Distributional inference

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Chapter 2
Trying to resolve the two-envelope problem

> *'If a question be put to [the wise man] about duty or about a number of other matters in which practice has made him an expert, he would not reply in the same way as he would if questioned as to whether the number of the stars is even or odd, and say that he did not know; for in things uncertain there is nothing probable, but in things where there is probability the wise man will not be at a loss either what to do or what to answer'*
>
> CICERO[1]

The two-envelope *paradox* is easily explained, but the corresponding *problem* is more difficult. Our discussion contains contributions from economy, psychology, logic, probability, and mathematical statistics, respectively, as well as an in-depth contribution from game theory. We conclude that the two-envelope problem does not allow a satisfactory solution. An interpretation is made for statistical science at large. The major part of this chapter coincides with ALBERS ET AL. (2003).

2.1 The two-envelope problem

In 1943, KRAITCHIK discussed the paradox of the neckties:

> 'Each of two persons claims to have the finer necktie. They call in a third person who must make a decision. The winner must give his necktie to the loser as consolation. Each of the contestants reasons as follows: 'I know what my tie is worth. I may lose it, but I may also win a better one, so the game is to my advantage'. How can the game be to the advantage of both?'

The snake in the grass is that equal 'probabilities' or 'chances' are assigned to winning and losing. 'In reality, however, the probability is not an objectively given fact, but depends on one's knowledge of the circumstances. In the present case it is wise not to try to estimate the probability.' (KRAITCHIK, 1943)

A similar paradox is the two-envelope paradox. It is unclear who gave the problem its modern form; Kraitchik already formulated a problem where two people compare

[1] Academica 2.110; taken from ZABELL, Symmetry and its discontents, **Causation, chance and crecedence**, Kluwer, 1988

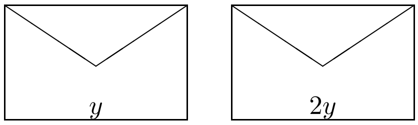

Figure 2.1: Two indistinguishable envelopes

the number of pennies in their purses. GARDNER (1982) called this the wallet game. ZABELL (1988a, 1988b) heard it from Budrys, see NALEBUFF (1989).

Two-envelope paradox.
Two indistinguishable envelopes, 1 and 2, see Figure 2.1, are submitted to 'you' (the decision maker; later the designation 'we' is used as the combination of 'you' and the statistician who tries to assist you). Envelope 2 contains a check worth twice the unknown value, say y, in Envelope 1. You choose one of the envelopes, say Envelope z, *at random* and, after opening it, you observe the value $x = zy$ contained by it. Finally you are allowed to decide between

a_1: keep the envelope you have;

a_2: return this envelope and take the other one.

Two variants will be discussed. In Variant 1, the discrete case, you are told that $y \in \mathbb{N}$; Variant 2, the continuous case, tells you $y \in \mathbb{R}^+$. A paradox appears if you would argue that choosing *at random* implies that, as the other envelope contains either $\frac{1}{2}x$ (namely if $z = 2$) or $2x$ (when $z = 1$), the expectation of the value contained by it is $\frac{1}{2} \cdot \frac{1}{2}x + \frac{1}{2} \cdot 2x = \frac{5}{4}x$, and that, hence, swapping is advantageous on the average. The snake in the grass is that you should not use the marginal or prior probability $\mathrm{P}(Z = 1)$, but the conditional or posterior probability $\mathrm{P}(Z = 1|X = x)$, given the knowledge you have, namely that $X = ZY$ has the outcome x observed. Unfortunately, the relevant posterior probabilities $\mathrm{P}(Z = z|X = x)$ $(z = 1, 2)$ are unknown to you. Moreover you might question whether y can really be regarded as the outcome of a random variable Y. The paradox has been explained, but you are still in need of a solution to the problem. Should you decide upon a_1 or upon a_2, given the outcome $X = x$? That is the question.

This problem has been discussed extensively in the literature, see, e.g., ZABELL (1988a, 1988b), NALEBUFF (1989), and the (probabilistic) references cited in Section 2.3.

2.2 Explorations

It is obvious that you are in need of some additional information, factual, contextual, or otherwise. To settle the issue 'scientifically' you might consult a variety of experts, such as an *economist*, *psychologist*, or *logician*. The economist will argue that the utility of an action is not necessarily proportional to its monetary value. The law of diminishing returns prescribes this relation usually to be a concave function. Henceforth, it is assumed that there is complete correspondence between utility and monetary value. The psychologist will tell you that some people are risk-averse while others are risk-prone. To proceed 'in the most general way' we assume that the decision maker has a risk-neutral attitude and, hence, will try to maximize expected utilities (see Section 2.6 for a more elaborate treatment of the economist, psychologist, etcetera). This, however, requires probabilistic terminology. The logician SMULLYAN (1997) maintained that 'probability is really quite inessential to the heart of the two-envelope paradox'. He presents it as a *logical paradox*, i.e. 'two contrary, or even contradictory, propositions to which we are led by apparently sound arguments. The arguments are considered sound because, when used in other contexts, they do not seem to create any difficulty' (VAN HEIJENOORT, 1967). Here are the propositions derived by Smullyan:

Proposition 1: The amount you will gain by trading, if you do gain, is greater than the amount you will lose, if you do lose.

Proposition 2: The two amounts are really the same.

He proves both of them in the following way[2]:'To prove Proposition 1, let x be the amount you are now holding. Then the other envelope either contains $2x$ or $\frac{x}{2}$. If you gain by trading, you will gain x dollars (moving from x to $2x$), whereas if you lose by trading, you will lose only $\frac{x}{2}$. Since x is greater than $\frac{x}{2}$, then Proposition 1 is established.

To prove Proposition 2, let d be the difference between the two amounts in the envelopes (or what is the same thing, the lesser of the two amounts). Well, if you gain on the trade, you will gain d dollars. If you lose on the trade, you will lose d dollars. Since d is equal to d, then Proposition 2 is established.' (SMULLYAN, 1997, p. 174)

Both proofs seem to be sound, but it cannot be the case that both are correct. The problem turns out to be that some of the terms that are used in both proofs are ambiguous, they can be interpreted in two different ways. If you read the proofs you tend to interpret these terms in each proof in such a way that the proof is correct. But if you take the interpretation that makes one of the proofs correct and use that interpretation in the other proof then the other proof is not correct.

The ambiguous terms are 'the amount you will gain by trading, if you do gain' and 'the amount you will lose by trading if you do lose'. What do these terms refer to? This all depends on what you mean by 'if you do gain' for example. Under what circumstances do you gain? In the first proof this is the case if the amount in the envelope you did not pick is double the amount you are now actually holding. In the second proof this is the case if you picked the envelope which contains the highest amount actually available.

The difference between these interpretation becomes clear if we look at the case where

[2] In his book, Smullyan uses the letter n in his proof of proposition 1. We have replaced it with x to keep a uniform notation. In a recent publication CHASE (2002) elaborates on Smullyan.

you lose by trading. Now in the case of proof one and the first interpretation, 'the amount you gain by trading, if you do gain' refers to twice the amount that is actually in your envelope, which is four times what is in the other envelope. In case of proof two, on the other hand, 'the amount you gain by trading, if you do gain' refers to the amount you would win by trading if you had picked the other envelope, which is one times what is in the other envelope. A similar analysis is provided in (CHASE, 2002). Regardless of which interpretation you choose, this does not 'resolve' the problem.

To make the choice between a_1 and a_2 *additional knowledge* will be needed. In this respect it is natural to refer, as we shall do in Section 2.3, to the probabilistic knowledge that $\mathrm{P}(Z = 1) = \mathrm{P}(Z = 2) = \frac{1}{2}$. It will turn out that this 'factual' knowledge does not yet settle the issue. That is why we will look for more additional knowledge, perhaps of a less factual kind, in the hope that after all some 'reasonable' solution will appear. It is in this respect that we will refer to the following logical paradox. The point we shall make is that the context of this paradox enforces a solution (at least temporarily).

> *The Protagoras paradox* (GELLIUS, 1946)
> 'Euathlus, a wealthy young man, was desirous of instruction in oratory and the pleading of causes. He became a pupil of Protagoras, the keenest of all sophists, and promised to pay him a large sum of money, as much as Protagoras had demanded. He paid him half of the amount at once, before beginning his lessons, and agreed to pay the remaining half on the day when he first pleaded before jurors and won his case. Afterwards, when he had been for some little time a pupil and follower of Protagoras, and had in fact made considerable progress in the study of oratory, he nevertheless did not undertake any cases. And when the time was already getting long, and he seemed to be acting thus in order not to pay the rest of the fee, Protagoras formed what seemed to him at the time a wily scheme; he determined to demand his pay according to the contract, and brought suit against Euathlus.
>
> And when they had appeared before the jurors to bring forward and to contest the case, Protagoras began as follows: "Let me tell you, most foolish of youths, that in either event you will have to pay what I am demanding, whether judgment be pronounced for or against you. For if the case goes against you, the money will be due me in accordance with the verdict, because I have won; but if the decision be in your favour, the money will be due me according to our contract, since you will have won a case."
>
> To this Euathlus replied: "I might have met this sophism of yours, tricky as it is, by not pleading my own cause but employing another as my advocate. But I take greater satisfaction in a victory in which I defeat you, not only in the suit, but also in this argument of yours. So let me tell you in turn, wisest of masters, that in either event I shall not have to pay what you demand, whether judgment be pronounced for or against me. For if the jurors decide in my favour, according to their verdict nothing will be due you, because I

> have won; but if they give judgment against me, by the terms of our contract I shall owe you nothing, because I have not won a case." '

This is the paradox. But the context is such that Protagoras went to court. This had the following consequences according to Gellius: 'Then the jurors, thinking that the plea on both sides was uncertain and insoluble, for fear that their decision, for whichever side it was rendered, might annul itself, left the matter undecided and postponed the case to a distant day. Thus a celebrated master of oratory was refuted by his youthful pupil with his own argument, and his cleverly devised sophism failed.'

We conclude that, due to the context within which it appeared, this paradox is solved, at least temporarily, in favor of Euathlus. Later authors, e.g. STEWART (2000), missed this point. Dealing with the two-envelope paradox we shall look for similar sources of additional information, using probability theory (Section 2.3), mathematical statistics (Section 2.4), and game theory (Section 2.5), unfortunately with less success.

2.3 Probability theory

Apart from Smullyan's, most attempts to solve the two-envelope problem are in a Bayesian spirit (e.g. ZABELL, 1988a, 1988b; NALEBUFF, 1988, 1989; BROOME, 1995; CLARK AND SHACKEL, 2000; JACKSON ET AL., 1994; LINZER, 1994; and MCGREW ET AL., 1997). The formulation of the two-envelope problem chosen in Section 2.1 is such that the gain or utility

$$\begin{aligned} U(x,y,z;a_1) &= x = yz \\ U(x,y,z;a_2) &= 2^{3-2z}x = y(3-z) = \begin{cases} 2x & \text{if } z=1 \\ \frac{1}{2}x & \text{if } z=2 \end{cases} \end{aligned}$$

depends on the true values x, y, and z governing the actual experiment (one of these three can be 'deleted' because $x = yz$). To incorporate the information that $\mathrm{P}(Z=1) = \mathrm{P}(Z=2) = \frac{1}{2}$, we regard (x, z) as the outcome of a pair of random variables (X, Z) which, in principle, may assume any value $(\xi, \zeta) \in \mathcal{X} \times \{1, 2\}$ where $\mathcal{X}$ is $\mathbb{N}$ or $\mathbb{R}^+$. Using the utility function in the form

$$U(\xi,\zeta;a) = \begin{cases} \xi & \text{if } a = a_1 \\ 2^{3-2\zeta}\xi & \text{if } a = a_2 \end{cases}$$

it is 'rational' to choose a such that the conditional (or posterior) expectation

$$\mathbf{E}\left(U(x,Z;a)|X=x\right) = \begin{cases} x & \text{if } a=a_1 \\ 2x\mathrm{P}(Z=1|x) + \frac{1}{2}x\mathrm{P}(Z=2|x) & \text{if } a=a_2 \end{cases}$$

is maximum or, equivalently, to use the 'procedure' $d^* : \mathcal{X} \to \{a_1, a_2\}$ defined by

$$d^*(\xi) = \begin{cases} a_1 & \text{if } \mathrm{P}(Z=1 \mid X=\xi) < \frac{1}{3} \\ a_2 & \text{if } \mathrm{P}(Z=1 \mid X=\xi) > \frac{1}{3}, \end{cases}$$

the choice in the case of equality being arbitrary. 'Procedure' d^* is such that the expected utility is maximum, both a priori (unconditionally) and a posteriori (conditionally, given X). Unfortunately d^* is not yet a workable procedure: the conditional probability $\mathrm{P}(Z=1|X=x)$ has not yet been specified. Such specification requires additional information, e.g. about the way y has come into being. *Can y be regarded as the outcome of a random variable Y and, if so, can the distribution of Y be specified?* The personalist Bayesian will answer both questions affirmatively and argues as follows

In the discrete case, let $f(\eta) = \mathrm{P}(Y=\eta)$ be specified ($\eta \in \mathcal{Y}$). The joint distribution of (X, Y, Z) is then determined by (1) $X = YZ$, (2) Y and Z are stochastically independent, (3) Y has density f, (4) $\mathrm{P}(Z=1) = \mathrm{P}(Z=2) = \frac{1}{2}$. We obtain

$$\begin{aligned}\mathrm{P}(Z=1|X=x) &= \frac{\mathrm{P}(Z=1, X=x)}{\mathrm{P}(X=x)} \\ &= \frac{\mathrm{P}(Z=1, Y=x)}{\mathrm{P}(Z=1, Y=x) + \mathrm{P}(Z=2, Y=\frac{1}{2}x)} \\ &= \frac{f(x)}{f(x) + f(\frac{1}{2}x)}.\end{aligned}$$

If x is odd, then $f(\frac{1}{2}x) = 0$ and $\{Z=1\}$ is sure: swapping provides you with $2x$. If x is even, then it depends on f whether (or not) $f(\frac{1}{2}x)/f(x)$ is smaller (or larger) than 2 and swapping (or not swapping) is most profitable.

In the continuous case the difference between odd and even disappears and we have

$$\begin{aligned}\mathrm{P}(Z=1|X=x) &= \lim_{\Delta\downarrow 0} \mathrm{P}(Z=1|x-\Delta < X < x+\Delta) \\ &= \lim_{\Delta\downarrow 0} \frac{\mathrm{P}(|x-Y|<\Delta, Z=1)}{\sum_{z=1}^{2} \mathrm{P}(|x-zY|<\Delta, Z=z)} \\ &= \frac{f(x)}{f(x) + \frac{1}{2}f(\frac{1}{2}x)}\end{aligned}$$

(see BROOME, 1995). Now the optimal procedure prescribes to decide upon a_2 if $f(\frac{1}{2}x)/f(x)$ is smaller than 4.

The Achilles' heel of this Bayesian solution is the assumption that y is the outcome of a random variable Y with *known* density f. In Section 2.4 we shall worry about this assumption (see also Section 2.6). In the remainder of this section, a matter of theoretical interest is discussed.

Suppose that f is such that, in the discrete case $f(\frac{1}{2}x)/f(x) \leq 2$ holds for all even values of x, or that, in the continuous case, $f(\frac{1}{2}x)/f(x) \leq 4$ holds for all x. Swapping is then always indicated according to the theory. This sounds paradoxical because if 'you' get Envelope z and 'I' get Envelope $3-z$, we will both believe to gain by swapping. The explanation is that this situation can only appear if $\mathbf{E}\,Y = \infty$ and we both may expect an infinite amount of money, no matter what we do. The existence of paradoxical f's is a matter of elementary analysis. In the discrete case,

$$f(\eta) = \begin{cases} (h-1)\, h^{(-s-1)}, \quad h > 0 & \text{if } \eta = 2^s, (s = 0, 1, 2, \ldots) \\ 0 & \text{otherwise} \end{cases}$$

works. NALEBUFF (1989, p. 189) introduced this density (with $h = 3/2$), and BROOME (1995) linked this density to Daniel Bernoulli's St. Petersburg Paradox (for an explanatory and interesting reference, see DEHLING, 1997). In the continuous case

$$f(\eta) = \begin{cases} 0 & \text{if } \eta < 0 \\ (h-1)(\eta+1)^{-h}, \quad h \in (1, 2] & \text{if } \eta \geq 0 \end{cases}$$

(the case $h = 2$ was mentioned by BROOME, 1995) will do.

2.4 Mathematical statistics

The previous section is based on the assumption that y is the outcome of a random variable Y and that the distribution of Y is specified by a known function f. In practice, this assumption is questionable. It might happen, of course, that previous experiences provided a reliable estimate of f. It might also be the case that certain theories imply 'partial knowledge' of f. The thrill of the two-envelope problem, however, is in the situation that the decision-maker has no other information than that the outcome of X is x and, of course, that $\mathrm{P}(Z = 1) = \mathrm{P}(Z = 2) = \frac{1}{2}$. Theoreticians are fascinated by such problems. They will try to enforce a solution by using their mind. In the present section we start out with two rationalizations both providing a 'unique' solution (though these solutions should not be regarded as satisfactory). At the end we establish that one of these solutions (namely 'swap, no matter x') is *inadmissible* from the viewpoint of Wald's theory of statistical decision functions.

Approach 1

At the beginning of Section 2.3 it was established that the optimal 'procedure' prescribes to assign a_1 (a_2) if the posterior probability $\mathrm{P}(Z = 1|X = x)$ is smaller (larger) than $\frac{1}{3}$. The question is how to estimate this posterior probability. Note that, in Kolmogorov's theory, this conditional probability is interpreted as the conditional expectation $\mathbf{E}\,(\mathbf{1}_{\{Z=1\}}|X = x)$ and that, in general, the conditional expectation $\mathbf{E}\,(\mathbf{1}_{\{Z=1\}}|X)$ is the projection of $\mathbf{1}_{\{Z=1\}}$ on the space of all functions of X, a subspace of $L_2(\Omega, \mathcal{F}, \mathcal{P})$. This suggests to estimate $\mathrm{P}(Z = 1|X = x)$ by constructing a procedure $d : \mathcal{X} \to [0, 1]$, this construction being such that the mean squared error of prediction $\mathbf{E}\,\left(\mathbf{1}_{\{Z=1\}} - d(X)\right)^2$ is minimum. The value $d(x)$ is then the estimate required.

The joint distribution of X and Z is determined by that of (X, Y, Z) discussed in Section 2.3. The density f of Y appears as the unknown parameter governing the risk. In the continuous case, absence of information about f allows an interpretation in the sense of invariance under scale transformations. If the predictor d is required to be scale invariant then it is constant and the solution $d \equiv \frac{1}{2}$ appears as *uniformly, best invariant predictor of* $\mathbf{1}_{\{Z=1\}}$. As $\frac{1}{2} > \frac{1}{3}$, its consequence is to swap, no matter the outcome x observed. The arguments involved are considerably manipulative because the real issue, namely that of deciding whether or not $\mathrm{P}(Z = 1|X = x)$ is smaller than $\frac{1}{3}$, is replaced by another one, namely the prediction of $\mathbf{1}_{Z=1}$. Moreover some information about f will exist: scale invariance is a mathematical idealization, not something factual to rely on.

Approach 2
All Bayesian and almost all non-Bayesian statisticians know that invariance considerations are elegant as well as dangerous. They would prefer to incorporate additional information, e.g. that $\mathbf{E}\ Y$ is approximately equal to some a priori value, say $\mathbf{E}\ Y = 100$, or, if such value is unavailable, an a posteriori value, say $\mathbf{E}\ Y = \frac{2}{3}x$ (because $\mathbf{E}\ X = \frac{3}{2}y$). Such information $\mathbf{E}\ Y = \mu$ (μ specified) can be incorporated elegantly by constructing f such that the entropy

$$I(f) = \begin{cases} -\sum_{y=1}^{\infty} f(y) \log f(y) & \text{(discrete case)} \\ -\int_0^{\infty} f(y) \log f(y)\ \mathrm{d}y & \text{(continuous case)} \end{cases}$$

is maximum under the restrictions that $\mathbf{E}\ Y = \mu$ and $\sum f = 1$ (discrete case) or $\int f = 1$ (continuous case). It is well known that the solution to this optimization problem is of the exponential form $f(y) = \exp(\theta y - \psi(\theta))$ where

$$\psi(\theta) = \begin{cases} \log(\sum_{y=1}^{\infty} \exp(\theta y)) = \theta - \log(1 - \mathrm{e}^{\theta}) & \text{(discrete case)} \\ \log(\int_0^{\infty} \exp(\theta y)\ \mathrm{d}y) = -\log(-\theta) & \text{(continuous case)} \end{cases}$$

and $\theta \in (-\infty, 0)$ is determined such that $\mathbf{E}\ Y = \psi'(\theta) = \mu$. This is easily obtained from the Lagrangian function $L(f) =$

$$= \begin{cases} -\sum f(u) \log f(u) + \lambda\left(\sum u f(u) - \mu\right) + \gamma\left(1 - \sum f(u)\right) & \text{(discrete case)} \\ -\int f(u) \log f(u) + \lambda\left(\int u f(u) - \mu\right) + \gamma\left(1 - \int f(u)\right) & \text{(continuous case)} \end{cases}$$

with optimality conditions $\frac{\partial L}{\partial f(u)} = \frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \gamma} = 0$ (cf. GOLAN ET AL., 1986, JAYNES, 1996). If one already 'knows' the solution, then a proof can be obtained by using the positivity of the Kullback-Leibler information number.

The interesting case is that where, in absence of further information, $\mu = \frac{2}{3}x$ is used and the ratio $f(\frac{1}{2}x)/f(x) = \exp(-\frac{1}{2}\theta x)$ is compared with the value 2 in the discrete case when x is even (if x is odd, then swapping is always indicated), and with the value 4 in the continuous case.

In the discrete case we have that $\psi'(\theta) = (1 - \mathrm{e}^{\theta})^{-1}$ is equal to $\frac{2}{3}x$ if $\theta = \log(1 - \frac{3}{2x})$ and $f(\frac{1}{2}x)/f(x) = (1 - \frac{3}{2x})^{-\frac{1}{2}x}$ is larger than 2 for all $x \in \{2, 4, 6, \ldots\}$ and, hence, swapping if x is odd and not swapping if x is even is indicated.

In the continuous case we have that $\psi'(\theta) = -\frac{1}{\theta} = \frac{2}{3}x$ implies $\theta = -\frac{3}{2x}$ and that $f(\frac{1}{2}x)/f(x) = \exp(\frac{3}{4})$ is smaller than 4 for all x and, hence, swapping is always indicated.

Inadmissibility considerations
The above mathematical-statistical discussions were considerably manipulative. Two procedures were suggested namely

$$d(\xi) = \begin{cases} a_1 & \text{if } \xi \text{ is even} \\ a_2 & \text{if } \xi \text{ is odd} \end{cases}$$

in the discrete case (see Section 2.5 for an extensive discussion), and

$$d(\xi) \equiv a_2$$

in the continuous case. It is easy to establish that the latter procedure, as well as that where $d(\xi) \equiv a_1$, is inadmissible in the sense that other procedures exist with expected utility never smaller and often larger than the expected utility $\frac{3}{2}\mathbf{E}\,Y$ of these constant procedures. An example is

$$d_k(\xi) = \begin{cases} a_1 & \text{if } \xi > k \\ a_2 & \text{if } \xi \leq k \end{cases}$$

with k a predetermined constant in $\mathbb{R}^+$. To establish that d_k and many other procedures provide expected utilities above $\frac{3}{2}\mathbf{E}\,Y$, it is of interest to consider 'randomized procedures' or, equivalently, test functions $\varphi : \mathcal{X} \to [0,1]$ with the interpretation that $\varphi(x)$ is the probability of a_1 and $\alpha(x) = 1 - \varphi(x)$ that in favor of a_2. The utility of such randomized decision is

$$U(x,y,z;a_1)\varphi(x) + U(x,y,z;a_2)\alpha(x) = x\varphi(x) + (3y - x)\alpha(x)$$

if the outcome of (X,Y,Z) is $(x = yz, y, z)$. The expected utility is

$$\begin{aligned} \mathbf{E}\,(X\varphi(X) + (3Y - X)\alpha(X)) &= \mathbf{E}\,(2X - 3Y)\varphi(X) + \mathbf{E}\,(3Y - X) \\ &= \mathbf{E}\,(2X - 3Y)\varphi(X) + \tfrac{3}{2}\mathbf{E}\,Y \end{aligned}$$

and the excess over the expected utility $\frac{3}{2}\mathbf{E}\,Y$ based on $d(\xi) \equiv a_1$ or $d(\xi) \equiv a_2$ is

$$\begin{aligned} \mathbf{E}\,(2X - 3Y)\varphi(X) &= \tfrac{1}{2}\mathbf{E}\,(2Y - 3Y)\varphi(Y) + \tfrac{1}{2}\mathbf{E}\,(4Y - 3Y)\varphi(2Y) \\ &= \tfrac{1}{2}\mathbf{E}\,Y(\varphi(2Y) - \varphi(Y)) \end{aligned}$$

and this is strictly positive if φ is strictly increasing.
If $\varphi = \mathbf{1}_{(k,\infty)}$ corresponds to d_k, then the excess is equal to

$$\tfrac{1}{2}\mathbf{E}\,Y(\mathbf{1}_{(k,\infty)}(2Y) - \mathbf{1}_{(k,\infty)}(Y) = \tfrac{1}{2}\mathbf{E}\,Y\mathbf{1}_{(k/2,k)}(Y)$$

which is strictly positive if $\mathrm{P}(\frac{1}{2}k \leq Y \leq k) > 0$.

Conclusion
We still are not in the situation that the problem can be regarded as solved. In this respect it is interesting to recall that the context of the Protagoras paradox mentioned in Section 2.2 enforced a (temporary) solution. A contextual ingredient already available but not yet explored is that illustrated by Kraitchik's question at the beginning of Section 2.1: how can the *game* be to the advantage of both? The next section is concerned with this perspective.

2.5 Game theory

We restrict the attention to the formulation of the problem involving only two players: 'you/we' and the mysterious player who completes the envelopes. This situation is very similar to the ones studied by WALD (1964) in his decision-theoretic approach to the problems of statistics. Our approach is in line with FERGUSON (1967), and will use the notation style customary in the theory of statistical decision functions.

A game being defined as a triple, the first game $(\mathcal{A}, \mathcal{Y}, U)$ to consider is that where Player 1 chooses an action from $\mathcal{A} = \{a_1, a_2\}$, Player 2 (the rich eccentric or Nature) chooses y from $\mathcal{Y}$ (either $\mathbb{N}$ or $\mathbb{R}^+$) and

$$U(y, z; a) = \begin{cases} yz & \text{if } a = a_1 \\ y(3 - z) & \text{if } a = a_2 \end{cases}$$

goes from Player 2 to Player 1 ('you/we'). This payoff depends on the outcome z of Z. This formulation is inadequate in the sense that Player 2 cannot be regarded as the 'minimizing' player and also in the sense that the information $x = yz$ available to Player 1 has not been used. To allow for the last mentioned statistical input, the game $(\mathcal{A}, \mathcal{Y}, U)$ is replaced by the game $(D, \mathcal{Y}, U)$ where we now have that Player 1 chooses a decision procedure $d : \mathcal{X} \to \mathcal{A}$ from the class D of all (nonrandomized) rules of this kind. The payoff is defined as

$$\begin{aligned} U(y; d) &= \mathbf{E}\, U(y, Z; d(x, Z)) \\ &= \begin{cases} y & \text{if } d(y) = a_1,\ d(2y) = a_2 \\ 2y & \text{if } d(y) = a_2,\ d(2y) = a_1 \\ \frac{3}{2}y & \text{if } d(y) = d(2y) \end{cases} \end{aligned}$$

This formulation still is irrelevant in the sense that Player 2 will choose y as small as possible. If in the discrete case the specification $\mathcal{Y} = \{1, 2, \ldots, n\}$ is made (for some given n) then taking $y = 1$ is the minimax strategy of Player 2 and any rule with $d(1) = a_2$ and $d(2) = a_1$ is such that the minimum payoff is maximum. The two-person zero-sum game $(D, \mathcal{Y}, U)$ is 'determined' in the sense that $\max_d \min_y U(y; d) = \min_y \max_d U(y; d) = 2$ while the saddle points are characterized in the above. Randomization can be dispensed with.

In his review of WALD (1964), SAVAGE (1951) explained the preoccupation by losses, risks, errors, etcetera in the Neyman-Pearson-Wald approach to statistics. In this approach we are not discussing utilities but regrets; shortcomings with respect to the best. Given y, the best we can do is to swap if we observe y and to keep what we have if we observe $2y$. Hence

$$\begin{aligned} L(y, z; a) &= \max_{h=1,2} U(y, z; a_h) - U(y, z; a) \\ &= 2y - U(y, z; a) \\ &= \begin{cases} (2 - z)y & \text{if } a = a_1 \\ (z - 1)y & \text{if } a = a_2 \end{cases} \end{aligned}$$

and the expected loss, given y, is equal to

$$R(y;d) \quad = \quad \begin{cases} y & \text{if } d(y)=a_1,\ d(2y)=a_2 \\ 0 & \text{if } d(y)=a_2,\ d(2y)=a_1 \\ \frac{1}{2}y & \text{if } d(y)=d(2y) \end{cases}$$

After this *Umwertung aller Werte*, the rich eccentric has become Player 1 while 'we' are now Player 2, the minimizing player in the two-person zero-sum game where we are allowed to choose from the set D and pay the amount

$$R(y,d) = \mathbf{E}\ L(y,Z;d(X))$$

where $X = yZ$ and this risk depends on the value y chosen by Player 1, the equivalent of 'Nature' in Wald's approach to the problems of statistics, initiated in his review of VON NEUMANN AND MORGENSTERN (1944) (see WALD, 1947). The game $(\mathcal{Y}, D, R)$ is not determined because the lower value of the game

$$\underline{v} = \sup_{y\in\mathcal{Y}} \inf_{d\in D} R(y,d) = 0,$$

is less than the upper value

$$\overline{v} = \inf_{d\in D} \sup_{y\in\mathcal{Y}} R(y,d) > 0.$$

In regular situations, it is a necessity that $\mathcal{Y}$ is finite for $\overline{v} < \infty$. Exceptions (irregular situations) are special constructions such as $\mathcal{Y} = \{1,3,5,\ldots\}$ where, of course, the optimal solution

$$d(x) \begin{cases} a_1 & \text{if } x \text{ is even} \\ a_2 & \text{if } x \text{ is odd} \end{cases}$$

yields $R(y,d) = 0$ for all y.

When no restrictions are put on $\mathcal{Y}$ then $\overline{v} = \infty$ because $\max(R(\frac{n}{2},d), R(n,d)) = \frac{n}{4}$ in the nonrandomized case, and $\max(R(\frac{n}{2},\delta), R(n,\delta)) = \frac{n}{6}$ after randomization. To overcome this difficulty and to establish a positive result we will discuss the specific cases where the *upper bound* n is specified a priori. We will review both the continuous $(\mathcal{Y} = [0,n])$ and the discrete case $(\mathcal{Y} = \{1,2,\ldots,n\})$.

The case $\mathcal{Y} = [0,n]$, n given

We are interested in constructing the minimax risk procedure d^* (if it exists) and start out by noting that Bayesian and Laplacian statisticians are attracted by the uniform prior with density $f(\eta) = n^{-1}$, $(0 \le \eta \le n)$. Section 2.3 provides that the corresponding Bayes rule prescribes

$$d^*(x) \quad = \quad \begin{cases} a_2 & \text{if } x \le n \\ a_1 & \text{if } x > n \end{cases}$$

which, quite surprisingly, is minimax as well, with respect to the class D of all nonrandomized Bayes rules. To establish this minimaxity of d^*, we note that

$$R(y, d^*) \quad = \quad \begin{cases} \frac{1}{2}y & \text{if } 0 < y \leq \frac{1}{2}n \\ 0 & \text{if } y > \frac{1}{2}n \end{cases}$$

and that, hence, $\sup_{y \in [0,n]} R(y, d^*) = \frac{1}{4}n$. Next we concentrate the attention on the two possibilities $y = \frac{1}{2}n$ and $y = n$ having $x \in \left\{\frac{1}{2}n, n, 2n\right\}$ as consequence. Any nonrandomized rule d assigns values $d(\frac{1}{2}n)$, $d(n)$, $d(2n)$ to these outcomes. The most appropriate assignments (a_2, a_1, a_1) and (a_2, a_2, a_1) lead to maximum risks equal to $\frac{1}{2}n$ and $\frac{1}{4}n$, respectively. Hence $\sup_y R(y, d) \geq \frac{1}{4}n$ holds for all d. Equality holds for d^* which, hence, is minimax in the sense that

$$\sup_{y \in \mathcal{Y}} R(y, d^*) = \inf_{d \in D} \sup_{y \in \mathcal{Y}} R(y, d) = \overline{v}.$$

Note that minimaxity holds with respect to the class D of nonrandomized mixed rules. If randomization is allowed, the upper value $\overline{v}$ of the game can be decreased to $\frac{n}{6}$ by taking

$$\varphi(x) = \begin{cases} \frac{1}{3}, & x \leq \frac{n}{2} \\ \frac{2}{3}, & \frac{n}{2} < x \leq n \\ 1, & x > n^3 \end{cases}$$

The case $\mathcal{Y} = \{1, \ldots, n\}$*, n given*
Following the suggestions in FERGUSON (1967), we consider the mixed extension $(\mathcal{Y}^*, \mathcal{D}, r)$. Here $\mathcal{Y}^*$ is the class of all probability measures τ on $\mathcal{Y}$, each one characterizable by an element f from the unit simplex S_n in $\mathbb{R}^n$, the coordinates f_η corresponding to the probabilities assigned to the possibilities η for y. The class $\mathcal{D}$ is that of behavioral randomized rules $\delta : \mathcal{X} \to \mathcal{A}^*$ which, as indicated before, prescribe that an action is taken from $\mathcal{A} = \{a_1, a_2\}$ according to a random mechanism which chooses a_1 with probability

$$\varphi(x) = \delta(x)(\{a_1\}),$$

where the test function $\varphi : \mathcal{X} \to [0, 1]$ with

$$\mathcal{X} = \begin{cases} \{1, 2, \ldots, n, n+2, n+4, \ldots, 2n\} & \text{if n is even} \\ \{1, 2, \ldots, n, n+1, n+3, \ldots, 2n\} & \text{if n is odd.} \end{cases}$$

The loss involved in such randomized decision is defined as

$$\begin{aligned} L(y, z, \delta(x)) &= \varphi(x)L(y, z; a_1) + (1 - \varphi(x))L(y, z, a_2) \\ &= \varphi(x)(3y - 2x) + x - y \end{aligned}$$

where, of course, $x = yz$. The risk (expected loss) given y is equal to

$$\begin{aligned} R(y, \delta) &= \mathbf{E}\, \varphi(yZ)(3y - 2yZ) + y\mathbf{E}\, (Z - 1) \\ &= \tfrac{1}{2}y(1 + \varphi(y) - \varphi(2y)). \end{aligned}$$

with special cases (needed later on)

$$\begin{aligned} R(\tfrac{n}{2},\delta) &= \tfrac{1}{2}\tfrac{n}{2}(1+\varphi(\tfrac{n}{2})-\varphi(n)) = \tfrac{n}{6} \\ R(n,\delta) &= \tfrac{1}{2}n(1+\varphi(n)-\varphi(2n)) = \tfrac{n}{6}, \end{aligned}$$

if n is even. In case n odd, we have $R(\frac{n-1}{2},\delta) = R(n-1,\delta) = \frac{n-1}{6}$. Note that there is no reason not to choose $\varphi(2n) = 1$. To minimize $\max_{y\in\{1,\ldots,n\}} R(y,\delta)$ (and for some other purposes as well) it is interesting to consider the Bayes risk

$$r(\tau,\delta) = \sum_{y=1}^{n} R(y,\delta)f(y) = \tfrac{1}{2}f^T(a+A\varphi)$$

where $a = (1,2,\ldots,n)^T$ and A is equal to

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & -3 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 & 0 & -4 \end{pmatrix}$$

in the cases $n = 2$, 3, 4. Extensions to $n \geq 5$ are obvious.

Any of the general minimax risk theorems of NIKAIDÔ (1953, 1959; see SCHAAFSMA, 1971 for more references) implies that

$$\max_f \min_\varphi r(f,\varphi) = \min_\varphi \max_f r(f,\varphi) = \nu$$

and that a saddle point (f^*, φ^*) exists such that

$$r(f,\varphi^*) \leq \nu = r(f^*,\varphi^*) \leq r(f^*,\varphi)$$

holds for all $f \in S_n$ and $\varphi \in \Phi$. Here we used that r: $S_n \times \Phi \to \mathbb{R}$ defined by

$$r(f,\varphi) = \tfrac{1}{2}f^T(a+A\varphi)$$

is linear in f and affine-linear in φ. The theorems imply that the procedure φ^* is minimax and the distribution τ^* with density f^* is least favorable. The saddle point (f^*,φ^*) is not necessarily unique and in our problem it is not unique (if $n \geq 2$). We are now, fortunately, well equipped to characterize the minimax risk procedures.

Theorem 2.1 *If n is* even, *then $\varphi^*(\frac{n}{2}) = 0$ and $\varphi^*(n) = \frac{1}{3}$ is necessary to have $\max_\eta R(\eta,\varphi) \leq \frac{n}{6}$. If we choose φ^* such that*

$$\varphi^*(i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \frac{n}{2} \\ \frac{1}{3} & \text{if } \frac{n}{2} < i \leq n \\ 1 & \text{if } n < i \leq 2n \end{cases}$$

accordingly, then we have $\max_\eta R(\eta, \varphi^*) = \frac{n}{6}$. *The class of all minimax risk procedures is characterized by*

$$\left\{ \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+2}, \ldots, \varphi_{2n})^T \mid \max_\eta \tfrac{1}{2}\eta(1 + \varphi(\eta) - \varphi(2\eta)) \leq \tfrac{n}{6} \right\}.$$

If n *is* odd, *then* $\varphi^*(\frac{n-1}{2}) = 0$ *and* $\varphi^*(n-1) = \frac{1}{3}$ *is necessary to have* $\max_\eta R(\eta, \varphi) \leq \frac{n-1}{6}$. *If we choose* φ^* *such that*

$$\varphi^*(i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \frac{n-1}{2} \\ \frac{1}{3} & \text{if } \frac{n-1}{2} < i \leq n-1 \\ 0 & \text{if } i = n \\ 1 & \text{if } n < i \leq 2n \end{cases}$$

then we have $\max_\eta R(\eta, \varphi^*) = \frac{n-1}{6}$. *The class of all minimax risk procedures is characterized by*

$$\left\{ \varphi = (\varphi_1, \ldots, \varphi_n, \varphi_{n+1}, \varphi_{n+3}, \ldots, \varphi_{2n})^T \mid \max_\eta \tfrac{\eta(1+\varphi(\eta)-\varphi(2\eta))}{2} \leq \tfrac{n-1}{6} \right\}.$$

Proof
It follows from $\max(R(\frac{n}{2}, \delta), R(n, \delta)) = \frac{n}{6}$ that $\min_{\delta \in \mathcal{D}} \max_\eta R(\eta, \delta) \geq \frac{n}{6}$. If we choose $\varphi = \varphi^*$ as mentioned, then indeed $\max_\eta R(\eta, \varphi^*) = \frac{n}{6}$. It is easily seen that

$$\begin{aligned} R(\eta, \varphi^*) &= \tfrac{1}{2}\eta(1 + \varphi^*(\eta) - \varphi^*(2\eta)) \\ &= \begin{cases} \frac{1}{2}\eta & \text{if } \eta \leq \frac{n}{4} \\ \frac{1}{3}\eta & \text{if } \frac{n}{4} < \eta \leq \frac{n}{2} \\ \frac{1}{6}\eta & \text{if } \frac{n}{2} < \eta \leq n \end{cases} \end{aligned}$$

The maximum risk will thus be $\frac{n}{8}$ in case $y \leq \frac{n}{4}$, and $\frac{n}{6}$ in cases $\frac{n}{4} < y \leq \frac{n}{2}$ and $\frac{n}{2} < y \leq n$. Hence, $\max_\eta R(\eta, \varphi^*) = \frac{n}{6}$.
That this solution is not unique is quickly seen by taking $\tilde{\varphi}^*$ equal to φ^*, with the exception that $\tilde{\varphi}^*(1) = 1$. Then

$$R(\eta, \tilde{\varphi}^*) = \begin{cases} 1 & \text{if } \eta = 1 \\ R(\eta, \varphi^*) & \text{elsewhere} \end{cases}$$

and (if $n > 6$) no harm is done to the maximum.
In case of n odd, then the necessity of $\varphi^*(\frac{n-1}{2}) = 0$, $\varphi^*(n-1) = \frac{1}{3}$ is easily shown, in the same way as for the even case. That $\max_\eta R(\eta, \varphi^*) \leq \frac{n-1}{6}$ is trivial after the observation that

$$\begin{aligned} R(\eta, \varphi^*) &= \tfrac{1}{2}\eta(1 + \varphi^*(\eta) - \varphi^*(2\eta)) \\ &= \begin{cases} \frac{1}{2}\eta & \text{if } \eta \leq \frac{n-1}{4} \\ \frac{1}{3}\eta & \text{if } \frac{n-1}{4} < \eta \leq \frac{n-1}{2} \\ \frac{1}{6}\eta & \text{if } \frac{n-1}{2} < \eta \leq n-1 \\ 0 & \text{if } \eta = n \end{cases} \end{aligned}$$

Analogue to the case n even, the solution and its non-uniqueness are trivial. □

2.6 Discussion: the limits of reason

Stripped to its logical essentials, the two-envelope problem is about choosing between the sure gain x and the unsure gain that is either $\frac{1}{2}x$ or $2x$. Issues like the one discussed in the two-envelope problem cannot be settled unless something additional, either factual or fictitional or contextual, is incorporated. The Protagoras paradox is of particular interest in this respect because, there, the *context* was such that a (temporary) solution was enforced. With respect to the two-envelope problem one has to go beyond the logical essentials because the *factual* knowledge $\mathrm{P}(Z = 1) = \mathrm{P}(Z = 2) = \frac{1}{2}$ has to be incorporated. The Bayesian solutions obtained, however, were not applicable because they depended on the unknown function f. In Section 2.4 an attempt was made to settle the issue mathematically. These attempts were not successful either. In Section 2.5 we tried to incorporate arguments from the theory of games and from Wald's theory of statistical decision functions. It was only after the specification of the 'upper bound' n for y, that we could arrive at 'something not unreasonable'.

We conclude that, if almost no information exists with respect to f, then it is wise *not* to try to estimate the posterior probability $\mathrm{P}(Z = 1|X = x)$. Note that half a century ago, Kraitchik made a similar statement, see Section 2.1. If an optimal decision or optimal procedure is advocated then a scrutiny will reveal that it largely is based on something fictitious. To put it in Whitehead's words, it would be a *fallacy of misplaced concreteness* if one completely accepts such result.

We are interested in the two-envelope problem, because there are some consequences for Statistical Science at large. It often happens that the statistician is asked to use data in order to compute some posterior probability, to make a distributional inference, or to suggest an optimal decision. Some, perhaps many, of these situations are such that the lack of relevant information is so large that it is wise *not* to try to settle the issue. This leaves us with the problem to draw a distinction between those situations where the factual information is too weak to make a meaningful statement and those where the factual information is sufficiently strong. The difficulty is, of course, in the area between.

The criticism mentioned here does not only refer to the Bayesian approach where it has to be considered unwise to specify a prior distribution unless relevant information is sufficiently abundant. Criticism refers also to the Neyman-Pearson-Wald approach where we are trying to discuss 'many possible worlds', one for every value of the parameter θ. Usually we assume that exactly one of these worlds is factually true, but we don't know which one. It is well-known from logic, and is exemplified by the two-envelope paradox, that paradoxes may appear if one compares arguments from different worlds, one factual, the other fictitious. We, perhaps, have to admit that our approaches are considerably manipulative and, as KARL PEARSON noticed in his paper about the Fundamental Problem of Practical Statistics (1920), *'in the nearness of an abyss'*.

In our approach to the two-envelope problem, the economist, psychologist and logician did not get as much attention as the probabilist, mathematical statistician and game theorist, because we considered $\mathrm{P}(Z = 1) = \mathrm{P}(Z = 2) = \frac{1}{2}$ to be *factual knowledge* which should not be ignored.

The two-envelope problem is an extremely simple example of the complicated socio-economic and other issues to be settled in practice. Sciences like economy, sociology, and psychology try to clarify these issues by emphasizing various aspects of the managerial use of information. The factual basis of everything is, of course, that data are collected representing the actual state of reality. Unfortunately, such statistical data leave room for uncertainties of various kinds. The sciences indicated have much more to say than is suggested in the context of the two-envelope problem. What they have to say, however, is largely of a qualitative 'existential' nature. The economist may infer from empirical data that assessments of utilities and probabilities are considerable subjective, i.e. different from person to person. The psychologist may use his data to characterize people as systematically risk averse or risk prone. The sociologist will argue that economic behaviour is much less 'rational' than a decision-theorist might hope. The two-envelope problem provides a nice illustration of what Kant called *die Grenzen der Vernunft* — the limits of reason — and the need of factual information to make inferences and decisions 'reasonable'.