Chapter 1

String theory and supergravity

In this chapter we will briefly review some basic aspects of string theory, supergravity, dualities and (membrane) solutions.

1.1 Free string theory

For obtaining the dynamics of a classical string it is natural to consider the higher dimensional generalization of the relativistic particle. The trajectory of a free relativistic point particle is described by the minimization of the length of its worldline. Equivalently, the action for a free classical string in D dimensions will be proportional to the area of its worldsheet, i.e. the two-dimensional surface it spans in space-time. The worldsheet can be parametrized by the spacelike variable σ ($0 \leq \sigma \leq \ell_s$), the coordinate along the string of length ℓ_s, and timelike variable τ. The embedding of the string worldsheet in Minkowski space-time is given by the functions $X^\mu(\sigma,\tau)$ ($\mu = 0,\ldots,D-1$). The action describing the string dynamics is called the Nambu-Goto action [2, 3],

$$S = -T \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{|\det(\partial_\alpha X^\mu \partial_\beta X_\mu)|}\,, \tag{1.1}$$

where T is the string tension given by $\frac{1}{2\pi\alpha'}$ with $\alpha' = \frac{\ell_s^2}{\hbar}$ the so-called Regge-slope. The indices α, β run over σ and τ. Although this form of the action is quite natural, there is a better formulation more suitable for e.g. quantization of the string, without the square root. This action was first discovered by Brink, Di Vecchia, Deser, Howe and Zumino [4, 5] but is better known as the Polyakov action [6]

$$S = -\frac{T}{2} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{|\gamma|}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu\,, \tag{1.2}$$

where an auxiliary worldsheet metric $\gamma_{\alpha\beta}$ has been introduced ($\gamma \equiv \det \gamma_{\alpha\beta}$). The two actions are equivalent after eliminating $\gamma_{\alpha\beta}$ by using its equation of motion. The reparametrization invariance of the worldsheet can be used to choose the so-called conformal gauge in which we take the worldsheet metric to be equal to the two-dimensional Minkowskian metric. Then

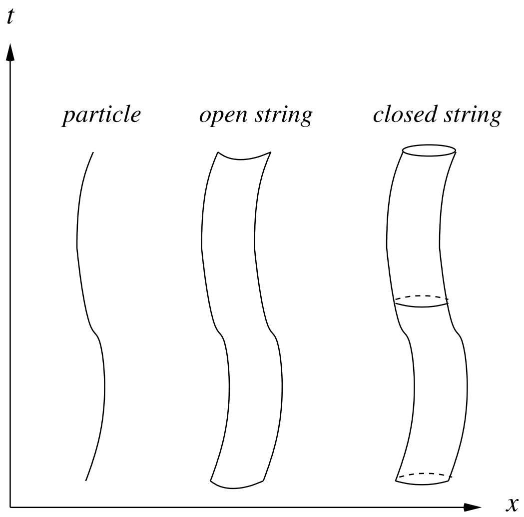

Figure 1.1: Open and closed string worldsheets.

Type of string	Boundary condition		
closed	periodic:	$X^\mu(\sigma + \ell_s) = X^\mu(\sigma)$	
open	Dirichlet:	$X^\mu(\sigma) = \text{constant}\,,$	$\sigma = 0, \ell_s$
	Neumann:	$\partial_\sigma X^\mu(\sigma) = 0\,,$	$\sigma = 0, \ell_s$

Table 1.1: Boundary conditions for open and closed strings

using (1.2) the equation of motion for the string can be easily found:

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0\,. \tag{1.3}$$

Note that this wave equation can be solved with two different types of boundary conditions (see table 1.1) describing either open or closed strings (see figure 1.1). The solutions are now fully determined in terms of oscillator expansions, both for left and right moving modes

$$X^\mu(\sigma,\tau) = X^\mu_{\rm L}(\sigma+\tau) + X^\mu_{\rm R}(\sigma-\tau)\,. \tag{1.4}$$

At this point, consistently quantizing the string turns out to restrict the space-time dimension to $D = 26$. The oscillation modes of the string behave as particles, having specific mass and energy. Studying the spectrum one finds:

open string: tachyon (scalar) T_1, massless vector A_μ, $\cdots$
closed string: tachyon (scalar) T_2, dilaton (scalar) ϕ, graviton $h_{\mu\nu}$ (symmetric, traceless), two-form $B_{\mu\nu}$ (antisymmetric), $\cdots$

The spin-2 graviton particle is believed to be the gauge-particle mediating the gravitational force. So as a surprising result we see that theories with closed strings (or self-interacting open strings) seem to contain gravity! This led people to believe that string theory could form the basis of a theory of quantum gravity. However, the open string spectrum still contains a tachyon as ground state, an unphysical particle with negative mass squared. Furthermore any unified theory of elementary particle physics also should contain fermions. It turns out that including fermions in our theory will provide us with a way to eliminate the tachyon from the spectrum. Also, consistency of the theory will further restrict the number of dimensions to $D = 10$. We can add fermions to (1.2) by again choosing the conformal gauge and adding a kinetic term for a two-component worldsheet Majorana spinor

$$\psi_\mu \equiv \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix} , \tag{1.5}$$

transforming as vectors under the space-time Lorentz group, giving [7]

$$S = -\frac{T}{2} \int \mathrm{d}\sigma \mathrm{d}\tau \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - \mathrm{i}\, \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) , \tag{1.6}$$

where ρ^α is a two dimensional representation of the Clifford algebra. This action turns out to have a worldsheet symmetry called *supersymmetry*, mapping the fermions to bosons and visa versa. Just like in the bosonic case we can have two types of boundary conditions for the open string

$$\begin{aligned} \text{Ramond (R):} \quad & \psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \quad \psi_+^\mu(\ell_s,\tau) = \psi_-^\mu(\ell_s,\tau) , \\ \text{Neveu-Scharz (NS):} \quad & \psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \quad \psi_+^\mu(\ell_s,\tau) = -\psi_-^\mu(\ell_s,\tau) . \end{aligned} \tag{1.7}$$

For the closed string the periodic Ramond or anti-periodic Neveu-Schwarz boundary conditions for left and right moving modes can be chosen independently, resulting in four different sectors: R-R, NS-NS, R-NS and NS-R. Demanding the spectrum of (1.6), apart from worldsheet supersymmetry, also to have space-time supersymmetry, will lead to the so-called Gliozzi-Scherk-Olive(GSO)-projection [8]. Since the fermionic spectrum does not have any negative mass-squared states and the massless sector has to be supersymmetric, this projection will successfully eliminate the tachyonic ground state from the spectrum. This theory, having manifest worldsheet supersymmetry, is called the Neveu-Schwarz-Ramond (NSR) formalism; a GSO-projection is needed to obtain space-time supersymmetry.

There is another formulation of superstring theory, called the Green-Schwarz (GS) formulation. This theory describes the embedding of the bosonic worldsheet in superspace and is therefore manifestly space-time supersymmetric. However, quantization of this theory until recently [9–11] was only possible in the light-cone gauge.

Using either the NSR or the GS formalism, choosing several combinations of the boundary conditions in the open and closed string case turns out to yield five different supersymmetric string theories: Type IIA, Type IIB, Type I, Heterotic $E_8 \times E_8$ and Heterotic SO(32). Type IIA and Type IIB are theories of closed strings and contain $\mathcal{N} = 2$ space-time supersymmetry. In Type IIA both supersymmetry parameters have opposite chirality, whereas in Type IIB they are equal. Type I is the only open string theory, and has $\mathcal{N} = 1$ supersymmetry. Both Heterotic theories also have $\mathcal{N} = 1$ and differ by their gauge groups, under which the massless vector fields transform.

1.2 Nonlinear sigma model

Until now we only considered non-interacting strings, moving in a flat Minkowski background. Next consider the closed bosonic string in a more general background consisting of the massless states $(\phi, h_{\mu\nu}, B_{\mu\nu})$, generated by vibrating closed strings in the bulk. The resulting action, invariant under worldsheet reparametrizations, is called the nonlinear sigma model

$$S = -\frac{T}{2}\int \mathrm{d}\sigma\mathrm{d}\tau \left\{\left(\sqrt{|\gamma|}\,\gamma^{\alpha\beta}g_{\mu\nu} - \varepsilon^{\alpha\beta}B_{\mu\nu}\right)\partial_\alpha X^\mu \partial_\beta X^\nu - \alpha'\sqrt{|\gamma|}\,\phi\,\mathcal{R}^{(2)}(\gamma)\right\}, \tag{1.8}$$

where the background metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\mathcal{R}^{(2)}$ is the Ricci-scalar of the worldsheet metric $\gamma_{\mu\nu}$. The last term in the action, with ϕ taken equal to 1, is proportional to a topological invariant quantity in two dimensions, called the Euler characteristic χ

$$\chi = \frac{1}{4\pi}\int \mathrm{d}\sigma\mathrm{d}\tau\,\sqrt{|\gamma|}\mathcal{R}^{(2)}(\gamma) = 2(1-g)\,, \tag{1.9}$$

where g denotes the genus of the Riemann surface swept out by the string. A redefinition of the dilaton in terms of its vacuum expectation value: $\phi \to \phi + \langle\phi\rangle$ gives a rescaling of the classical path integral with a factor $e^{\langle\phi\rangle\chi}$. As a consequence, every interaction vertex will have an associated string coupling constant

$$g_s \equiv e^{\langle\phi\rangle}\,. \tag{1.10}$$

Therefore a worldsheet with genus g can be seen as the g-th loop correction for string theory. In contradistinction to the first two terms the topological term is not classically invariant under the worldsheet Weyl symmetry $\gamma_{\alpha\beta} \to \Lambda^2(\sigma,\tau)\gamma_{\alpha\beta}$. It has been included to enable us to get a consistent conformal invariant theory at the quantum level, provided the β-functionals associated to the "coupling constants" ϕ, $h_{\mu\nu}$ and $B_{\mu\nu}$ vanish. In lowest non-trivial approximation in α' one obtains [12]

$$\begin{aligned}
0 &= \beta^{(h)}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + 2\nabla_\mu\partial_\nu\phi + \mathcal{O}(\alpha')\,,\\
0 &= \beta^{(B)}_{\mu\nu} = \tfrac{1}{2}\nabla^\rho H_{\rho\mu\nu} - H_{\mu\nu\rho}\nabla^\rho\phi + \mathcal{O}(\alpha')\,,\\
0 &= \beta^{(\phi)} = R + \tfrac{1}{12}H^2 - 4\nabla^\rho\partial_\rho\phi + 4\partial^\rho\phi\,\partial_\rho\phi + \mathcal{O}(\alpha')\,,
\end{aligned} \tag{1.11}$$

where $R_{\mu\nu}(g)$ is the Ricci tensor of space-time, R the corresponding Ricci scalar, and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the field strength of the two-form. The form of these equations suggests they can be interpreted as equations of motion for the background fields. Indeed they can also be obtained from the following low energy effective action

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\,\sqrt{|g|}e^{-2\phi}\left(R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha')\right), \tag{1.12}$$

where κ_0 can be related to the gravitational coupling constant, defined in terms of Newton's constant in $D = 26$ as

$$\kappa = \kappa_0 e^{\langle\phi\rangle} = \sqrt{8\pi G_{26}}. \tag{1.13}$$

Observe that the leading order term is not the conventional Einstein-Hilbert kinetic term, due to the dilaton pre-factor. This is because we are currently in the so-called string frame $g = g^{(S)}$. Performing the Weyl-rescaling $g^{(S)}_{\mu\nu} = e^{\phi/2} g^{(E)}_{\mu\nu}$ we get the action in the Einstein frame

$$S = \frac{1}{2\kappa_0^2} \int \mathrm{d}^{26}x \sqrt{|g^{(E)}|} \left(R(g^{(E)}) - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2\cdot 3!} e^{-\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + O(\alpha') \right). \tag{1.14}$$

The analysis above can be repeated for open strings, having an extra massless vector field A_μ in their background, coupling to the string endpoints. This interaction is described by the boundary term

$$S = -\frac{T}{2} \int_{\partial\Sigma} \mathrm{d}\tau \, A_\mu \partial_\tau X^\mu , \tag{1.15}$$

which gives rise to the following contribution to the low energy effective action for open strings

$$S = \frac{1}{2\kappa_0} \int \mathrm{d}^{26}x \sqrt{|g^{(S)}|} \left(-\tfrac{1}{4} e^{-\phi} F_{\mu\nu} F^{\mu\nu} \right) , \tag{1.16}$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. This analysis, thus far purely bosonic and in $D = 26$, can be extended to the supersymmetric case in $D = 10$, and it then turns out that the low energy effective descriptions of all five superstring theories, except Type I, have one part in common, the so-called "common sector"; namely the NS-NS sector given by the ten dimensional analog of (1.14). The low energy limits of these superstring theories coincide with known supersymmetric extensions of Einstein gravity, called supergravities, which will be described in more detail below.

1.3 Supergravity effective actions

As mentioned in the Introduction, supergravity (sugra), as a gauge theory for supersymmetry, was first introduced in 1976 [13] as an extension of Einstein's theory of general relativity. Although it was not shown to be a finite perturbation theory in all orders, these effective actions are still useful for many applications. Especially because of the remarkable fact that they turned out to describe the low energy effective behavior of string theories. Several different methods can be used to formulate supergravity. One approach is to directly gauge the supersymmetry algebra, comparable to the procedure we will follow in chapters 5, 6 and 7 for constructing off-shell supergravity in five dimensions. In this section we will give some more details about the five different supergravity/superstring theories living in ten dimensions. Also see table 1.2.

1.3.1 Type II

Type II theory gives a description of oriented closed superstrings moving in a background consisting of massless closed string vibration modes. It is called Type II since the theory contains two space-time supersymmetries. Since the left and right moving modes of closed superstrings are decoupled, the states are described by tensorial products of two open string states. The

open	closed	String theory	Low energy limit
	x	IIA	$\mathcal{N}=2$ IIA sugra
	x	IIB	$\mathcal{N}=2$ IIB sugra
x	x	Type I	$\mathcal{N}=1$ sugra coupled to SO(32) YM multiplet
	x	Heterotic SO(32)	$\mathcal{N}=1$ sugra coupled to SO(32) YM multiplet
	x	Heterotic $E_8 \times E_8$	$\mathcal{N}=1$ sugra coupled to $E_8 \times E_8$ YM multiplet

Table 1.2: superstring theories and their low energy limits.

massless string states transform under the little group SO(8) of the ten dimensional Lorentz group SO(9, 1). These irreducible representations are given by the trivial irrep **1** (dilaton), the fundamental vector $\mathbf{8_v}$, the spinor reps $\mathbf{8_c}, \mathbf{8_s}$ (two gauginos with opposite chirality), **28** (anti-symmetric two-form), $\mathbf{35_v}$ (graviton), $\mathbf{35_s}$ (self-dual four-form), $\mathbf{56_c}, \mathbf{56_s}$ (two gravitinos with opposite chirality). Since the Ramond sector of an open string state transforms under a spinor representation we can distinguish two possibilities for the closed string states. The Ramond sectors of left and right moving string states can either have opposite or equal chirality, leading to two different superstring theories, respectively called IIA and IIB:

$$\begin{aligned} \text{IIA}: &\quad (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s})\,, \\ \text{IIB}: &\quad (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c})\,. \end{aligned} \tag{1.17}$$

These direct product states can be decomposed into SO(8) irreps to give the full massless spectrum. Both theories have a common NS-NS sector

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} = \phi \oplus B_{\mu\nu} \oplus h_{\mu\nu}\,. \tag{1.18}$$

The other bosonic degrees of freedom reside in the R-R sector

$$\begin{aligned} \text{IIA}: &\quad \mathbf{8_c} \otimes \mathbf{8_s} = \mathbf{8_v} \oplus \mathbf{56_v} = C_{(1)} \oplus C_{(3)}\,, \\ \text{IIB}: &\quad \mathbf{8_c} \otimes \mathbf{8_c} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c} = C_{(0)} \oplus C_{(2)} \oplus C^{+}_{(4)}\,, \end{aligned} \tag{1.19}$$

and are therefore called RR gauge fields. The zero-form $C_{(0)}$ is called the axion. The fermionic fields are found in the NS-R and R-NS sectors (chirality denoted by α or $\dot{\alpha}$)

$$\begin{aligned} \text{IIA/IIB}: &\quad \mathbf{8_v} \otimes \mathbf{8_c} = \mathbf{8_s} \oplus \mathbf{56_c} = \lambda^{\dot{\alpha}} \oplus \psi^{\alpha}_{\mu}\,, \\ \text{IIA}: &\quad \mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{8_c} \oplus \mathbf{56_s} = \lambda^{\alpha} \oplus \psi^{\dot{\alpha}}_{\mu}\,. \end{aligned} \tag{1.20}$$

The bosonic truncations of the IIA and IIB actions are given below (in the string frame). For IIA we have

$$\begin{aligned} S_{\text{IIA}} = &\ \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{|g|} \left\{ e^{-2\phi} \left[R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!} H^2_{(3)} \right] - \tfrac{1}{2\cdot 2!} G^2_{(2)} \right. \\ & \left. - \tfrac{1}{2\cdot 4!} G^2_{(4)} \right\} - \frac{1}{4\kappa^2} \int \mathrm{d}^{10}x\, \mathrm{d}C_{(3)} \wedge \mathrm{d}C_{(3)} \wedge B_{(2)}\,, \end{aligned} \tag{1.21}$$

where the field strengths are defined as follows (see appendix A for our conventions on form notation)

$$H_{(3)} = \mathrm{d}B_{(2)}\,, \quad G_{(2)} = \mathrm{d}C_{(1)}\,, \qquad G_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)}\,. \tag{1.22}$$

For IIB we have

$$\begin{aligned} S_{\mathrm{IIB}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\,\Big\{e^{-2\phi}\Big[R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{(3)}^2\Big] - \tfrac{1}{2}G_{(1)}^2 - \tfrac{1}{2\cdot 3!}G_{(3)}^2 \\ &\qquad -\tfrac{1}{2\cdot 5!}G_{(5)}^2\Big\} - \frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\, C_{(4)} \wedge \mathrm{d}C_{(2)} \wedge H_{(3)}\,, \end{aligned} \tag{1.23}$$

with

$$H_{(3)} = \mathrm{d}B_{(2)}\,, \quad G_{(1)} = \mathrm{d}C_{(0)}\,, \qquad G_{(3)} = \mathrm{d}C_{(2)} - H_{(3)} \wedge C_{(0)}\,. \tag{1.24}$$

The above IIB action is called the non-self-dual action, since the self-duality condition for the four-form gauge field does not follow from the action [14, 15]. The equations of motion have to be supplemented by

$$G_{(5)} = {}^{*}G_{(5)}. \tag{1.25}$$

1.3.2 Type I

Type I string theory is a theory with unoriented open strings and having $\mathcal{N} = 1$ supersymmetry. It also contains a closed string sector due to open string interactions. Since the open string endpoints can interact with a one-form gauge field $A^I_{(1)}$, we can assign charges to them. The only corresponding consistent gauge group turns out to be SO(32). The spectrum can be derived from the IIB spectrum by a specific parity projection Ω on the left and right moving sectors, keeping the left-right symmetric states, i.e. projecting out the NS-NS two-form. The action is given by

$$S_{\mathrm{I}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\,\Big\{e^{-2\phi}\Big[R(g) + 4(\partial\phi)^2\Big] - \tfrac{1}{2\cdot 3!}H_{(3)}^2 + \tfrac{1}{4}e^{-\phi}F^I_{(2)}F_{(2)I}\Big\}\,, \tag{1.26}$$

where

$$F^I_{(2)} = \mathrm{d}A^I_{(1)} + [A_{(1)}, A_{(1)}]^I\,, \qquad H_{(3)} = \mathrm{d}C_2 + A^I_{(1)} \wedge \mathrm{d}A_{(1)I}\,, \tag{1.27}$$

and the trace runs over all the group generators.

1.3.3 Heterotic

The last two theories are Heterotic superstring $\mathcal{N} = 1$ theories with gauge groups $\mathrm{E}_8 \times \mathrm{E}_8$ and SO(32) respectively. These theories contain oriented closed strings. The action is given by

$$S_{\mathrm{Het}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}e^{-2\phi}\Big\{R(g) + 4(\partial\phi)^2 - \tfrac{1}{2\cdot 3!}H_{(3)}^2 + \tfrac{1}{4}F^I_{(2)}F_{(2)I}\Big\}\,, \tag{1.28}$$

where

$$F^I_{(2)} = \mathrm{d}A^I_{(1)} + [A_{(1)}, A_{(1)}]^I\,, \qquad H_{(3)} = \mathrm{d}B_2 + \tfrac{1}{2}A^I_{(1)} \wedge \mathrm{d}A_{(1)I}\,. \tag{1.29}$$

1.4 Dualities

Although a lot information can be obtained from superstring theories by making use of perturbative techniques, non-perturbative studies of superstring theories turn out to be extremely difficult. Furthermore, we saw in the previous section there are five different consistent theories of quantum gravity at first sight.

The concept of duality could be used to solve this unification problem, by showing that the five superstring theories are connected, suggesting that each of these five theories are merely different vacua of a single theory called M-theory.

Some examples of dualities in physics have been known for a long time. For example, assuming the existence of magnetic monopoles, Maxwell's equations were found to be invariant under the transformation

$$\vec{E} \to \vec{B}\,, \qquad \vec{B} \to -\vec{E}\,, \qquad e \leftrightarrow g\,, \tag{1.30}$$

with e the electric charge and g the magnetic monopole charge. Dirac's theory of monopoles [16] showed that the following quantization condition has to hold

$$e \cdot g = 2\pi n\,, \qquad n \in \mathbb{Z}\,, \tag{1.31}$$

connecting a strongly coupled theory of electrodynamics to a weakly coupled theory of monopoles! Similarly it was found [17] that $e \leftrightarrow g$ is an exact symmetry of $\mathcal{N} = 4$ Yang-Mills theory. In the following sections we will analyze dualities in the context of superstring theory. However, first we will briefly introduce the concept of compactification.

1.4.1 T-duality

This duality, also called target space duality, connects different theories, compactified on inverse radii. If we compactify one dimension, the periodicity along this coordinate y implies that fields can be expanded into their eigenfunctions on the circle

$$\Phi(x^\mu, y) = \sum_n \Phi_n(x^\mu) e^{\mathrm{i}\, p_y y}\,, \tag{1.32}$$

where $p_y \equiv \frac{n}{R}$ is the quantized conjugate momentum of y, and Φ_n are the so-called Kaluza-Klein modes. For more details on compactification and dimensional reduction, see chapter 3. To demonstrate this type of duality, let us consider a theory with coordinate x^9 compactified on a circle of radius R. In the simple case of a theory with only point particles, there are two clearly discernable limits. $R \to \infty$ will lead to a continuous conjugate momentum spectrum in the compact direction, restoring the uncompactified theory. On the other hand, when R shrinks to zero, the momentum will be zero or infinite, effectively decoupling the compact coordinate. However, in the case of string theory, closed strings can wrap w times around the circle, generalizing the periodicity condition to

$$X^\mu(\tau, \sigma + \ell_s) = X^\mu(\tau, \sigma) + 2\pi w R, \tag{1.33}$$

for the string coordinates X^μ, where $w \in \mathbb{N}$ is called the winding number. Inspecting the conjugate momenta of left and right movers along the circle, and the altered mass spectrum, one observes a new 'symmetry' of the theory, called T-duality. It relates a theory compactified on a circle with radius R, winding number w and momentum labelled by n, to another theory compactified on a circle with inverse radius α'/R, and interchanged momentum and winding numbers [18]. Furthermore, the right moving modes are changed by a parity transformation. For the fermionic modes this means that the right moving Ramond ground state alters its chirality; a theory with two opposite chiralities maps to a theory with equal chiralities, i.e. it has been found [19, 20] that the massless sectors of IIA and IIB supergravity are each others T-dual [21]. Also Heterotic SO(32) and Heterotic $E_8 \times E_8$ supergravity turn out to be T-dual. Both dualities are conjectured to hold in the corresponding (non-perturbative) string theory limit.

In the case of open strings it can be shown that the boundary condition changes from Neumann to Dirichlet under T-duality, i.e. their endpoints are localized on the circle. The hyperplane given by $x^9 = c$ turns out to describe a solitonic object in string theory, called a D-brane. This particular class of solutions will be discussed in more detail in section 1.5.2.

Note that T-duality, because of its perturbative nature, does not give us any more insights into the non-perturbative behavior of string theory.

1.4.2 S-duality

Another type of duality is the strong-weak duality. Similarly to the EM-duality it maps between strongly and weakly coupled regimes of different theories, making it particularly useful for obtaining non-perturbative information in one theory, using perturbative methods in the S-dual theory. From (1.10) we see that this duality generally corresponds to changing the sign of the dilaton: $\phi \to -\phi$.

In the supergravity approximation some simple examples are given by the S-duality between Type I and Heterotic SO(32) [22], which can be easily observed after scaling the metric to the Einstein frame. Secondly IIB turns out to be self-dual [23]. To see this, we write the IIB action (1.23) in a manifestly $S\ell(2,\mathbb{R})$ covariant manner. The dilaton and the RR-scalar can be combined into a complex scalar $\tau = C_{(0)} + \mathrm{i}\, e^{-\phi}$, which transforms under the Möbius transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{1.34}$$

with

$$\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,. \tag{1.35}$$

The NS-NS and R-R two-forms transform as a doublet under $S\ell(2,\mathbb{R})$

$$\begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} = \begin{pmatrix} aC_{(2)} + bB_{(2)} \\ cC_{(2)} + dB_{(2)} \end{pmatrix}, \tag{1.36}$$

and the four-form transforms as a singlet. An S-duality transformation can now be seen as a specific $S\ell(2,\mathbb{R})$ transformation with $a = d = 0$ and $b = -c = 1$

$$\phi \to -\phi\,, \qquad C_{(2)} \to B_{(2)}\,, \qquad B_{(2)} \to -C_{(2)}\,, \tag{1.37}$$

mapping IIB onto itself. Since quantum mechanics requires the charge, with respect to the NS-NS two-form of the basic object of string theory, the fundamental string, to be quantized, the symmetry group is broken to the discrete subgroup $S\ell(2,\mathbb{Z})$. Due to the non-perturbative nature of this duality type, $S\ell(2,\mathbb{Z})$ has been conjectured as being the symmetry group of non-perturbative IIB superstring theory.

$D = 11$ supergravity

In $D = 11$ dimensions there is only one possible (physical) supergravity theory, having 32 supercharges. This $\mathcal{N} = 1$ supergravity theory was found by Cremmer, Julia and Scherk in 1978 [24], with the bosonic part given by

$$S_{D=11} = \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x \sqrt{|g|} \left\{ R(g) - \tfrac{1}{2\cdot 4!} G_{(4)}^2 + \tfrac{1}{144^2} \varepsilon^{(4)(4')(3)} G_{(4)} G_{(4')} C_{(3)} \right\} , \tag{1.38}$$

with field strength $G_{(4)} = \mathrm{d}C_{(3)}$. The supersymmetric version of this action will be the starting point of chapter 4.

It was first realized by [25] that compactification of $D = 11$ supergravity onto a circle with radius $R_{11} = (g_s)^{2/3}$ exactly yields $D = 10$ IIA supergravity. It was also found in [26, 27] that compactification onto an interval $R_{11} = S^1/\mathbb{Z}_2$ (called an orbifold) yields Heterotic $E_8 \times E_8$ supergravity. Led by these observations, a unified theory was conjectured, called M-theory, of which the low energy approximation is given by $D = 11$ supergravity. The strong coupling limit (large g_s) of IIA/Heterotic string theory is given by M-theory.

The earlier mentioned $S\ell(2,\mathbb{Z})$ symmetry of IIB supergravity can now be easily explained. Since we know that IIA and IIB are T-dual, a compactification of $D = 11$ supergravity onto two circles $S^1 \times S^1$ with radii R_{11} and R_{10}, should be equal to IIB supergravity compactified on a circle with radius $1/R_{10}$. This is true if $g_{IIB} = R_{11}/R_{10}$. However, since the compact manifold $S^1 \times S^1$ forms a torus, with modular group $S\ell(2,\mathbb{Z})$, the IIB coupling constant g_{IIB} is related to its inverse g_{IIB}^{-1}. It follows that the symmetry group of IIB is given by $S\ell(2,\mathbb{Z})$, and therefore the theory is self-dual.

Some models of M-theory have been proposed, i.e. the matrix model [28], but until now there is still little known about M-theory. However, it is believed that all five superstring theories in ten dimensions somehow should follow from taking some particular low energy limit of M-theory, leading to a web of dualities, as depicted in figure 1.2.

1.5 Solutions

In this section we will discuss several kinds of solutions of the supergravity equations of motion. These solutions have played an essential role in strengthening our belief in dualities in the non-perturbative limit. For reviews on this subject see [29, 30].

1.5.1 p-Branes

The existence of higher rank gauge fields in string theory suggest a further generalization of strings is possible, namely p-branes, $(p + 1)$-dimensional objects in space-time, coupling to a

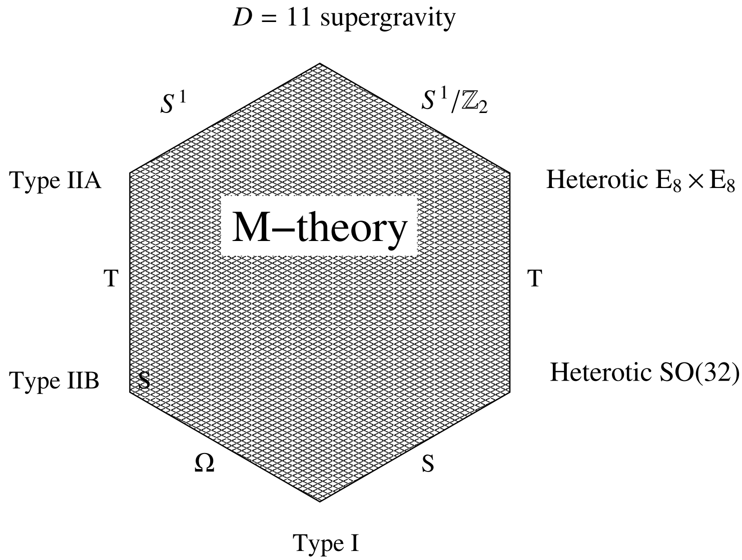

Figure 1.2: M-theory and the web of string theories and their dualities.

$(p+1)$-rank gauge field $A_{\mu_1 \ldots \mu_{p+1}}$ as follows:

$$\int \mathrm{d}^{p+1}\xi \, \partial_{\alpha_1} X^{\mu_1} \ldots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} A_{\mu_1 \ldots \mu_{p+1}} \varepsilon^{\alpha_1 \ldots \alpha_{p+1}} \,, \tag{1.39}$$

in the same way we know a point particle ($p = 0$) couples to a one-form gauge field, and the NS-NS two-form $B_{\mu\nu}$ couples to a string worldsheet. The electrical charge of such an object can be found by a generalization of Gauss' law to be

$$Q_e \sim \int_{S^{D-p-2}} {}^*F_{(p+2)} \,, \tag{1.40}$$

where ${}^*F_{(p+2)}$ is the Hodge-dual (see appendix A) of the $A_{(p+1)}$ field strength, and S^{D-p-2} is a sphere surrounding the p-brane. This charge is conserved due to the equation of motion for the gauge field. Associated with this electrically charged p-brane solution is a dual magnetic $(D-p-4)$-brane, coupling to $\tilde{A}_{(D-p-3)}$, the dualization of the gauge field $A_{(p+1)}$. Its topological magnetic charge is given by

$$Q_m \sim \int_{S^{p+2}} F_{(p+2)} \,, \tag{1.41}$$

which is conserved due to the Bianchi identity. Here we integrated over the transverse space of the p-brane. These charges satisfy

$$Q_e \cdot Q_m = 2\pi n \,, \qquad n \in \mathbb{Z} \,, \tag{1.42}$$

generalizing Dirac's quantization condition for electric and magnetic monopoles. In order to explicitly find solitonic p-brane solutions in a given supergravity theory, let us consider a consistent bosonic truncation, only containing one $(n-1)$-form gauge field

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|g|}\left(R - \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2\cdot n!}e^{a\phi}F^2_{(n)}\right). \tag{1.43}$$

In order to solve the equations of motion following from (1.43), a convenient Ansatz is given by

$$\begin{aligned} &\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} + e^{2B(r)}\mathrm{d}y^m\mathrm{d}y^n\delta_{mn}\,, \qquad \phi = \phi(r)\,,\\ &\mu,\nu = 0\ldots p \qquad m,n = p+1,\ldots,D-1\,, \end{aligned} \tag{1.44}$$

with $r \equiv \sqrt{y^m y^n \delta_{mn}}$ the isotropic radial distance in the transverse space. The above Ansatz is consistent with a $P_{(p+1)} \times \mathrm{SO}(D-p-1)$ symmetry of space-time, with Poincaré symmetry along the worldvolume directions. There are two possible solutions of the equations of motion, leading to an electric ($p=n-2$) or magnetic ($p=D-p-2$) p-brane [31, 32]:

$$\left\{\begin{array}{lcll} \mathrm{d}s^2 &=& H^{\frac{-4(D-p-3)}{\Delta}(D-2)}\mathrm{d}x^2_{(p+1)} + H^{\frac{4(p+1)}{\Delta}(D-2)}\mathrm{d}y^2_{(D-p-1)}\,, & \\ e^\phi &=& H^{\frac{2a}{\zeta\Delta}}\,, \qquad \zeta = \left\{\begin{array}{ll} +1 & \text{electric} \\ -1 & \text{magnetic}\end{array}\right.\,, & \\ F_{m\mu_1\ldots\mu_{n-1}} &=& \frac{2}{\sqrt{\Delta}}\varepsilon_{\mu_1\ldots\mu_{n-1}}\partial_m(H^{-1}) & \text{electric}\,,\\ F_{m_1\ldots m_n} &=& -\frac{2}{\sqrt{\Delta}}\varepsilon_{m_1\ldots m_n r}\partial_r H & \text{magnetic}\,, \end{array}\right. \tag{1.45}$$

where the harmonic function H satisfies $\nabla^2 H = 0$. For $D-p-1 > 2$, H can be written as $H(r) = 1 + (\frac{r_0}{r})^{D-p-3}$, where r_0 is an integration constant related to the charge in the magnetic case. The constant Δ is given by

$$\Delta = a^2 + \frac{2(p+1)(D-p-3)}{D-2}\,. \tag{1.46}$$

Examples

The simplest example in Type II theories is the electric one-brane, coupling to the NS-NS two-form, called the fundamental string (F1). This solution can be obtained from (1.45) by using $p=1$, $a=-1$ and $D=10$. Its magnetic dual is called the Neveu-Schwarz five-brane (NS5). Type II theories also contain RR-gauge fields, allowing for a separate class of solutions as described in the following section.

The $D=11$ supergravity theory only contains one three-form gauge field, and no dilaton (take $a=0$). The only sources we can have for a three-form are a two-brane or five-brane solution, so we take $\Delta = 4$ in (1.45). The resulting solutions are called the electric M2-brane [33] and magnetic M5-brane [34]. The compactification of the M2-brane along its spatial direction was found to give exactly the F1 solution of IIA supergravity. The NS5 solution can be obtained by compactifying the M5 brane along a transverse direction.

A special case of p-brane ($p = D-2$) is the so-called domain-wall, a brane with one transverse direction, separating space-time into two regions. As we will see later these objects play an important role in so-called brane-world scenarios.

1.5.2 D-branes

In section 1.4.1 we already encountered D-branes as hyperplanes where open strings can end. They turn out to be a special class of p-brane solutions, coupling to RR potentials, and satisfying Dirichlet boundary conditions along their spacelike coordinates [35], i.e. they are fixed in space. Their dynamics is generated by the open string modes. In the string frame the Dp-brane geometry takes the following simple form

$$\begin{cases} \mathrm{d}s^2 & = & H^{-\frac{1}{2}}\mathrm{d}x^2_{(p+1)} - H^{\frac{1}{2}}\mathrm{d}x^2_{(D-p-1)}\,, \\ e^{-2\phi} & = & H^{\frac{p-3}{2}}\,, \\ F^{RR}_{012\ldots pm} & = & \partial_m H^{-1} \qquad (m = p+1,\ldots,9)\,, \end{cases} \tag{1.47}$$

Since IIA / IIB supergravity only contains odd / even-form gauge potentials, it only contains D$2p$ / D$(2p+1)$-branes. In IIB there are two special cases. There is a D(-1)-brane called the D-instanton, whose position is fixed in space-time, coupling to the axion. There is also a self-dual dyonic D3 brane solution.

The D-brane low energy effective worldvolume action was found by Leigh [36] by using the same technique used in section 1.2, and is called the Dirac-Born-Infeld action

$$S_{\mathrm{DBI}} = -T_p \int \mathrm{d}^{p+1}\xi\, e^{\phi}\, \sqrt{|g+\mathcal{F}|}\,, \tag{1.48}$$

where T_p is the tension of the Dp-brane, and $\mathcal{F}_{ij} = 2\pi\alpha' F_{ij} - B_{ij}$ ($F = \mathrm{d}A$). When considering D-brane actions in Type II supergravity it turns out one also has to include a Wess-Zumino term

$$S_{\mathrm{WZ}} = T_p \int \left(e^{\mathcal{F}} \wedge C\right)_{(p+1)}\,, \tag{1.49}$$

where $\mathcal{F}$ is given as a formal sum over all odd (IIA) or even (IIB) RR-forms, and the expansion picks out only forms of rank $(p+1)$.

1.5.3 Brane dualities

A lot of evidence for the conjectured dualities has been obtained by inspecting the solutions. The solutions described in the last section are all related by the same dualities, and by dimensional reduction. This is depicted in figure 1.3. Some other solutions that have not been mentioned are Kaluza-Klein ($\mathcal{KK}_D$) monopoles and gravitational waves ($\mathcal{W}_D$).

1.6 BPS states

The presence of p-branes allows the D-dimensional supertranslation algebra to be extended with central charges $Z_{(p)}$ related to the p-brane charges.

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \delta^{ij}\,(\Gamma^\mu C)_{\alpha\beta}\,P_\mu + \sum_p (\Gamma^{\mu_1\ldots\mu_p}C)_{\alpha\beta}\,Z^{ij}_{\mu_1\ldots\mu_p}\,, \tag{1.50}$$

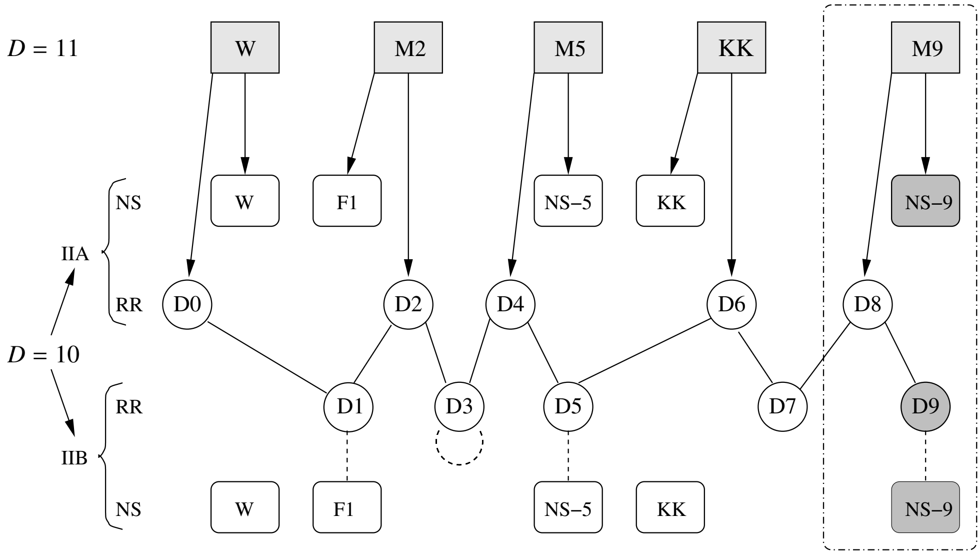

Figure 1.3: Web of dualities between supergravity solutions in $D = 10$ and $D = 11$ [37].

with $i = 1, \ldots, \mathcal{N}$. Positivity of the Q^2 operator on the left hand side gives rise to the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) bound [38, 39], symbolically relating the mass and charge by

$$M \geq c|Z|\,. \tag{1.51}$$

States saturating this bound are called BPS states. These states are stable against decay since they minimize the energy for a given charge. Supersymmetry protects these states from renormalization by quantum effects; the mass-charge relation also holds non-perturbatively, therefore these states have played an important role in the study of dualities.

The BPS states we consider are purely bosonic configurations, where the background fermionic fields have been put to zero. Stability and consistency of this solution of the field equations requires the supersymmetry variations of the fermions to vanish, leading to the BPS equations. This provides a convenient way to explicitly derive BPS states. For example, if we consider a Type II background with only one $(p+1)$-form present, the supersymmetry transformation rules can be written as [40]

$$\begin{aligned} \delta\psi^i_\mu &= (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\,\epsilon^i + \frac{(-1)^p}{8(p+2)!}e^\phi\Gamma\cdot F^{(p+2)}\Gamma_\mu P(p)\epsilon^i \equiv D_\mu\epsilon^i = 0\,, \\ \delta\lambda^i &= \Gamma^\mu(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\,\phi\epsilon^i + \frac{3-p}{4(p+2)!}e^\phi\Gamma\cdot F^{(p+2)}P(p)\epsilon^i = 0\,, \end{aligned} \tag{1.52}$$

where $P(p)$ is a p-dependent projection operator and ω^{ab}_μ the spin-connection. The first equation is called the Killing spinor equation. These differential constraints on the background fields are called the BPS equations. Substituting a p-brane Ansatz into these equations allows us to solve for the parameters. One also finds the following algebraic constraint on ϵ

$$\epsilon + \Gamma_{01\dots p} P(p) \epsilon = 0\,. \tag{1.53}$$

As a consequence this breaks half of the supersymmetry, which is generally true for objects saturating the BPS-bound. The above procedure is applied in chapter 4 in order to find $\frac{1}{2}$-BPS solutions of $D = 9$ gauged supergravity.