Chapter 2

Scalar potential: Domain-walls and other applications

In this chapter we will give some motivations for the research described in the remainder of this thesis. We will briefly show the relevance of scalar potentials and domain-walls to supergravity, field theories and cosmology. In the next chapters we will give two methods to obtain scalar potentials from supergravity, dimensional reduction and conformal supergravity.

2.1 Gauged and massive supergravities

Gauged supergravities have played an important role during the past 25 years in a broad range of applications. In most of these cases the key factor is the so-called scalar potential. Scalar potentials e.g. in bosonic scalar-gravity models are essentially non-restricted massive deformations; for example ϕ^4 theory coupled to gravity. However, in supersymmetric models, like gauged and massive supergravities, the form of the potential is fully determined. In that case the gauge coupling constant g can be related to the mass parameter m by: $m = \kappa g$. Supersymmetry in general not only restricts the form of the potential; it also imposes a geometrical structure on the collection of scalars in the theory, called the scalar manifold. We will see explicit examples of this in chapters 6 and 7.

Strictly speaking gauged supergravities are defined as supergravity theories where either a subgroup of or the full R-symmetry group is gauged, using one or more vectors present in the theory. In some cases this will involve the coupling of extra matter multiplets, e.g. vector multiplets. In practice the term gauged supergravity is often used to denote a gauging of an arbitrary global symmetry group.

The general procedure of gauging a supergravity theory consists of

- choosing an appropriate gauge group G.
- performing a minimal substitution, i.e. coupling vector fields A^I_μ to matter fields Φ^I by

introducing covariant derivatives

$$D_\mu \Phi^I = \partial_\mu \Phi^I + g\, A_\mu^J\, f_{JK}{}^I \Phi^K\,, \tag{2.1}$$

locally invariant under

$$\delta_G \Phi^I = -g\, \Lambda^J f_{JK}{}^I \Phi^K\,, \qquad \delta_G A_\mu^I = \partial_\mu \Lambda^I - g\, \Lambda^J f_{JK}{}^I A_\mu^K\,, \tag{2.2}$$

where g is the gauge coupling constant, Λ^I is the gauge parameter, and $f_{IJ}{}^K$ are the structure constants encoding the properties of the specific gauge group. Note that non-Abelian gaugings also allow for self-couplings between the vector fields.

- restoring supersymmetry by adding terms to the action and/or transformation rules, making use of the Noether method. This procedure generally gives rise to mass-terms for the fermions and contributes to the scalar potential.

In some specific cases a wide range of possible gauge groups have been classified, e.g. for $\mathcal{N} = 2$, $D = 5$ supergravity in [41]. More details on this subject will be given in chapter 7.

Note that not all potentials necessarily have to come from gauging. A good example of a so-called massive supergravity theory is Romans' [42] deformation of IIA (1.21) in ten dimensions with one mass parameter, consistent with supersymmetry. Although the string theory origin of these theories is somewhat unclear, their importance should not be underestimated. E.g. Romans' theory contains the D8-brane as a natural solution, coupling to the 'zero-form' mass parameter.

2.1.1 Vacua

In this section we will illustrate the way in which the vacua of gauged/massive supergravities are determined by the extrema of the scalar (super)potential. In conventional gravity or supergravity theories the vacua are those solutions of the field equations with maximal symmetry, i.e. the largest number of isometries. Let us first consider the D-dimensional Einstein-Hilbert action, with cosmological constant Λ

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g|}\, (R - 2\Lambda)\ . \tag{2.3}$$

The corresponding field equation is given by the vacuum Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0\,, \tag{2.4}$$

which, after taking the trace, gives an expression of the cosmological constant in terms of the Ricci-scalar

$$\Lambda = \tfrac{(D-2)}{2D}\, R\,. \tag{2.5}$$

Depending on the curvature of space-time, the vacua correspond to de Sitter (dS), anti-de Sitter (AdS) or flat Minkowski space:

anti-de Sitter (AdS)	:	negative curvature
Minkowski	:	zero curvature
de Sitter (dS)	:	positive curvature

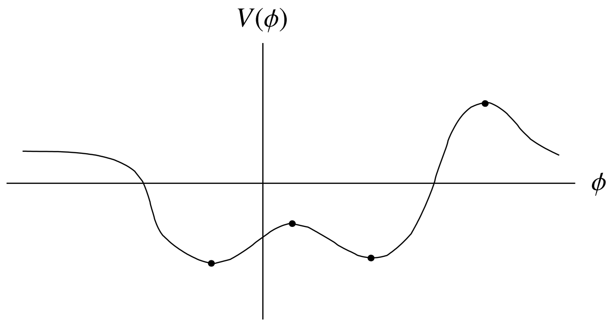

Figure 2.1: Critical points of the scalar potential.

Next consider a slight generalization corresponding to a minimal truncation of a gauged/massive sugra action. The action for a scalar field coupled to gravity is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g|} \Big(R - \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \Big) . \tag{2.6}$$

The equations of motion of (2.6) are given by

$$\begin{aligned} \nabla_\mu\,\partial^\mu\phi &= \frac{\partial V}{\partial\phi} , \\ R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \Big(\tfrac{1}{4}\partial_\rho\phi\,\partial^\rho\phi + \tfrac{1}{2}V(\phi)\Big)\,g_{\mu\nu} . \end{aligned} \tag{2.7}$$

In order to be consistent with maximum symmetry, the vacuum expectation value (v.e.v.) of the scalar field has to be constant, and should correspond to a local extremum of the potential called a critical point, see figure 2.1:

$$\langle\phi\rangle = \phi_c , \qquad \frac{\partial V}{\partial\phi}\Big|_{\phi=\phi_c} = 0 . \tag{2.8}$$

At these extrema the equations (2.7) reduce to the field equation describing three different vacuum solutions, depending on the value and sign of the cosmological constant. In the (A)dS cases the cosmological constant is given by

$$\Lambda = \tfrac{1}{2}V(\phi_c) . \tag{2.9}$$

In section 2.3 a specific class of vacuum solutions of (2.6) will be discussed, having $(D-1)$-dimensional Poincaré invariance and scalar v.e.v. that are dependent on the D-th coordinate. The geometry of these half-supersymmetric solitons, called domain-walls, interpolates between two conventional vacua with different cosmological constants.

2.1.2 Scalar (super)potential

Supergravity models generically consist of a basic supergravity multiplet coupled to any number of supermultiplet representations of the underlying supersymmetry algebra. Interactions in those models are usually described by three types of potentials for the scalar fields in the theory: the superpotential W and the potential V, derived from W.

The connection between V and W can be made clear by the following toy model. Let us consider the same scalar-gravity model as given in (2.6) and describe scalar fluctuations φ around the AdS vacuum associated with the critical point ϕ_c with $V(\phi_c) < 0$

$$\varphi = \phi - \phi_c\,. \tag{2.10}$$

Expanding the action (2.6) to lowest order around this critical point yields

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|g|}\left(R - \tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}M^2\varphi^2 - V(\phi_c)\right)\,, \tag{2.11}$$

with the following equations of motion

$$\begin{aligned}\left(\nabla_\mu\,\partial^\mu - M^2\right)\varphi &= 0\,,\\ R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} &= \mathcal{O}(\varphi^2)\,,\end{aligned} \tag{2.12}$$

describing a scalar particle of mass M in an AdS background with cosmological constant Λ

$$M^2 \equiv \left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\phi_c}\,, \qquad \Lambda \equiv \tfrac{1}{2}V(\phi_c)\,. \tag{2.13}$$

A more general case was considered by Townsend [43]: supergravity coupled to vector multiplets, also called Einstein-Maxwell supergravity. This theory contains multiple scalars ϕ^x that can be interpreted as coordinates on some manifold described by metric g_{xy}

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|g|}\left(R - \tfrac{1}{2}g_{xy}(\phi)\,\partial_\mu\phi^x\,\partial^\mu\phi^y - V(\phi)\right)\,. \tag{2.14}$$

In the bosonic case it was shown from stability requirements of the AdS vacua that the potential can be expressed in terms of the superpotential

$$V(\phi) = 4(D-2)^2\left[2g^{xy}\,\frac{\partial W}{\partial\phi^x}\frac{\partial W}{\partial\phi^y} - \frac{D-1}{D-2}\,W(\phi)^2\right]\,. \tag{2.15}$$

A similar result can be obtained by requiring supersymmetry invariance, where the superpotential can be read off from the transformation rules of the fermions

$$\begin{aligned}\delta\psi_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab})\epsilon + W(\phi)\,\Gamma_\mu\,\epsilon\,,\\ \delta\lambda_x &= g_{xy}\,\partial\!\!\!/\phi^y\epsilon - (D-2)\,\frac{\partial W}{\partial\phi^x}\epsilon\,.\end{aligned} \tag{2.16}$$

These contributions to the scalar potential are called the fermion-shifts. Note that not all potentials in supersymmetric theories can be written in terms of a superpotential. We will see examples of this in chapter 4.

2.1.3 Applications

From the above it should be clear that it is very useful to study the properties of gauged supergravities. In particular the scalar potential can provide crucial information about the vacua, solutions and dynamics of supergravity theories, that can be used in many applications, some of which will be further explained in the following sections.

- The DW/QFT correspondence is a conjectured duality between supergravity on a domain-wall background and a quantum field theory. A special case of this duality is the AdS/CFT duality. Useful properties of field theories can be obtained by studying domain-wall solutions of supergravity, which are fully determined by the form of the scalar potential. By using this duality, the domain-wall geometries describe renormalization group flows in the dual field theory. See sections 2.2 and 2.3 for more details.

- Brane-world scenarios try to describe our four-dimensional world by assuming that we live on the worldvolume of a domain-wall solution in five dimensions. Whether a supersymmetric embedding of these scenarios is possible or not depends on the vacuum solutions of the scalar potential in gauged $\mathcal{N} = 2$ supergravity in five dimensions.

- Inflationary models are used to study several issues in cosmology like the smallness of the cosmological constant, the horizon problem and the isotropy of the universe. These models try to explain the dynamical properties of the universe by studying the scalar-potentials occurring in specific scalar-gravity systems. For certain values of the so-called "slow-roll parameter" cosmic inflation occurs, as the result of the "rolling" of the scalar field towards the minimum of the potential. For a review see [44, 45].

2.2 AdS/CFT

One of the most important developments of the past few years has been the conjecture of Maldacena in 1997, called the AdS/CFT correspondence [46]. This was later generalized to the DW/QFT correspondence [47, 48]. Before giving a brief explanation of this conjecture, let us first give some relevant information about the geometry of anti-de Sitter.

2.2.1 (A)dS geometry

The geometry of AdS_D is given by the SO(2, $D-1$) invariant hyperboloid in $(D+1)$-dimensional space

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{D-1} X_i^2 = -\Re^2 \,, \tag{2.17}$$

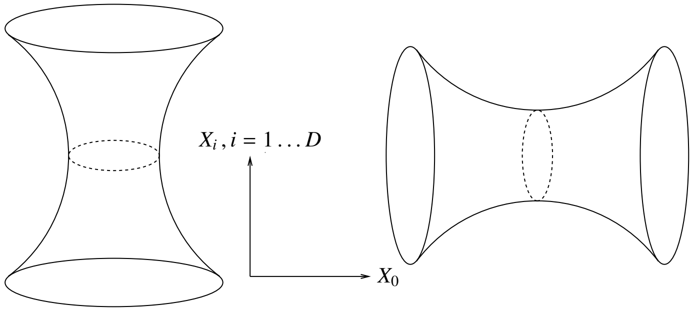

Figure 2.2: AdS_D and dS_D as hyperboloids in $\mathbb{R}^{2,D-1}$.

where $\Re$ denotes the AdS-radius, see figure 2.2. This induces the following line element in terms of so-called horospherical coordinates $\{x_\alpha, U, V\}$:

$$\begin{aligned} U &= X_{-1} + X_{D-1}\,, \\ x_\alpha &= \frac{X_\alpha \Re}{U}\,, \qquad (\alpha = 0, \dots, D-2)\,, \\ V &= X_{-1} - X_{D-1} = \left(\frac{U}{\Re^2}\right)x^2 + \left(\frac{\Re^2}{U}\right)\,, \\ \mathrm{d}s^2 &= \left(\frac{U}{\Re}\right)^2 \mathrm{d}x^2 + \left(\frac{\Re}{U}\right)^2 \mathrm{d}U^2\,. \end{aligned} \tag{2.18}$$

A more convenient parametrization in the context of brane-world scenarios, that we will encounter further on, are the so-called Poincaré coordinates

$$\mathrm{d}s^2 = e^{-2r/\Re}\mathrm{d}x^2 + \mathrm{d}r^2\,, \qquad e^{-r/\Re} = \frac{U}{\Re}\,. \tag{2.19}$$

2.2.2 Maldacena conjecture

In 1997 a remarkable connection between string theory and gauge theory was conjectured by Maldacena [46], proposing an equivalence between string theory on an $AdS_p \times S^{D-p}$ and a conformal field theory (CFT) in $(p-1)$ dimensions, on the boundary of the AdS space. We will illustrate this statement by briefly describing Maldacenas original motivation.

First imagine we have an open string with both endpoints ending on a single D3-brane. As the lowest mode is given by a vector field with U(1) gauge invariance, this induces a four-dimensional U(1) gauge theory on the brane. Since the D3-brane is a half-BPS solution, breaking half of the total number of supersymmetries, the $D = 4$ U(1) theory has $\mathcal{N} = 4$ supersymmetry.

Now extend this to a system of N parallel D3-branes, separated by a distance r. The open strings stretching between the various branes again induce a U(1) gauge theory on each brane. In the limit of $r \to 0$ we have a stack of coinciding branes and the gauge symmetry on the branes is enhanced from $U(1)^N$ to $U(N)$, which in the low energy limit describes a four-dimensional CFT with gauge group $SU(N)$, known as $D = 4$ $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. The bosonic symmetry group of this gauge theory is given by the product of the conformal group in four dimensions, $SO(4, 2)$ and the R-symmetry group $SO(6)$.

On the other hand we know that the stack of D3-branes, like any massive object, causes the space-time to curve. Far away from the branes the space-time is given by Minkowski space but in the near-horizon limit, i.e. near the branes, the geometry can be shown to resemble that of the space $AdS_5 \times S^5$. Since the radii of the sphere and of the AdS space are proportional to N, the resemblance gets better for increasing N. The isometry group of this background geometry is given by $SO(4, 2) \times SO(6)$.

Consequently it was conjectured that IIB string theory on a $AdS_5 \times S^5$ background in the large N limit is dual to a CFT on the boundary of AdS_5, given by $\mathcal{N} = 4$ SYM. This statement was later generalized to the so-called Domain-wall/Quantum field theory (DW/QFT) correspondance, that relates supergravity on a near-horizon geometry of a p-brane to a (non-conformal) QFT on the brane. The most striking result of these correspondances is that they relate a gravitational theory, like supergravity or string theory, to a non-gravitational conformal field theory.

In the context of this conjecture, it is useful to study $\mathcal{N} = 8$, $D = 5$ gauged supergravity; the dimensional reduction of IIB on $AdS_5 \times S^5$ gives $SO(6)$ gauged $D = 5$, $\mathcal{N} = 8$ sugra [49, 50]. This reduction is believed to be a consistent nonlinear truncation, meaning that all classical solutions of the five-dimensional theory can be uplifted to IIB solutions. For example, the $SO(6)$-invariant AdS_5 groundstate can be uplifted to an $AdS_5 \times S^5$ vacuum. Therefore the five-dimensional theory should contain all relevant deformations of $\mathcal{N} = 4$ SYM, and all domain-wall solutions in this theory can be uplifted.

During the past few years a considerable amount of evidence has been gathered in many different applications to support the AdS/CFT conjecture. For more details we refer to the reviews [46, 51].

2.3 Domain-walls

Domain-walls are (D-1)-dimensional solutions of the sugra equations of motion, separating two regions of space-time. In the case of a (singular) (D-2)-brane solution, also called 'thin' domain-wall, the brane couples to a volume form which can be dualized to a cosmological constant. The value and/or sign of the cosmological constant usually is different when passing both sides of the domain-wall. The other type of domain-wall is the 'smooth' or 'thick' domain-wall.

2.3.1 Solution

The 'thin' domain-wall solution is a δ-function singularity, given by (1.45) with $p = D - 2$

$$\text{domain-wall} = \begin{cases} \mathrm{d}s^2 &= H(y)^{2\alpha}\mathrm{d}x^2_{(D-1)} + H(y)^{2\beta}\mathrm{d}y^2\,, \\ e^{\varphi} &= H(y)^{-\frac{2a}{\Delta}}\,, \\ F_{(D)} &= \sqrt{\frac{4}{\Delta}}\,\mathrm{d}^d x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + Q|y|\,. \end{cases} \tag{2.20}$$

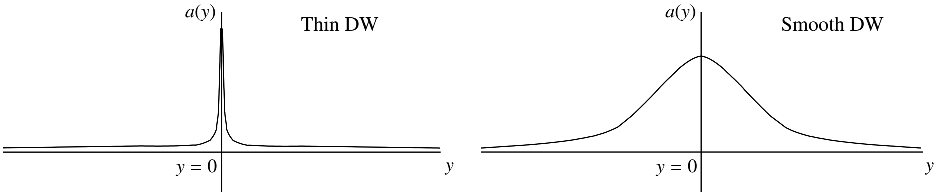

Figure 2.3: Warp-factor for singular and smooth branes.

with

$$\alpha = \tfrac{2}{\Delta(D-2)}\,, \qquad \beta = \tfrac{2(D-1)}{\Delta(D-2)}\,, \qquad Q = \sqrt{\Lambda\Delta}\,, \tag{2.21}$$

where the parameter Δ given by

$$\Delta = a^2 - 2\tfrac{D-1}{D-2}\,. \tag{2.22}$$

This expression for Δ is bounded from below by the value Δ_{AdS} corresponding to the AdS vacuum solution of pure supergravity[1]

$$\Delta \geq \Delta_{AdS} \equiv -2\tfrac{D-1}{D-2}\,. \tag{2.23}$$

It was indeed shown [52] that the corresponding domain-wall solution describes two regions of AdS-space. More generally it can be shown that the near-brane limit of solutions of this type are flat Minkowski, and the asymptotic limit far away from the brane describes AdS geometry. Domain-walls therefore are solutions interpolating between two vacua of the theory.

In phenomenological and cosmological models people are usually more interested in non-singular solutions, where there is no δ-function source. A more general domain-wall Ansatz can be written as

$$ds^2 = a(y)^2 dx^2_{(D-1)} + dy^2\,, \tag{2.24}$$

where the function a is called the warp factor; see figure 2.3. For $a(y) = e^{-|y|/L}$ this corresponds to the domain-wall consisting of two slices of AdS in Poincaré coordinates. Depending on the properties of the scalar potential smooth solutions for $a(y)$ can exist corresponding to so-called thick domain-walls.

2.3.2 Toy model: domain-walls as RG-flows

In this section we will show how domain-walls can be associated with renormalization group flows (RG-flows). The application of the AdS/CFT duality shows that supergravity flow equations, connecting critical points of the scalar potential, describe (holographic) RG-flows of quantum field theories, connecting different fixed points.

An exact analysis of the scalar potential is in general not possible, due to the non-trivial geometry of the scalar manifold and the large number of scalars appearing in the potential. Instead of trying to solve the minimization problem at the level of the second order equations of motion, there is a more appealing method.

[1] Corresponding to the case $a = 0$, i.e. constant dilaton; the metric therefore describing AdS-space.

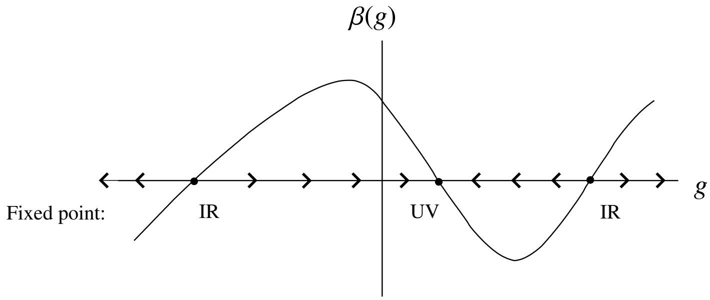

Figure 2.4: Fixed points of the beta-function.

Our starting point is the smooth domain-wall Ansatz in the context of the scalar-gravity toy model (2.6)

$$\mathrm{d}s^2 = e^{2A(y)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2\,, \quad \phi = \phi(y)\,. \tag{2.25}$$

As we saw before, at the critical points of $V(\phi)$ the scalar ϕ is constant and the geometry becomes AdS with cosmological constant given by $\Lambda = \frac{1}{2}V(\phi_c)$. However, we also saw that these AdS vacua are dual to a conformal field theory on the boundary of the AdS space-time. Relevant deformations of these CFTs on the field theory side give rise to so-called RG-group flows to other conformal theories. Mapped to its gravity dual this corresponds to scalar fluctuations around AdS space-time. The RG-flow of the coupling constants is described by the U dependence of the scalar fields. These scalars ϕ can be interpreted as coupling constants g and the warp-factor $a(y) = e^{A(y)}$ behaves as a renormalization group scale or energy scale U in the dual field theory side. The expression of the field theory beta-function is conventionally given by

$$\beta(g) \equiv U\frac{\partial g(U)}{\partial U}\,, \tag{2.26}$$

and is depicted in figure 2.4. The arrows denote the flow-direction of the coupling constant g with increasing energy U. The zeroes of the beta-function, called *fixed points*, correspond to scale-independent conformal field theories. These fixed points correspond to critical points of the scalar potential on the supergravity side. There are two types of fixed points: IR points corresponding to low energy scales and UV points at high energy scales, behaving as attractors. A small dictionary mapping between objects in gauge/gravity theory is given in table 2.1.

Returning to equation (2.25), we see that this geometry describes anti-de Sitter space if we take $A(y) = -\frac{y}{L}$. Using the metric-Ansatz (2.25) the equations of motion (2.7) become [53]

$$\begin{aligned} \phi''(y) + (D-1)A'(y)\phi'(y) &= \tfrac{\partial V}{\partial \phi}\,, \\ (D-2)A''(y) + \tfrac{(D-1)(D-2)}{2}A'(y)^2 &= -\tfrac{1}{4}\phi'(y)^2 - \tfrac{1}{2}V(\phi)\,, \\ \tfrac{(D-1)(D-2)}{2}A'(y)^2 &= \tfrac{1}{4}\phi'(y)^2 - \tfrac{1}{2}V(\phi)\,. \end{aligned} \tag{2.27}$$

These equations can be interpreted as Euler-Lagrange equations for the energy functional

$$E = \int_{-\infty}^{\infty} \mathrm{d}y \frac{e^{(D-1)A(y)}}{D-2}\left(-(D-1)(D-2)A'(y)^2 + \frac{1}{2}\phi'(y)^2 + V(\phi)\right)\,. \tag{2.28}$$

Sugra on AdS_D	$(D-1)$-dim. gauge theory
Critical point: AdS space-time	Fixed point: CFT $\beta = 0$
Warp-factor $a(y)$	Energy scale U
Scalar $\phi(y)$	Coupling constant $g(U)$
Domain-wall flow-equations	RG-flow

Table 2.1: A domain-wall/RG-flow dictionary.

Substituting equation (2.15), and using the Bogomol'nyi trick to write E as a sum of squares we obtain

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \mathrm{d}y \frac{e^{(D-1)A(y)}}{D-2} \left(\frac{1}{2} \left[\phi'(y) \mp (D-2) \frac{\partial W}{\partial \phi} \right]^2 - (D-1)(D-2) \left[A'(y) \pm \frac{1}{2} W(\phi) \right]^2 \right) \\ & \pm \left[e^{(D-1)A(y)} W(\phi) \right]_{-\infty}^{\infty} . \end{aligned} \tag{2.29}$$

Written in this form the equations minimizing the energy are easily read off to be

$$\begin{aligned} \phi'(y) &= \mp(D-2)\frac{\partial W}{\partial \phi} , \\ A'(y) &= \pm\frac{1}{2} W(\phi) . \end{aligned} \tag{2.30}$$

These equations describe gradient-flows on the hypersurface given by the functional $W(\phi^i)$ in the scalar manifold. Contrary to the second order equations of motion, the analysis of these first order equations is much simpler. Solutions of the flow-equations are automatically solutions of the equations of motion.

Remarkably, in gauged supergravity theories the flow equations could also have been obtained by plugging the same Ansätze (2.25) and (2.15) into the BPS-equations corresponding to the domain-wall solution

$$\begin{aligned} \delta\psi_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab})\epsilon + W(\phi)\,\Gamma_\mu\,\epsilon = 0 , \\ \delta\lambda &= \not{\partial}\phi\epsilon - (D-2)\,\frac{\partial W}{\partial \phi}\epsilon = 0 . \end{aligned} \tag{2.31}$$

In any theory there are generally different relevant deformations possible, all describing certain RG-flows: IR-UV, UV-UV, IR-IR. For instance, in [54, 55] a flow was constructed from $\mathcal{N} = 4$ SYM to $\mathcal{N} = 1$ SYM by studying the scalar potential of $\mathcal{N} = 8$, $D = 5$ supergravity. Flows of the type IR-IR are of particular interest in the context of supersymmetric brane-world scenarios as we will see in the following section. Figure 2.5 gives an example of a flow between two IR-IR fixed points. The big question however still remains. . . does there exist a corresponding domain-wall? The answer to this question can be given by studying the scalar potential of the most general matter-coupled gauged supergravity theory.

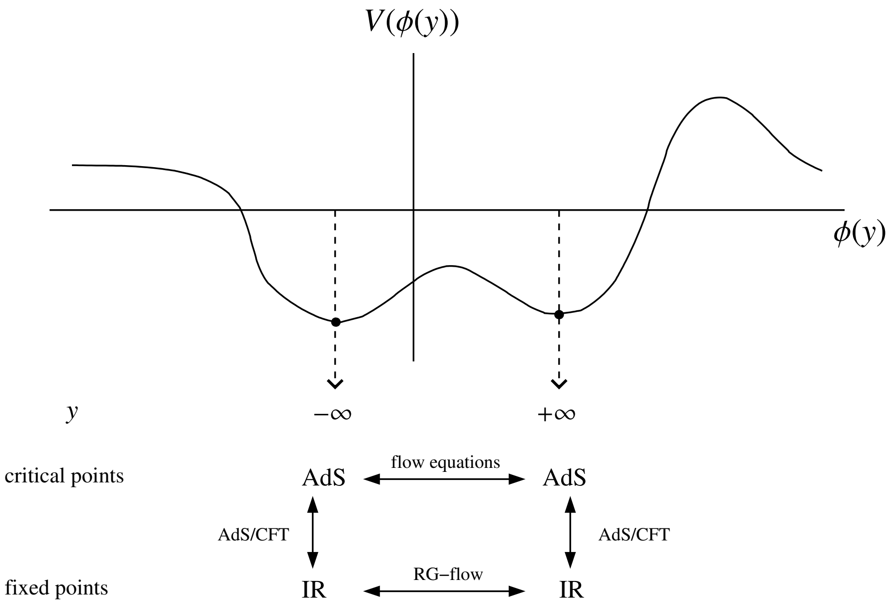

Figure 2.5: Domain-walls as Renormalization Group Flows.

2.4 Brane-world scenarios

The idea of brane-worlds rests on the assumption that our four-dimensional space-time is given by an infinitesimally thin three-brane floating in $(4+n)$ dimensions. Standard model particles are living on the brane but gravity extends in the transverse dimensions. In 1999 Randall and Sundrum proposed two specific brane-world models, motivated to solve a couple of long standing problems in theoretical physics: the hierarchy problem and the cosmological constant problem. The hierarchy problem covers the huge difference of order of magnitude between the Planck scale and the weak scale. Some of the older models tried to explain this using large extra dimensions [56, 57]. Although the idea by Randall and Sundrum is not completely new [58], their approach came with remarkable new insights that stimulated further research until the current moment. In their original two papers they gave two different models, RS I [59] and RS II [60] which will be schematically described below. For more details we refer to the original papers or the reviews [61–63].

2.4.1 Randall-Sundrum I: two branes

The two-brane scenario is a model of five-dimensional gravity on an orbifold $\mathcal{M}_4 \times S^1/\mathbb{Z}_2$ with two three-branes located on both $\mathbb{Z}_2$ fixed points, separated by a distance πr_c. The brane at $y = 0$ is called the "hidden" or "Planck" brane and the one at $y = \pi r_c$ the "visible" or "Standard model" brane (see figure 2.6). The idea is simple:

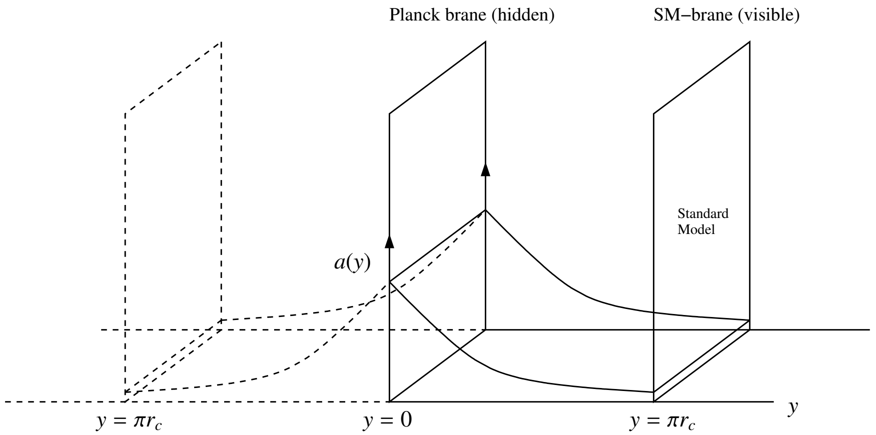

Figure 2.6: The two-brane Randall-Sundrum scenario.

- Symbolically write down an action for the combined system

$$\begin{aligned} S &= S_{\text{gravity}} + S_{\text{vis}} + S_{\text{hid}}\,, \\ S_{\text{gravity}} &= \int \mathrm{d}^4x\,\mathrm{d}y\,\sqrt{|G|}\left(2M^3R - \Lambda\right), \\ S_{\text{brane}} &= \int \mathrm{d}^4x\,\sqrt{|g_{\text{ind}}|}\,(\mathcal{L} - V_{\text{brane}})\,, \qquad \text{(for both branes)} \end{aligned} \tag{2.32}$$

  where g_{ind} is the induced metric on the brane, V_{brane} the vacuum energy of the brane, and M the five-dimensional Planck mass.

- Write down an Ansatz for the background metric, possessing four-dimensional Poincaré invariance

$$\mathrm{d}s^2 = a(y)^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + r_c^2\mathrm{d}y^2\,, \qquad a(y) = e^{-\sigma(y)}\,. \tag{2.33}$$

- Deduce the modified Einstein equations from (2.32) and (2.33). These equations are solved by

$$\sigma(y) = r_c|y|\sqrt{\frac{-\Lambda}{24M^3}}\,, \qquad V_{\text{hid}} = -V_{\text{vis}} = -\Lambda/k\,, \qquad \Lambda = -24M^3k^2\,, \tag{2.34}$$

  where k is some integration constant. We see that the solution of the warp-factor requires the background to consist of two slices of AdS in Poincaré coordinates.

As a result of the above procedure the effective Planck scale on the brane can be calculated to be

$$M_{\text{pl}}^2 = \frac{M^3}{k}\left(1 - e^{-2\pi r_c k}\right). \tag{2.35}$$

The hierarchy problem can now be solved by taking the five-dimensional Planck scale to be of the order of the weak scale, and considering the effective theory on the visible brane at $y = \pi r_c$. If

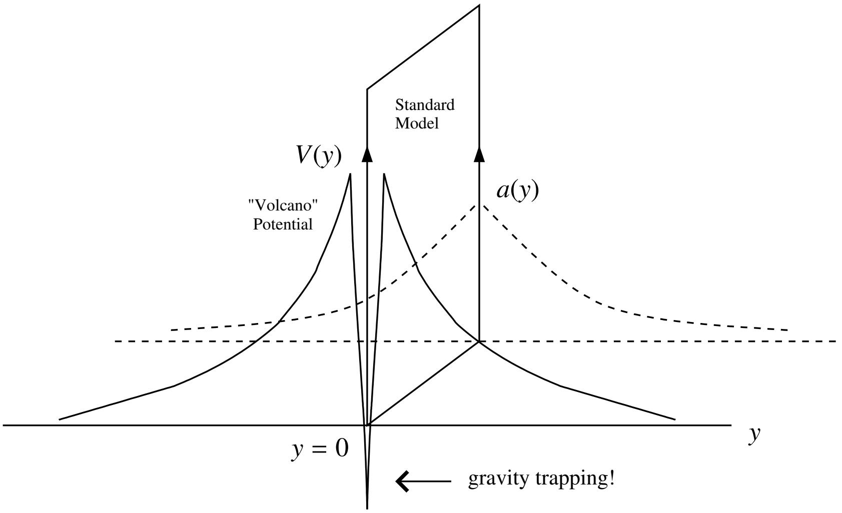

Figure 2.7: The single-brane Randall-Sundrum scenario.

we take $r_c k \approx 50$, this results in a scale hierarchy due to the exponential form of the warp-factor. This was concluded by considering matter fields living on the visible brane. Although solving the hierarchy problem, nevertheless this scenario is still problematic; it lacks the possibility of localization of gravity on the visible brane. Also the presence of a negative tension brane was required. Soon after the RS I model was proposed, another model was suggested, with only one brane, to resolve these problems.

2.4.2 Randall-Sundrum II: one brane

The one-brane scenario initially starts off with the same setup as the one described in the previous section , but the invisible brane is sent to infinity, and is therefore physically removed from the model, see figure 2.7. The brane-tension of the remaining brane is positive and again fine-tuned against the bulk cosmological constant. Instead of solving the hierarchy problem, the warp-factor is now used for the localization of the graviton to the brane. By studying fluctuations of the metric G it was shown that they are effectively described by Newton's equation on the brane, predicting higher order corrections to the Newtonian potential

$$V_N(r) = G_N \frac{m_1 m_2}{r}\left(1 + \frac{1}{r^2 k^2}\right). \tag{2.36}$$

Although both these models have appealing properties, fermions will have to be included in order to obtain phenomenologically interesting models.

2.4.3 Supersymmetric Randall-Sundrum scenario

The simplest way of including fermions in the theory is by trying to embed the model in a supersymmetric theory. The best candidate for this theory is thought to be $D = 5$, $\mathcal{N} = 2$ supergravity. As we saw in the previous sections some parameter tweaking was necessary for obtaining a consistent model. The main obstruction of a supersymmetric analog however is that the scalar potential is now more restricted, not leaving a lot of room for tweaking the parameters of the solution. Furthermore, the three-brane used in the model should be a valid supergravity solution, namely a domain-wall in five dimensions. Several possible solutions were suggested to resolve this issue.

One solution was given by [64, 65] who considered the insertion of singular brane sources in order to restore supersymmetry in spaces with singularities such as the thin domain-walls. This scenario is conjectured to be the dimensional reduction of the eleven-dimensional Hořava-Witten model [27], on some six-dimensional Calabi-Yau manifold [66, 67].

Another, more appealing, solution would be to find a smooth soliton solution interpolating between two AdS vacua. For such solutions to be compatible with a supersymmetric Randall-Sundrum scenario, the scalar potential should have at least two connected stable IR critical points with the same value of the cosmological constant. Secondly, the flow-equations should be solvable for the smooth domain-wall Ansatz and the corresponding warp-factor should be exponentially decreasing for $y \to \pm\infty$. In order to find such solutions a thorough investigation of the most general gauged $\mathcal{N} = 2$, $D = 5$ sugra is needed. Note that brane-world models can be given a place in string theory, by requiring this five-dimensional theory to follow from a specific Calabi-Yau compactification of M-theory. Alternatively one could try to find an explicit embedding of $\mathcal{N} = 2$ in $\mathcal{N} = 8$ sugra in $D = 5$, which could be related to string theory by the AdS/CFT conjecture.

Let us give a brief description of the field content of ungauged $\mathcal{N} = 2$, $D = 5$ supergravity and its relevant matter multiplets (I labelling the representation of the gauge group):[2]

- (8 + 8) Gravity multiplet: vielbein $e_\mu{}^a$, two gravitinos ψ^i_μ, graviphoton A_μ
- (8 + 8) Vector multiplets: vector A^I_μ, two gauginos ψ^{iI}, scalar σ^I
- (4 + 4) Hyper multiplets: four quaternions q^X, two hyperinos ζ^A

In five dimensions we can also have self-dual tensor multiplets, provided the vectors are in the adjoint representation. Otherwise the tensors can be dualized into vectors. Normally these tensors are self-dual in the sense as explained in [68].

Pure ungauged $\mathcal{N} = 2$, $D = 5$ sugra was constructed by Cremmer in 1980 [69]. A few years later Günaydin, Sierra and Townsend constructed U(1)- and SU(2)-gauged $\mathcal{N} = 2$, $D = 5$ sugra coupled to an arbitrary number of vector multiplets [70–72]. Several years ago Günaydin and Zagermann added tensor-couplings for specific gauge groups [73–75]. Finally, in 2000, Ceresole and Dall'Agata constructed gauged $\mathcal{N} = 2$, $D = 5$ sugra coupled to n_V vectors, n_T tensors and n_H hyper multiplets [76].

The analysis of the scalar potential in these such theories is highly non-trivial. Several simplified cases therefore have been considered in the literature. Many NO-GO theorems have

[2]The bosonic and fermionic degrees of freedom are denoted between parenthesis.

been posed [77–81]. It was found that without hyper-couplings no IR critical points could be found [55, 77, 78, 82]. After including hypermultiplets several solutions were found yielding only one single IR critical point [54, 55, 83]. In [79] multiple critical points were found, having at least one IR direction [3], but they were not connected.

Last year however a possible solution of a smooth domain-wall was found, by Behrndt and Dall'Agata [84], admitting a supersymmetric extension of the Randall-Sundrum scenario. They considered $\mathcal{N} = 2$ sugra coupled to a single hypermultiplet. The crucial ingredient was the restriction to a specific class of non-homogeneous quaternionic manifolds. A generalization to more general non-homogeneous quaternionic manifolds was recently considered by Anguelova and Lazaroiu [85]. Although solutions already have been found, they are not by far the most general solutions possible. First of all because only the coupling of one hypermultiplet was analyzed. Secondly, because a specific type of tensor-couplings was overlooked in the literature, corresponding to non-compact gaugings, which could have surprising implications. We constructed this extension in the context of $\mathcal{N} = 2$, $D = 5$ conformal supergravity [86]; this will be discussed in chapters 5, 6 and 7.

[3] Saddle points of the scalar potential.