Chapter 3

Dimensional reduction

In this chapter we will explain how to obtain massive deformations, i.e. scalar potentials and cosmological constants from dimensional reduction. We start by reviewing some aspects of standard Kaluza-Klein reduction. Then we will introduce a mechanism to generate masses in lower dimensions, called Scherk-Schwarz dimensional reduction, which will be used in chapter 4 to construct gauged and massive supergravities in ten and nine dimensions. For reviews on the subject of dimensional reduction see [87–90].

3.1 Kaluza-Klein dimensional reduction

Even before the advent of string theory, the possibility of extra dimensions was discussed. As early as 1921, a few years after Einstein wrote down his theory of general relativity [91], Kaluza attempted to unify gravitation with electromagnetism by assuming that we live in a five-dimensional universe [92]. By ignoring the extra dimension[1] he managed to obtain the four-dimensional field equations of both gravity and electromagnetism from a five-dimensional theory of pure gravity. Several years later Klein reformulated this theory using the action-principle [93]. He also assumed that the extra coordinate was curled up as a circle with radius of the order of ℓ_p, explaining why this coordinate had never been observed in experiment.

The same mechanism can now also be used for ten-dimensional string theories, in order to try to obtain the four-dimensional world as we observe it and make contact with experiment. For this purpose, one assumes that the ten-dimensional space-time can be written in the form $\mathcal{M}_4 \times K_6$, where $\mathcal{M}_4$ is our four-dimensional space-time, and K_6 a compact sub-manifold of Planckian size. Dimensional reduction also plays an important role in the context of dualities in string theory; many of these results have been obtained by using torus or sphere reductions. In this chapter we will mostly restrict ourselves to compactifications of the type $\mathcal{M}_{D+1} = \mathcal{M}_D \times S^1$, splitting the coordinates $x^{\hat{\mu}}$ into x^{μ} and the compact coordinate z.[2] In section 3.2.2 we will

[1] Kaluza considered our universe to be an isolated four-dimensional subspace of $\mathbb{R}^5$ where all derivatives with respect to the fifth coordinate vanish, the so-called "cylinder condition".

[2] We use hats to denote the dimensionality of a certain object or index. In this section hatted fields are living in $(D+1)$ dimensions and hatted indices run from $0\dots D$. See appendix B.1 for our conventions.

generalize this to compactifications on higher dimensional compact manifolds.

First of all we observe that all fields in this $(D+1)$-dimensional space have to satisfy the boundary condition

$$\hat{\Phi}(x^\mu, z + 2\pi R_z) = \hat{\Phi}(x^\mu, z)\,, \tag{3.1}$$

and therefore can be Fourier-expanded in terms of the eigenfunctions of the circle

$$\hat{\Phi}(x^\mu, z) = \sum_n \Phi_n(x^\mu) e^{\mathrm{i}\, n\, z/R_z}\,, \tag{3.2}$$

where R_z is the circle-radius. When we insert this Ansatz into the massless Klein-Gordon equation in flat space-time we obtain the equation of motion for a field with mass $M = |\frac{n}{R_z}|$ in the non-compact subspace

$$\hat{\Box}\hat{\Phi}(x^\mu, z) = 0 \quad \rightarrow \quad [\Box + \partial_z\partial^z]\,\Phi_n(x^\mu) \equiv \left[\Box + \left(\tfrac{n}{R_z}\right)^2\right]\Phi_n(x^\mu) = 0\,. \tag{3.3}$$

The infinite set of fields Φ_n is called the tower of massive Kaluza-Klein (KK) modes. Taking the limit where the radius of the circle goes to zero, these KK-modes become infinitely massive and decouple from the massless theory. These modes can be neglected in any effective field theory. In this limit the higher dimensional fields simply become independent of the compact coordinate. This is equivalent to the assumption that the $(D+1)$-dimensional space-time has an isometry along z, with associated Killing vector $K^z = \partial_z$. The process of compactification combined with consistently truncating away the massive modes is called Kaluza-Klein reduction. In the next sections we will give some explicit examples, demonstrating the KK-mechanism.

Metric

Let us first consider pure gravity in $(D+1)$ dimensions, given by the Einstein-Hilbert action

$$\hat{S} = \frac{1}{2\kappa^2_{D+1}} \int \mathrm{d}^{D+1}\hat{x}\,\sqrt{|\hat{g}|}\,\hat{R}\,. \tag{3.4}$$

The metric of $(D+1)$-dimensional space-time can be decomposed into the following components: $\hat{g}_{\mu\nu}, \hat{g}_{\mu z}$ and $\hat{g}_{zz}$ naively looking like a metric, vector and scalar in D dimensions respectively. In order to have all components behave correctly under D-dimensional general coordinate transformations (g.c.t.'s) one arrives at the following Ansatz for the metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix}, \tag{3.5}$$

with α and β arbitrary constants. This Ansatz corresponds to the line element

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}\left(\mathrm{d}z + A_\mu \mathrm{d}x^\mu\right)^2\,. \tag{3.6}$$

The scalar ϕ is called the dilaton and A_μ the Kaluza-Klein vector.

Since we will perform reductions of supergravity theories in the next chapter, it is convenient to use the Palatini formalism. Namely in order to covariantly describe spinors we need to define a flat tangent space in each point of space-time. Vielbeins are the orthonormal basis vectors defined over the manifold. The metric can be written in terms of these vielbeins as follows

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{a}}\hat{e}_{\hat{\nu}}{}^{\hat{b}}\hat{\eta}_{\hat{a}\hat{b}}\,. \tag{3.7}$$

We can use the internal Lorentz gauge degrees of freedom to write the vielbein and inverse vielbein in upper triangular form

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{\alpha\phi}e_\mu{}^a & e^{\beta\phi}A_\mu \\ 0 & e^{\beta\phi} \end{pmatrix}, \qquad \hat{e}_{\hat{a}}{}^{\hat{\mu}} = \begin{pmatrix} e^{-\alpha\phi}e_a{}^\mu & -e^{-\alpha\phi}A_a \\ 0 & e^{-\beta\phi} \end{pmatrix}. \tag{3.8}$$

Using the above Ansätze we obtain the following action (for details see appendix B.2)

$$S = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\, e\left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}F(A)^2\right). \tag{3.9}$$

Note that we had to use the following values for α and β to obtain the canonical Einstein-Hilbert action in the Einstein-frame

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\,, \qquad \beta = -(D-2)\alpha\,. \tag{3.10}$$

The D-dimensional gravitational constant is related to the $(D+1)$-dimensional one by the volume of the compact space

$$\kappa_{D+1}^2 = \kappa_D^2\int \mathrm{d}z = 2\pi R_z\,\kappa_D^2\,. \tag{3.11}$$

The above mechanism demonstrates how Kaluza and Klein partly succeeded in unifying Maxwell's theory with gravity, by reducing pure gravity from five to four dimensions. The only awkward feature they were not able to explain was the presence of the scalar field. Naively one would put it equal to zero, but this would not be consistent with the field equations; it would imply that the KK-vector should vanish as well.

Forms

Next consider antisymmetric forms in a gravitational background. The generic KK-Ansatz for a $(n-1)$-form gauge field is given by[3]

$$\hat{A}_{\mu_1\ldots\mu_{n-1}} \equiv A_{\mu_1\ldots\mu_{n-1}}\,, \qquad \hat{A}_{\mu_1\ldots\mu_{n-2}\,z} \equiv A_{\mu_1\ldots\mu_{n-2}}\,. \tag{3.12}$$

Using these Ansätze, we obtain the lower-dimensional gauge invariant field-strengths

$$\begin{aligned}
\hat{F}_{\alpha_1\ldots\alpha_{n-1}\underline{z}} &= \hat{e}_{\alpha_1}{}^{\hat{\mu}_1}\ldots\hat{e}_{\alpha_{n-1}}{}^{\hat{\mu}_{n-1}}\hat{e}_{\underline{z}}{}^{z}\hat{F}_{\mu_1\ldots\mu_{n-1}\,z} \equiv e^{(D-n-1)\phi}F_{\alpha_1\ldots\alpha_{n-1}}\,,\\
\hat{F}_{\alpha_1\ldots\alpha_n} &= \hat{e}_{[\alpha_1}{}^{\mu_1}\ldots\hat{e}_{\alpha_{n-1}}{}^{\mu_{n-1}}\left(\hat{e}_{\alpha_n]}{}^{\mu_n}\hat{F}_{\mu_1\ldots\mu_n} + n\,\hat{e}_{\alpha_n]}{}^{z}\hat{F}_{\mu_1\ldots\mu_{n-1}\,z}\right) \equiv e^{-n\alpha\phi}F_{\alpha_1\ldots\alpha_n}\,,\\
F_{(n-1)} &= \mathrm{d}A_{(n-2)}\,, \qquad F_{(n)} = \mathrm{d}A_{(n-1)} - n\,F_{(n-1)}\wedge A_{(1)}\,.
\end{aligned} \tag{3.13}$$

Usually the reduction Ansätze will be of a slightly different form. This is because lower-dimensional field redefinitions are already applied on the level of the Ansätze, in order to get a nice expression for the reduced theory.

[3] Flat compact directions are denoted by $\underline{z}$. These differ from the non-flat ones by the vielbein $\hat{e}_z{}^{\underline{z}}$.

3.1.1 Symmetries

Let us now have a look at the symmetries of the lower-dimensional theory, and how they are obtained from the higher-dimensional symmetries. The $(D+1)$-dimensional theory of gravity contains two symmetries:

- General coordinate transformations (g.c.t.)
  The g.c.t. can be written infinitesimally as

$$\delta_{g.c.t.} x^{\hat\mu} = -\hat\xi^{\hat\mu} \,, \qquad \delta_{g.c.t.}\hat g_{\hat\mu\hat\nu} = \hat\xi^{\hat\rho}\partial_{\hat\rho}\hat g_{\hat\mu\hat\nu} + 2\hat g_{\hat\rho(\hat\mu}\partial_{\hat\nu)}\hat\xi^{\hat\rho} \,. \tag{3.14}$$

  Demanding all fields (e.g. $\hat g_{\mu z}$) to stay independent of z after a g.c.t., one derives the following constraints on the g.c.t.-parameter

$$\partial_z\hat\xi^{\mu} = 0 \,, \qquad \partial_z\partial_{\hat\mu}\hat\xi^{z} = 0 \,, \tag{3.15}$$

  which are solved by

$$\hat\xi^{\mu} = \xi^{\mu}(x^{\mu}) \,, \qquad \hat\xi^{z} = cz + \lambda(x^{\mu}) \,. \tag{3.16}$$

  The g.c.t. corresponding to these parameters also leave invariant the KK-Ansatz (3.6), and give rise to the following transformations after substituting (3.5) and (3.16) into (3.14).

$$\begin{array}{lcllll}
\delta g_{\mu\nu} &=& \xi^\rho\partial_\rho g_{\mu\nu} + 2g_{\rho(\mu}\partial_{\nu)}\xi^\rho & - & 2\alpha\beta^{-1}c\, g_{\mu\nu} & \\
\delta A_\mu &=& \xi^\rho\partial_\rho A_\mu + A_\rho\partial_\mu\xi^\rho & - & c\,A_\mu & + \ \partial_\mu\lambda(x) \\
\delta\phi &=& \underbrace{\xi^\rho\partial_\rho\phi}_{\text{g.c.t.}} & + & \underbrace{\beta^{-1}c}_{\text{scale symmetry}} & \underbrace{}_{\text{U(1)}}
\end{array} \tag{3.17}$$

  The emerging of a U(1) associated with z-independent reparametrizations of the compact coordinate is in fact a generic feature of dimensional reduction. The KK-vector transforms as a true gauge vector. As we will see in section 3.2.2, in general a lower-dimensional gauge group G is generated by the Killing vectors on the internal manifold. The relevance of the scale-symmetry will become clear after discussing the second symmetry.

- Global Weyl symmetry
  Actually this is not a symmetry of the action, but of the equations of motion. Under this symmetry the metric scales with a constant factor. Infinitesimally this becomes

$$\delta\hat g_{\hat\mu\hat\nu} = 2a\hat g_{\hat\mu\hat\nu} \,, \tag{3.18}$$

  reducing to

$$\delta g_{\mu\nu} = 2a\left[1-\alpha\beta^{-1}\right]g_{\mu\nu} \,, \qquad \delta A_\mu = 0 \,, \qquad \delta\phi = a\beta^{-1} \,. \tag{3.19}$$

By taking two linearly independent combinations of both scale-symmetries we obtain a global dilaton shift symmetry of the lower-dimensional action, and a uniform scaling symmetry[4] that is only valid at the level of the equations of motion

$$a = -\tfrac{c}{D-1} \quad : \quad \delta g_{\mu\nu} = 0 \,, \qquad \delta A_\mu = -cA_\mu \qquad \delta\phi = -\tfrac{c}{\alpha(D-1)} \,, \tag{3.20}$$

$$a = -c \quad : \quad \delta g_{\mu\nu} = 2a g_{\mu\nu} \,, \qquad \delta A_\mu = aA_\mu \qquad \delta\phi = 0 \,. \tag{3.21}$$

[4] Also called "trombone" symmetry in the literature [87].

Both types of symmetries will turn out to be of use in chapter 4 when we consider scale symmetries in supergravity.

3.2 Scherk-Schwarz dimensional reduction

Even though the KK-mechanism has some very appealing features, it is clearly not the most general way of dimensional reduction; all massive modes are truncated away and consequently we can never obtain masses for gauge particles in lower dimensions. It also does not provide a natural way of breaking some part of the supersymmetry, which clearly is needed to obtain physically plausible theories. In 1979 Scherk and Schwarz posed an interesting alternative in a series of two papers [94, 95], called generalized dimensional reduction, or also Scherk-Schwarz (SS) reduction.

The general feature of SS-reduction is the usage of symmetries of the higher-dimensional theory to introduce masses in lower dimensions. The generalization consisted of allowing the higher-dimensional fields to depend on the compact coordinate z, in a way prescribed by the symmetries of the action. This assures us that the z-dependence will be completely removed from the equations of motion of the reduced action.

Two types of symmetries can be used:

1. global/internal symmetries: phase, scale and shift symmetries or other global symmetries
2. local/external symmetries: space-time symmetries, such as translations or rotations in the compact manifold

3.2.1 Scherk-Schwarz I

In this section we will restrict ourselves to symmetries of the first kind; the latter type will be briefly explained in section 3.2.2.

In the case of a global U(1) phase-symmetry $\hat{\Phi} \to e^{\mathrm{i}\Lambda}\hat{\Phi}$, we generalize the periodicity condition (3.1) by identifying the two fields up to an extra global phase-transformation, or "twist"

$$\hat{\Phi}(x^\mu, z + 2\pi R_z) = e^{2\pi \mathrm{i} m R_z}\hat{\Phi}(x^\mu, z)\,, \tag{3.22}$$

resulting in the following mode-expansion

$$\hat{\Phi}(x^\mu, z) = e^{\mathrm{i} m z}\sum_n \Phi_n(x^\mu) e^{\mathrm{i} n z/R_z}\,. \tag{3.23}$$

In the limit $R_z \to 0$ the massive modes again decouple, and we are left with the effective Ansatz

$$\hat{\Phi}(x^\mu, z) = e^{\mathrm{i} m z}\Phi_0(x^\mu)\,, \tag{3.24}$$

which can also be obtained by replacing the global symmetry parameter Λ by mz. More generally if there is a global symmetry group G acting on the fields: $\hat{\Phi} \to g(\hat{\Phi})$, we allow for a specific z-dependence in our reduction Ansatz through a symmetry transformation dependent on the compact coordinate(s)

$$\hat{\Phi}(x^\mu, z) = g_z(\hat{\Phi}(x^\mu))\,, \qquad g_z = g(z) \in G\,. \tag{3.25}$$

This particular form of the Ansatz guarantees that the reduced theory will be independent of z, and results in the gauging of the group G, producing a scalar potential or cosmological constant. When a mass term is produced for the gravitino, there is even a spontaneous breaking of supersymmetry. Since the z-dependent transformation in general will not be periodic, as before, going once around the compact coordinate will produce a twist, the so-called *monodromy*

$$\mathcal{M}(g) = g(2\pi R_z)g(0)^{-1}\,, \qquad \mathcal{M} \in G\,. \tag{3.26}$$

Writing the group element in terms of the generators of the Lie algebra

$$g(z) = e^{Mz}\,, \qquad M \in \mathrm{Lie}(G)\,, \tag{3.27}$$

we obtain an expression for the monodromy in terms of M

$$\mathcal{M} = e^{M}\,, \qquad M = g^{-1}\partial_z g\,. \tag{3.28}$$

In practice it turns out that the object M can be interpreted as the mass matrix of the reduced theory, as we will see in chapter 4. The specific function g can now be determined by demanding that M is independent of z.

One might wonder at this point whether the reduced theory we obtain this way is unique. Not every choice for g will necessarily lead to a new reduced theory; provided these functions are in the same conjugacy class, their reduced theories will only differ by a field redefinition. Independent reductions can therefore be classified by the conjugacy classes of the mass matrix M [96–98]. We will see examples of this in the next chapter.

Let us first look at some simple examples of the mechanism described above.

Complex scalar field in a gravitational background

The action for a complex scalar field in a curved background is given by

$$\hat{S} = \frac{1}{2\kappa^2_{D+1}} \int \mathrm{d}^{D+1}\hat{x}\, \hat{e}\left(\hat{R} - \tfrac{1}{2}\,\partial_{\hat{\mu}}\hat{\varphi}(\hat{x})\partial^{\hat{\mu}}\hat{\varphi}^*(\hat{x})\right)\,. \tag{3.29}$$

This action contains two global symmetries: invariance under phase transformations and under shifts.

(1) Phase symmetry: $\quad \hat{\varphi} \to e^{\mathrm{i}\,c}\hat{\varphi}$

Following the above prescription, the corresponding SS-Ansatz becomes:

$$\hat{\varphi}(\hat{x}) = e^{\mathrm{i}\,m_1 y}\varphi(x)\,, \qquad (m_1 \text{ real})\,. \tag{3.30}$$

The reduction of the scalar part of the action then gives:

$$\begin{aligned} \hat{e}\,\partial\hat{\varphi}(\hat{x})\partial\hat{\varphi}^*(\hat{x}) &= \hat{e}\,\partial_{\hat{\mu}}\hat{\varphi}(\hat{x})\partial_{\hat{\nu}}\hat{\varphi}^*(\hat{x})\hat{e}_{\hat{a}}{}^{\hat{\rho}}\hat{e}_{\hat{b}}{}^{\hat{\mu}}\hat{\eta}^{\hat{a}\hat{b}} \\ &= e\left(|\mathcal{D}\varphi(x)|^2 + m_1^2 e^{2(\alpha-\beta)\phi}\,|\varphi|^2\right)\,, \end{aligned} \tag{3.31}$$

where the covariant derivative is defined as: $\quad \mathcal{D}_\mu = \partial_\mu - \mathrm{i}\,m_1 A_\mu$.

This we recognize as the usual expression for the covariant derivative, associated with the gauging of a U(1) phase transformation of an Abelian gauge theory. So a straightforward interpretation of m_1 is that of a charge for the complex scalar field. Alternatively m_1 can be interpreted as mass parameter. Added up to (3.9), the complete reduced action describes a real scalar ϕ (the dilaton) and a charged complex scalar φ in a curved background. Most importantly, this procedure seems to have produced a scalar potential, describing the interactions between both scalar fields. Finally note that the same result, except for the scalar potential, could have been obtained by a KK-reduction, followed by the gauging of the generated global U(1) symmetry. The scalar potential, however, is gauge-invariant by itself and cannot be constructed by gauging alone. In supersymmetric theories however the scalar potential can always be reconstructed by demanding invariance of the action.

(2) Shift symmetry: $\hat{\varphi} \to \hat{\varphi} + c$.
As SS-Ansatz we now take

$$\hat{\varphi}(\hat{x}) = \hat{\varphi}(x) + m_2 z \qquad (m_2 \text{ complex})\,. \tag{3.32}$$

This time the scalar part of the action reduces as

$$\hat{e}\,\partial\hat{\varphi}(\hat{x})\partial\hat{\varphi}^*(\hat{x}) \;=\; e\left(|\mathcal{D}\varphi(x)|^2 + |m_2|^2\, e^{2(\alpha-\beta)\phi}\right), \tag{3.33}$$

with covariant derivative $\mathcal{D}_\mu\varphi = \partial_\mu\varphi - m_2 A_\mu$, invariant under the so-called massive gauge transformations

$$\begin{cases} \delta\varphi &= m_2\,\lambda(x)\,, \\ \delta A_\mu &= \partial_\mu\lambda(x)\,, \end{cases} \tag{3.34}$$

induced by a general coordinate transformation along z in $(D+1)$ dimensions, as we saw earlier in the Kaluza-Klein reduction. The complete reduced action becomes

$$S = \int \mathrm{d}^D x\, e\left[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\,e^{-2(D-1)\alpha\phi}F^2(A) - \tfrac{1}{2}\,|\mathcal{D}\varphi|^2 - \tfrac{1}{2}\,m_2^2 e^{2(\alpha-\beta)\phi}\right]. \tag{3.35}$$

If we now fix the massive gauge transformations, by taking $\Re e(\varphi) = 0$, the action consists of the following parts:

1. a D-dimensional Einstein-Hilbert action + kinetic term for the dilaton
2. $\int \mathrm{d}^D x\, e\left[-\frac{1}{4}e^{-2(D-1)\alpha\phi}F(A)^2 - m_2^2 A^2\right]$,
   which is the well known Proca action for a massive vector field, coupled to gravity, with additional dilaton interaction terms.
3. a scalar potential.

Compared to the KK-reduction, obtained by taking $m_1 = m_2 = 0$, we gained a scalar potential and masses for either the scalar or vector field.

Instead of the two separate global symmetries, we could have used the combination of both symmetries for the Scherk-Schwarz reduction. The Ansatz then becomes

$$\hat{\varphi} = e^{\,\mathrm{i}\, m_1 z}\,(\varphi + m_2 z)\;. \tag{3.36}$$

The reduced action corresponds to the sum of the separate actions corresponding to the phase- and shift symmetry.

$O(N)$ scalar fields

The $O(N)$ N-scalar model in $(D+1)$-dimensional flat space-time, is described by the Lagrangian $\mathcal{L}_{D+1}$

$$\mathcal{L}_{D+1} = -\tfrac{1}{2}\partial_{\hat\mu}\hat{\vec{\Phi}}^T\partial^{\hat\mu}\hat{\vec{\Phi}} - \tfrac{1}{2}m^2\hat{\vec{\Phi}}^T\hat{\vec{\Phi}}\,, \tag{3.37}$$

invariant under global orthogonal transformations working on the scalar fields

$$\hat{\vec{\Phi}} \to \hat{\vec{\Phi}}' = O\hat{\vec{\Phi}}\,, \qquad O \in O(N)\,, \quad O^T O = O\,O^T = \mathbb{1}\,. \tag{3.38}$$

In order to perform a generalized reduction of this theory, we again use the transformation parameter of the global symmetry (group) to put the z-dependence into, yielding the following Ansatz

$$\hat{\vec{\Phi}}(x,z) = O(z)\hat{\vec{\Phi}}(x)\,, \tag{3.39}$$

resulting in the action

$$\mathcal{L}_D = -\tfrac{1}{2}\partial_\mu\vec{\Phi}^T \underbrace{O^T\,O}_{=\mathbb{1}}\partial^\mu\vec{\Phi} - \tfrac{1}{2}\Big(m^2 + [\underbrace{O^T\,\partial_z O}_{\equiv O^{-1}\partial_z O}]^2\Big)\vec{\Phi}^T\,\vec{\Phi} \tag{3.40}$$

The only term in the lower-dimensional action, still containing the matrix $O(z)$, is proportional to $O^{-1}(z)\partial_z O(z)$. Since we do not want this term to contain any z-dependence, the term $O^{-1}(z)\partial_z O(z) \equiv M$ again can be interpreted as a mass matrix, just like in (3.28). The exact form of $O(z)$ is not important since it does not explicitly appear in the reduced theory, but using the properties of $O(z)$, the constraint can in principle be solved.

3.2.2 Non-Abelian reductions and Scherk-Schwarz II

In order to generalize the five-dimensional KK-theory we replace the internal space S^1 with some other (compact) space of higher dimension. The first generalization of the KK-mechanism was first considered by Pauli in 1953 [99]. His starting point was the six-dimensional space-time $\mathcal{M}_4 \times S^2$. The extra dimensions form a two-sphere S^2 with internal symmetry group SO(3) $\simeq$ SU(2)/ U(1). Using an appropriate Ansatz (given below for general case) he constructed a non-Abelian theory with gauge group SU(2), one year before Yang and Mills published their famous paper [100].

In the years after that further generalizations were proposed [101–106], which can be roughly classified in the following possibilities for the compact space E_n:

1. $E_n = T^n$: Torus reduction from $(D+n)$ to D dimensions on T^n, i.e. n successive circle reductions. Each reduction-step will give rise to a KK-vector and a dilaton. Also, p-form gauge fields will reduce to a p-form and $(p-1)$-form gauge field one dimension lower. Further reductions will also create 0-form potentials or *axions* coming from the KK-vector(s) in the compact directions. The reduced theory is ungauged and will finally consist of a plethora of scalars and vectors, in the adjoint of the gauge group U(1)^n.

2. $E_n = G$: Group manifold reduction, where G is the compact Lie-group associated with general coordinate transformations on the compact manifold, which will become the gauge group of the reduced theory.

3. $E_n = G/H$: Coset space reduction, where H is the maximal compact subgroup of G. The most common examples are sphere-reductions: $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Note that in most cases coset reductions are preferred above group manifold reductions, since less extra dimensions are needed to obtain a certain gauge group. E.g. in order to obtain the SO(8) gauge group one could use the SO(8) group manifold or the coset SO(8)/ SO(7) corresponding to the seven-sphere. The first case would require $dim(E_n) = 28$ whereas for the coset reduction one only needs $dim(E_n) = 7$.

4. inhomogeneous spaces or spaces without any isometries.

Note that not all these reductions can be performed in a consistent way. The only exceptions where consistency can be understood from group-theoretic arguments are the circle, torus or group-manifold reductions. All other reductions will have to satisfy certain requirements to get a full decoupling of the massless and massive modes in the KK-spectrum [107].

For cases 2 and 3 the generic line-element Ansatz is given by

$$\mathrm{d}\hat{s}^2 = \mathrm{d}s^2 + g_{\alpha\beta}\,[\mathrm{d}z^\alpha + K_I^\alpha(z)A_{I\mu}\mathrm{d}x^\mu][\mathrm{d}z^\beta + K_J^\beta(z)A_{J\nu}\mathrm{d}x^\nu]\,, \tag{3.41}$$

where $g_{\alpha\beta}$ is the internal metric on E_n. The coordinates $x^{\hat{\mu}}$ have been split into x^μ and compact coordinates z^α. The functions $K_I^\alpha(z)$ are the Killing vectors of the internal metric, generating the isometry group G with structure constants $f_{IJ}{}^K$

$$K_I \equiv K_I^\alpha \partial_\alpha \rightarrow \qquad \mathcal{L}_{K_I} K_J = [K_I, K_J] = f_{IJ}{}^K K_K\,. \tag{3.42}$$

These same structure constants also define the lower-dimensional gauge group since they end up in the covariant derivatives of the lower-dimensional vector fields, after reducing the general coordinate transformation of the metric under

$$\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}}\,, \qquad \hat{\xi}^\mu = 0\,, \quad \hat{\xi}^\alpha = K_I^\alpha(z)\lambda_I(x)\,. \tag{3.43}$$

The second type of Scherk-Schwarz reductions (SS2) makes use of these z-dependent diffeomorphisms on the group manifold, to assign to some fields a specific dependence on the compact coordinates. For details we refer to [95].