_a	U_{ij}	$Q_{\alpha i}$	$S_{\alpha i}$
Fields	$e_\mu{}^a$	ω_μ^{ab}	b_μ	$f_\mu{}^a$	V_μ^{ij}	ψ_μ^i	ϕ_μ^i
Parameters	ξ^a	λ^{ab}	Λ_D	Λ_K^a	Λ^{ij}	ϵ^i	η^i

Table 5.1: The gauge fields and parameters of the superconformal algebra $F^2(4)$.

5.2.2 The gauge fields and their curvatures

The $D = 5$ conformal supergravity theory is based on the superconformal algebra $F^2(4)$ whose generators are those in table 5.1. As a first step we assign to every generator of the superconformal algebra a gauge field. These gauge fields and the names of the corresponding gauge parameters are given in table 5.1.

The transformations are generated by operators according to

$$\delta = \xi^a P_a + \lambda^{ab} M_{ab} + \Lambda_D D + \Lambda_K^a K^a + \Lambda^{ij} U_{ij} + \mathrm{i}\,\bar\epsilon Q + \mathrm{i}\,\bar\eta S\,. \tag{5.23}$$

Gauge fields h_μ^A in general transform as

$$\delta_B(\epsilon^B) h_\mu^A = \partial_\mu \epsilon^A + \epsilon^C h_\mu^B f_{BC}{}^A\,, \tag{5.24}$$

where the structure constants $f_{BC}{}^A$ can be read off from the algebra (5.19). We find

$$\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \lambda^{ab} e_{\mu b} - \Lambda_D e_\mu{}^a + \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta\omega_\mu{}^{ab} &= \mathcal{D}_\mu\lambda^{ab} - 4\xi^{[a} f_\mu{}^{b]} - 4\Lambda_K^{[a} e_\mu{}^{b]} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\phi_\mu - \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\gamma^{ab}\psi_\mu\,,\\
\delta b_\mu &= \partial_\mu\Lambda_D - 2\xi^a f_{\mu a} + 2\Lambda_K^a e_{\mu a} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu\,,\\
\delta f_\mu{}^a &= \mathcal{D}_\mu\Lambda_K^a - \lambda^{ab} f_{\mu b} + \Lambda_D f_\mu{}^a + \tfrac{1}{2}\bar\eta\gamma^a\phi_\mu\,,\\
\delta V_\mu^{ij} &= \partial_\mu\Lambda^{ij} - 2\Lambda^{(i}{}_\ell V_\mu^{j)\ell} - \tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)}\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\xi^a\gamma_a\phi_\mu^i - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\psi_\mu^i - \tfrac{1}{2}\Lambda_D\psi_\mu^i - \Lambda^i{}_j\psi_\mu^j - \mathrm{i}\,e_\mu^a\gamma_a\eta^i\,,\\
\delta\phi_\mu^i &= \mathcal{D}_\mu\eta^i - \tfrac{1}{4}\lambda^{ab}\gamma_{ab}\phi_\mu^i + \tfrac{1}{2}\Lambda_D\phi_\mu^i - \Lambda^i{}_j\phi_\mu^j - \mathrm{i}\,\Lambda_K^a\gamma_a\psi_\mu^i + \mathrm{i}\,f_\mu^a\gamma_a\epsilon^i\,,
\end{aligned} \tag{5.25}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to dilatations, Lorentz rotations and SU(2) transformations:

$$\begin{aligned}
\mathcal{D}_\mu\xi^a &= \partial_\mu\xi^a + b_\mu\xi^a + \omega_\mu{}^{ab}\xi_b\,,\\
\mathcal{D}_\mu\lambda^{ab} &= \partial_\mu\lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c}\,,\\
\mathcal{D}_\mu\Lambda_K^a &= \partial_\mu\Lambda_K^a - b_\mu\Lambda_K^a + \omega_\mu{}^{ab}\Lambda_{Kb}\,,\\
\mathcal{D}_\mu\epsilon^i &= \partial_\mu\epsilon^i + \tfrac{1}{2}b_\mu\epsilon^i + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon^i - V_\mu^{ij}\epsilon_j\,,\\
\mathcal{D}_\mu\eta^i &= \partial_\mu\eta^i - \tfrac{1}{2}b_\mu\eta^i + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\eta^i - V_\mu^{ij}\eta_j\,.
\end{aligned} \tag{5.26}$$

The corresponding curvatures can be calculated by using the general rule

$$R_{\mu\nu}{}^A = 2\partial_{[\mu}h_{\nu]} + h_\nu^C h_\mu^B f_{BC}{}^A\,. \tag{5.27}$$

The structure constants can again be read off from the (anti)commutator expressions (5.19) and we obtain the following curvatures (terms proportional to vielbeins are underlined for later use):

$$\begin{aligned}
R_{\mu\nu}{}^{a}(P) &= 2\partial_{[\mu}e_{\nu]}{}^{a} + \underline{2\omega_{[\mu}{}^{ab}e_{\nu]b}} + \underline{2b_{[\mu}e_{\nu]}{}^{a}} - \tfrac{1}{2}\bar{\psi}_{[\mu}\gamma^{a}\psi_{\nu]}\,,\\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} + \underline{8f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}} + \mathrm{i}\,\bar{\phi}_{[\mu}\gamma^{ab}\psi_{\nu]}\,,\\
R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} - \underline{4f_{[\mu}{}^{a}e_{\nu]a}} - \mathrm{i}\,\bar{\phi}_{[\mu}\psi_{\nu]}\,,\\
R_{\mu\nu}{}^{a}(K) &= 2\partial_{[\mu}f_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{ab}f_{\nu]b} - 2b_{[\mu}f_{\nu]}{}^{a} - \tfrac{1}{2}\bar{\phi}_{[\mu}\gamma^{a}\phi_{\nu]}\,,\\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} - 2V_{[\mu}{}^{k(i}V_{\nu]k}{}^{j)} - 3\,\mathrm{i}\,\bar{\phi}^{(i}_{[\mu}\psi^{j)}_{\nu]}\,,\\
R_{\mu\nu}{}^{i}(Q) &= 2\partial_{[\mu}\psi^{i}_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi^{i}_{\nu]} + b_{[\mu}\psi^{i}_{\nu]} - 2V_{[\mu}{}^{ij}\psi_{\nu]j} + \underline{2\,\mathrm{i}\,\gamma_{a}\phi^{i}_{[\mu}e_{\nu]}{}^{a}}\,,\\
R_{\mu\nu}{}^{i}(S) &= 2\partial_{[\mu}\phi^{i}_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\phi^{i}_{\nu]} - b_{[\mu}\phi^{i}_{\nu]} - 2V_{[\mu}{}^{ij}\phi_{\nu]j} - 2\,\mathrm{i}\,\gamma_{a}\psi^{i}_{[\mu}f_{\nu]}{}^{a}\,.
\end{aligned} \tag{5.28}$$

Since the transformation laws given above satisfy the $F^2(4)$ superalgebra, we have constructed a gauge theory of $F^2(4)$. However, this is not a gauge theory of diffeomorphisms of space-time yet; this can only be realized if we take the spin connection as a composite field that depends on the vielbein. So far we have it as an independent field.[2]

Furthermore, we see that the number of bosonic and fermionic degrees of freedom do not match. The gauge fields together have 96 + 64 degrees of freedom. Therefore, we can not have a supersymmetric theory with invertible general coordinate transformations generated by the square of supersymmetry operations.

5.2.3 Curvature constraints

The solution to the problems described above is well known. In order to convert the P-gauge transformations into general coordinate transformations and to obtain irreducibility we need to impose curvature constraints. This will define some gauge fields to be dependent fields.

We will consider the fünfbein as an invertible field. Then some of the curvatures in (5.28) are linear in some gauge fields. This is shown by the underlined terms in (5.28). Therefore, we can impose constraints on these curvatures that are solvable for these gauge fields. Such constraints are called conventional constraints, and imposing them reduces the Weyl multiplet, such that we get closer to an irreducible multiplet. The conventional constraints are

$$\begin{aligned}
R_{\mu\nu}{}^{a}(P) &= 0 \quad (50)\,,\\
e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(M) &= 0 \quad (25)\,,\\
\gamma^{\mu}R_{\mu\nu}{}^{i}(Q) &= 0 \quad (40)\,.
\end{aligned} \tag{5.29}$$

In brackets we denoted the number of restrictions each constraint imposes. These constraints are similar to those for other Weyl multiplets in four dimensions with $\mathcal{N} = 1$ [127,130], $\mathcal{N} = 2$ [145] or $\mathcal{N} = 4$ [146], or in six dimensions for the $(1,0)$ [147] or $(2,0)$ [148] Weyl multiplets.

[2] One might think that the field equations can determine the spin connection as a dependent gauge field. This can indeed be done for the spin connection, but it is not known how to generalize this for the gauge fields of special (super)conformal symmetries, which we also want to be dependent gauge fields.

Field	#	Gauge	SU(2)	w	Field	#	Gauge	SU(2)	w
		Elementary gauge fields					Dependent gauge fields		
${e_\mu}^a$	9	P^a	1	-1	$\omega_\mu^{[ab]}$	–	$M^{[ab]}$	1	0
b_μ	0	D	1	0	${f_\mu}^a$	–	K^a	1	1
$V_\mu^{(ij)}$	12	SU(2)	3	0					
ψ_μ^i	24	Q_α^i	2	$-\frac{1}{2}$	ϕ_μ^i	–	S_α^i	2	$\frac{1}{2}$
		Dilaton Weyl multiplet					Standard Weyl multiplet		
$B_{[\mu\nu]}$	6	$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$	1	0	$T_{[ab]}$	10		1	1
A_μ	4	$\delta A_\mu = \partial_\mu \Lambda$	1	0					
σ	1		1	1	D	1		1	2
ψ^i	8		2	$\frac{3}{2}$	χ^i	8		2	$\frac{3}{2}$

Table 5.2: Fields of the Weyl multiplets, and their roles. The upper half contains the fields that are present in all versions. They are the gauge fields of the superconformal algebra (see section 5.2). The fields at the right-hand side of the upper half are dependent fields. The symbol # indicates the off-shell degrees of freedom. The lower half contains the extra matter fields that appear in the two versions of the Weyl multiplet. In the left half we have those of the Dilaton Weyl multiplet, and at the right those of the Standard Weyl multiplet. We also indicated the (generalized) gauge-symmetries of the fields A_μ and $B_{\mu\nu}$.

5.2.4 Adding matter fields

After imposing the constraints we are left with 21 bosonic and 24 fermionic degrees of freedom. The independent fields are those in the upper left part of table 5.2. In order to get matching bosonic and fermionic degrees of freedom, we have to introduce extra matter fields in the multiplet, to obtain the full Weyl multiplet. There turns out to be two possibilities for a $D = 5$ Weyl multiplet, each with 32 + 32 degrees of freedom. The auxiliary fields $(A_\mu, B_{\mu\nu}, \sigma, \psi^i)$ lead to the Dilaton Weyl multiplet, whereas the set (T_{ab}, D, χ^i) leads to the Standard Weyl multiplet. The latter type is the Weyl multiplet one would expect when comparing to four and six dimensional theories with eight supercharges. Furthermore, since both Weyl multiplets are related by field redefinitions [131] we will restrict ourselves to the Standard Weyl multiplet from now on.

Modified constraints. The extra matter fields will change the transformations of the gauge fields. In fact, for the transformation of a general gauge field h_μ^I we will have (apart from the general coordinate transformations):

$$\delta_J(\epsilon^J)h_\mu^I = \partial_\mu \epsilon^I + \epsilon^J {h_\mu}^A {f_{AJ}}^I + \epsilon^J {M_{\mu J}}^I \,, \tag{5.30}$$

where we use the index I to denote all gauge transformations apart from general coordinate transformations, and an index A includes the translations.

The last term depends on the matter fields, and its explicit form has to be determined below. But also the second term has contributions from matter fields. This is due to the fact that the structure 'functions' of the final algebra $f_{IJ}{}^K$ are modified from those of the $F^2(4)$ algebra which was used for (5.25). These extra terms also lead to modified curvatures

$$\widehat{R}_{\mu\nu}{}^I = 2\partial_{[\mu}h_{\nu]}{}^I + h_\nu{}^B h_\mu{}^A f_{AB}{}^I - 2h_{[\mu}{}^J M_{\nu]J}{}^I \,. \tag{5.31}$$

The commutator of two supersymmetry-transformations will also change. In particular we will find transformations with field-dependent parameters. They can be conveniently written as so-called covariant general coordinate transformations, which are defined as

$$\delta_{cgct}(\xi) = \delta_{gct}(\xi) - \delta_I(\xi^\mu h_\mu{}^I) \,, \tag{5.32}$$

namely a combination of general coordinate transformations and all the other transformations whose parameter ϵ^I is replaced by $\xi^\mu h_\mu{}^I$.

Note that the curvature modifications also lead to modified curvature constraints:[3]

$$R_{\mu\nu}{}^a(P) = 0 \,, \qquad e^\nu{}_b \widehat{R}_{\mu\nu}{}^{ab}(M) = 0 \,, \qquad \gamma^\mu \widehat{R}_{\mu\nu}{}^i(Q) = 0 \,. \tag{5.33}$$

In general one can add extra terms to the constraints (5.33), which just amount to redefinitions of the composite fields. By choosing suitable terms simplifications were obtained in four and six dimensions. In this case one could e.g. add a term $T_{\mu b}T^{ba}$ to the second constraint rendering all the constraints invariant under S-supersymmetry, but in five dimensions this turns out to be impossible. Therefore we keep the constraints as written above.

Due to these constraints the fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ^i are no longer independent, but can be expressed in terms of the other fields. In order to write down the explicit solutions of these constraints, it is useful to extract the terms which have been underlined in (5.28). We define $\widehat{R}'$ as the curvatures without these terms. Formally,

$$\widehat{R}'_{\mu\nu}{}^I = \widehat{R}_{\mu\nu}{}^I + 2h^J_{[\mu} e_{\nu]}{}^a f_{aJ}{}^I \,, \tag{5.34}$$

where $f_{aJ}{}^I$ are the structure constants in the $F^2(4)$ algebra that define commutators of translations with other gauge transformations. Then the solutions to the constraints are

$$\begin{aligned}
\omega_\mu^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c + 2e_\mu{}^{[a}b^{b]} - \tfrac{1}{2}\bar{\psi}^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar{\psi}^b\gamma_\mu\psi^a \,, \\
\phi_\mu^i &= -\tfrac{1}{12}\,\mathrm{i}\,(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})\widehat{R}'_{ab}{}^i(Q) \,, \\
f_\mu^a &= \tfrac{1}{6}\mathcal{R}_\mu{}^a - \tfrac{1}{48}e_\mu{}^a\mathcal{R} \,, \qquad \mathcal{R}_{\mu\nu} \equiv \widehat{R}'_{\mu\rho}{}^{ab}(M)e_a{}^\rho e_{\nu b} \,, \qquad \mathcal{R} \equiv \mathcal{R}_\mu{}^\mu \,.
\end{aligned} \tag{5.35}$$

The constraints imply further relations between the curvatures through Bianchi identities.

[3]Note that the third constraint implies that $\gamma_{[\mu\nu}\widehat{R}_{\rho\sigma]}{}^i(Q) = 0$.

Modified transformation rules. To obtain all the extra transformations one imposes the superconformal algebra, but at the same time allowing modifications of the algebra by field-dependent quantities. The techniques are the same as already used in four and six dimensions in [145, 146], and were described in detail in [147].

For the fields in the upper left corner of table 5.2, we now have to specify the extra parts M in (5.30). This will in fact only apply to Q-supersymmetry. The other transformations are as in (5.25). The full supersymmetry transformations of these fields are

$$\begin{aligned}
\delta_Q e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta_Q\psi^i_\mu &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i\,,\\
\delta_Q V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi^{j)}_\mu + \mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)}_\mu + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)}\,,\\
\delta_Q b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu - 2\bar\epsilon\gamma_\mu\chi\,,
\end{aligned} \tag{5.36}$$

where $\mathcal{D}_\mu\epsilon$ is given in (5.27).

The modification M in (5.30) is the last term of the transformations of ψ^i_μ, V^{ij}_μ and b_μ. The second term in the transformation of V^{ij}_μ on the other hand is due to the fact that the structure constants have become structure functions, and in particular there appears a new T-dependent SU(2) transformation in the anti-commutator of two supersymmetries. We will give the full new algebra in section 5.3.

The transformation rules for the matter fields of the Standard Weyl multiplet are as follows (Q and S supersymmetry)

$$\begin{aligned}
\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon\widehat{R}_{ab}(Q)\,,\\
\delta\chi^i &= \tfrac{1}{4}\epsilon^i D - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac{1}{8}\,\mathrm{i}\,\gamma^{ab}\not{D}T_{ab}\epsilon^i - \tfrac{1}{8}\,\mathrm{i}\,\gamma^a D^b T_{ab}\epsilon^i\\
&\quad -\tfrac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \tfrac{1}{6}T^2\epsilon^i + \tfrac{1}{4}\gamma\cdot T\eta^i\,,\\
\delta D &= \bar\epsilon\not{D}\chi - \tfrac{5}{3}\,\mathrm{i}\,\bar\epsilon\gamma\cdot T\chi - \mathrm{i}\,\bar\eta\chi\,.
\end{aligned} \tag{5.37}$$

There are no explicit gauge fields here, as should be the case for 'matter', i.e. non-gauge fields. These are all hidden in the covariant derivatives and covariant curvatures. The covariant derivative for any matter field Φ is given by the rule

$$D_a\Phi = e^\mu_a\left(\partial_\mu - \delta_I(h^I_\mu)\right)\Phi\,. \tag{5.38}$$

The last term represents a sum over all transformations except general coordinate transformations, with parameters replaced by the corresponding gauge fields. In practice, the Lorentz transformations and SU(2) transformations follow directly from the index structure and lead to additions similar to those in (5.27). For the Weyl transformations there is a term $-w\,b_\mu\Phi$, where w is the Weyl weight of the field that can be found in table 5.2, and then there remain the terms for Q and S supersymmetry. There are no K transformations for any matter field in five dimensions.

The covariant curvatures in (5.37) are given by the general rule (5.31), e.g.

$$\begin{aligned}
\widehat{R}_{\mu\nu}{}^i(Q) &= R_{\mu\nu}{}^i(Q) + 2\,\mathrm{i}\,\gamma\cdot T\gamma_{[\mu}\psi^i_{\nu]}\,,\\
\widehat{R}_{\mu\nu}{}^{ij}(V) &= R_{\mu\nu}{}^{ij}(V) - 8\bar\psi^{(i}_{[\mu}\gamma_{\nu]}\chi^{j)} - \mathrm{i}\,\bar\psi^{(i}_{[\mu}\gamma\cdot T\psi^{j)}_{\nu]}\,,
\end{aligned} \tag{5.39}$$

where $R_{\mu\nu}{}^{i}(Q)$ and $R_{\mu\nu}{}^{ij}(V)$ are those given in (5.28). Given the transformation rules in (5.36) and (5.37), we could calculate the transformations of the dependent fields. Their transformation rules are now determined by their definition due to the constraints. An equivalent way of expressing this is that their transformation rules are modified w.r.t. (5.25) due to the non-invariance of the constraints under these transformations. We have chosen the constraints to be invariant under all bosonic symmetries without modifications. Therefore, only the Q- and S-supersymmetries of the dependent fields are modified to get invariant constraints.

This finishes our discussion of the Standard Weyl multiplet. The final results for this multiplet have been collected in section 5.3.

5.3 Results for the Weyl multiplet

In this section we collect the essential results of the previous sections, and give the supersymmetry algebra, which is modified by field-dependent terms. The transformation under dilatation is for each field $\delta_D\Phi = w\Lambda_D\Phi$, where the Weyl weight w can be found in table 5.2. The Lorentz, and SU(2) transformations are evident from the index structure, and our normalizations can be found in (5.25).

5.3.1 The Standard Weyl multiplet

The Q- and S-supersymmetry and K-transformation rules for the independent fields of the Standard Weyl multiplet are

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac12\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i - \mathrm{i}\,\gamma_\mu\eta^i\,,\\
\delta V_\mu{}^{ij} &= -\tfrac32\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)} + \mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi_\mu^{j)} + \tfrac32\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)}\,,\\
\delta T_{ab} &= \tfrac12\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon\widehat{R}_{ab}(Q)\,,\\
\delta\chi^i &= \tfrac14\epsilon^i D - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac18\,\mathrm{i}\,\gamma^{ab}\not{D}T_{ab}\epsilon^i - \tfrac18\,\mathrm{i}\,\gamma^a D^b T_{ab}\epsilon^i\\
&\quad -\tfrac14\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \tfrac16 T^2\epsilon^i + \tfrac14\gamma\cdot T\eta^i\,,\\
\delta D &= \bar\epsilon\not{D}\chi - \tfrac53\,\mathrm{i}\,\bar\epsilon\gamma\cdot T\chi - \mathrm{i}\,\bar\eta\chi\,,\\
\delta b_\mu &= \tfrac12\,\mathrm{i}\,\bar\epsilon\phi_\mu - 2\bar\epsilon\gamma_\mu\chi + \tfrac12\,\mathrm{i}\,\bar\eta\psi_\mu + 2\Lambda_{K\mu}\,.
\end{aligned}\tag{5.40}$$

The covariant derivative $\mathcal{D}_\mu\epsilon$ is given in (5.27). For other covariant derivatives, see the general rule (5.38), with more explanation below that equation. The covariant curvatures $\widehat{R}(Q)$ and $\widehat{R}(V)$ are given explicitly in (5.39). The expressions for the dependent fields are given in (5.35), where the prime indicates the omission of the underlined terms in (5.28).

5.3.2 Modified superconformal algebra

The original algebra given in (5.19) is no longer satisfied on the Weyl multiplet; the algebra closes up to matter field modifications. These modifications can be written as superconformal transformations, with field dependent parameters. The algebra realized on the Weyl multiplet therefore is also called a 'soft' algebra. This is also the algebra that all matter multiplets will have to satisfy, apart from possible additional transformations under which the fields of the Weyl multiplet do not transform, and possibly field equations if these matter multiplets are on-shell.

The full commutator of two supersymmetry transformations is given by

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{cgct}(\xi_3^\mu) + \delta_M(\lambda_3^{ab}) + \delta_S(\eta_3) + \delta_U(\lambda_3^{ij}) + \delta_K(\Lambda_{K3}^a)\,. \tag{5.41}$$

Note that the Dilaton Weyl multiplet also gives rise to field dependent gauge transformations, which have been omitted here. The covariant general coordinate transformations have been defined in (5.32). The parameters appearing in (5.41) are

$$\begin{aligned}
\xi_3^\mu &= \tfrac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\,,\\
\lambda_3^{ab} &= -\mathrm{i}\,\bar\epsilon_2\gamma^{[a}\gamma\cdot T\gamma^{b]}\epsilon_1\,,\\
\lambda_3^{ij} &= \mathrm{i}\,\bar\epsilon_2^{(i}\gamma\cdot T\epsilon_1^{j)}\,,\\
\eta_3^i &= -\tfrac{9}{4}\,\mathrm{i}\;\bar\epsilon_2\epsilon_1\chi^i + \tfrac{7}{4}\,\mathrm{i}\;\bar\epsilon_2\gamma_c\epsilon_1\gamma^c\chi^i\\
&\quad +\tfrac{1}{4}\,\mathrm{i}\;\bar\epsilon_2^{(i}\gamma_{cd}\epsilon_1^{j)}\left(\gamma^{cd}\chi_j + \tfrac{1}{4}\widehat{R}^{cd}{}_j(Q)\right),\\
\Lambda_{K3}^a &= -\tfrac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1 D + \tfrac{1}{96}\bar\epsilon_2^i\gamma^{abc}\epsilon_1^j\widehat{R}_{bcij}(V)\\
&\quad +\tfrac{1}{12}\,\mathrm{i}\,\bar\epsilon_2\left(-5\gamma^{abcd}D_bT_{cd} + 9D_bT^{ba}\right)\epsilon_1\\
&\quad +\bar\epsilon_2\left(\gamma^{abcde}T_{bc}T_{de} - 4\gamma^cT_{cd}T^{ad} + \tfrac{2}{3}\gamma^aT^2\right)\epsilon_1\,.
\end{aligned} \tag{5.42}$$

For the Q, S commutators we find the following algebra:

$$\begin{aligned}
[\delta_S(\eta), \delta_Q(\epsilon)] &= \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\eta) + \delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\eta) + \delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\eta^{j)}) + \delta_K(\Lambda_{3K}^a)\,,\\
[\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_K(\tfrac{1}{2}\bar\eta_2\gamma^a\eta_1)\,.
\end{aligned} \tag{5.43}$$

with

$$\Lambda_{3K}^a = \tfrac{1}{6}\bar\epsilon\left(\gamma\cdot T\gamma_a - \tfrac{1}{2}\gamma_a\gamma\cdot T\right)\eta\,. \tag{5.44}$$

This concludes our description of the Standard Weyl multiplet.

In this chapter we have taken the first step in the superconformal tensor calculus by constructing the Standard Weyl multiplet for $\mathcal{N} = 2$ conformal supergravity theory in five dimensions. We explained how the Weyl multiplet could be obtained by gauging the superconformal algebra $F^2(4)$. The results of this chapter will be our starting point for the construction of general supergravity/matter couplings in five dimensions.