Chapter 4

Scherk-Schwarz reductions and gauged supergravities

In this chapter we will make use of the techniques explained in chapter 3 to construct five different two-parameter massive deformations of the unique nine-dimensional $\mathcal{N} = 2$ supergravity. All of these deformations have a higher-dimensional origin via SS-reduction and correspond to gauged supergravities. Although the ultimate goal is to do a full analysis of the scalar potentials in lower-dimensional gauged supergravities, $D = 4$ and $D = 5$ in specific, in this chapter we will study dimensional reductions from $D = 11$ via $D = 10$ down to $D = 9$; nine-dimensional supergravity shares some of the complexities of the lower-dimensional cases, but is still simple enough to study in full detail. Based on these results we will conclude by making a systematic search for half-supersymmetric domain-walls and non-supersymmetric de Sitter space solutions. Furthermore, we discuss in which sense the supergravities we have constructed can be considered as low-energy limits of compactified superstring theory.

Appendix B.1 contains our conventions and in appendix B.3 we discuss some manipulations with spinors and gamma-matrices in ten and nine dimensions.

This chapter is based on the work published in [108]. In this chapter we will only treat the generalized reduction from $D = 11$ to $D = 10$ in some detail; a detailed description of the reductions to $D = 9$ is given in [108].

4.1 $D = 11$ supergravity

Our starting point is eleven-dimensional supergravity [24], which field content is given by[1]

$$D = 11: \qquad \{\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}, \hat{\hat{\psi}}_{\hat{\hat{\mu}}}\}\,. \tag{4.1}$$

[1] In order to distinguish between $D = 11, D = 10$ and $D = 9$ we indicate $D = 11$ fields and indices with a double hat, $D = 10$ fields and indices with a single hat and $D = 9$ fields and indices without hat.

$\mathbb{R}^+$	$\hat{\hat{e}}_{\hat{\mu}}{}^{\hat{\hat{a}}}$	$\hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}$	$\hat{\hat{\psi}}_{\hat{\mu}}$	$\hat{\hat{\epsilon}}$	$\hat{\hat{\mathcal{L}}}$
$\hat{\hat{\alpha}}$	1	3	$\frac{1}{2}$	$\frac{1}{2}$	9

Table 4.1: The $\mathbb{R}^+$-weights of the $D = 11$ supergravity fields, the supersymmetry parameters $\hat{\hat{\epsilon}}$ and the Lagrangian $\hat{\hat{\mathcal{L}}}$.

The Einstein-frame action and the corresponding supersymmetry transformations, up to quartics, are given by

$$\mathcal{L} = \frac{\hat{\hat{e}}}{2\kappa_{11}^2}\Big[\hat{\hat{R}}(\hat{\omega}) - \bar{\hat{\hat{\psi}}}_{\hat{\mu}}\hat{\hat{\Gamma}}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\hat{\mathcal{D}}}_{\hat{\nu}}(\hat{\omega})\hat{\hat{\psi}}_{\hat{\rho}} - \tfrac{1}{92}\hat{\hat{G}}_{(4)}\hat{\hat{G}}^{(4)} - \tfrac{1}{92}\Big(\bar{\hat{\hat{\psi}}}_{\hat{\mu}}\hat{\hat{\Gamma}}^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\hat{\hat{\psi}}_{\hat{\nu}} + 12\bar{\hat{\hat{\psi}}}^{\hat{\alpha}}\hat{\hat{\Gamma}}^{\hat{\gamma}\hat{\delta}}\hat{\hat{\psi}}^{\hat{\beta}}\Big)\hat{\hat{G}}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$$
$$+\tfrac{1}{(144)^2}\hat{\hat{\varepsilon}}^{(4)(4')(3)}\hat{\hat{G}}_{(4)}\hat{\hat{G}}_{(4')}\hat{\hat{C}}_{(3)}\Big], \tag{4.2}$$

$$\begin{aligned}
\delta\hat{\hat{e}}_{\hat{\mu}}{}^{\hat{\hat{a}}} &= \bar{\hat{\hat{\epsilon}}}\hat{\hat{\Gamma}}^{a}\hat{\hat{\psi}}_{\hat{\mu}}\,,\\
\delta\hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= -3\,\bar{\hat{\hat{\epsilon}}}\hat{\hat{\Gamma}}_{[\hat{\mu}\hat{\nu}}\hat{\hat{\psi}}_{\hat{\rho}]}\,,\\
\delta\hat{\hat{\psi}}_{\hat{\mu}} &= \hat{\hat{\mathcal{D}}}_{\hat{\mu}}(\hat{\omega})\hat{\hat{\epsilon}} + \tfrac{1}{192}(\hat{\hat{\Gamma}}^{(4)}\hat{\hat{\Gamma}}_{\hat{\mu}} - \tfrac{1}{3}\hat{\hat{\Gamma}}_{\hat{\mu}}\hat{\hat{\Gamma}}^{(4)})\hat{\hat{G}}_{(4)}\hat{\hat{\epsilon}}\,,
\end{aligned} \tag{4.3}$$

with the field strength $\hat{\hat{G}}_{(4)} = \mathrm{d}\hat{\hat{C}}_{(3)}$ and $\hat{\hat{\mathcal{D}}}_{\hat{\mu}} = \partial_{\hat{\mu}} + \frac{1}{4}\hat{\hat{\omega}}_{\hat{\mu}}{}^{\hat{\hat{a}}\hat{\hat{b}}}\hat{\hat{\Gamma}}_{\hat{\hat{a}}\hat{\hat{b}}}$. The 11D fermionic field content consists solely of a 32-component gravitino, whose field equation reads

$$X_0(\hat{\hat{\psi}}^{\hat{\mu}}) \equiv \hat{\hat{\Gamma}}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\hat{\mathcal{D}}}_{\hat{\nu}}\hat{\hat{\psi}}_{\hat{\rho}} = 0\,, \tag{4.4}$$

where we have set the three-form equal to zero for simplicity.[2] Under supersymmetry this fermionic field equation transforms into

$$\delta_0 X_0(\hat{\hat{\psi}}^{\hat{\mu}}) = \tfrac{1}{2}\hat{\hat{\Gamma}}^{\hat{\nu}}\hat{\hat{\epsilon}}\,[\hat{\hat{R}}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{\hat{R}}\,\hat{\hat{g}}^{\hat{\mu}}{}_{\hat{\nu}}]\,, \tag{4.5}$$

which implies the bosonic Einstein equation for the metric. The supersymmetry rules and field equations are covariant under an $\mathbb{R}^+$ symmetry with parameter $\hat{\hat{\alpha}}$ [109]. A generic field $\hat{\hat{\Phi}}$ with weight w scales as $\hat{\hat{\Phi}} \to e^{w\hat{\hat{\alpha}}}\hat{\hat{\Phi}}$ under this symmetry. The weights of the $D = 11$ fields under this $\mathbb{R}^+$ are given in table 4.1. Note that the Lagrangian is not invariant but scales with weight $w = 9$. Therefore this $\mathbb{R}^+$ is a symmetry of the equations of motion only.

No massive deformation of the eleven-dimensional supergravity theory is known; in particular, no cosmological constant can be added [110]. One problem with a $D = 11$ supersymmetric cosmological constant is that its reduction gives rise to a $D = 10$ cosmological constant with a dilaton coupling that differs from Romans' massive deformation. A general deformation of $D = 11$ supergravity involving the use of extra Killing vectors has been considered in [111], but we will not pursue this possibility here.

[2]This is because we are only interested in solutions coupling to the metric and the dilaton.

4.2 Massive deformations of $D = 10$ IIA supergravity

As already mentioned in section 1.4.2, the Kaluza-Klein reduction of eleven-dimensional supergravity yields the effective IIA theory in ten dimensions. This gives us the following field content of the $D = 10$ IIA supergravity theory

$$D = 10 \ \ \text{IIA:} \qquad \{\hat{e}_{\hat{\mu}}{}^{\hat{a}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}, \hat{A}_{\hat{\mu}}, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{\psi}_{\hat{\mu}}, \hat{\lambda}\}\,. \tag{4.6}$$

For this reduction we use the reduction Ansätze[3]

$$\begin{aligned}
&\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}} = \begin{pmatrix} e^{-\hat{\phi}/12}\hat{e}_{\hat{\mu}}{}^{\hat{a}} & -e^{2\hat{\phi}/3}\hat{A}_{\hat{\mu}} \\ 0 & e^{2\hat{\phi}/3} \end{pmatrix}, \qquad \hat{\hat{\psi}}_{\hat{a}} = e^{\hat{\phi}/24}(\hat{\psi}_{\hat{a}} - \tfrac{1}{24}\hat{\Gamma}_{\hat{a}}\hat{\lambda})\,, \qquad \hat{\hat{\epsilon}} = e^{-\hat{\phi}/24}\hat{\epsilon}\,, \\
&\hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \qquad \hat{\hat{C}}_{\hat{\mu}\hat{\nu}x} = -\hat{B}_{\hat{\mu}\hat{\nu}}\,, \qquad \hat{\hat{\psi}}_{\underline{x}} = \tfrac{1}{3}e^{\hat{\phi}/24}\hat{\Gamma}_{\underline{x}}\hat{\lambda}\,.
\end{aligned} \tag{4.7}$$

Applying these to the 11D transformation rules (4.3), we obtain the IIA transformation rules in the Einstein frame and up to quartics[4]:

$$\begin{aligned}
\delta_0\hat{e}_{\hat{\mu}}{}^{\hat{a}} &= \bar{\hat{\epsilon}}\hat{\Gamma}^{\hat{a}}\hat{\psi}_{\hat{\mu}}\,, \\
\delta_0\hat{\psi}_{\hat{\mu}} &= \Big(\hat{\mathcal{D}}_{\hat{\mu}} + \tfrac{1}{48}e^{-\hat{\phi}/2}(\hat{\not{H}}\hat{\Gamma}_{\hat{\mu}} + \tfrac{1}{2}\hat{\Gamma}_{\hat{\mu}}\hat{\not{H}})\Gamma_{11} \\
&\qquad + \tfrac{1}{16}e^{3\hat{\phi}/4}(\hat{\not{F}}\hat{\Gamma}_{\hat{\mu}} - \tfrac{3}{4}\hat{\Gamma}_{\hat{\mu}}\hat{\not{F}})\Gamma_{11} + \tfrac{1}{192}e^{\hat{\phi}/4}(\hat{\not{G}}\hat{\Gamma}_{\hat{\mu}} - \tfrac{1}{4}\hat{\Gamma}_{\hat{\mu}}\hat{\not{G}})\Big)\hat{\epsilon}\,, \\
\delta_0\hat{B}_{\hat{\mu}\hat{\nu}} &= 2e^{\hat{\phi}/2}\bar{\hat{\epsilon}}\Gamma_{11}\hat{\Gamma}_{[\hat{\mu}}(\hat{\psi}_{\hat{\nu}]} + \tfrac{1}{8}\hat{\Gamma}_{\hat{\nu}]}\hat{\lambda})\,, \\
\delta_0\hat{A}_{\hat{\mu}} &= -e^{-3\hat{\phi}/4}\bar{\hat{\epsilon}}\Gamma_{11}(\hat{\psi}_{\hat{\mu}} - \tfrac{3}{8}\hat{\Gamma}_{\hat{\mu}}\hat{\lambda})\,, \\
\delta_0\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= -3e^{-\hat{\phi}/4}\bar{\hat{\epsilon}}\hat{\Gamma}_{[\hat{\mu}\hat{\nu}}(\hat{\psi}_{\hat{\rho}]} - \tfrac{1}{24}\hat{\Gamma}_{\hat{\rho}]}\hat{\lambda}) + 3\hat{A}_{[\hat{\mu}}\delta_0\hat{B}_{\hat{\nu}\hat{\rho}]}\,, \\
\delta_0\hat{\lambda} &= \Big(\hat{\not{\partial}}\hat{\phi} + \tfrac{1}{12}e^{-\hat{\phi}/2}\hat{\not{H}}\Gamma_{11} + \tfrac{3}{8}e^{3\hat{\phi}/4}\hat{\not{F}}\Gamma_{11} + \tfrac{1}{96}e^{\hat{\phi}/4}\hat{\not{G}}\Big)\,\hat{\epsilon}\,, \\
\delta_0\hat{\phi} &= \tfrac{1}{2}\bar{\hat{\epsilon}}\hat{\lambda}\,,
\end{aligned} \tag{4.8}$$

with the field strengths:

$$\hat{F} = \mathrm{d}\hat{A}\,, \qquad \hat{H} = \mathrm{d}\hat{B}\,, \qquad \hat{G} = \mathrm{d}\hat{C} + \hat{A}\wedge\hat{H}\,, \tag{4.9}$$

and $\hat{\mathcal{D}}_{\hat{\mu}} = \partial_{\hat{\mu}} + \frac{1}{4}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\Gamma}_{\hat{a}\hat{b}}$. For later purposes we indicate these (undeformed) supersymmetry transformations by δ_0. Upon (massless) reduction with the Ansätze (4.7) the 11D field equation (4.4) splits up into two field equations for the 10D IIA fermionic field content, a gravitino and a dilatino:

$$\begin{aligned}
X_0(\hat{\psi}^{\hat{\mu}}) &\equiv \hat{\Gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{\mathcal{D}}_{\hat{\nu}}\hat{\psi}_{\hat{\rho}} - \tfrac{1}{8}(\hat{\not{\partial}}\hat{\phi})\hat{\Gamma}^{\hat{\mu}}\hat{\lambda} = 0\,, \\
X_0(\hat{\lambda}) &\equiv \hat{\Gamma}^{\hat{\nu}}\hat{\mathcal{D}}_{\hat{\nu}}\hat{\lambda} - \hat{\Gamma}^{\hat{\nu}}(\hat{\not{\partial}}\hat{\phi})\hat{\psi}_{\hat{\nu}} = 0\,,
\end{aligned} \tag{4.10}$$

[3]The flat x^{11}-direction is denoted by $\underline{x}$, and the curved x^{11}-direction by x. The particular dilaton prefactors were conveniently chosen to get the standard form of the IIA transformation rules.

[4]An additional field dependent 10D Lorentz transformation is needed to get the correct transformation rule for e.g. the vielbein: $\delta_Q(\epsilon) = \delta_Q(\hat{\epsilon}) + \delta_M$.

$\mathbb{R}^+$	$\hat{e}_{\hat{\mu}}{}^{\hat{a}}$	$\hat{B}_{\hat{\mu}\hat{\nu}}$	$e^{\hat{\phi}}$	$\hat{A}_{\hat{\mu}}$	$\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$	$\hat{\psi}_{\hat{\mu}}$	$\hat{\lambda}$	$\hat{\epsilon}$	$\hat{\mathcal{L}}$	Origin
$\hat{\alpha}$	$\frac{9}{8}$	3	$\frac{3}{2}$	0	3	$\frac{9}{16}$	$-\frac{9}{16}$	$\frac{9}{16}$	9	$\hat{\hat{\alpha}}$
$\hat{\beta}$	0	$\frac{1}{2}$	1	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0	0	0	

Table 4.2: The $\mathbb{R}^+$-weights of the $D = 10$ IIA supergravity fields, the supersymmetry parameter $\hat{\epsilon}$ and the Lagrangian $\hat{\mathcal{L}}$.

where we have set the vector, two- and three-form equal to zero. Under supersymmetry these fermionic field equations transform into

$$\begin{aligned}\delta_0 X_0(\hat{\psi}^{\hat{\mu}}) &= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}]\,,\\ \delta_0 X_0(\hat{\lambda}) &= \hat{\epsilon}\,[\Box\hat{\phi}]\,,\end{aligned} \tag{4.11}$$

which imply the usual graviton-dilaton field equations. The corresponding action can be deduced from (4.2), using the same reduction Ansätze (4.7), and is e.g. given in [112] (in the string frame). The transformation rules have two $\mathbb{R}^+$-symmetries, one with parameter $\hat{\alpha}$ that scales the Lagrangian and one with parameter $\hat{\beta}$ that leaves the Lagrangian invariant. The first symmetry follows via dimensional reduction from the $D = 11$ $\mathbb{R}^+$-symmetry with parameter $\hat{\hat{\alpha}}$. The weights of these two $\mathbb{R}^+$-symmetries are given in table 4.2. The gauge symmetry associated to the Ramond-Ramond vector, with parameter $\hat{\lambda}$, reads

$$\hat{A} \to \hat{A} - \mathrm{d}\hat{\lambda}\,, \qquad \hat{C} \to \hat{C} - \mathrm{d}\hat{\lambda}\wedge\hat{B}\,. \tag{4.12}$$

The $D = 10$ IIA supergravity theory allows two massive deformations which we discuss one by one below.

4.2.1 Deformation m_{R}: $D = 10$ massive supergravity

The first massive deformation, with mass parameter m_{R}, is due to Romans [42]. In this case (the same is true for all other cases) the supersymmetry transformations receive two types of massive deformations: explicit and implicit ones. The explicit deformations are terms, at most linear in m_{R}, that are added to the original supersymmetry rules. These explicit deformations are denoted by $\delta_{m_{\mathrm{R}}}$ and define the fermion-shifts, used for determining the scalar potential. They are given in terms of a superpotential $W(\hat{\phi})$ and derivatives thereof by

$$m_{\mathrm{R}}:\left\{\begin{array}{lcl}\delta_{m_{\mathrm{R}}}\hat{\psi}_{\hat{\mu}} &=& -\tfrac{1}{8}W\hat{\Gamma}_{\hat{\mu}}\hat{\epsilon}\,,\\ \delta_{m_{\mathrm{R}}}\hat{\lambda} &=& 4\tfrac{\delta W}{\delta\hat{\phi}}\hat{\epsilon}\,,\end{array}\right. \qquad \text{with } W = \tfrac{1}{4}e^{5\hat{\phi}/4}m_{\mathrm{R}}\,. \tag{4.13}$$

There are further implicit massive deformations to the original supersymmetry rules δ_0, which are given in (4.8), due to the fact that in these rules one must replace all field strengths by corresponding *massive* field strengths which are given by

$$\hat{F} = \mathrm{d}\hat{A} + m_{\mathrm{R}}\hat{B}\,, \qquad \hat{H} = \mathrm{d}\hat{B}\,, \qquad \hat{G} = \mathrm{d}\hat{C} + \hat{A}\wedge\hat{H} + \tfrac{1}{2}m_{\mathrm{R}}\hat{B}\wedge\hat{B}\,. \tag{4.14}$$

The Lagrangian contains terms linear and quadratic in $m_{\rm R}$. Again there are implicit deformations, via the massive field strengths, and explicit deformations. The explicit deformations quadratic in the mass parameter define the scalar potential which can be written in terms of the superpotential $W(\hat\phi)$ and derivatives thereof by using (2.15) and (2.16).

The linear deformations of the fermionic (gravitino and dilatino) field equations of Romans' theory can be found by requiring closure of the supersymmetry algebra:

$$m_{\rm R}: \left\{ \begin{array}{lcl} X_{m_{\rm R}}(\hat\psi^{\hat\mu}) & \equiv & m_{\rm R} e^{5\hat\phi/4}\hat\Gamma^{\hat\mu\hat\nu}(\frac{1}{4}\hat\psi_{\hat\nu} + \frac{5}{288}\hat\Gamma_{\hat\nu}\hat\lambda)\,, \\ X_{m_{\rm R}}(\hat\lambda) & \equiv & m_{\rm R} e^{5\hat\phi/4}\hat\Gamma^{\hat\nu}(-\frac{5}{4}\hat\psi_{\hat\nu} - \frac{21}{160}\hat\Gamma_{\hat\nu}\hat\lambda)\,. \end{array} \right. \tag{4.15}$$

The undeformed equations, $X_0(\hat\psi^{\hat\mu})$ and $X_0(\hat\lambda)$, are given in eqs. (4.10). Under supersymmetry these fermionic field equations, $X_0 + X_{m_{\rm R}}$, transform into the deformed bosonic equations of motion. Since we will only be interested in finding solutions that are carried by the metric and the scalars it is convenient to truncate away all bosonic fields except the metric and the dilaton.[5] After this truncation we find that under supersymmetry the fermionic field equations transform into

$$\begin{aligned} &(\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\hat\psi^{\hat\mu}) = \tfrac{1}{2}\hat\Gamma^{\hat\nu}\hat\epsilon\,[\hat R^{\hat\mu}{}_{\hat\nu} - \tfrac{1}{2}\hat R\hat g^{\hat\mu}{}_{\hat\nu} - \tfrac{1}{2}(\partial^{\hat\mu}\hat\phi)(\partial_{\hat\nu}\hat\phi) + \tfrac{1}{4}(\partial\hat\phi)^2\hat g^{\hat\mu}{}_{\hat\nu} + \tfrac{1}{4}m_{\rm R}^2 e^{5\hat\phi/2}\hat g^{\hat\mu}{}_{\hat\nu}]\,, \\ &(\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\hat\lambda) = \hat\epsilon\,[\Box\hat\phi - \tfrac{5}{4}m_{\rm R}^2 e^{5\hat\phi/2}]\,. \end{aligned} \tag{4.16}$$

At the right-hand side we find the Romans' bosonic field equations for the metric and the dilaton, one solution of which is the D8-brane. Note that the bosonic field equations contain terms quadratic in the mass parameter.

Romans' theory is not known to have a higher-dimensional supergravity origin; neither is it a gauged supergravity. A candidate symmetry of the Lagrangian to be gauged is the $\hat\beta$ symmetry of table 4.2. However, the candidate gauge field $\hat A_{\hat\mu}$ has a nontrivial weight under $\hat\beta$. This means that the curl $\mathrm{d}\hat A$ transforms with a non-covariant term proportional to $\hat A \wedge \mathrm{d}\hat\lambda$. Such a term cannot be cancelled by adding an extra term, such as $\hat B$, to the definition of the $\hat A$ curvature. In short, the $\hat\beta$-symmetry cannot be gauged [113]. The same table shows that on the other hand $\hat A_{\hat\mu}$ has weight zero under the $\hat\alpha$-symmetry which is a symmetry of the equations of motion only. This $\hat\alpha$-symmetry can indeed be gauged at the level of the equations of motion. This gauging leads to the $D = 10$ gauged supergravity discussed below.

4.2.2 Deformation m_{11}: $D = 10$ gauged supergravity

The second massive deformation, with mass parameter m_{11}, has been considered in [114, 115] and is a gauged supergravity. It can be obtained by generalized Scherk-Schwarz reduction of $D = 11$ supergravity using the $\mathbb{R}^+$ symmetry $\hat{\hat\alpha}$ of table 4.1 [115]. The corresponding reduction Ansätze can be obtained by adding the appropriate factors $e^{w m_{11} x}$ to the Ansätze in (4.7), using the corresponding weights in table 4.1. This reduction leads to the following explicit massive deformations of the $D = 10$ IIA supersymmetry rules

$$m_{11}: \left\{ \begin{array}{lcl} \delta_{m_{11}}\hat\psi_{\hat\mu} & = & \frac{9}{16}m_{11}e^{-3\hat\phi/4}\hat\Gamma_{\hat\mu}\Gamma_{11}\hat\epsilon\,, \\ \delta_{m_{11}}\hat\lambda & = & \frac{3}{2}m_{11}e^{-3\hat\phi/4}\Gamma_{11}\hat\epsilon\,. \end{array} \right. \tag{4.17}$$

[5] Note that a further truncation to $\phi = c$ is inconsistent.

The implicit massive deformations of the original supersymmetry rules δ_0 are given by the massive bosonic field strengths

$$\mathrm{D}\hat{\phi} = \mathrm{d}\hat{\phi} + \tfrac{3}{2}m_{11}\hat{A}\,, \quad \hat{F} = \mathrm{d}\hat{A}\,, \quad \hat{H} = \mathrm{d}\hat{B} + 3m_{11}\hat{C}\,, \quad \hat{G} = \mathrm{d}\hat{C} + \hat{A}\wedge\hat{H}\,, \tag{4.18}$$

while the covariant derivative of the supersymmetry parameter $\hat{\mathcal{D}}_{\hat{\mu}}\hat{\epsilon}$ is replaced by

$$\hat{D}_{\hat{\mu}}\hat{\epsilon} = (\partial_{\hat{\mu}} + \tfrac{1}{4}\hat{\omega}_{\hat{\mu}}{}^{\hat{a}\hat{b}}\hat{\Gamma}_{\hat{a}\hat{b}} + \tfrac{9}{16}m_{11}\hat{\Gamma}_{\hat{\mu}}\hat{\not{A}})\hat{\epsilon}\,. \tag{4.19}$$

The gauge vector in the definition of the covariant derivative is required to make the derivative of the supersymmetry parameter *and* the spin connection $\mathbb{R}^+$-covariant.

The linear deformations of the fermionic field equations read in this case

$$m_{11}: \begin{cases} X_{m_{11}}(\hat{\psi}^{\hat{\mu}}) & \equiv \quad m_{11}e^{-3\hat{\phi}/4}\Gamma_{11}\hat{\Gamma}^{\hat{\mu}\hat{\nu}}(-\tfrac{9}{2}\hat{\psi}_{\hat{\nu}} + \tfrac{17}{48}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,, \\ X_{m_{11}}(\hat{\lambda}) & \equiv \quad m_{11}e^{-3\hat{\phi}/4}\Gamma_{11}\hat{\Gamma}^{\hat{\nu}}(\tfrac{3}{2}\hat{\psi}_{\hat{\nu}} - \tfrac{9}{16}\hat{\Gamma}_{\hat{\nu}}\hat{\lambda})\,. \end{cases} \tag{4.20}$$

We first consider the truncation where all bosonic fields except the metric and the dilaton are set equal to zero. Under supersymmetry the fermionic field equations transform into

$$\begin{aligned}(\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\hat{\psi}^{\hat{\mu}}) &= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} \\ &\quad + 36m_{11}^2e^{-3\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}] + \Gamma_{11}\hat{\epsilon}[3m_{11}e^{-3\hat{\phi}/4}\partial^{\hat{\mu}}\hat{\phi}]\,, \\ (\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\hat{\lambda}) &= \hat{\epsilon}\,[\Box\hat{\phi}] + \hat{\Gamma}^{\hat{\nu}}\Gamma_{11}\hat{\epsilon}[9m_{11}e^{-3\hat{\phi}/4}\partial_{\hat{\nu}}\hat{\phi}]\,.\end{aligned} \tag{4.21}$$

The terms involving Γ_{11} are part of the vector field equation. Therefore, to obtain a consistent truncation, we must further truncate the dilaton to zero. One is then left with only the metric satisfying the Einstein equation with a positive cosmological constant, a solution of which is 10D de Sitter space [115].

The reduced theory is a gauged supergravity where the $\mathbb{R}^+$ symmetry $\hat{\alpha}$ of table 4.2 has been gauged. In particular, the gauge parameter and transformation of the Ramond-Ramond potentials read as follows[6]

$$\hat{\alpha}: \qquad \Lambda = e^{w_{\hat{\alpha}}m_{11}\hat{\lambda}} \quad \text{with} \quad \hat{A}\to\hat{A} - \mathrm{d}\hat{\lambda}\,, \quad \hat{C}\to e^{3m_{11}\hat{\lambda}}(\hat{C} - \mathrm{d}\hat{\lambda}\wedge\hat{B})\,, \tag{4.22}$$

where $w_{\hat{\alpha}}$ are the weights under $\hat{\alpha}$. We note that one can take two different limits of the $\hat{\alpha}$ gauge transformations. First, the limit $m_{11}\to 0$ leads to the massless gauge transformations (4.12). Note that $\hat{C}$ transforms trivially under this gauge symmetry in the sense that $\hat{C}$ can be made gauge-invariant after a simple field-redefinition. Secondly, one can take the limit that $\hat{\alpha}$ is constant. This leads to the ungauged $\mathbb{R}^+$ $\hat{\alpha}$-symmetry of table 4.2.

A noteworthy feature of the $D = 10$ gauged supergravity is that no Lagrangian can be defined for it. In the search for supersymmetric domain-wall solutions in five dimensions other examples of gauged supergravity theories without a Lagrangian have been found [86]; we will encounter these in chapter 6. Note that one can write down a Lagrangian for the ungauged theory. The reason that one cannot write down a Lagrangian after gauging is that the symmetry that is gauged is not a symmetry of the Lagrangian but only of the equations of motion. It would be instructive to construct the $D = 10$ gauged supergravity from the ungauged theory by gauging the $\hat{\alpha}$-symmetry. Apparently, it shows that one can gauge symmetries that leave a Lagrangian invariant up to a scale factor.

[6] It is understood that each field with $w_{\hat{\alpha}} \neq 0$ is multiplied by Λ.

4.3 $D = 10$ IIB supergravity

The other ten-dimensional supergravity theory is chiral IIB, which field content is given by

$$D = 10 \ \ \text{IIB}: \qquad \{\hat{e}_{\hat{\mu}}{}^{\hat{a}}, \hat{\phi}, \hat{\chi}, \hat{B}^{(1)}_{\hat{\mu}\hat{\nu}}, \hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}, \hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \hat{\psi}_{\hat{\mu}}, \hat{\lambda}\}\,. \tag{4.23}$$

The supersymmetry transformation rules of ten-dimensional IIB supergravity read (in complex notation)

$$\begin{aligned}
\delta\hat{e}_{\hat{\mu}}{}^{\hat{a}} &= \tfrac{1}{2}\bar{\hat{\epsilon}}\hat{\Gamma}^{\hat{a}}\hat{\psi}_{\hat{\mu}} + \text{h.c.}\,,\\
\delta\hat{\psi}_{\hat{\mu}} &= \hat{D}_{\hat{\mu}}\hat{\epsilon} - \tfrac{1}{16\cdot 5!}\,\mathrm{i}\,\hat{\not{G}}^{(5)}\hat{\Gamma}_{\hat{\mu}}\hat{\epsilon}\\
&\quad + \tfrac{1}{16\cdot 3!}\,\mathrm{i}\,e^{\hat{\phi}/2}\left(\hat{\Gamma}_{\hat{\mu}}\hat{\Gamma}^{(3)} + 2\hat{\Gamma}^{(3)}\hat{\Gamma}_{\hat{\mu}}\right)\left(\hat{H}^{(1)} - \hat{\tau}\hat{H}^{(2)}\right)_{(3)}\hat{\epsilon}^{*}\,,\\
\delta\hat{\lambda} &= -e^{\hat{\phi}}\hat{\not{\partial}}\hat{\tau}\hat{\epsilon}^{*} - \tfrac{1}{2\cdot 3!}e^{\hat{\phi}/2}\hat{\Gamma}^{(3)}\left(\hat{H}^{(1)} - \hat{\tau}\hat{H}^{(2)}\right)_{(3)}\hat{\epsilon}\,,\\
\delta\hat{B}^{(1)}_{\hat{\mu}\hat{\nu}} &= -e^{\hat{\phi}/2}\hat{\tau}^{*}\left(\bar{\hat{\epsilon}}^{*}\hat{\Gamma}_{[\hat{\mu}}\hat{\psi}_{\hat{\nu}]} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\hat{\epsilon}}\hat{\Gamma}_{\hat{\mu}\hat{\nu}}\hat{\lambda}\right) + \text{h.c.}\,,\\
\delta\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}} &= -e^{\hat{\phi}/2}\left(\bar{\hat{\epsilon}}^{*}\hat{\Gamma}_{[\hat{\mu}}\hat{\psi}_{\hat{\nu}]} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\hat{\epsilon}}\hat{\Gamma}_{\hat{\mu}\hat{\nu}}\hat{\lambda}\right) + \text{h.c.}\,,\\
\delta\hat{D}_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}} &= 2\,\mathrm{i}\,\bar{\hat{\epsilon}}\hat{\Gamma}_{[\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\psi}_{\hat{\rho}]} - \tfrac{3}{2}\,\varepsilon_{ij}\hat{B}^{(i)}_{[\hat{\mu}\hat{\nu}}\delta\hat{B}^{(j)}_{\hat{\lambda}\hat{\rho}]} + \text{h.c.}\,,\\
\delta\hat{\chi} &= -\tfrac{1}{4}e^{-\hat{\phi}}\bar{\hat{\epsilon}}\hat{\lambda}^{*} + \text{h.c.}\,,\\
\delta\hat{\phi} &= \tfrac{1}{4}\,\mathrm{i}\,\bar{\hat{\epsilon}}\hat{\lambda}^{*} + \text{h.c.}\,,
\end{aligned} \tag{4.24}$$

with the complex scalar $\hat{\tau} = \hat{\chi} + ie^{-\hat{\phi}}$ and the field strengths

$$\vec{\hat{H}} = \mathrm{d}\vec{\hat{B}}\,, \qquad \hat{G} = \mathrm{d}\hat{D} + \tfrac{1}{2}\vec{\hat{B}}^{T}\eta\vec{\hat{H}}\,, \qquad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.25}$$

The field strength $\hat{G}$ is subject to a self-duality constraint:

$$\hat{G}_{\hat{\mu}_1\cdots\hat{\mu}_5} = -\tfrac{1}{5!}\hat{\varepsilon}_{\hat{\mu}_1\cdots\hat{\mu}_{10}}\hat{G}^{\hat{\mu}_{10}\cdots\hat{\mu}_6}\,, \tag{4.26}$$

which can be used to eliminate the four form potential $C_{(4)}$, after a dimensional reduction to $D = 9$.

The covariant derivative of the IIB Killing spinor reads

$$\hat{D}_{\hat{\mu}}\hat{\epsilon} = (\partial_{\hat{\mu}} + \tfrac{1}{4}\hat{\omega}_{\hat{\mu}}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} + \tfrac{1}{4}\,\mathrm{i}\,e^{\hat{\phi}}\partial_{\hat{\mu}}\hat{\chi})\hat{\epsilon}\,. \tag{4.27}$$

The corresponding action can be found in [112]. The IIB supersymmetry rules transform covariant under the $\mathrm{S}\ell(2,\mathbb{R})$ transformations (omitting indices)

$$\begin{aligned}
&\hat{\tau} \to \frac{a\hat{\tau} + b}{c\hat{\tau} + d}\,, \qquad \vec{\hat{B}} \to \Omega\vec{\hat{B}}\,, \qquad \hat{D} \to \hat{D}\,, \qquad \text{with } \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{S}\ell(2,\mathbb{R})\,,\\
&\hat{\psi}_{\hat{\mu}} \to \left(\frac{c\,\hat{\tau}^{*} + d}{c\,\hat{\tau} + d}\right)^{1/4}\hat{\psi}_{\hat{\mu}}\,, \qquad \hat{\lambda} \to \left(\frac{c\,\hat{\tau}^{*} + d}{c\,\hat{\tau} + d}\right)^{3/4}\hat{\lambda}\,, \qquad \hat{\epsilon} \to \left(\frac{c\,\hat{\tau}^{*} + d}{c\,\hat{\tau} + d}\right)^{1/4}\hat{\epsilon}\,.
\end{aligned} \tag{4.28}$$

Here we have used the vector notation $\vec{\hat{B}} = (\hat{B}^{(1)}, \hat{B}^{(2)})^T$. The group $S\ell(2,\mathbb{R})$ contains a set of three one-parameter conjugacy classes defining one compact and two non-compact subgroups. We will describe them shortly. Each of the subgroups is generated by a $S\ell(2,\mathbb{R})$ group element Ω with $\det\Omega = 1$. As a global symmetry group of IIB supergravity, $S\ell(2,\mathbb{R})$ is suitable for performing a Scherk-Schwarz reduction to $D = 9$. There are three different cases to consider, corresponding to the three different subgroups listed below.

1. One non-compact subgroup $\mathbb{R}$ is generated by

$$\Omega_p = e^{\frac{1}{2}\hat{\zeta}(\sigma_1 + \mathrm{i}\sigma_2)} = \begin{pmatrix} 1 & \hat{\zeta} \\ 0 & 1 \end{pmatrix}. \tag{4.29}$$

   Each element defines a parabolic conjugacy class with $\mathrm{Tr}\,\Omega = 2$. These parabolic transformations leave the combination $(\hat{B}^{(2)})^2$ invariant. Therefore the invariant metric is given by diag (0,1). The action of the $\mathbb{R}$ $\hat{\zeta}$-symmetry on the fields can not be expressed by assigning weights to the standard basis of fields given in (4.23).

2. An $\mathrm{SO}(1,1)^+$ subgroup which is generated by elements

$$\Omega_h = e^{\hat{\gamma}\sigma_3} = \begin{pmatrix} e^{\hat{\gamma}} & 0 \\ 0 & e^{-\hat{\gamma}} \end{pmatrix}. \tag{4.30}$$

   Each element defines a hyperbolic conjugacy class with $\mathrm{Tr}\,\Omega > 2$. These hyperbolic transformations leave the combination $\hat{B}^{(1)}\hat{B}^{(2)}$ invariant. After diagonalization this leads to an invariant metric given by diag (1,-1). The weights corresponding to the $\mathrm{SO}(1,1)^+$ $\hat{\gamma}$-symmetry are given in table 4.3.

3. There is a SO(2) subgroup which is generated by elements Ω of $S\ell(2,\mathbb{R})$ with

$$\Omega_e = e^{\mathrm{i}\hat{\theta}\sigma_2} = \begin{pmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}. \tag{4.31}$$

   Each element defines an elliptic conjugacy class with $\mathrm{Tr}\,\Omega < 2$. The elliptic transformations leave $(\hat{B}^{(1)})^2 + (\hat{B}^{(2)})^2$ invariant. After diagonalization this leads to an invariant metric given by diag (1,1). The action of the SO(2) $\hat{\theta}$-symmetry on the fields can not be expressed by assigning weights to the standard real basis of fields given in (4.23).

Table 4.3 contains the weights of the $\hat{\gamma}$-symmetry defined above[7] and of a new $\mathbb{R}^+$ symmetry $\hat{\delta}$ which is *not* a subgroup of $S\ell(2,\mathbb{R})$ and that does not leave the Lagrangian invariant. One could combine $S\ell(2,\mathbb{R})$ with this new $\mathbb{R}^+$ into a $G\ell(2,\mathbb{R})$ symmetry that leaves the IIB equations of motion invariant. Its action is the product of the two separate transformations: $\tilde{\Omega} = \Omega\Lambda_{\hat{\delta}}$. This exhausts all the symmetries of $D = 10$ IIB supergravity.

The IIB supergravity theory is not known to have massive deformations. One of the reasons for this is that there is no candidate vector field like in the IIA case.

[7]The other two symmetries defined above cannot be defined in terms of weights of real fields only.

$\mathbb{R}^+$	$\hat{e}_{\hat{\mu}}{}^{\hat{a}}$	$e^{\hat{\phi}}$	$\hat{\chi}$	$\hat{B}^{(1)}_{\hat{\mu}\hat{\nu}}$	$\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}$	$\hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$	$\hat{\psi}_{\hat{\mu}}$	$\hat{\lambda}$	$\hat{\epsilon}$	$\hat{\mathcal{L}}$	symmetry
$\hat{\gamma}$	0	−2	2	1	−1	0	0	0	0	0	$SO(1,1)^+$
$\hat{\delta}$	1	0	0	2	2	4	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	8	$\mathbb{R}^+$

Table 4.3: The scaling weights of the $D = 10$ IIB supergravity fields, the supersymmetry parameter $\hat{\epsilon}$ and the Lagrangian $\hat{\mathcal{L}}$.

$\mathbb{R}^+$	$e_\mu{}^a$	e^ϕ	e^φ	χ	A_μ	$A^{(1)}_\mu$	$A^{(2)}_\mu$	$B^{(1)}_{\mu\nu}$	$B^{(2)}_{\mu\nu}$	$C_{\mu\nu\rho}$	ψ_μ	λ	$\tilde{\lambda}$	ϵ	$\mathcal{L}$	Origin
α	$\frac{9}{7}$	0	$\frac{6}{\sqrt{7}}$	0	3	0	0	3	3	3	$\frac{9}{14}$	$-\frac{9}{14}$	$-\frac{9}{14}$	$\frac{9}{14}$	9	11D
β	0	$\frac{3}{4}$	$\frac{\sqrt{7}}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0	0	IIA
γ	0	−2	0	2	0	1	−1	1	−1	0	0	0	0	0	0	IIB
δ	$\frac{8}{7}$	0	$-\frac{4}{\sqrt{7}}$	0	0	2	2	2	2	4	$\frac{4}{7}$	$-\frac{4}{7}$	$-\frac{4}{7}$	$\frac{4}{7}$	8	IIB

Table 4.4: The scaling weights of the nine-dimensional supergravity fields, the supersymmetry parameter ϵ and the Lagrangian $\mathcal{L}$.

4.4 Massive deformations of $D = 9$, $\mathcal{N} = 2$ supergravity

The Kaluza-Klein reduction of either (massless) IIA or IIB supergravity gives the unique $D = 9$, $\mathcal{N} = 2$ massless supergravity theory. Its field content is given by

$$D = 9: \qquad \{e_\mu{}^a, \phi, \varphi, \chi, A_\mu, A^{(1)}_\mu, A^{(2)}_\mu, B^{(1)}_{\mu\nu}, B^{(2)}_{\mu\nu}, C_{\mu\nu\rho}, \psi_\mu, \lambda, \tilde{\lambda}\}\,. \tag{4.32}$$

The supersymmetry rules are given in [108]. The massless nine-dimensional theory inherits several global symmetries from its parents: two $\mathbb{R}^+$ symmetries α,β from IIA supergravity and one $\mathbb{R}^+$ symmetry δ plus a full $S\ell(2,\mathbb{R})$ symmetry from IIB supergravity. The latter leads in particular to an SO(2) symmetry θ, an $SO(1,1)^+$ symmetry γ and an $\mathbb{R}$-symmetry ζ. The weights of all these symmetries, except for the SO(2) θ-symmetry and $\mathbb{R}$ ζ-symmetry, and their higher-dimensional origin are given in table 4.4 (see also [109]).

It turns out that only three out of the four scalings given in table 4.4 are linearly independent, due to the relation

$$\tfrac{4}{9}\alpha - \tfrac{8}{3}\beta = \gamma + \tfrac{1}{2}\delta\,. \tag{4.33}$$

We observe the following pattern. Using (4.33) to eliminate one of the scaling-symmetries we are left with three independent scaling-symmetries. Each of the three gauge fields $A_\mu, A^{(1)}_\mu, A^{(2)}_\mu$ has weight zero under the linear combination of *two* out of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only.

mass parameters	$S\ell(2,\mathbb{R})$
(m_1, m_2, m_3)	triplet
$(m_4, \tilde{m}_4)$	doublet
$(m_{11}, m_{\rm IIA})$	doublet
$m_{\rm IIB}$	singlet

Table 4.5: This table indicates the different multiplets that the $D = 9$ mass parameters form under $S\ell(2,\mathbb{R})$.

The $D = 9$ $S\ell(2,\mathbb{R})$ symmetry acts in the following way:

$$\begin{aligned}
&\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \vec{A} \to \Omega\vec{A}, \qquad \vec{B} \to \Omega\vec{B}, \qquad \text{with } \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{R}), \\
&\psi_\mu \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \psi_\mu, \qquad \lambda \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{3/4} \lambda, \\
&\tilde{\lambda} \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{-1/4} \tilde{\lambda}, \qquad \epsilon \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \epsilon,
\end{aligned} \tag{4.34}$$

while φ and C are invariant. We have used a vector notation for the two vectors and two anti-symmetric tensors, like in $D = 10$. Again one can combine $S\ell(2,\mathbb{R})$ with an $\mathbb{R}^+$ symmetry to form $G\ell(2,\mathbb{R})$ with parameter $\tilde{\Omega} = \Omega\Lambda_{\mathbb{R}^+}$.

In addition to the global symmetries there is a number of local symmetries. In particular, the gauge transformations of the vectors read

$$\begin{aligned}
A^{(1)} &\to A^{(1)} - \mathrm{d}\lambda^{(1)}, \qquad & A^{(2)} &\to A^{(2)} - \mathrm{d}\lambda^{(2)}, \\
A &\to A - \mathrm{d}\lambda, & \vec{B} &\to \vec{B} - \vec{A} \wedge \mathrm{d}\lambda.
\end{aligned} \tag{4.35}$$

We now turn to massive deformations of the 9D theory. Applying a SS dimensional reduction of the higher-dimensional supergravities we obtain a number of massive deformations in nine dimensions, as illustrated in figure 4.1. By employing the different global symmetries of 11D, IIA and IIB supergravity we obtain seven deformations of the unique $D = 9$ supergravity. Since the procedure is quite straightforward – though tedious – we will not give any details here; these can be found in [108].

Note that the different massive deformations can be related. Symmetries of the massless theory become field redefinitions in the massive theory that only act on the massive deformations. This means that the mass parameters transform under such transformations: they have a scaling weight under the different scaling symmetries and fall in multiplets of $S\ell(2,\mathbb{R})$. In table 4.5 the multiplet structure of the massive deformations under $S\ell(2,\mathbb{R})$ is given. The mass parameter $\tilde{m}_4$ is defined as the S-dual partner of m_4 and can not be obtained by a SS reduction of IIA supergravity.

All these deformations correspond to a gauging of a 9D global symmetry. In particular, it is always the symmetry that is employed in the SS reduction Ansatz that becomes gauged upon

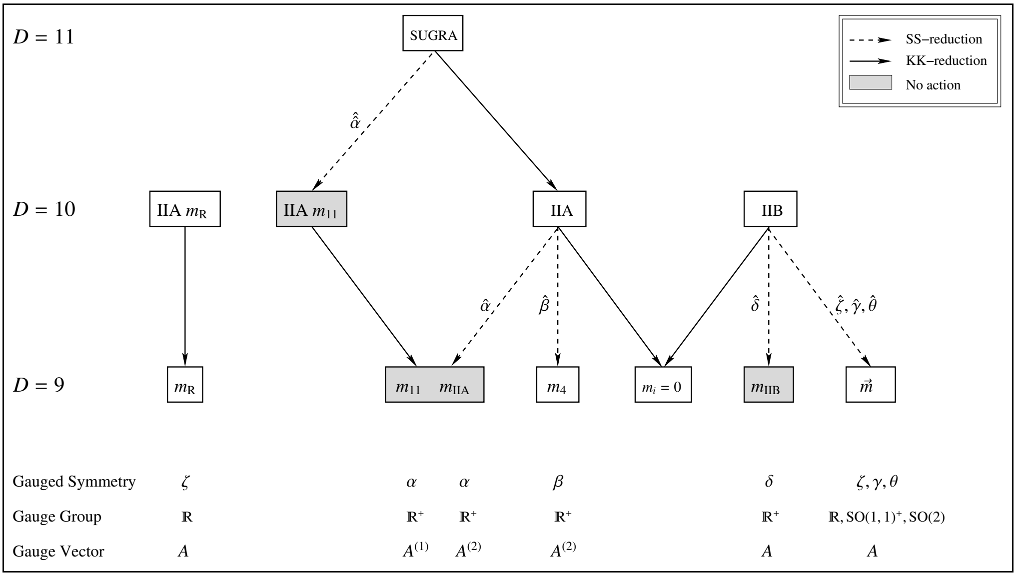

Figure 4.1: Overview of all reductions performed in [108]. These cases can all be interpreted as gauged supergravities, with gauged symmetry and corresponding gauge field as given in the figure. Mass parameters in the same box, such as $m_{11}, m_{\rm IIA}$ or m_1, m_2, m_3, form a multiplet under $S\ell(2,\mathbb{R})$. Further details of these cases will be given below. Note that the two ways of obtaining the $\mathbb{R}$-gauging give rise to the massive T-duality of [116], provided that $m_1 = m_2 = m_{\rm R}$ and $m_3 = 0$.

reduction. The corresponding gauge vector is always provided by the metric, i.e. it is the Kaluza-Klein vector of the dimensional reduction. In all but one case this is the complete story and one finds an Abelian gauged supergravity. It turns out that there is one exception where we find a *non-Abelian* gauge symmetry. This can be understood from the following general rule.[8] As we noted, the Kaluza-Klein vector gauges the symmetry employed in the SS reduction Ansatz. The fate of either of the remaining two gauge vectors is restricted to three possibilities:

- The vector is a singlet under the gauge symmetry and its field strength acquires no modification, e.g. $A^{(1)}$ in the $m_{\rm IIA}$ deformation.
- The vector transforms under the gauge symmetry and its field strength acquires a massive deformation proportional to a two-form. The degrees of freedom of the vector are eaten up by the two-form via the Stückelberg mechanism, e.g. A in the $m_{\rm IIA}$ deformation.
- The vector transforms under the gauge symmetry and its field strength acquires no massive deformation proportional to a two-form. In this case we must have gauge enhancement to preserve covariance, e.g. $A^{(1)}$ in the m_4 deformation.

All cases we find in $D = 9$ are consistent with this rule of thumb. Details can be found in [108].

[8]We thank Sergio Ferrara for clarifying discussions on this issue.

4.5 Combining massive deformations

In this section we would like to consider combining the massive deformations discussed in the previous section. The resulting theories will have more mass parameters characterizing the different deformations. However, not all combinations will turn out to be consistent with supersymmetry. This inconsistency only appears when turning to the bosonic field equations: the supersymmetry algebra with a combination of massive deformations always closes, as can be seen from the following argument.

Suppose one has a supergravity with one massive deformation m and supersymmetry transformations $\delta_0 + \delta_m$. In all cases discussed in this chapter the massive deformation of the supersymmetry rules satisfies the following property: $\delta_m(\text{boson}) = 0$. In other words, only the supersymmetry variations of the fermions receive massive corrections. This implies that the issue of the closure of the supersymmetry algebra is a calculation with m-independent parts and parts linear in m but no parts of higher order in m.[9] On the one hand $[\delta(\epsilon_1), \delta(\epsilon_2)]$ has no terms quadratic in m since one of the two δ's acts on a boson. On the other hand the supersymmetry algebra closes modulo fermionic field equations which also have only terms independent of and linear in m. Therefore, given the closure of the massless algebra, the closure of the massive supersymmetry algebra only requires the cancellation of terms linear in m.

In the previous sections we have not checked the closure of the massive supersymmetry algebras since this was guaranteed by the higher-dimensional origin, i.e. Scherk-Schwarz reduction of supergravity leads to a gauged supergravity. However, the argument of linearity allows us to combine different massive deformations. Suppose one has two massive supersymmetry algebras with transformations $\delta_0 + \delta_{m_a}$ and $\delta_0 + \delta_{m_b}$. Both supersymmetry algebras close modulo fermionic field equations with (different) massive deformations. Then the combined massive algebra with transformation $\delta_0 + \delta_{m_a} + \delta_{m_b}$ also closes modulo fermionic field equations whose massive deformations are given by the sum of the separate massive deformations linear in m_a and m_b. The closure of the combined algebra is guaranteed by the closure of the two massive algebras since it requires a cancellation at the linear level.

Under supersymmetry variation of the fermionic field equations, one in general finds linear *and* quadratic deformations of the bosonic equations of motion. In addition to these corrections, we find that there are also 'non-dynamical' equations posing constraints on the mass parameters. Solving these equations generically excludes the possibility of combining massive deformations by requiring mass parameters to vanish. At first sight, one might seem surprised that the supersymmetry variation of the fermionic equations of motion leads to constraints other than the bosonic field equations. However, one should keep in mind that the multiplets involved cannot be linearized around a Minkowski vacuum solution. Therefore, the usual rules for linearized (Minkowski) multiplets do not apply here.

We find that generically adding massive deformations is possible whenever the $D = 10$ symmetries, giving rise to the separate massive deformations, can be combined in $D = 10$ as symmetries of IIA or IIB supergravity only. The combined $D = 9$ supergravity is then a gauged supergravity which just follows by performing a SS reduction on the combined $D = 10$ symmetry.

[9]That is, up to cubic order in fermions. We have not checked the higher-order fermionic terms but, based upon dimensional arguments, we do not expect that these rule out the possibility of combining massive deformations.

In the first subsection we will discuss the situation in $D = 10$ and in the next subsection we will review the $D = 9$ situation; see [108] for details.

4.5.1 Combining massive deformations in 10D

The 10D IIA supergravity theory has two massive deformations parameterized by m_R and m_{11}. Can we combine these two massive deformations? Based on the linearity argument presented above one would expect a closed supersymmetry algebra. The bosonic field equations (with up to quadratic deformations) can be derived by applying the supersymmetry transformations (with only linear deformations) to the fermionic field equations (containing only linear deformations). For simplicity, we truncate all bosonic fields to zero except the metric and the dilaton. We thus find

$$\begin{aligned}
&(\delta_0 + \delta_{m_\mathrm{R}} + \delta_{m_{11}})(X_0 + X_{m_\mathrm{R}} + X_{m_{11}})(\hat{\psi}^{\hat{\mu}}) \\
&= \tfrac{1}{2}\hat{\Gamma}^{\hat{\nu}}\hat{\epsilon}\,[\hat{R}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} - \tfrac{1}{2}(\partial^{\hat{\mu}}\hat{\phi})(\partial_{\hat{\nu}}\hat{\phi}) + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} + \tfrac{1}{4}m_\mathrm{R}^2 e^{5\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}} + 36 m_{11}^2 e^{-3\hat{\phi}/2}\hat{g}^{\hat{\mu}}{}_{\hat{\nu}}] \\
&\quad + \Gamma_{11}\hat{\epsilon}[3m_{11}e^{-3\hat{\phi}/4}\partial^{\hat{\mu}}\hat{\phi}] + \Gamma_{11}\hat{\Gamma}^{\hat{\mu}}\hat{\epsilon}\,[\tfrac{15}{4}m_\mathrm{R}m_{11}e^{\hat{\phi}/2}]\,, \qquad (4.36) \\
&(\delta_0 + \delta_{m_\mathrm{R}} + \delta_{m_{11}})(X_0 + X_{m_\mathrm{R}} + X_{m_{11}})(\hat{\lambda}) \\
&= \hat{\epsilon}\,[\Box\hat{\phi} - \tfrac{5}{4}m_\mathrm{R}^2 e^{5\hat{\phi}/2}] + \hat{\Gamma}^{\hat{\nu}}\Gamma_{11}\hat{\epsilon}[9m_{11}e^{-3\hat{\phi}/4}\partial_{\hat{\nu}}\hat{\phi}] + \Gamma_{11}\hat{\epsilon}\,[\tfrac{33}{2}m_\mathrm{R}m_{11}e^{\hat{\phi}/2}]\,.
\end{aligned}$$

At the right-hand side we find four different structures. Three of them correspond to the field equations of the metric, dilaton and RR vector. The vector field equation corresponds to the terms linear in m_{11} and containing Γ_{11}. They show us that truncating the RR vector to zero forces us to further truncate the dilaton to $\phi = c$. More interesting is the fourth structure which is bilinear in $m_\mathrm{R}m_{11}$. It leads to the constraint $m_\mathrm{R}m_{11} = 0$. This constraint cannot be a remnant of a higher-rank form field equation due to its lack of Lorentz indices. It could only fit in the scalar field equation but the Γ_{11} factor prevents this. It is an extra constraint which does not restrict degrees of freedom but rather restricts mass parameters.

We conclude that, even though the closure of the algebra is a linear calculation and therefore always works for combinations, the bosonic field equations exclude the possibility of the combination of massive deformations in $D = 10$ dimensions.

4.5.2 Combining massive deformations in 9D

Repeating the above analysis – i.e. requiring that the fermionic field equations transform under supersymmetry to a complete set of bosonic field equations – restricts us to five cases, each containing two non-zero mass parameters:

- **Case 1** with $\{m_\mathrm{IIA}, m_4\}$: this combination can also be obtained by Scherk-Schwarz reduction of IIA employing a linear combination of the symmetries $\hat{\alpha}$ and $\hat{\beta}$, guaranteeing its consistency. It is also a gauging of both this symmetry and (for $m_4 \neq 0$) the parabolic subgroup of $\mathrm{S}\ell(2,\mathbb{R})$ in 9D, giving the non-Abelian gauge group A(1).
- **Case 2,3,4** with $\{\vec{m}, m_\mathrm{IIB}\}$: as in the case with $m_\mathrm{IIB} = 0$ and only $\vec{m}$ this combination contains three different, inequivalent cases depending on $\vec{m}^2$ (depending crucially on the fact that m_IIB is a singlet under $\mathrm{S}\ell(2,\mathbb{R})$):

  - **Case 2** with $\{\vec{m}, m_{\text{IIB}}\}$ and $\vec{m}^2 = 0$.
  - **Case 3** with $\{\vec{m}, m_{\text{IIB}}\}$ and $\vec{m}^2 > 0$.
  - **Case 4** with $\{\vec{m}, m_{\text{IIB}}\}$ and $\vec{m}^2 < 0$.

  All these combinations can also be obtained by Scherk-Schwarz reduction of IIB employing a linear combination of the symmetries $\hat{\delta}$ and (one of the subgroups of) $S\ell(2,\mathbb{R})$, guaranteeing its consistency. All cases (assuming that $m_{\text{IIB}} \neq 0$) correspond to the gauging of an Abelian non-compact symmetry in 9D. Only the special case $\{\vec{m}^2 < 0, m_{\text{IIB}} = 0\}$ corresponds to a SO(2)-gauging.

- **Case 5** with $\{m_4 = -\frac{12}{5} m_{\text{IIA}}, m_2 = m_3\}$: this case can be understood as the generalized dimensional reduction of Romans' massive IIA theory, employing the $\mathbb{R}^+$ symmetry that is not broken by the m_{R} deformations: $\hat{\beta} - \frac{5}{12}\hat{\alpha}$. It gauges both this linear combination of $\mathbb{R}^+$'s and the parabolic subgroup of $S\ell(2,\mathbb{R})$ in 9D, giving the non-Abelian gauge group A(1).

Another solution to the quadratic constraints has parameters $\{m_{\text{IIA}}, m_{11}\}$, but this combination does not represent a new case. It can be obtained from only m_{IIA} (and thus a truncation of case 1) via an $S\ell(2,\mathbb{R})$ field redefinition (since they form a doublet). Thus the most general deformations are the five cases given above, all containing two mass parameters. All five of these are gauged theories and have a higher-dimensional origin. Both case 1 and case 5 have a non-Abelian gauge group provided $m_4 \neq 0$.

4.6 Solutions

In the first part of this chapter we constructed a gauged supergravity with 32 supersymmetries in $D = 10$; after that we illustrated how to obtain a variety of gauged supergravities in $D = 9$, using the same methods. They all have in common that there is a scalar potential. Our next goal is to make a systematic search for solutions that are based on this scalar potential. In the next subsections we will search for two types of solutions: (i) 1/2 BPS domain-wall (DW) solutions and (ii) maximally symmetric solutions with constant scalars, i.e. de Sitter (dS), Minkowski (Mink) or anti-de Sitter (AdS) solutions.

4.6.1 1/2 BPS domain-wall solutions

The authors of [117] already made a systematic search for half-supersymmetric DW solutions of the gauged supergravities corresponding to the cases 3, 4 and 5. Due to a one-to-one relationship with seven-branes in $D = 10$ dimensions [111] they could even make a systematic investigation of the quantization of the mass parameters by using the results of [118, 119].

The goal of this subsection is to investigate whether the five massively deformed supergravities we found in subsection 4.5.2 allow new half-supersymmetric DW solutions. In other words, we will derive all 1/2 BPS seven-brane solutions to the nine-dimensional supergravities described in the previous sections. This analysis should lead, as a check of our calculations, to at least all the solutions of [117]. Since we are looking for 1/2 BPS solutions it is convenient to solve the Killing spinor equations, which are obtained by setting the supersymmetry variation of the gravitino and dilatinos to zero. In this way we solve first order equations instead of second order equations which we would encounter if we were to solve the field equations directly. For static configurations a solution to the Killing spinor equation is also a solution to the field equations, so we do not have to check explicitly that the field equations are satisfied. The projector[10] for a DW is given by $\frac{1}{2}(1 \pm \gamma_y)$, where y denotes the transverse direction. We find that, in order to make a projection operator in the Killing spinor equations, we are forced to set all mass parameters to zero except for $\vec{m}$, which corresponds to cases 3, 4 and 5 of section 4.5. This is a consistent combination of masses and we obtain three classes of domain-wall solution which were discussed in detail in [117]. As it turns out, there are no more half-supersymmetric DW solutions.

To summarize, we find that there are no new codimension-one 1/2 BPS solutions to the $D = 9$ supergravity theories we obtained in the previous sections, as compared to the three classes of domain-wall solutions given in [117].

4.6.2 Solutions with constant scalars

In this subsection we will consider solutions with all three scalars constant. This is a consistent truncation in two cases, both of which have two mass parameters. In this truncation one is left with the metric only satisfying the Einstein equation with a cosmological term

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \,, \tag{4.37}$$

with Λ quadratic in the two mass parameters. Depending on the sign of this term one then has anti-de Sitter, Minkowski or de Sitter geometry.

We find that solutions with constant scalars are possible in the following massive supergravities:

- $D = 10$ with $\{m_{11}\}$ has $\Lambda = 36 {m_{11}}^2 e^{-3\hat{\phi}/2}$, which gives rise to de Sitter_{10} [115], breaking all supersymmetry. The $D = 11$ origin of this solution is Mink_{11} written in a basis where the x-dependence is of the required form [115]

$$\text{Mink}_{11}: \qquad \mathrm{d}s^2 = e^{2m_{11}x}(-\mathrm{d}t^2 + e^{2m_{11}t}\mathrm{d}x_9^2 + \mathrm{d}x^2) \,. \tag{4.38}$$

- $D = 9$, **Case 1** with $\{m_{\text{IIA}} = -\frac{2}{3} m_4\}$ has $\Lambda = \frac{63}{4} {m_4}^2 e^{\phi - 3\varphi/\sqrt{7}}$, which gives rise to de Sitter_9, breaking all supersymmetry. This case follows from the reduction of Mink_{10} by using a combination of IIA scale symmetries that leave the dilaton invariant (since Minkowski has vanishing dilaton) so that, after reduction, one is left with a non-trivial geometry only.

- $D = 9$, **Case 4** with $\{m_{\text{IIB}}, m_3\}$ has $\Lambda = 28 {m_{\text{IIB}}}^2 e^{4\varphi/\sqrt{7}}$, which gives rise to de Sitter_9 for non-vanishing m_{IIB}. This case follows from the reduction of Mink_{10} by using a combination of IIB scale symmetries that leave the dilaton invariant. Note that for vanishing m_{IIB} this reduces to Mink_9, despite the presence of m_3 [120]. For either m_{IIB} or m_3 non-zero this solution breaks all supersymmetry.

[10] From a general analysis of the possible projectors in nine dimensions, i.e. demanding that the projector squares to itself and that its trace is half of the spinor dimension, in order to yield a 1/2 BPS state, we find that there is a second projector given by $\frac{1}{2}(1 \pm \mathrm{i}\,\gamma_t)$. This projector would give a euclidean DW, i.e. a DW having time as a transverse direction. Note that such a Euclidean DW can never be 1/2 BPS since if there existed a Killing spinor it would square to a Killing vector in the *transverse* direction, i.e. time, which is not an isometry of the euclidean DW.

4.7 Conclusions

In this chapter we have illustrated how to construct five different $D = 9$ massive deformations with 32 supersymmetries, each containing two mass parameters. We found in [108] that all these five theories have a higher-dimensional origin via SS reduction from $D = 10$ dimensions. Furthermore, the massive deformations gauge a global symmetry of the massless theory. The gauge groups we obtained are the Abelian groups SO(2), $SO(1, 1)^+$, $\mathbb{R}$, $\mathbb{R}^+$ and the unique two-dimensional non-Abelian Lie group A(1) of scalings and translations on the real line.

We have analyzed the possibility of combining massive deformations to obtain more general massive supergravities that are not gauged or do not have a higher-dimensional origin. Our analysis shows that the only possible combinations are the five two-parameter deformations, which are all gauged and can be uplifted. We have not made a systematic search for massive $D = 9$ supergravities that are not the combination of gaugings and we cannot exclude that there are more possibilities; this requires a separate calculation. In this context, it is of interest to point out that examples of massive supergravities like Romans have been found in lower dimensions, e.g. [121, 122]. In these cases the compactification manifolds are such that the candidate gauge fields are truncated away.

It is intriguing that some of the gauged supergravities we have constructed result from gauging an $\mathbb{R}^+$ scale symmetry that does not leave the Lagrangian invariant but scales it with a factor. Apparently, it is possible to gauge such symmetries at the level of the equations of motion.

Finally we would like to address the question of whether the gauged supergravities we constructed can be interpreted as the leading terms in a low-energy approximation to (compactified) superstring theory. The nine-dimensional massive deformations split up in two categories: those where only the theory to lowest order in α' has a higher-dimensional origin and those where also the higher-derivative corrections can be obtained from 10D. The latter category can be derived using symmetries that extend to all orders in α'. We have two such symmetries:

- The $S\ell(2,\mathbb{R})$ (or rather its $S\ell(2,\mathbb{Z})$ subgroup) symmetry of IIB. Thus the $\vec{m} = (m_1, m_2, m_3)$ deformations correspond to the low-energy limits of three different sectors of compactified IIB string theory (depending on $\vec{m}^2 = \frac{1}{4}({m_1}^2 + {m_2}^2 - {m_3}^2)$). In [117] DW solutions were constructed for all three sectors. Of these only the D7-brane has a well-understood role in IIB string theory.

- The linear combination $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ of $\mathbb{R}^+$-symmetries of IIA. Thus one can define a massive deformation m_s within case 1 with $\{m_{\text{IIA}} = \frac{1}{12}m_s, m_4 = m_s\}$ which corresponds to the low-energy limit of a sector of compactified IIA string theory. No vacuum solution has been constructed for this sector. It would be very interesting to try to find a vacuum solution and understand which role it plays in IIA string theory.

In fact, one can have a better understanding of the m_s massive deformation and the $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ symmetry of IIA from the following point of view. The combination $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ of IIA can be understood from its 11D origin as the general coordinate transformation $x^{11} \to \lambda\, x^{11}$; one can easily check that this is indeed the case by comparing with (3.17). This explains why all α' corrections transform covariantly under this specific $\mathbb{R}^+$: the higher-order corrections in 11D are invariant under general coordinate transformations and upon reduction they must transform covariantly under the reduced g.c.t.'s, among which is the $\frac{1}{12}\hat{\alpha} + \hat{\beta}$ scaling-symmetry.