Distributional inference

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Chapter 7

Making statistical inferences about a frequency unseen: an application to ornithology

'To count or to think, that is the question'

P.R. Halmos[1]

'Whenever you can, count'

Sir F. Galton[2]

The ornithologist G.Th. de Roos is involved in a study of catching, measuring, ringing and colourringing, counting and indentifying individual *Ruddy Turnstones*, a wader species belonging to the *Charadriiformes* order. Part of his program deals with resight observations. On a particular day he visually inspects a fairly constant population to identify the ringed birds by reading their ring-number. Some ringed birds will be missed. That is why observations are repeated on other days. The issue of interest is whether, after some repetitions, De Roos can be fairly sure that he has identified all ringed birds in his population or, equivalently, that the frequency of unseen birds is zero.

There is, of course, extensive literature about the estimation of abundances, like that of the ringed birds. Most theory is concerned with an asymptotic setting. In our context the emphasis is upon the determination of the 'probability' that the frequency unseen is zero. The methods of inference we develop are based on the assumption of a bird-independent probability p_i of identifying a ringed bird on day i. In Section 7.3 results are derived on the additional assumption of equality of the p_i, an assumption which seems reasonable for the data set submitted. In Section 7.4 different p_i's are allowed. In Section 7.5 we will review the different approaches, and discuss in what cases our theory may be used.

7.1 Introduction

De Roos has collected and is still collecting biometric and moult data on Ruddy Turnstones (*Arenaria interpres*, see the drawing in Figure 7.1[3]). Having their breeding

[1]To count or to think, that is the question, *Nieuw Archief voor Wiskunde*, **13**:1, 1995.

[2]In: J.R. Newman, The World of Mathematics, Simon and Schuster, 1956.

[3]Taken from Naumann, Naturgeschichte der Vögel Mitteleuropas, Band VIII, Table 5, Gera, 1902. Digital rendering by Peter von Sengbusch, Hamburg University.

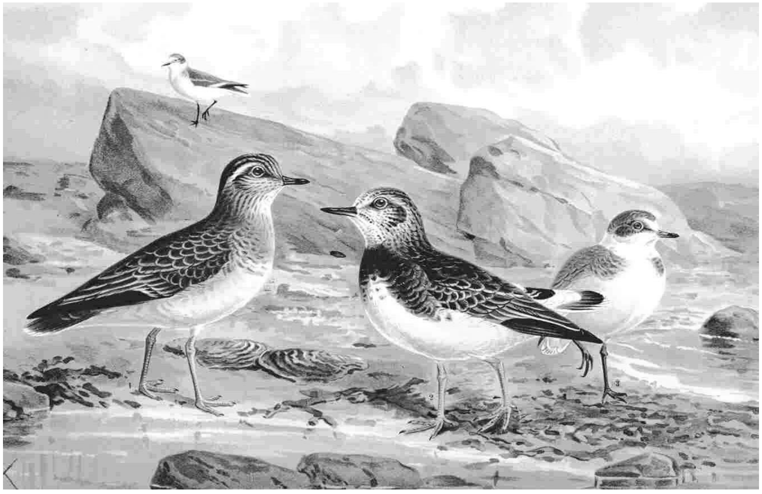

Figure 7.1: Ruddy turnstones

habitat in the high-artic tundras of Siberia and Greenland, these waders may migrate via the Frisian island Vlieland which provides good opportunities for De Roos to study them. The reason is that this place provides optimal opportunities for catching Turnstones and Purple Sandpipers because these species have their feeding areas and high tide roosts on breakwaters, stretching into the North Sea, because these act like artificial rocky shores for these birds. There the birds can be caught by torch at night during dark moon, cloudy sky and favourable tide conditions depending on the wind direction and wind force.

The birds that have been caught, were ringed and colourringed (in case they have not been ringed before), sexed, classified according to age, weighted and measured by determining various aspects of size and shape.

During the day, at high tide, the birds can be counted from a distance and, individually, identified by reading the ring-number with a telescope. Measurements were performed during the period August 1985 until April 1999. In total 1132 different Turnstones were caught and measured at least once. Also, rich spotting data are available. An analysis of the morphology (size and shape) of the birds is performed by DE BRUIN (2003). In this paper we shall restrict ourselves to the following problem: *after how many days of identifying ringed birds can one safely assume that all ringed birds being present in the population have been actually seen?*

We are fascinated by this seemingly simple problem. On a number of days, mostly three days apart, De Roos inspected a stable group of about seventy birds, some of which were ringed. Each day not all ringed birds will be identified. But by continuing the procedure regularly, the observer may feel certain that, after a certain

Date	a	b	c	d	e	f	g	h	i	j	k	l	m	n	Total
20-8		1		1								1			$s_1 = 3$
23-8		1										1		1	$s_2 = 3$
26-8		1		1					1			1	1		$s_3 = 5$
29-8									1	1	1	1	1		$s_4 = 5$
2-9			1				1		1	1	1	1			$s_5 = 6$
5-9		1	1	1					1		1	1			$s_6 = 6$
8-9		1	1	1	1	1	1				1	1			$s_7 = 8$
11-9		1	1	1			1		1			1	1	1	$s_8 = 8$
14-9			1	1	1	1	1		1	1	1		1		$s_9 = 9$
17-9	1	1				1		1	1				1		$s_{10} = 6$
20-9		1	1	1	1	1	1			1		1		1	$s_{11} = 9$
Total	1	8	6	7	3	4	5	1	7	4	5	9	5	3	68

Table 7.1: A one denotes that the bird indicated is spotted on the day indicated. All dates are in 1992. Note that $r_1 = 2$, $r_2 = 0$, $r_3 = 2$, $r_4 = 2$, $r_5 = 3$, $r_6 = 1$, $r_7 = 2$, $r_8 = 1$, and $r_9 = 1$ (see the text). Note also that $n = 68$ identifications have been made involving $m = 14$ birds on $k = 11$ days.

day (day 11 in our case), all ringed birds have been identified. A statistical approach to characterize the degree of (un)certainty requires a mathematical model.

Using the letters a, b, ..., n to indicate birds identified, the data appear in the form presented in Table 7.1. Apart from s_i, the number of birds seen on day i ($i = 1, \ldots, k = 11$) and reported in the table, we use the notation r_h for the number of birds identified on (exactly) h different days. Note that r_0 is the unknown value of interest: the number of ringed birds unseen. Finally we introduce $m = r_1 + \ldots + r_9 = 14$ as the total number of birds seen, $r = r_0 + m$ as the total number of ringed birds, and $n = \sum_{i=1}^{11} s_i = \sum_{h=1}^{11} h r_h = 68$ as the number of identifications made. Note that it takes until day 10 before each one of these 14 birds has been seen.

The mathematical-statistical problem

Given are the data presented in Table 7.1, the outcomes $r_1, \ldots, r_9$ in particular. Required is a distributional inference about the number $y = r_0$ of ringed birds, available in the group but not identified by the ornithologist. In particular an assessment of the probability that $r_0 = 0$, i.e. all ringed birds have been seen, is requested.

Before starting the analysis, we first note that it is *not* necessarily most appropriate to solve the second problem by identifying the epistemic probability of $r_0 = 0$ with the value $\{Q(x)\}(0)$ assigned by the distributional inference $Q(x)$ to $\{R_0 = 0\}$. The reason is that the principles to be involved in deriving $Q(x)$ are not necessarily in complete agreement with those involved in deriving the probability that the hypothesis H_0: $r_0 = 0$ is true. (Some 'fuzziness' is involved in any statistical analysis.)

Secondly we note that our problem is closely related to the proofreaders problem studied extensively in the literature. POLYA (1975) studied this problem for two proofreaders ('days' in our context) who read, independently of each other, the same manuscript. Let $A+C$ and $B+C$ denote the number of misprints found by each reader, where C is the number of commonly found misprints and let M be the (unknown) total number of misprints, then $M-A-B-C$ is the number of undiscovered misprints. Polya's estimate for this number is AB/C, the statistical uncertainties involved are derived using the δ-method. In YANG ET AL. (1982) an 'optimal stopping rule' for rereading the manuscripts is discussed. In comparison with the proofreading problem, our problem has the advantage that probabilistic independence assumptions are less awkward.

At the time we developed our theory, we did not yet have access to the data reported in Table 7.1. The examples suggested to us were such that the number k of days is so small (yet larger than 2) that it is practically impossible to falsify the hypothesis of bird-independent 'experimental' probabilities. The capture-mark-recapture literature (e.g. OTIS ET AL., 1978, and WHITE ET AL., 1982) and the software package `MARK`[4] (suggested to us by J.B. Hulscher, personal communication) emphasize that 'tests for equal catchability' or 'equal identifiability' should be performed. While we worried about day-effects we somewhat overlooked the possibility (evidently present in the proofreaders problem) of bird-effects. We shall return to this issue in Section 7.5.

7.2 Primary approaches

We have the outcomes $(r_1, \ldots, r_{11}) = (2, 0, 2, 2, 3, 1, 2, 1, 1, 0, 0)$ and need the latent outcome r_0 or, more precisely, the 'probabilities' that $r_0 = 0$, 1, 2, … respectively. There are, of course, many approaches to perform such extrapolation. One such approach is to assume that $r_0, r_1, \ldots, r_k$ are outcomes of independent Poisson variables with parameters λ_h satisfying some model $\lambda_h = \lambda_\theta(h)$, e.g. with $\log \lambda_h = \alpha + \beta h$. The extensive analysis to be made in the sequel will contain also some ad hoc modelling as well.

The probabilistic context
In Table 7.1 the letters a, b, …, n are used to indicate the 14 birds. It could be, for instance, that the older birds have a central position in the group of birds on the peer, and are therefore less easily identifiable. Other explanations for deviant stochastic behaviour are also available. Nevertheless we will regard the birds as probabilistically homogeneous, and the ringed ones as 'red balls in an urn'. Also time effects are, at least for the moment, ignored, and the assumption is made that the population on each of the 11 days is the same (actually, the total number of birds, ringed and unringed, varies between 60 (on day 1) and 72 (on day 10)). We would like to assume that there is an unknown ('experimental') probability p, independent of i and j, that the j-th one of the ringed birds is seen on day i. Here $j = 1, \ldots, r = \sum_{h=0}^{k} r_h$ and $i = 1, \ldots, k(= 11)$. Note that $r = r_0 + m = r_0 + 14$. Making some independence

[4] Developed by G.C. White, Department of Fishery and Wildlife Biology, Colorado State University. Available at `http://www.cnr.colostate.edu/~gwhite/mark/mark.htm`. Documentation can be found at `http://www.cnr.colostate.edu/class_info/fw663/Mark.html`.

assumption in addition, the essence of Table 7.1 is captured in the Kolmogorovian setting $(\Omega, \mathcal{F}, \mathrm{P})$ where $\mathcal{F}$ is the space of all $k \times r$ matrices ω where

$$\omega_{i,j} = \begin{cases} 1 & \text{if bird } j \text{ is seen on day } i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{P}(\{\omega\}) = p^n (1-p)^{kr-n}$$

where, as indicated before, $n = \sum_{h=1}^{k} h r_h = 68$ is the total number of identifications made from the theoretical maximum kr which appears if all r ($r \geq 14$) birds are seen on all $k = 11$ days. The random variables (some are 'statistics' in the sense that their outcome is available, some are not) we are interested in are

1. $S_i(\omega) = \sum_{j=1}^{r} \omega_{i,j}$ the number of birds identified on day i;
2. $T_j(\omega) = \sum_{i=1}^{k} \omega_{i,j}$ the number of times bird j is seen;
3. $R_h = \{j | T_j = h\}$ the number of birds seen on h of the k days;
4. $M = \sum_{h=1}^{k} R_h$ the number of birds seen at least once;
5. $N = \sum_{h=1}^{k} h R_h = \sum_{j=1}^{k} S_j$ the total number of identifications made.

For this probabilistic model, the following statements are trivial:

1. $S_1, \dots, S_k$ are i.i.d., $S_i \sim B(r, p)$;
2. $T_1, \dots, T_r$ are i.i.d., $T_j \sim B(k, p)$;
3. $(R_0, \dots, R_k) \sim \text{Multinomial}(r; (1-p)^k, \binom{k}{1} p (1-p)^{k-1}, p^k)$;
4. $M \sim B(r, 1-(1-p)^k)$;
5. $N \sim B(rk, p)$;
6. Given $\{T_k \geq 1\}$, the conditional distribution of T_j is defined by the probabilities

$$\mathrm{P}(T_j = c | T_j \geq 1) = \frac{\binom{k}{c} p^c (1-p)^{k-c}}{1-(1-p)^k} \quad (c = 1, \dots, k);$$

7. The conditional distribution of N, given $\{M = m\}$ corresponds to that of the sum $\sum_{g=1}^{m} N_g$ of independent random variables $N_1, \dots, N_m$, all having the distribution specified under 6.

7.3 Obtaining a solution by ignoring day effects

We have already stated that we like to assume the existence of a *constant* experimental probability p. Is this realistic? It would apply if the numbers $s_1, s_2, \dots, s_{11}$ are outcomes of independent $B(r, p)$ variables, with both r and p unknown. Our observations $s_1, \dots, s_{11}$ provide no convincing evidence to reject the null hypothesis that they came from a $B(14, \frac{\bar{s}}{14} = .44)$ (or, possibly, $B(15, .44)$) distribution. In the current context the assumption of constant probability seems reasonable. However, there are also various arguments against the hypothesis. Even if the ornithologist proceeds in equivalent manners on different days, weather conditions, for instance, are likely to play a role, providing the possibility that the birds are on one day less likely to present themselves to the telescope in the appropriate position than on the other. This suggests that the assumption of fixed p might be unrealistic. In this section, however, we will use the assumption of a constant probability and it seems reasonable to concentrate the attention on the outcome $x = (m, n)$ of (M, N) and on the number of days k taken into account. From Table 7.1 we have $k = 11$ and $x = (14, 68)$.

For these outcomes we shall estimate p by equating

$$\mathbf{E}\,(N|M = m) = m\mathbf{E}\,(T_j|T_j \geq 1) = mkp/(1 - (1 - p)^k)$$

to the outcome n or, equivalently, by computing the estimate $\hat{p}$ as the solution of

$$p = \frac{n}{mk}(1 - (1 - p)^k).$$

Finally, ignoring the uncertainties involved in the estimation of p and concentrating the attention on the outcome m of

$$M \sim B(r, 1 - (1 - \hat{p})^k),$$

the distributional inference $Q(m)$ about $r_0 = r - m$ is composed by taking

$$Q(m) = \text{NegBin}(m, q).$$

where

$$q = 1 - (1 - \hat{p})^k$$

The steps behind this construction are as follows. We concentrate the attention on constructing $Q(m)$ such that the corresponding value $G_m(\theta)$ of its distribution function equals the degree of belief in H_θ: $r \leq \theta$. This fiducial approach (see SALOMÉ, 1998 and KROESE ET AL., 1999) yields the $Q(m)$ mentioned above. KARDAUN AND SCHAAFSMA (2003) show that the distributional inference

$$\tilde{Q}(m) = (\tfrac{1}{2} + \tfrac{1}{2}q)\text{NegBin}(m, q) + (\tfrac{1}{2} - \tfrac{1}{2}q)\text{NegBin}(m + 1, q)$$

provides a slight refinement (in the sense that $\tilde{Q}$ is weakly similar, whilst Q is not). Above methods provide $\hat{p} = .441$ as the solution of $14 \cdot 11p = 68(1 - (1 - p)^{11})$ and $Q(m) = \text{NegBin}(14, .998)$ which assigns probability

$$\{Q(m)\}(0) = .998^{14} = .977$$

k_k	2	3	4	5	6	7	8	9	10	11
m_k	4	6	8	10	10	12	12	12	14	14
n_k	6	11	16	22	28	36	44	53	59	68
$\hat{p}$	.667	.559	.456	.408	.454	.419	.455	.490	.420	.441
$\{Q\}$	.624	.584	.480	.470	.765	.762	.910	.972	.941	.977
$\{\tilde{Q}\}$	.613	.573	.469	.460	.759	.758	.909	.972	.940	.977

Table 7.2: First three rows: number of birds spotted at least once (m_k) and total number of observations (n_k); both after days k_k. Last three rows: estimate $\hat{p}$ and the epistemic probabilities $\{Q(x)\}(0)$ and $\{\tilde{Q}(x)\}(0)$.

to the hypothesis H_0: $r_0 = 0$ (or, equivalently, $r = 14$). Note that $\tilde{Q}(m)$ assigns the same probability up to three digits (it is trivial that Q and $\tilde{Q}$ converge to each other when $m \to \infty$ or $p \to 1$).

Of course, also after the first, second, ..., etcetera, day, De Roos could have tried to make a distributional inference about the number of unseen birds. The outcomes (k_k, m_k, n_k) $(k_k = 1, \ldots, k = 11)$ of interest as well as the epistemic probabilities $\{Q(x)\}(0) = (1 - (1 - \hat{p}^k)^m)$ are displayed in Table 7.2. The last two rows of this table contains the epistemic probabilities $\{Q(x)\}(0)$ and $\{\tilde{Q}(x)\}(0)$assigned in favor of H_0: $r_0 = 0$, i.e. the degree of belief in the statement that all ringed birds have been identified at least once. It is not until after the ninth day that we are sufficiently certain (at $\alpha = 5\%$) that De Roos has seen all ringed birds available. Nevertheless the next day (day 10), two new birds appeared (birds a and h).

Though we are completely satisfied by this approach in the present context, we are not in general, because the existence of a fixed probability p, independent of day and bird, is often questionable. Another drawback is that the statistical uncertainties involved in the construction of p have been ignored. In the next section theory will be presented to deal with the case that day effects are not ignorable.

7.4 Obtaining a solution by taking day effects into account

Concerning the question whether the s_i (see Table 7.1) are sufficiently alike to accept H_0: $p_1 = \ldots = p_{11}$, we decided to accept H_0 at the beginning of Section 7.3. However, as stated earlier, there are reasons to question the absence of day effects. It would be nice if we could adapt the theory of the previous section to the situation where on day i a probability p_i is involved, and no assumptions are made about the equality of these p_i.

To cope with this situation we shall apply a 'dirty trick': by conditioning to $S_1 = s_1, \ldots, S_k = s_k$ we can get rid of $p_1, \ldots, p_k$. This is very convenient and natural, but not compelling because $s_1, \ldots, s_k$ contains some information about r, e.g. the logical fact that $r \geq s = \max(s_1, \ldots, s_k)$.

Given the number s_i of ringed birds on day i, all $\binom{r}{s_i}$ combinations of s_i birds from the r ringed birds available have the same probability $1/\binom{r}{s_i}$ to be the s_i ones seen. Following a recommendation by A.J. Stam (personal communication), Ω consists of the sets

$$\{v_{i,1}, \ldots, v_{i,11}\} \subset \{1, \ldots, r\}$$

of the index numbers of the s_i birds seen on day i. Assuming independence between days (no bird-effect), the conditional probability distribution P on $(\Omega, \mathcal{F})$ is determined by

$$\mathrm{P}(\{\omega\}) = \begin{cases} 1/\left(\binom{r}{s_1}\binom{r}{s_2}\cdots\binom{r}{s_k}\right) & \text{if } S_i(\omega) = s_i \ (i = 1, \ldots, k) \\ 0 & \text{otherwise.} \end{cases}$$

The distribution of the relevant observable $M = \sum_{j=1}^{r} \mathbf{1}_{\{T_j \geq 1\}} = r - R_0$ can now be studied for any a priori possible value $\theta \in \Theta = \{s, s+1, \ldots\}$ of r (remember that $s = \max(s_1, s_2, \ldots)$). Standard theory developed by LOÈVE (1955, see also PARZEN, 1960) provides that

$$\mathrm{P}(M = m) = \sum_{j=m}^{r} (-1)^{j-m} \binom{j}{m} Q_j$$

and that

$$\mathrm{P}(M \geq m) = \sum_{j=m}^{r} (-1)^{j-m} \binom{j-1}{m-1} Q_j$$

where

$$\begin{aligned} Q_0 &= 1 \\ Q_1 &= \sum_{j=1}^{r} \mathrm{P}(A_j) = r \prod_{i=1}^{k} (1 - s_i/r) \\ Q_2 &= \sum_{j_1=1}^{r} \sum_{j_2=j_1+1}^{r} \mathrm{P}(A_{j_1} A_{j_2}) \\ &\vdots \\ Q_r &= \mathrm{P}(A_1 A_2 \ldots A_r) \end{aligned}$$

and $A_j = \{T_j = 0\}$ $(j = 1, \ldots, r)$.

As r is unknown, we introduce the auxiliary random variable M_θ such that M_θ has the distribution which M would have had if $r = \theta$. The result just mentioned provides the relevant 'physical' probabilities $p_\theta(\mu) = \mathrm{P}(M_\theta = \mu)$ $(\mu = s, s+1, \ldots, \theta)$ whether or not a priori probabilities are specified, we are interested in the construction of (epistemic) posterior probabilities

$$q_m(\theta) \quad (\theta = m, m+1, \ldots)$$

specifying the opinion we should have about r after observing the outcome m of M. The posterior probability $q_m(m)$ is of particular interest because it refers to the (epistemic) probability that De Roos has identified all ringed birds in the population. There, obviously, are two issues involved:

1. The determination of $p_\theta(\mu)$;
2. How to convert the $p_\theta(\mu)$, given the outcome m of M, into the $q_m(\theta)$.

Finally, there is a minor issue with respect to the determination of $q_m(m)$ or, equivalently, of the 'probability' that $r_0 = 0$ (and $r = m$). These issues are now successively dealt with. (The last mentioned minor issue does not seem of much interest in this application.) The distribution of M has been derived in the above. Its expectation is given by

$$\begin{aligned} \mathbf{E}\,(M) &= \sum_{j=1}^{r} \mathrm{P}(T_j \geq 1) \\ &= \sum_{j=1}^{r} (1 - \mathrm{P}(T_j = 0)) \\ &= r - r \prod_{i=1}^{k} \left(1 - \frac{s_i}{r}\right) \end{aligned}$$

because $\{T_j = 0\} = \{\omega | \omega_{ij} = 0 \ i = 1, \ldots, k\}$ is the event which occurs if bird j is never seen. The (conditional) probability that bird j is not seen on day i is equal to $1 - s_i/r$.

Thus having obtained

$$\mathbf{E}\,(M_\theta) = \theta \left(1 - \prod_{i=1}^{k} (1 - \frac{s_i}{\theta})\right)$$

we shall content ourselves by providing approximate $p_\theta(\mu)$'s by equating $\mathcal{L}\,(M_\theta)$ to the distribution on $\{s, s+1, \ldots, \theta\}$ which maximizes the entropy

$$-\sum_{\mu=s}^{\theta} p(\mu) \log p(\mu)$$

under the restrictions

$$p(\mu) \geq 0, \quad \sum_{\mu=s}^{\theta} p(\mu) = 1, \quad \sum_{\mu=s}^{\theta} \mu p(\mu) = \mathbf{E}\,(M_\theta)$$

having $\tilde{p_\theta}(\mu) = e^{c_\theta \mu - \psi(\theta)}$ $(\mu = s, \ldots, \theta)$ as its solution with $\psi(\theta) = \log \sum_{\mu=s}^{\theta} e^{c_\theta \mu}$ and with c_θ such that $\sum \mu \tilde{p_\theta}(\mu) = \mathbf{E}\, M_\theta$. We believe that it is not reasonable, in the context of Table 7.1 with $k = 11$, $s = 9$, $m = 14$, $n = 68$, to convert the $\tilde{p_\theta}(\mu)$ into posterior probabilities $q_m(\theta)$ by normalizing the likelihood function $l_m(\theta) = \tilde{p_\theta}(m)$ or, equivalently, by using the formal Bayes approach with improper prior $w(\theta) = 1$ $(\theta = s, s+1, \ldots)$. It seems more reasonable to try to comply with some requirement of weak similarity by using some form of Fisher's fiducial argument, e.g. that where the distribution function

$$G_m(z) = \sum_{\theta=m}^{z} q_m(\theta)$$

of the distributional inference about r is equated to the P-value $\alpha_z(m) = \mathrm{P}(M_z \geq m)$ or to the symmetrized P-value $\tilde{\alpha_z}(m) = \frac{1}{2}\mathrm{P}(M_z \geq m) + \frac{1}{2}\mathrm{P}(M_z \geq m+1)$where, of course, the approximate values

$$\begin{aligned} \mathrm{P}(M_z \geq m) &\approx \sum_{\mu=m}^{z} \tilde{p_z}(m) \\ &= e^{-\psi(z)} \sum_{\mu=m}^{z} e^{c_z \mu} \\ &= e^{-\psi(z)} \left(e^{c_z(z+1)} - e^{c_z(m)} \right) / (1 - e^{c_z}) \end{aligned}$$

are used. With respect to the determination of $q_m(m)$, we believe that it is appropriate, in the present context, to use a Bayesian approach where the data-dependent prior $w(\theta) = \frac{1}{2}$ $(\theta = m, m+1)$ is used. It provides us with

$$\begin{aligned} q_m(m) &= \frac{p_m(m)}{p_m(m) + p_{m+1}(m)} \\ &\approx 1/(1 + e^{(c_{m+1} - c_m)m - \psi(m+1) + \psi(m)}) \end{aligned}$$

Application of above theory to our data yields the following. Remember that $s = \max(s_i) = 9$, $k = 11$, and $m = 14$. Taking $\theta = 14$ provides $\mathbf{E}\, M_{14} = 14(1 - \prod(1 - s_i/14)) = 13.984$, and the maximum entropy solution $\tilde{p}_{14}(\mu) \propto \exp(c_{14}\mu)$ with $c_{14} = 4.175$, hence

$$(\tilde{p}_{14}(9), \ldots, \tilde{p}_{14}(14)) = (0, 0, 0, 0, .015, .985).$$

For the denominator of $q_m(m)$, given above, it is also necessary to look at the situation $\theta = m + 1$. This provides $\mathbf{E}\, M_{15} = 14.968$, and $\tilde{p}_{15}(\mu) \propto \exp(c_{15}\mu)$ with $c_{15} = 3.470$, hence

$$(\tilde{p}_{15}(9), \ldots, \tilde{p}_{15}(15)) = (0, 0, 0, 0, .001, .030, .969).$$

k	2	3	4	5	6	7	8	9	10	11
m	4	6	8	10	10	12	12	12	14	14
s	3	5	5	6	6	8	8	9	9	9
$\tilde{p}_m(m)$	.750	.750	.681	.664	.836	.839	.940	.984	.958	.985
$\tilde{p}_{m+1}(m)$	.323	.302	.256	.250	.195	.186	.099	.036	.069	.030
$q_m(m)$	.700	.713	.727	.726	.811	.819	.905	.965	.933	.970

Table 7.3: First three rows: number of birds spotted at least once (m) at day k and maximum number of observations at a day s until day k. Last rows: the derivation of the epistemic probabilities $\{Q(x)\}(0)$.

This provides us with *a* probability that $r = 14$ (which implies $r_0 = 0$), namely

$$q_{14}(14) = \frac{\tilde{p}_{14}(14)}{\tilde{p}_{14}(14) + \tilde{p}_{15}(14)} = \frac{.985}{.985 + .030} = .970,$$

so, according to this model, the hypothesis that all ringed birds available have been seen is not rejected at $\alpha = 5\%$. (The minor issue of whether $q_m(m)$ is appropriate is skipped over here.)

Note that, similar to the previous section, also after the first, second, etcetera, day, we could have made a statement about the number of unseen birds. The 'probabilities' $q_m(m)$ = expressed in Table 7.3 denote the probabilities assigned in favor of H_0: $r_0 = 0$, i.e. the degree of belief in the statement that are ringed birds are spotted at least once, after the certain day. The agreement between the statements $Q(x)(0)$ (or $\{\tilde{Q}(x)\}$) made in Section 7.3, Table 7.2 and $q_m(m)$ made in Section 7.4, Table 7.3 is very satisfying.

The theory in this section is not completely compelling, since much more information is needed about the accuracy of the maximum-entropy approximation to the true distribution $\mathcal{L}(M_\theta)$.

7.5 Discussion

In 1994 P.R. HALMOS gave a lecture in Groningen in honor of Johann Bernoulli. The lecture was entitled *'To count or to think, that is the question'*. He made it very clear that he was a rationalist and an idealist. He wanted to think: he was a mathematician. Empiricists like Francis Galton, prefer to count. It is easy to conclude from everything in this chapter that the answer is that one has to count *and* to think. This combination of activities, however, is not as easy as it sounds.

While De Roos was involved in his experiments, we were developing the theory for the problem indicated without having access to the real data. We were thinking about a small number of consecutive days (say, $k = 6$) such that falsification of the hypothesis of bird-independent experimental probabilities would not be feasible. In such situations the research worker may decide to make the assumption of 'no bird effects'. If the hypothesis $p_1 = \ldots = p_k$ is acceptable (like in Table 7.1), the theory of Section 7.3 is applicable. If this hypothesis is not reasonable, then one may use the theory of Section 7.4. Note that the results reported in Tables 7.2 and 7.3 are not much different. This could be expected on the basis of the acceptability of the hypothesis $p_1 = \ldots = p_k$. In practice, it may very well happen that day-effects are present. Ornithologist J.B. Hulscher was dealing with counting all ringed Oystercatchers (*Haematopus ostralegus*) on Schiermonnikoog (another Frisian island) in winter, the birds being ringed in spring. In his experience (personal communication) the frequencies $s_1, \ldots, s_k$ of birds counted on k consecutive days were too much different to assume a common p. This implies that the theory in Section 7.4 is of practical interest as well.

However, if we study the frequencies $(r_1, \ldots, r_{11}) = (2, 0, 2, \ldots, 0)$ of Turnstones with 1, 2, …, 11 identifications then these frequencies are too much dispersed to satisfy the probabilistic assumption of no bird-effect. This leaves us with an awkward issue. We have developed and applied theory (more exact than that of, e.g. OTIS ET AL. 1978, and WHITE ET AL., 1982) to a table which is in conflict with the necessary assumption of 'equal watchability'. In less extensive applications this assumption will not be rejected on the basis of available data. If one does not accept the existence of 'experimental' probabilities p_i (possibly day-dependent) then only ad-hoc methods are possible. E.g. those applied by STAM in his article about numismatics (1987). He fitted a truncated negative binomial distribution instead of a Poisson one to observations of the kind $(r_1, r_2, \ldots)$. If we have agreed that we have to count *and* to think, then the next question is how to combine these two activities. The conclusion from this chapter is that this may be awkward.