# CHAPTER TWO
# DEMOCRITUS AND THE DIFFERENT LIMITS TO DIVISIBILITY

## § 0. Introduction

In the previous chapter I tried to give an extensive analysis of the reasoning in and behind the first arguments in the history of philosophy in which problems of continuity and infinite divisibility emerged. The impact of these arguments must have been enormous. Designed to show that rationally speaking one was better off with an Eleatic universe without plurality and without motion, Zeno's paradoxes were a challenge to everyone who wanted to salvage at least those two basic features of the world of common sense. On the other hand, sceptics, for whatever reason weary of common sense, could employ Zeno-style arguments to keep up the pressure.

The most notable representative of the latter group is Gorgias, who in his book *On not-being* or *On nature* referred to 'Zeno's argument', presumably in a demonstration that what is without body and does not have parts, is not. It is possible that this followed an earlier argument of his that whatever is one, must be without body.[1] We recognize here what Aristotle calls Zeno's principle, that what does not have bulk or size, is not. Also in the following we meet familiar Zenonian themes:

> Further, if it moves and shifts [as] one, what is, is divided, not being continuous, and there [it is] not something. Hence, if it moves everywhere, it is divided everywhere. But if that is the case, then everywhere it is not. For it is there deprived of being, he says, where it is divided, instead of 'void' using 'being divided'.[2]

Gorgias is talking here about the situation that there is motion within what is. If there is to be such motion, parts moving relative to each other must be distinguished. Gorgias seems to be arguing, following Parmenides and Zeno in his second paradox of plurality, that in a homogeneous whole, parts cannot be distinguished in any real sense.[3] Therefore if there is to be motion, what is, is not continuous, but has gaps in it, where it is not. Now Gorgias makes the further assumption that what is, is everywhere moving. To make that possible, what is needs to be divided everywhere, that is, to have gaps

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1 We cannot be sure, because the manuscripts of our source, the pseudo-Aristotelian *De Melisso Xenophane Gorgia* are damaged and corrupted: 979b35-980a1. Here I follow B. Cassin, *Si Parménide. Le traité anonyme* De Melisso Xenophane Gorgia*. Edition critique et commentaire* (Lille, 1980) 499-503.

2 *MXG* 980a3-8, following the text as edited by Cassin, *Si Parménide* 504.

3 Cf. Th. Buchheim (ed., transl. & comm.), *Gorgias von Leontinoi: Reden, Fragmente und Testimonien* (Hamburg, 1989) 184-185.

everywhere. Thus, one may conclude in the Zenonian way, the whole of what is, is not at all.

Perhaps the most famous example of those who sought to meet the Zenonian challenge by giving an account of continuity is Aristotle, but he was not the first, nor did he develop an account in complete independence. He could draw upon debates in the Academy, where, to judge from what we know about Xenocrates and Plato and especially from what we read in Plato's *Parmenides*, issues of infinite divisibility were subject of study and discussion. But above all, if explicit disagreement is to be the criterion, he responded to the ideas of the first philosophers to attempt to escape from the Zenonian conclusions, those of the Presocratic atomists Leucippus and especially Democritus.

Among the Presocratics Democritus stands out as the only one – as far as we know from the sources – to have paid close attention to the arguments set up by Zeno and his followers. This statement, however, can be understood in several ways. 'Having paid close attention to Zenonian arguments' may involve for Democritus nothing more than taking up the Zenonian heritage, making some accommodations in order to save the phenomena of change and plurality, but for the rest leaving much of the heritage intact. This is a familiar picture: atomism is continuous with the philosophy of Elea in that atomism in a way pluralizes the Eleatic one. Thus the atoms still have many of its characteristics, like unchangeability, eternal existence and fullness. A more important feature of this picture is that the atomists did not question the validity of the Eleatic arguments, but ducked their nasty consequences by denying some of their premisses. Aristotle was the first to express his disapproval of this, according to him, *ad hoc* procedure. Discussing Eleatic doctrines he says, clearly referring to the atomists:

> Some gave in to both arguments, to the one [claiming] that everything is one if what is (τὸ ὄν) signifies one, [by saying] that there is what is not, to the one from the dichotomy, by positing atomic magnitudes.[4]

With respect to the atomistic acceptance of the paradoxical statement that what is not, is, it may seem difficult to challenge Aristotle's verdict that there is something *ad hoc* about it.[5] But as regards Aristotle's accusation that the atomists gave in to the argument from the dichotomy, I could not disagree more. As I will present it in this chapter, we should understand the phrase 'having paid close attention to Zenonian arguments', as applied to Democritus, in a more active way. He took up some of the Zenonian arguments, accepted one important principle, but rejected another as inconsistent with the first. There is nothing like 'giving in' here, nor are there any *ad hoc* attempts to escape the Zenonian argument. Instead there is philosophical analysis and the introduction of important distinctions.

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4 *Physica* 1.3; 187a1-3. From here on most of the text of § 1 has been taken from my article 'The Foundations of Presocratic Atomism', *Oxford Studies in Ancient Philosophy* 17 (1999) 1-14. The present text, however, supersedes that article.

5 This is not to say that it is clear how the introduction of what is not, that is, of the void, would be a way of giving in to the argument that everything is one. Of course, there is an interpretation – mistaken, as I shall argue – which suggests itself immediately: what is, is divided into parts, because there is the void to separate them. It is, however, unclear in what sense the atomists would thus commit themselves to the Eleatic assumptions that what is, signifies one, nor indeed is it clear how exactly that assumption led to the Eleatic conclusion, that everything is one.

In this chapter I shall first discuss Democritus' argument for the existence of atoms and defend it against two objections, one of which we are already familiar with from Zeno. The results of this discussion are used to establish that Democritus, while setting a limit to physical divisibility, did not mind divisibility without end when it concerned conceptual or mathematical division. So was he then prepared to let lengths dissolve into sizeless limit entities? I shall address that issue by discussing part of Democritus' work in mathematics.

## § 1. An argument for the existence of atoms

Atomism, Democritus believed, was something one could prove. Of course he uses atoms and especially their shapes and sizes in order to explain the phenomena, such as the qualities and changes of sensible objects, for example when he refers to the hooked shape of some atoms in order to explain the experience of a bitter taste, or the round shape and small size of atoms involved in fire. However, for him these explanatory considerations are secondary; they do not constitute his reason for believing in the existence of atoms. Rather, he offered a metaphysical argument.[6] We find it in Aristotle's report in *De Generatione et Corruptione* 1.2, from which I quote the most important passages:

> Concerning there being atomic magnitudes .. Democritus would appear to be convinced by appropriate, that is, physical arguments. ... For there is a problem if someone would claim that a body, that is a magnitude, is divisible everywhere, and that this is possible. What will be there which escapes division? For if it is divisible everywhere and this is possible, then it may be divided [thus] at the same time, even if it has not been divided at the same time. And if that were to happen, there would be nothing impossible. Thus, if it is of such a kind as to be divisible everywhere – whether in the middle in the same way or in general [by whatever method] –, when it has been divided, nothing impossible will have happened, since even when it has been divided ten thousand times into ten thousand [parts] (εἲς μυρία μυριάκις διῃρημένα),[7] [there is] nothing impossible, although perhaps no-one would [ever] divide [it so].
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> As body, then, is such [*scil.* divisible] everywhere, let it be divided. What will then be left? Magnitude? For that is not possible, for something will not be divided, whereas it was divisible everywhere. However, if there will not be any body or magnitude, and [still] there will be a

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[6] In addition to the argument to be discussed below, there is another argument which seems to be meant as an existence argument. According to Simplicius the atomists – he may be including Epicurus as well – argued from the non-availability of evidence for unlimited cutting:

> Those who rejected cutting to infinity, on the grounds that we are not able to cut to infinity and thus to convince ourselves of the incompletability of cutting, said that bodies consist of indivisibles and divide into indivisibles. (*In Physica* 925.10-13 = DK 67 A13 = Luria [*Democritea* (Leningrad, 1970)] fr. 113)

This is a bad argument, and, more importantly, quite unlike Democritus, who had sceptical inclinations. Perhaps, though, Barnes, *Presocratic Philosophers* 350-351, is too hasty in suggesting it to be 'an invention by Simplicius'. It might be that Democritus used such an argument, not in order to conclude that there are atoms, but in order to make room for their existence. Against those who objected that experience shows us that we can divide a body anywhere we like, he would then point out the limits of our ability to get evidence – it is reason which has to decide the issue. Also in the argument to be presented in the main text we will see a similar reference to a gap between what we can do and verify and what is in fact possible (*DGC* 1.2; 316a22-23). (Cf. perhaps also Aetius, *De placitis philosophorum* 1.3.18ff = Luria fr. 217, who twice calls the atoms λόγῳ θεωρητά, that is, 'to be studied by reason'.)

[7] Thus I do not accept the emendation of H.H. Joachim (ed. and comm.), *Aristotle. On Coming-to-be and Passing-away (De Generatione et Corruptione)* (Oxford, 1922) ad locum and 78, who writes <διαιρεθ>ῇ instead of ᾖ.

> division, either it will consist of points, and [there are] sizeless things of which it is composed, or [there will be] nothing at all, so that it would come to be from nothing and be composed [of nothing], and the whole would be nothing but appearance. Similarly if it consists of points, it will not be a quantity. For whenever they touched and there was one magnitude, and they were together, they did not make the whole any larger. For when divided into two and more, the whole is not anything smaller, nor [indeed] larger, than before. Therefore even if all [points] are put together, they will not produce any magnitude.[8]

After going through some further elaborations of possible ways of composing a magnitude from sizeless entities, Aristotle concludes on behalf of the atomist:

> Therefore since it is impossible that magnitudes consist of .. points, it is necessary that there are indivisible bodies and magnitudes.[9]

Later on in the same chapter, Aristotle restates the argument for the existence of atoms:

> It would seem to be impossible to be potentially divisible everywhere at the same time. For if it were possible, it could also happen (not so that at the same time it is both actually, indivisible and divided, but divided at any point). There will then be nothing left, and the body will have passed away into something incorporeal, and would come to be again either from points or from nothing at all. And how is that possible?
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> However, it is clear that it divides into separable and into ever smaller magnitudes and into magnitudes coming apart and separated. But neither may someone dividing in successive stages bring about an infinite process of breaking, nor is it possible for the magnitudes to be divided at every point at the same time (for it is not possible), but [only] up to a limit. It is necessary, therefore, that there are invisible atomic magnitudes in it, especially if, that is, coming to be and passing away are to occur by segregation and aggregation.[10]

Though there is a marked difference in style and vocabulary between the two passages, they more or less give us the same argument. The heart of that argument is a *reductio ad absurdum*, starting with the following supposition about the object of discussion, *M*:

(1) *M* is divisible everywhere.

In the restatement this supposition is unpacked as saying that *M* could happen to be divided at any point, that is:

(2) It is possible that *M* is divided everywhere.

The same analysis of (1) we find also in the first passage, only now in a few more words. There Aristotle is at great pains to point out that it does not matter how such a state of being divided everywhere is to be reached. That is why he stresses twice that there is nothing impossible in such a state, given (1). It is likely that the emphatic addition after the statement of (1): ‘and that this is possible’, serves the same purpose: to indicate that we should be merely concerned with the possibility that this, *viz.* the situation of *M* being

[8] 316a11-34
[9] 316b14-16
[10] 316b21-34

divided everywhere, is the case, and not with all the possibilities of division which the actualization of this possibility requires to be actualized.[11]

The situation, however, declared possible in (2), that *M* is divided everywhere, is problematic, as Aristotle says in two sentences in the restatement. In the first passage it again takes him more words to point this out. First, Aristotle makes clear that in the state of *M* being divided everywhere there cannot be anything of size left, because that is against the supposition. So *M*, in its completely divided state, must consist of points, that is, sizeless entities, or of nothing. The second alternative he dismisses immediately, but on the first he spends an argument which is rather unclear; I shall discuss it in detail in § 2.2.2. Its conclusion, however, is unequivocal: it is impossible that something with size consists of sizeless entities. So (2) and therefore (1) are not true:

(3) *M* is not divisible everywhere.

In the first passage this is taken to be enough – if we leave Aristotle's subsequent elaborations, which do not seem to be Democritean anyway, out of account[12] – to conclude that there are indivisible magnitudes:

(C) *M* consists of atoms.

The restatement, however, is more careful. For (C) does not have to be true given (3), as (3) leaves open the possibility that *M* is indivisible. This gap is closed explicitly in the restatement, by the observation that *M* divides into magnitudes:

(D) *M* is divisible somewhere.

With (D) the atomist seems entitled to conclude (C), and thus to have proven the existence of atoms.

### *§ 1.1. Two problems*

The reductio part of this argument, up to (3), we have seen before, in § 4.1 of Chapter One. There we encountered it in the so-called Porphyry-fragment, which I ascribed to Zeno himself. On the one hand, this makes clear how closely Democritus must have studied the Zenonian arguments, but on the other it also seems to get him into trouble. For in that fragment we do not have (C), but the alternative conclusion that M is indivisible; the observation (D) is rejected. What is more, this rejection is supported with an argument: because M is similar everywhere, if M is divisible anywhere, then M is divisible everywhere. The same 'argument from homogeneity', as I called it in the previous chapter, Aristotle ascribes to 'some of the ancients', who can only be the Eleatics.[13] According to this argument from homogeneity, it is impossible to accept (D)

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11 *Contra* Barnes, *Presocratic Philosophers* 358-359, who thinks that its point is to indicate that '*you* can actually effect the division' (my italics). For another passage where an equivalent phrase is used, see *DGC* 1.9; 327a14; it has the same function.

12 316a34-b14

13 *DGC* 1.8; 325a9-12, quoted below, p. 65.

without accepting (1). This rules out the atomist's escape from the dilemma between divisibility everywhere and divisibility nowhere.

Even if one were to pass over this objection, there would be a second, yet more serious problem: the argument does not seem cogent, for, as Aristotle may be taken to point out, it can be refuted as trading on an ambiguity. Though the passage, which immediately follows Aristotle's exposition of the atomist argument, is very difficult to understand (*DGC* 1.2; 317a1-12; for an interpretation see Chapter Three § 4.2.2 and Chapter Four § 3.3.2), it seems reasonably clear that Aristotle in effect tries to distinguish between two senses of 'divisible everywhere', namely between (α) 'everywhere able to be divided' and (β) 'able to be divided everywhere'. That is, Aristotle distinguishes between:

(α) $\forall x$ ($x$ is a point on line $l \rightarrow \Diamond\, l$ is divided at $x$)
(β) $\Diamond\, \forall x$ ($x$ is a point on line $l \rightarrow l$ is divided at $x$)

Usually Aristotle is taken to have brought to light the fallaciousness of the atomist argument, since for the transition from (1) to (2) to be justified one must read (1) as in (β), while in order to reach (C) one must read (3), the denial of (1), as the denial of (α).[14]

### *§ 1.2. Problems solved*

Fallacious and violating an Eleatic objection: apparently the existence argument of *De Generatione et Corruptione* 1.2 does not provide a solid foundation to the atomist theory. Therefore one might be tempted to go along with those commentators who propose what is in fact an alternative account as to why there are atoms, which does seem to provide a compelling argument.[15] Though this interpretation has been more or

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14 Barnes, *Presocratic Philosophers* 403-404; C.J.F. Williams (transl. and comm.), *Aristotle's* De Generatione et Corruptione [DGC] (Oxford, 1982) 75; W. Charlton, 'Aristotle's Potential Infinites', in: L. Judson (ed.) *Aristotle's* Physics*. A Collection of Essays* (Oxford, 1991) 129-149, there 135; White, *Continuous* 201-202; cf. R. Sorabji, *Time, Creation and the Continuum. Theories in Antiquity and the Early Middle Ages* [*TCC*] (London, 1983) 340-341. It is probably for the same reason that C.C.W. Taylor, 'Anaxagoras and the Atomists', in: idem (ed.), *Routledge History of Philosophy* I *From the Beginning to Plato* (London and New York, 1997) 208-243, there 221, calls the argument unsound.
I shall ignore the problem that the negation of (α) is strictly speaking not sufficient for atomism, since a division, for example, according to which one does not divide a line at all points, but only at the points corresponding to the rational numbers or even at the points generated by dividing a line recursively in the middle, is incompatible with atomism. However, the basic line of argument of the atomists remains intact, as any division which, though compatible with the denial of (α), rules out atomism, is one at the end of which no line segments of any length have been left. It is possible, though for present purposes unnecessarily complicated, to rephrase (α) and (β) in such a way that this problem is avoided. For a discussion see D. Bostock, 'Time and the Continuum. A Discussion of Richard Sorabji, *Time, Creation, and the Continuum*', *Oxford Studies in Ancient Philosophy* 6 (1988) 255-270, there 260-263.

15 One might be confirmed in taking such a course by considerations which seem to cast doubt on the idea that in *DGC* 1.2 Aristotle is actually reporting a Democritean argument. For example, ascribing the argument to Democritus Aristotle uses the optative: 'Democritus *would appear* to be convinced by appropriate, that is, physical arguments' (316a13-14). Also, the whole argument is pervaded by Aristotle's own vocabulary and ideas, such as the use of δυνάμει and ἐντελεχείᾳ in the restatement, and the way in which all kinds of possibilities are explored. (For such considerations see J. Mau, *Zum Problem des Infinitesimalen bei den antiken Atomisten* (Berlin, 1954) 25-26 and F. Solmsen, 'Abdera's Arguments for the Atomic Theory', *Greek, Roman and Byzantine Studies* 29 (1988) 59-73, at 62-63 and n.7.) Though it is certainly true that Aristotle presents the argument in such a way as to suit his own philosophical purposes – I shall discuss in Chapter Four § 3.3.2 how he

less standard, I hope to show here that it is seriously mistaken, since it ignores part of the evidence. Moreover, the analysis of this evidence will unearth a principle which will dissolve the two objections raised above against the argument of *DGC* 1.2.

### *§ 1.2.1. Can atoms touch?*

One of the major attractions of this alternative account is that it makes Democritus invulnerable to the Eleatic argument from homogeneity. This had best be explained by starting from a passage in Aristotle, *De Generatione et Corruptione* 1.8:

> Some of the ancients thought that what is, is by necessity one and immovable. For the void is not and it would not be possible to move while a separate void is not, nor again are there many, as what keeps apart is not. And if someone holds that the whole is not continuous, but, though divided, <consists of parts which> touch – this does not differ from saying that [the whole] is many, and not one, and void. For if it is divisible everywhere, there is no unit (οὐθὲν εἶναι ἕν), so that the whole is not many, but void. If [the whole is divisible] here but not there, this looks like something contrived. For up to what extent and why is this [part] of the whole like this and full, and that [part] divided?[16]

It is possible to recognize in this passage both versions of the considerations from homogeneity as distinguished in § 4.2 of the previous chapter. For first the assumption seems to be that the mere absence of a separating 'entity' (namely the void, which, unlike the real entities in Zeno's second paradox of plurality, would give us real separation and thus a ground for distinguishing the units) is enough to conclude that what is, is one. Then an objection is imagined: what if the whole consists of divided but touching parts. This objection is dismissed on the basis of the other version, the argument from homogeneity in the strict sense.

Though these are two distinct versions of the considerations from homogeneity,[17] their conjunction also suggests a way out for the atomist. For if there were 'what keeps apart', there could be many. But the atomists have something which could do duty for what keeps apart: the void. Moreover, the presence of some void could provide a simple answer to the question 'up to what extent' something is divisible. So the presence of void between two atoms is a necessary condition for there being two atoms. Conversely, then, an atom is indivisible because it is homogeneous and does not contain any void.[18]

Based upon this account it is easy to construct an argument for the existence of atoms. The atomist accepts Zeno's argument for the indivisibility of what is: every

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does so –, I do not think there is enough reason to doubt Aristotle's ascription. Moreover, this argument is the only type of argument which Aristotle could be referring to when he says that the atomists answered the argument 'from the dichotomy by positing atomic magnitudes' (*Physica* 1.3; 187a1-3).

16 325a2-12

17 In Chapter One I expressed lack of certainty with regard to the question whether Zeno really did distinguish between them. This uncertainty should be maintained because Zeno does not formulate the target of the argument from homogeneity in the strict sense in terms of a whole consisting of divided though touching parts, that is, not in such a way that it is clear that the 'divisibility here but not there' is about the separability of already divided parts. When Zeno employs the argument from homogeneity in the Porphyry-fragment, he merely talks in terms of divisibility, which leaves us guessing whether he is aware of the distinction between separability and a more liberal conception of divisibility, between physical parts, which are found in the division, and conceptual parts, which are created in the division.

18 See especially Makin, *Indifference* 49-53.

homogeneous chunk of matter is indivisible, that is, an atom. However, we see every day that for some things (D) is true: there are objects which are divisible. Those objects thus cannot be homogeneous, but must contain void where they are divisible. They cannot be divisible everywhere, as they would need to contain void everywhere, which is absurd. Therefore they consist of atoms.

Though it has seldom been described as providing an argument for the existence of atoms, the underlying account has in fact been the standard interpretation since Philoponus.[19] It must be said that its neatness is very attractive. Moreover, there is textual evidence for it, notably from Simplicius:

> The atomists thought that the principles are atomic, indivisible and impassive because they are solid and do not participate in the void (ἀμοίρους τοῦ κενοῦ).[20]

Also the terms ναστός ('solid') and especially πλήρης ('full'), so often applied to atoms, suggest that only atoms are completely full, without any void.[21] In this respect they could serve as synonyms for ὁμοῖος παντῇ.[22]

All of this is difficult to reconcile with the argument Aristotle ascribes to Democritus in *DGC* 1.2, since there no mention is made of the void or homogeneity. In fact, that argument rather pictures one magnitude which is only divisible between the touching atoms out of which it is composed.[23] But since on the alternative argument for the existence of atoms atoms should always be kept apart by a layer of void, however thin, in order to avoid forming one homogeneous chunk of matter, it seems impossible that they should touch.

It is, however, not only this point about the argument of *DGC* 1.2 which makes the impossibility of touching atoms troublesome. First, how else but by touch can atoms interact with each other?[24] And second, there is a wealth of references in Aristotle and

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19 Makin, *Indifference* 49-53; Taylor, 'Anaxagoras and the Atomists' 221-222; Furley, Two Studies 99; Barnes, *Presocratic Philosophers* 349; Stokes, *One and Many* 219ff.; R.W. Baldes, 'Divisibility and Division in Democritus', *Apeiron* 12 (1978) 1-12, at 6 (though just stated as an assumption); A.-J. Voelke, 'Vide et non-être chez Leucippe et Démocrite' ['Vide'], *Revue de Théologie et de Philosophie* 122 (1990) 341-352, at 351; W.K.C. Guthrie, *A History of Greek Philosophy* II *The Presocratic Tradition from Parmenides to Democritus* (Cambridge, 1965) 396 and 506; Mau, *Infinitesimalen* 19; cf. Philoponus, *In De Generatione et Corruptione* e.g. 159.3.

20 Luria fr. 214 = DK 67 A14; see also Luria fr. 217.

21 E.g. Aristotle, *On Democritus* (apud Simplicium, *In De Caelo* 294.23-295.26 = Luria fr. 293); Luria fr. 151 = DK 67 A10; Luria fr. 173 = DK 67 A6; Luria fr. 176; Luria fr. 188; Luria fr. 192 = DK 68 A44; Luria fr. 194; Luria fr. 197; Luria fr. 199; Luria fr. 214 = DK 68 A46 & 125.

22 Cf. S. Makin, 'The Indivisibility of the Atom', *Archiv für Geschichte der Philosophie* 71 (1989) 125-149, at 126-127; *Indifference* 54-55

23 I know of only one attempt, by Baldes, 'Divisibility' 6-8, to square the argument of *DGC* 1.2 with the necessary presence of void between the atoms. He imports into the argument the distinction between an object as it appears and as it really is. An object which appears to be a continuous magnitude cannot be divisible everywhere, for then nothing grounds the appearance. As it really is, however, the object consists of atoms separated by void, and is thus not continuous, nor divisible everywhere; the only really continuous entities are the imperceptible atoms, while the object is only divisible at the interstices of void. This attempt looks very strained.

24 It has been suggested to me, however, that there is one testimony that might indicate that the void has some causal power of its own. It is DK 68 A156 = Luria frr. 7 and 78 (Plutarch, *Adversus Coloten* 4, 1108f):

> Colotes is mistaken about the saying of [Democritus] in which he determines that the thing is not more than the nothing (μὴ μᾶλλον τὸ δὲν ἢ τὸ μηδὲν εἶναι), taking body as a thing, void as nothing, because, as he claims (ὡς), that has some character and a reality of its own (φύσιν τινὰ καὶ ὑπόστασιν ἰδίαν).

others to the possibility of touching and colliding atoms.[25] Even more importantly, in three of these passage the possibility of touching atoms is affirmed as if it were something contested or problematic. In *DGC* 1.8 Aristotle says:

> From the atoms then generations and segregations arise. For according to Leucippus there would be two ways, through void and through touch (*for there each is divisible*).[26]

Earlier on in the same chapter he reports:

> By coming together the atoms produce generation, by dissolving they produce corruption. And they act and are acted upon where they happen to touch (*for they are not one there*) (ᾗ τυγχάνουσιν ἁπτόμενα (ταύτῃ γὰρ οὐχ ἓν εἶναι))[27], and by being put together and intertwining

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This is slender evidence for such far-reaching suggestions. It seems more likely that Plutarch wants to give some sense to the paradoxical τὸ μηδὲν εἶναι. Also Voelke, 'Vide' 348-352, wants to ascribe a kind of causal power to the void, but it is not clear to me what exactly he is claiming. It is remarkable, finally, that Philoponus, who repeatedly insists on the necessary presence of void between atoms, denies that the void has a nature of its own (*In De Generatione et Corruptione* 156.24-157.1). Would he be prepared to ascribe a kind of action at a distance to atoms? Perhaps he would not, but Taylor, 'Anaxagoras and the Atomists' 222-223, in fact does suggest, on the basis of Philoponus' evidence, that atoms only interact at a distance. This seems to me an inescapable consequence of the standard interpretation, which I would take as its *reductio ad absurdum*.

25 E.g. DK 67 A1 = Luria frr. 318 & 323; DK 67 A10 = Luria fr. 318; DK 67 A14 = Luria fr. 323; DK 68 A43 = Luria fr. 299; DK 68 A49 = Luria frr. 298 & 323; DK 68 A56 = Luria fr. 180; DK 68 A57 = Luria frr. 42 & 179; cf. also *DGC* 1.8; 326a31-33.
Of course this evidence has been noted as well by proponents of the standard interpretation. Often their solution is to invoke Philoponus, who in *In De Generatione et Corruptione* 158.27-159.3 argues that what Democritus calls 'touch' is not strictly speaking touch (e.g. Barnes, *Presocratic Philosophers* 349; Guthrie, *History* 396; Voelke, 'Vide' 351). Philoponus, however, does not seem to be a very reliable witness with regard to atomism, for in his commentary on Aristotle's *De Generatione et Corruptione* he is neither consistent in pointing out that atoms do not really touch nor can he avoid some strained interpretations.
With regard to the former point, there are examples where it would have been most opportune for Philoponus to refer to the impossibility of touching atoms. One case concerns Aristotle's final criticism of the atomist theory at *DGC* 1.8; 326a31-33, where Aristotle wonders why touching atoms do not become one; Philoponus refrains from saying anything about the void which keeps the atoms apart (175.23-176.5). Another example, though somewhat less clear, would be his comments on *DGC* 1.2; 316a29-31 (*In DGC* 30.2-4):

> When [the points] were together in the magnitude which was undivided, and were one, which he called 'touching', the magnitude did not become anything larger by them.

Since Philoponus takes the whole passage from which these lines have been taken to be a faithful report of the arguments which induced Democritus to atomism (cf. *In DGC* 27.29), with 'he' he seems to refer to Democritus.
For an example of the latter complaint, I refer foremost to note 27. To the passage dealt with there one could perhaps add Philoponus' comments on *DGC* 1.8; 325b30-32 (*In DGC* 163.7-26), though it is not clear to me what exactly Philoponus is arguing – his comments make a rather incoherent impression.

26 325b29-32 – reading Λευκίππῳ μὲν γὰρ with the Latin translation; cf. Joachim, *On Coming-to-be and Passing-away* 164, and Williams, DGC 29.

27 ᾗ and ταύτῃ make most sense if they are local (cf. Williams, DGC 130). Philoponus comments on ποιεῖν δὲ καὶ πάσχειν ᾗ τυγχάνουσιν ἁπτόμενα (325a32) as follows:

> That is, through the void; for by means of the void they touch each other (Τουτέστι διὰ τοῦ κενοῦ· τούτῳ ἅπτονται ἀλλήλων) (*In De Generatione et Corruptione* 158.27-28)

thus taking ᾗ as a kind of dative of means referring to the void. However, ταύτῃ should have the same sense as ᾗ, which clearly does not work with Philoponus' interpretation: the way in which the void plays a part in the touching of atoms (through the void of by means of the void) is not the way in which the void plays a part in the not being one of the atoms, i.e. according to Philoponus the being kept apart of the atoms. This is an example of the strained interpretations Philoponus is forced to adopt because he thinks that atoms cannot touch. The only other alternative to a local interpretation of ᾗ and ταύτῃ I can imagine suffers from a similar defect. For if one reads: 'And they act and are acted upon *in the way* they happen to touch (for *in that way* they are not one)', then the way the atoms happen to touch, which determines the way they interact, cannot be the way they are not one, half of (a), and (a) is used as a principle to establish a correspondence with respect to (in)divisibility between media of motion, then (a) may also be used to derive the (in)divisibility of motion from the (in)divisibility of the path moved over. Thus we do have in (2) an argument for at least half of the initial thesis.

But (a) does not establish a mere correspondence with respect to (in)divisibility between path, moving and motion; it establishes a *temporal* correspondence. In (b) a relation of simultaneity is posited between making motion *m* and moving over *p*. And with regard to (a) with 'being present' a temporal element is clear in 'being present', even though only an absolute genitive and a conditional conjuction are used in stating the biimplication.[245] If then the motion, the moving and the path correspond through (a) with respect to (in)divisibility, and by the same principle (a) also correspond temporally, it may not be too bold to assume that (a) can also be used to argue from the (in)divisibility of any of these three media to the (in)divisibility of time. For suppose the 'when' of the correspondence given by (a) were divisible: then we should be able to distinguish between at least two parts of the moving over *p* as well. In this way we would have a complete argument for the initial thesis, an argument which, unlike the proportional arguments, does not merely assume that magnitude, motion and time are connected, but ultimately bases such a connection on the concept of *moving over* p *in time* t.

### *§ 5.2.2. Motion without motion*

In (1) and (2), motion and moving are treated as media which come in certain quantities, in this case in equal indivisible measures, but without any internal structure. In (3)-(5), however, Aristotle leaves this what one might call homogeneous or continuous perspective on motion and moving, and states that each stretch of motion and moving has an internal structure: it is from somewhere to somewhere, and within it one has to distinguish between the process of moving, indicated by the present and imperfect tenses, and the completion and result of moving, indicated by the perfect tense. From this discrete perspective he redescribes the result of (2). The indivisible stretch of moving, which occurred over an indivisible path by making an indivisible motion, is split up temporally into a process of moving over an indivisible path and a state of completion of motion over that indivisble path. Aristotle here, just as elsewhere, takes it for granted that we have to accept such a division; from the second half of the dilemma presented in (3), it is obvious that he thinks the simultaneity of process and state is just impossible.

Distinguishing within the time of motion a time for a process of moving and a later time for a state of having moved, on the other hand, is also unacceptable, because it leads to the divisibility of the indivisible. The situation, however, envisaged by Aristotle in the derivation of that absurdity is rather difficult to comprehend: what does it mean to

internal object, as Wagner, Physikvorlesung 150 and 618, does. On the latter translation the biimplication (a) seems devoid of any content, while on the former there is the additional problem that then there is no link with (b). That such a link is intended also appears from (3) where in 232a1-2 the scheme of (b) is followed, but in an even clearer reference to principle (a): 'Z was moving over partless A, by which motion D was present.'

245 Moreover, the difference between on the one hand the correspondence with respect to indivisibility between path and motion stated in (1) – where the motion DEF is said to contain 'an indivisible for each part [of the path ABC]', but nothing more – and on the other hand the correspondence argued for in (2) seems to lie exactly in the temporal nature of the correspondence in (2).

> they generate. But *from the really one a plurality cannot come to be nor from the really many a unity – that is impossible.*[28]

And in the only fragment left of his lost work *De Democrito* Aristotle explains:

> As the atoms move they collide with each other and interlock in such a way that, *though they touch and get close to each other* (περιπλέκεσθαι περιπλοκὴν τοιαύτην, ἣ συμψαύειν μὲν αὐτὰ καὶ πλησίον ἀλλήλων εἶναι ποιεῖ), *yet a single substance is never in reality produced from them; for it would be very simple-minded to suppose that two or more things could ever become one.*[29]

These passages show that Democritus did consider the difficulty of touching atoms. The italicized lines cannot but be taken as an explicit rejection of the Eleatic argument that it is impossible that a homogeneous entity be divisible in one place (between the touching atoms) and not in another (within the atoms).[30] Thus they make the standard interpretation untenable.

### § 1.2.1.1. Interpretational interlude

Is this too strong a claim? Was there not also other evidence mentioned in favour of the standard interpretation, claiming or suggesting that atoms are indivisible because they do not participate in the void? In what follows I shall argue that we should not rely on this evidence, so that there is no reason whatsoever to doubt that atoms can touch.

First, some distinctions will need to be drawn which are crucial to the atomist position. For what does it mean to form a continuous whole or to be divided? The Eleatics would say that the only possible meaning of 'division' is separation by a gap; if entities form one homogeneous whole of touching parts, they are in fact one and indivisible from each other because of the 'similar everywhere' argument. The reasoning is thus from continuity to indivisibility and unity, but as the only form of discontinuity admitted is separation, continuity is a broad notion with the Eleatics.

The atomists, on the other hand, insist on there being two ways of being divided. In the passage from *DGC* 1.2 quoted at the beginning of § 1 which I called the restatement, Aristotle carefully mentions divisions 'into separable magnitudes' (εἰς χωριστὰ μεγέθη) which are touching and those 'into magnitudes coming apart and separated' (εἰς ἀπέχοντα καὶ κεχωρισμένα).[31] Thus the atomists posit a third possibility between the Eleatic extremes of separation and unity, that of being divided while touching, which is the same as that of being separable. But of the distinctions between these three states, being separated, being merely divided and being one, that between the last two is clearly of the greater importance to the atomists. This also appears from their use of 'continuous'. The clearest testimony on this is from Simplicius:

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for there is only one way of their not being one, whereas the first half of the translated sentence suggests that there are several ways of touching and thus of interacting. Finally, for a verbally close parallel which does distinguish between the local and instrumental uses of the dative, see *Physica* 8.4; 255a13-15:

> For where (ᾗ) it is one and continuous, but not by touch (ἁφῇ), there (ταύτῃ) it is impassive; but where (ᾗ) it has been divided, there (ταύτῃ) the one is of such a nature as to act, the other as to be acted upon.

28 *DGC* 1.8; 325a31-36

29 Translation by J. Barnes and G. Lawrence in: J. Barnes (ed.), *The Complete Works of Aristotle* [*CWA*] II (Princeton, 1984) 2446. For the reference see note 21.

30 Cf. Williams, DGC 130.

31 316b28-29

> Only the atoms they called continuous. For the other things which appear to be continuous become neighbours to each other by contact (ἁφῇ προσεγγίζειν ἀλλήλοις). That is why they also abolished the cut as the dissolution of touching things – they called it the apparent cut. And because of that they said that many do not come to be from one, for the atom does not divide. Nor does a really continuous one come to be from many, but each thing appears to become one by the intertwining of atoms.[32]

Again continuity goes with unity; but the continuity between touching atoms is not real[33] for the same reason that separation of touching atoms is not real – the cut is only apparent. For there is no real dividing going on; the atoms are already divided from each other, thus forming a plurality.[34]

Thus the term συνεχής is much stricter for the atomists than for the Eleatics. The Eleatics call everything actually not separated by a gap continuous, whereas the atomists call something continuous only if it is not merely *actually* not separated by a gap, but also not separated in any *non-actual* state of affairs which (might have) obtained in the past or might obtain in the future. Now in order to accommodate the testimonies which seem to suggest that atoms have to be separated by void, I propose to read the terms πλήρης, ναστός and ἄμοιρος τοῦ κενοῦ with this much stricter atomist criterion in mind.[35] Thus these testimonies do not suggest an *explanation* of the indivisibility of atoms, but they state what this indivisibility *consists in*.[36]

### § 1.2.1.2. Differentiating what is homogeneous

Up to now I have considered an argument for the existence of atoms which could serve as an alternative to the argument of *DGC* 1.2 and which is not vulnerable to objections on the basis of the argument from homogeneity. That alternative argument did prove to be unacceptable because it ruled out touching atoms, whereas the evidence shows that

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32 Simplicius, *In De Caelo* 609.19-24 = Luria fr. 237

33 Aristotle knows of a distinction identical to that drawn by Simplicius between real and apparent continuity. For in *Physica* 8.4; 255a13 he refers to what is 'one and continuous, not by contact', whereas in *Physica* 3.4; 203a22 he employs the term 'continuous by contact'; both times he does so in connection with atomistic doctrines. By Aristotle's own standards, as set forth in *Physica* 5.3; 227a10-17, being continuous merely by contact would probably not qualify as real continuity, since he implies that only what is continuous is one.

34 Perhaps this is the background of the shift from being divisible to being divided in Aristotle's report in *DGC* 1.8; 325a10-12 (for the context see above, p. 65): 'Up to what limit is it divisible and why is one part of the whole like this and full, yet another divided?' It seems as if divisibility, i.e. separability, is grounded in a situation of being divided.

35 Perhaps even Simplicius' explanation of the atoms being ἄμοιροι τοῦ κενοῦ can be made to fit my proposal:

> For they said that there comes to be a division along void in the bodies (τὴν γὰρ διαίρεσιν κατὰ τὸ κενὸν τὸ ἐν τοῖς σώμασιν ἔλεγον γίνεσθαι). (*In De Caelo* 242.20-21)

if we understand by 'division' separation, which always involves void entering the larger body consisting of several atoms ('a division in such a way that there is void in the bodies'). If it cannot, however, we will have to say that Simplicius is just wrong.

36 Testimony which seems to be close to such an understanding of the terms as applied to atoms, is supplied by Aetius, who says:

> [The atoms are] not participative of void (ἀμέτοχα κενοῦ), ungenerated, eternal, indestructible, and cannot be broken (οὔτε θραυσθῆναι δυνάμενα), receive a reshaping in its parts or alter. .. And [the principle] is called atomic not because it is the smallest, but because it cannot be cut, being impassive and not participative of void. Hence, if <someone> calls an atom unbreakable, he also calls it impassive [and] not participative of void. (*De placitis philosophorum* 1.3.18ff = Luria fr. 217)

Here the indivisibility of the atoms is not so much explained as glossed by their being not participative of void. Moreover, given the abundance of modal terms applied to the atoms, and because it is here the very first term applied to them, it seems unlikely that ἀμέτοχα κενοῦ can be read in a non-modal way.

the atomists thought that atoms do touch, indeed that they affirmed the possibility of touching atoms in such a way that it appeared to be something contested, thus rejecting the argument from homogeneity. But that does not seem to bring us very far: rejecting an objection is not refuting an objection. The two problems with the argument of *DGC* 1.2 still stand.

However, if we look more carefully at the second and the third of the three passages quoted in evidence that the atomists did consider the difficulty of touching atoms, then we see that they also gave a reason to underpin their rejection of the argument from homogeneity. This reason – which could be rephrased as 'No unity from a plurality, no plurality from a unity' – I shall call henceforth the Atomistic Principle (AP).[37]

If we want to take seriously this idea that the atomists used the Atomistic Principle against the argument from homogeneity, it is incumbent on us to analyse how it could serve that purpose. Initially that seems easy. The atomist may set up the following counterexample to the argument from homogeneity. First two atoms are separated by a void; in a second situation they are touching. Because of the conservation principle (AP) they remain two. Still that does not explain how exactly (AP) weakens the argument from homogeneity. For suppose there are two entities, $E_1$ and $E_2$, exactly similar in all respects except that $E_1$ is indivisible and $E_2$ consists of two atoms; are not both homogeneous? And should both not therefore be indivisible (given that they cannot be divided everywhere), as there is nothing to explain the difference? This Eleatic example pulls us into a direction opposite to that of the atomist counterexample.

It is the atomists themselves who come to our rescue in providing an adequate analysis of the Eleatic example. We recall that, according to a testimony from Simplicius, they drew a distinction between real and apparent continuity, where something is really continuous if and only if it consists of parts which are not merely actually not separated, but are also not separated in any non-actual situation which (might have) obtained in the past or might obtain in the future; apparently continuous are then things which do not meet the second condition. This distinction Simplicius, moreover, links to the principle (AP), and it is not difficult to understand him: because the separation of touching atoms is only an apparent cut in an apparent continuum, a separation does not create a new plurality, since the atoms are already divided from each other, forming a plurality; conversely a real unity cannot be divided, as there is no real cut in a real continuum.[38]

With this distinction in mind, including its close connection with the principle (AP), we may respond to the Eleatic example by drawing a very similar distinction: both $E_1$ and $E_2$ are homogeneous in one sense, but only $E_1$ is homogeneous in another. And just like the distinction between real and apparent continuity this distinction too depends on whether or not one excludes reference to non-actual states of affairs which (might have) obtained in the past or which might obtain in the future. For the sense in which even $E_2$ is homogeneous is one which, against the point of the atomist counterexample, rules out such reference; that is, $E_2$ is described in terms which only refer to the changeless

[37] Aristotle refers to (AP) also in *Physica* 3.4; 203a22-23 and 33-34, and *De Caelo* 3.4; 303a6-7; cf. *Metaphysica* Z.13; 1039a7-11 as well.

[38] It may seem that according to Simplicius (AP) has a logically derivative status ('because of that'), but as Simplicius is here commenting upon *De Caelo* 3.4; 303a3-10, which merely mentions (AP), Simplicius can here just as well, and even preferably so, be taken to be explaining the *purpose* of (AP).

situation at one moment. I call this homogeneity of $E_2$ (which automatically also applies to $E_1$) 'geometrical homogeneity',[39] for it is only by using geometrical terms that one can avoid making any implicit reference to non-actualities.[40] On the other hand, the kind of homogeneity possessed solely by $E_1$ one can only ascribe to something when also having taken into account all (possible) states of affairs different from the present one, for $E_1$ is one because there is no possible situation in which it is divided. This homogeneity I call 'dispositional homogeneity', for the properties one ascribes in this way to something are, like (in)divisibility, dispositional properties.

The distinction between geometrical and dispositional homogeneity enables us to reveal the underlying assumption of the Eleatic use of the argument from homogeneity, for instance in the example of $E_1$ and $E_2$: the Eleatic argues from geometrical homogeneity to dispositional homogeneity, *in casu* from geometrical homogeneity to either divisibility everywhere or divisibility nowhere.[41] For their example only works if from the geometrical homogeneity of $E_2$ it also follows that $E_2$ is dispositionally homogeneous. But it is not, because at the point where it is divisible $E_2$ is different from elsewhere. It is this conflation of geometrical with dispositional homogeneity which the atomist denies implicitly when he justifies the possibility of touching atoms by appeal to (AP). For a principle like (AP) is indispensable for ascribing dispositional properties to something, as it allows one to conclude to the presence of such a property from what is the case in non-actual states of affairs, like in the atomist counterexample.[42] So (AP) enables the atomist to differentiate between geometrical and dispositional homogeneity and thus to question the validity of the Eleatic use of the argument from homogeneity.

### *§ 1.2.2. Validating the argument of DGC 1.2*

Two things have been shown in the previous section. First, the attempt to come up with an argument for the existence of atoms which could serve as an alternative to the beleaguered argument of *DGC* 1.2, failed because the underlying assumption, that atoms are indivisible because there is no void in them, cannot be squared with the evidence. Thus we are forced to return to the foundation provided by the argument of *DGC* 1.2. Second, in rejecting the argument from homogeneity by appeal to (AP), Democritus has refuted one of the two objections to that argument as stated in § 1.1. What I want to show next is that if (AP) is included among its premisses, the other objection to the argument will disappear as well.

This objection was that the argument at one stage needs 'divisible everywhere' to be understood as 'possibly divided everywhere' (β), while at another stage 'divisible every-

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39 Strictly speaking the whole universe, including all the void, may be called geometrically homogeneous; as used here the term only concerns materially instantiated geometrical homogeneity.

40 Cf. J. Franklin, 'Reply to Armstrong on Dispositions', *Philosophical Quarterly* 38 (1988) 86-87.

41 The former possibility is ruled out later on separate grounds. Because something is also dispositionally homogeneous if it is divisible everywhere, the distinction between geometrical and dispositional homogeneity does not coincide with that between apparent and real continuity, for a magnitude which is divisible everywhere would not be really continuous.

42 I ignore here the problem of possibilities of separation which are never realized during the whole of infinite history of the world. This problem would be solved for Democritus if he adopted a version of the principle of plenitude: every possibility of separation must be actualized at least once during this infinite history. There are some reasons, both philosophical and textual, to ascribe a version of the principle of plenitude to Democritus: see Makin, *Indifference* 209-224.

where' must be understood as 'everywhere possibly divided' (α). However, if one brings to bear upon this issue the atomistic doctrines based upon (AP), namely that there are no real divisions, but only apparent ones, that is, separations, and that in order for a separation to be possible an object has to be already divided, then one can see that in the argument of *DGC* 1.2 by 'divisible' the atomist means 'separable' and by 'divided' he means 'separated'. That again means that the weaker interpretation (α) of 'divisible everywhere' – read: separable everywhere – in fact implies 'divided everywhere': for *M* to be everywhere separable it must already consist of sizeless entities, which is absurd. Thus there is no ambiguity in the argument any more.

So it is very profitable to make (AP) a premiss of the argument for the existence of atoms. However, it is one thing to show that Democritus had the conceptual resources to answer the second problem, and another to argue that he did actually consider (AP) to be a premiss. Fortunately, there is evidence he did, for Aristotle says in the conclusion of his restatement:

> It is necessary, therefore, that there are invisible atomic magnitudes in it, especially if, that is (ἄλλως τε καὶ εἴπερ), coming to be and passing away are to occur by segregation and aggregation.[43]

The identification of generation and aggregation Aristotle thus takes to be a reason supportive of the atomist conclusion. How can that be? If one reads the definition of generation in terms of aggregation as saying that what we call the generation of an object is in fact nothing more than the aggregation of smaller entities, then this identification implies and is implied by the principle (AP). According to this reading it is very easy to understand Aristotle, along the lines discussed above. Now it has to be admitted that the identification of generation and aggregation need not be interpreted in this way. If 'aggregation' means the coming together and being fused of entities, so that a new unity comes into being and the old entities cease to be, then the principle (AP) would be irreconcilable with the identification of generation and aggregation. However, giving sense to Aristotle's inference from the identification to atomism seems to me to have become impossible in that way, for every candidate for the status of atom would then be dissolvable into equally new pluralities (by the concomitant identification of ceasing to be and segregation).[44] Moreover, also in another context Aristotle takes the identification of aggregation and coming to be as being inferred from (AP). The passage is from *De Caelo*:

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[43] 316b33-34

[44] Philoponus, who does not accept the inference from the identification to atomism (see *In De Generatione et Corruptione* 24.26-29; 38.24-27), offers the following interpretations to clear up this 'darkly expressed statement' (*ibidem* 38.22; it should be noted that Philoponus' manuscript did not include ἄλλως τε καὶ):

> One must, then, understand the statement more simply, saying thus: It is necessary, therefore, that there are atomic magnitudes', and punctuating there, as from another beginning: 'If, then, there is generation and ceasing to be, the one will be by segregation, the other by aggregation.' Or thus: 'It is necessary, therefore, that there are atomic magnitudes, from which, if there in fact is generation and ceasing to be (which is clear), the one will be by segregation, the other by aggregation' (*ibidem* 38.30-39.4).

In either of these ways he tries to save Aristotle from an inference which he thinks is mistaken. However, both his proposed interpretations are impossible if one retains ἄλλως τε καὶ. Moreover, even if one were to follow Philoponus' manuscript, it would be quite unclear what the function of the second part, from εἴπερ onwards, is in the context.

> For [Leucippus and Democritus] say that there are primary magnitudes which are unlimited in number and indivisible in magnitude, and that from one many do not come to be nor from many one, *but* that all things are generated (γεννᾶσθαι) by the intertwining and collision (περιπαλάξει) of these.[45]

By the use of 'but' (my italics) the identification of generation and intertwining, that is, aggregation becomes a positive restatement of the denial inherent in (AP).

Though I think that this is rather impressive evidence, there are problems with my use of the concluding lines of Aristotle's restatement for the purpose of making (AP) a premiss of Democritus' argument for the existence of atoms. (We will consider this passage, as well as some other passages from *DGC* 1.2, in more detail later; see Chapter Four § 3.3.) But one can give also indirect reasons which should lead us to the conclusion that Democritus did indeed adopt (AP) as a premiss of the argument. A first indication that he did so, can be derived from the obvious structural similarity of the atomistic argument and Zeno's argument in the Porphyry fragment. We have seen that the only difference between them was that whereas the atomist drew the conclusion that *M* is not divisible everywhere, but at least somewhere, Zeno came to the stronger conclusion that *M* is nowhere divisible. As Zeno ruled out the atomistic conclusion by the argument from homogeneity, Democritus can only have taken his weapon against that argument, (AP), as a premiss of his own argument.

Apart from this rather specific consideration there is also a more general point to be made. One result of the account of the previous section is that (AP) is not a consequence of atomism,[46] but rather an independent principle on which the impossibility of atoms being divided or merged is based.[47] Given this independent status, it would be quite strange if Democritus had not presupposed it in his argument for the existence of atoms. For this there are three closely related reasons. The first reason is that there is an inseverable link between arguing for the existence of entities exhibiting a feature *F*, in this case indivisibility, and arguing that these entities exhibit *F*, in this case arguing that they do so because of (AP). But defending himself against the argument from homogeneity Democritus does base the indivisibility (and the impossibility of being merged) of the atoms upon (AP).[48] Secondly and more specifically, the argument of *DGC* 1.2 starts with the hypothesis that *M* is *divisible* everywhere and assumes that *M* is at least *divisible* somewhere. Can we imagine that Democritus, when setting up this argument, had forgotten about his ban on real divisions, a ban based upon (AP)? Therefore he must be talking about mere *separability* rather than divisibility, thus making the argument invulnerable to Aristotle's so-called refutation.[49] And thirdly, since an argument based upon (AP) for the existence of atoms is very simple, almost trivial, it would be

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45 *De Caelo* 3.4; 303a5-8

46 The view that (AP) is a consequence of atomism is most clearly stated by Stokes, *One and Many* 225-234; it rests squarely on what I have called the standard interpretation as to why atoms are indivisible.

47 That (AP) enjoyed this independent status may also appear from the way Aristotle mentions it in the passages referred to in note 37.

48 I.M. Bodnár, in his critical review of proposed explanations for atomic indivisibility 'Atomic Independence and Indivisibility', *Oxford Studies in Ancient Philosophy* 16 (1998) 35-61, at 37-38 (cf. 61), still maintains that accounting for the indivisibility and unmergeability of particular pieces of matter, is accounting for (AP), not the other way round. Barnes, *Presocratic Philosophers* 350, seems to think that it is attractive to ascribe to Democritus an account of the indivisibility of atoms based upon (AP), but he hesitates for lack of evidence. I hope I have shown that one should be more confident.

49 Cf. the remark in note 34.

surprising if Democritus had denied such an argument to himself. For (AP) just says that everything which is divisible is not really one, but consists of indivisible basic units.

### *§ 1.3. Concluding remarks*

I hope to have shown that by including the principle (AP) among the premisses of the argument for the existence of atoms in *De Generatione et Corruptione* 1.2, the two objections raised against that argument can be removed. The accusation that it trades on the ambiguity of the crucial term 'divisible everywhere' between 'possibly divided everywhere' and 'everywhere possibly divided', appeared to be unfounded, since it follows from (AP) that every possibility of separation is based on an already existing division between atoms. Similarly the atomist could reject the complaint that it is completely arbitrary that a homogeneous body consisting of atoms without any void between them would be divisible only at the points of contact between the atoms. This complaint proved to be based on too broad a criterion of homogeneity, as the atomist could argue on the basis of the principle (AP) that a body is only homogeneous if it cannot be divided.

The conclusion must be that there is nothing wrong with the atomistic existence-proof. Rather the fact that the very same principle can be invoked by the atomist to answer both objections to his proof, shows that it is the Eleatic who has the impossible position. For since the core of the Eleatic argument from homogeneity as presented both by Aristotle in *DGC* 1.8 and in the Porphyry fragment is identical to the core of the atomistic argument against which the charge of ambiguating between two senses of 'divisible everywhere' was levelled, the Eleatic can only uphold his argument from homogeneity if he himself adopts the principle (AP). But at the same time he has to reject (AP) if he wants the argument from homogeneity to have the desired effect. This double bind for the Eleatic can be described in terms of the failure to distinguish between geometrical homogeneity and dispositional homogeneity, but also in terms of the failure to distinguish clearly the two versions of the considerations from homogeneity as I analysed them in Chapter One. For the criterion the Eleatic uses to establish his cherished homogeneity belongs to the second version: internal indistinguishability, the lack of gaps in one changeless situation. In Chapter One I pointed out that on this second version, according to which indistinguishability leads to unity, it is incompatible with the principle that pluralities stay pluralities, and unities unities – the atomistic principle (AP). The contents, however, the Eleatic subsequently gives to this homogeneity belongs to the first version, to the argument from homogeneity in the strict sense. For there homogeneity is not so much a matter of whether or not there are internal distinctions to be made, but of how the possibilities of distinctions are distributed over the entity.

The ultimate source of the failure to distinguish between the two, I take to be a kind of presentism. If one thinks, with Zeno, that everything real is confined to the present, then the absence of internal distinctions is enough reason to conclude that the (im)possibilities of future separation are distributed without internal distinctions as well. The atomist, on the other hand, argues the other way round, from the totality of possible situations over the whole of time to the present, actual situation, thus giving the present a derived status. On the basic issues Zeno and Democritus only seem to be able to dis-agree.

# § 2. Atomism and mathematics

## *§ 2.1. The problem*

Up to now we have considered two objections against the argument for the existence of atoms as ascribed to Democritus in *De Generatione et Corruptione* 1.2. These objections were of an internal nature, being aimed at parts of the internal mechanism of the argument, and thus contesting the correctness of its outcome. There is, however, a third objection which has been raised against the argument, one which does (provisionally) accept that it establishes the existence of atoms, but alleges that it even would prove too much for the atomist in that it proves that atoms are conceptually indivisible.

In the previous chapter we saw that Zeno did not distinguish between conceptual parts and physical parts, between conceptual divisions (divisions into parts defined by their length and position only) and physical divisions (divisions into parts which are separable by a gap). It is the opinion of quite a few scholars that Democritus in the argument at hand in fact did not distinguish between them either, because it seems to work equally well for conceptual divisions and physical divisions.[50] The idea is that it does not matter in what sense of division a magnitude is divided everywhere: whether it consists of physically separated points or of merely conceptually distinguished points, its ultimate parts are still without size.[51] The atoms then are mathematical minima.

Apart from the evidence provided by the existence argument of *DGC* 1.2, other reasons have been advanced for turning Democritus into a mathematical atomist. One idea is that Democritus wanted to respond to Zeno's first paradox of plurality or to its moving version, the Runner, by stopping in its tracks the infinite division they presuppose.[52] And that division is certainly of a conceptual kind. Then there is Aristotle's accusation that Democritus' atomism clashes with mathematics:

> [I]t is necessary that [Democritus and Leucippus], by saying that there are atomic bodies, are in conflict with the mathematical sciences and do away with many of the reputable opinions and

50 See Guthrie, *History* 503-504; cf. 394, especially note 2; Furley, *Two Studies* 94; Sorabji, *TCC* 356; S. Luria, 'Die Infinitesimaltheorie der antiken Atomisten', *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* B2 (1933) 106-185, at 136; cf. his *Democritea* 46-49 (pages!). The issue is fudged, however, because of the very different ways in which these authors express their views. Thus Guthrie talks about 'the logical indivisibility' of atoms, Sorabji about their 'conceptual indivisibility' and Furley about their 'theoretical indivisibility'. Moreover, some of them go out of their way to distinguish between all kinds of indivisibility. I am not going into that debate; I do not think there is anything more to the issue than the distinction between conceptual division and physical division as described in the main text. Luria, finally, does recognize that distinction, but ascribes to Democritus an Epicurus-style two layer-theory, according to which there are physical atoms made up of mathematical, that is, conceptual, atoms; he takes the present argument as evidence that Democritus assumed the existence of the latter kind.

51 Against the objection stated thus, it is pointless to argue on the basis of the language of the argument that Democritus *assumed* that he was talking about physical divisibility. (For such an attempt, see for example Barnes, *Presocratic Philosophers* 358-359.) The issue at this point is not whether he thought he was arguing for conceptual atomism, but whether he is, as a matter of fact, committed to it.

52 See D.J. Furley, *The Greek Cosmologists* I *The Formation of the Atomic Theory and its Earliest Critics* (Cambridge, 1987) 126.

> the phenomena according to perception, about which there has been a discussion earlier in the works about time and motion.[53]

Since 'the works about time and motion' must refer to the discussion in the *Physics*, notably *Physica* 6, I take it that with 'the reputable opinions and the phenomena according to perception' Aristotle means the ideas that bodies touch by having their boundaries together and that there is really motion, in the fluent way, rather than that motion consists of quirks, first being at one indivisible place and then at another.[54] Atomism can only do away with these points, if it posits mathematically indivisible bodies. The same line of thought Aristotle seems to follow in his reference to his discussion of *Physica* 6.1 in *De Generatione et Corruptione* 1.2 itself:

> Also for those who posit [indivisible bodies and magnitudes] no less impossible consequences follow; but there has been an inquiry into these issues elsewhere.[55]

Moreover, Aristotle's accusation seems to be confirmed by a testimony that calls Democritus' atoms partless (ἀμερῆ), which supposedly means that even mathematical parts cannot be distinguished within atoms.[56]

Finally, one reconstruction of the way Democritus arrived at the theorems that a pyramid and a cone are one third of a prism and a cylinder with the same base and equal height, credits him with the idea that a cone, and generally every solid figure, consists of infinitesimal laminae; these laminae are then identified with mathematically indivisible atoms. We know from Archimedes that Democritus was the first to state these theorems. For in the introduction to his *Methodus*, Archimedes says, praising the heuristical value of his method employing such infinitesimals, that:

> It is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge. That is the reason why, in the case of the theorems the proofs of which Eudoxus was the first to discover, viz. on the cone and the pyramid, that the cone is one-third of the cylinder and the pyramid one-third of the prism having the same base and equal height, no small share of the credit should be given to Democritus, who was the first to state the fact about the said figure, though without proof.[57]

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53 *De Caelo* 3.4; 303a20-24; for the point about the conflict with the mathematical sciences see also *De Caelo* 1.5; 271b9-11.

54 These issues are dealt with in *Physica* 6.1 and 6.9, where Aristotle argues against the idea that a continuum, whether a magnitude or a motion, consists of indivisibles. For an analysis, see Chapter Three, notably § 4.1.

55 316b16-18

56 DK 67 A13 = Luria fr. 113; thus Furley, *Two Studies* 95-96. Also S. Makin, 'The Indivisibility of the Atom', *Archiv für Geschichte der Philosophie* 71 (1989) 125-149, seems to suppose that partlessness implies mathematical indivisibility. With Furley, *Two Studies* 98-99 and Guthrie, *History* 505, I reject Alexander, *In Metaphysica* 36.25-27, as being mistaken about Democritus when he says that there are partless entities which are parts of atoms.

57 *Methodus* 428.29-430.9 (edited by J.L. Heiberg (Leipzig, 1913)); translation taken from E.J. Dijksterhuis, *Archimedes* (Princeton, 1987) 314. Since any geometrical reasoning, even if it falls short of a formal proof, leading to the claim that a cone is one-third of a corresponding cylinder, must make use of the proposition that a pyramid is one-third of a corresponding prism, we should not worry about the singular 'the said figure', which probably refers to the cone; Archimedes may have summarized the two crucial claims, of which, he says, Eudoxus was the first to give a real proof, with the final claim of the whole chain of reasoning. (Thus we do not have to resort to 'grammatical irregularities', as H.-J. Waschkies, *Von Eudoxos zu Aristoteles. Das Fortwirken*

The idea now of this reconstruction is that Democritus arrived at the final claim, that the cone is one-third of the cylinder (with equal height and equal base), by first assuming that (i) any two circles are to each other as the squares on their diameters; (ii) a pyramid is one-third of a prism with the same base; (iii) a cylinder and a prism with equal height and equal base are equal;[58] (iv) any solid consists of infinitesimal laminae; and then arguing that a cone and a pyramid with equal base and equal height are equal because for every pair of laminae of height h of the cone and the pyramid it is true that they are equal.[59]

This reconstruction seems to find confirmation in Democritus' famous cone-paradox:

> Further, have a look at how [Chrysippus] answered Democritus, when he in a physical and lively way raised the difficulty what one should think if a cone would be cut by a plane (ἐπιπέδῳ) parallel to the base: do the surfaces (ἐπιφανείας) of the segments turn out to be equal or unequal? For if they are unequal, they will render the cone irregular, as it gets many step-like incisions and roughnesses. If they are equal, however, the segments will be equal and the cone will, as it is composed of equal and not of unequal circles, appear to have the property of a cylinder – this is most absurd.[60]

Again we have here the assumption that a cone is composed of circles, stated as an assumption in the second horn of the dilemma, but which would also explain the first horn; it is therefore often thought that Democritus would opt for the first horn.[61]

However, if Democritus was indeed committed to holding there are mathematical minima, he must have held blatantly contradictory ideas. For there are many testimonies stating that his atoms have different shapes and sizes; there is even an infinite variety of them.[62] Already Aristotle was in some way complaining about this inconsistency:

> [Leucippus and Democritus] claim that since the bodies differ by shapes, and the shapes are unlimited, the simple bodies are also unlimited. But of what kind [of shapes they are] and what the shapes of each of the elements are, they have not explained anything further about, except that they assigned the sphere to fire. ... Further, not even on their own assumption the elements would seem to become unlimited, if indeed the bodies differ by shapes, and the shapes are all composed from pyramids, the rectilinear shapes from rectilinear pyramids and the sphere from

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*der Eudoxischen Proportionentheorie in der Aristotelische Lehre vom Kontinuum* (Amsterdam, 1977) 268-269, note 3, does.)

58 Of these three assumptions, (i) was known in Democritus' time, and (iii) was probably considered unproblematic. Waschkies, *Von Eudoxus* 271-276 thinks that for (ii) Democritus argued by the same principle as for the final conclusion: because any prism with a triangular base is divisible into three pyramids with equal base and equal height, and pyramids with equal base and equal height are equal, since for every pair of laminae parallel to the base with the same height *h* it is true that they are equal.

59 See Luria, 'Infinitesimaltheorie' 142-145

60 Plutarch, *De Communibus Notitiis* 1079e-f

61 The cone-paradox has been interpreted as an argument for mathematical atomism by Guthrie, *History* 487-488; Furley, *Two Studies* 100 (though very tentatively); S. Sambursky, *Physics of the Stoics* (London, 1959) 92-93; Luria, 'Infinitesimaltheorie' 138-148; Mau, *Infinitesimalen* 22-23; cf. also Waschkies, *Von Eudoxos* 276-284, R. Seide, 'Zum Problem des geometrischen Atomismus bei Demokrit', *Hermes* 109 (1981) 265-280, at 273-274, and T. Heath, *A History of Greek Mathematics* I *From Thales to Euclid* (Oxford, 1921) 180. D.E. Hahm, 'Chrysippus' Solution to the Democritean Dilemma of the Cone', *Isis* 63 (1972) 205-220, is neutral on the issue of its purpose and leaves it at the claim that mathematical atomism is presupposed in the construction of the paradox itself.

62 E.g. Aristotle, *On Democritus*; Aristotle, *DGC* 1.2; 315b11; 1.8; 325b25-27; Simplicius, *In Physica* 166.6-10

> eight parts. For there must be some principles of the shapes, so that, whether there is one principle, two, or more, the simple bodies too will be as many in number.[63]

Here Aristotle states two ideas clearly held by the atomists: (a) the (simple) bodies, that is, the atoms, differ only by their shapes, of which there is an infinite variety; (b) the shapes of the atoms are all composed from pyramids or, in the case of spheres, of pyramide-like parts which result from a division through the middle along the three perpendicular planes. He then argues that these two ideas are in contradiction, on the further premiss (c) that if the shape of a simple body is composed from other entities, these entities are in fact the simple bodies.

So on the one hand Democritus needs atoms which have different shapes and sizes, and thus are mathematically divisible, and on the other hand there is evidence suggesting that he was, or had better be, committed to mathematically indivisible atoms. Indeed, it was to solve this problem that Luria proposed that Democritus, just like Epicurus, distinguished between the merely physically indivisible atoms and the mathematically indivisible units of which they consist. His interpretation, however, has been universally rejected for lack of evidence.[64] Thus the problem remains in all its urgency.

### *§ 2.2. Against mathematical atomism*

To get rid of the problem, I want to argue in this section that we do not have any reason to assume that Democritus was a mathematical atomist. In itself, there is nothing new in that, but the way in which I shall deal with the issue is partly new, based as it is on the results obtained in § 1. After having thus defused some of the evidence pointing to mathematical atomism, I will go further, and try to make a case for the claim that Democritus even argued that there is a distinction between mathematical division and physical division to be observed. Again it will be the fact that atoms can touch which will be at the centre of of Democritus' ideas on this issue. Finally I will discuss, in a separate sub-section, almost all the evidence we have about Democritus' mathematics in the light of the problem of the status of limits as raised by Zeno.[65]

#### *§ 2.2.1. Defusing evidence*

We arrived at the problem of mathematical atomism because the argument for the existence of atoms in *DGC* 1.2 seemed to apply equally well in the case of mathematical divisibility. A perceptive reader, however, may already have had suspicions about this, in the light of my contention that only if we add (AP) as a premiss to that argument is it

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63 *De Caelo* 3.4; 303a11-14; a29-b3

64 Furley, *Two Studies* 97-98; Guthrie, *History* 504 (though at 394 and n.2 Guthrie seems to think that not all the atoms are minimal quanta. Thus he implicitly endorses Luria's interpretation!); Mau, *Infinitesimalen* 24; Barnes, *Presocratic Philosophers* 628 n.12.

65 I should mention here that reading an unpublished paper by H. Mendell, 'Democritus on Mathematical and Physical Shapes and the Emergence of Fifth Century Geometry' (1992), has helped me enormously in structuring my thoughts about a large part of the (quasi-)mathematical material to be dealt with in this section, as well as in discovering new lines of interpretation, even if sometimes because his paper forced me to make my own ideas clearer. The somewhat limited number of references to it in the footnotes does in this respect an unjustice to this rich paper, whose scope goes beyond the concerns in the present study.

a valid argument. Now already in the context of the argument itself Aristotle mentions this principle in an exclusively physical version:

> It is necessary, therefore, that there are invisible atomic magnitudes in it, especially if, that is, coming to be and passing away are to occur by segregation and aggregation.[66]

Also the use to which (AP) is put by the atomist against the Eleatic argument from homogeneity turns it into a physical principle, since on the basis of (AP) the atomist argues that the physical aggregation of two entities, that is, the disappearance of all separating void between them, does not make them one. The point of (AP) as used against the argument from homogeneity would have been completely different, if (AP) had also applied to mathematical divisions. For then the atomist would have posited mathematical atoms, whose inside was completely inaccessible to analysis. Thus there would not even have been a geometrical homogeneity, as distinct from dispositional homogeneity, for the Eleatic to base his argument on. Similarly the atomists could not have called their atoms really continuous, so that their distinction between real and apparent continuity would have lost its point as well.

This suggests that Democritus could not possibly have applied the principle (AP) in the case of merely mathematical divisions. Something like (AP) for mathematical parts is nevertheless used by Aristotle in the passage quoted above about the shapes being composed from pyramids. For from the claim explicitly ascribed to Leucippus and Democritus that all of the infinite variety of shapes which the atoms exhibit are composed from other shapes – namely pyramids –, Aristotle deduces that the atoms can only exhibit these other shapes. Thus he assumes for the atomists a principle (c) which says that the geometrical analysis of a shape exhibited by a body into other shapes, implies a pre-existing real division of the body exhibiting the original shape.

Rather than taking such a passage from Aristotle as indicating a very deep confusion on the part of Democritus, I see this as a case of Aristotle being not a completely trustworthy source, particularly not with regard to the propositions which he thinks the atomists (should) endorse and which he ascribes to them by way of an argument. He disagrees with atomism, and likes to assemble as many arguments as possible against its proponents. In the same category we should put Aristotle's accusation that atomism clashes with mathematics. Aristotle himself does not draw the distinction between kinds of divisibility, and this fact allows us to explain away any statement on his side implying that atoms are indivisible in every respect as a statement that is biased and made from his own perspective.[67]

The testimony according to which the atoms are ἀμερή is usually explained away as a misunderstanding of Democritus, stemming from the influence of Aristotle and Epicurus.[68] This might well be a very reasonable way of getting rid of the testimony, but it is also possible that 'partless' for Democritus did not refer to not having parts of just any kind, but to not having a specific kind of part.[69] For in a sense an atom is partless: it is not made up of parts, it is not an aggregate of parts. (AP) does not apply to mathematical divisions.

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66 316b33-34

67 Cf. Barnes, *Presocratic Philosophers* 357-358; *contra* Furley, *Greek Cosmologists* 130.

68 E.g. Barnes, *Presocratic Philosophers* 358; Sorabji, *TCC* 356 n.27.

69 Cf. e.g. Baldes, 'Divisibility' 4.

After the useful work done with the help of (AP) in disposing of unwelcome evidence, one might be tempted to deal in the same way with the two of the three remaining reasons given above for turning Democritus into a mathematical atomist, namely the ideas that Democritus thought were an answer to Zeno's first paradox of plurality and his Runner-paradox, and that he gave an argument for claims about cones and pyramids on the basis of decomposing them into infinitesimal laminae. In Chapter One we saw that the primacy of part over whole was crucial to Zeno in his argument that there is no limit to an entity or a run, so that they turn out to be unlimited in length. As this primacy of part over whole is enshrined in (AP), we could argue that by rejecting (AP) for mathematical divisions, Democritus did make room for a way in which limits could exist, namely as dependent parts of the whole. Similarly he could have taken the infinitesimal laminae as nothing more than dependent parts of the whole figure, so that the whole figure is not really composed from them.

In itself there is definitely something to this approach, but it is at once too much and too little. It is too little, because there is a lot more to be said about limits and limit entities – as I shall try to do below. It is too much, because there is no evidence for either of the two ideas being true. The only possible evidence for the first idea, Aristotle's statement that the atomists answered the argument 'from the dichotomy by positing atomic magnitudes' (quoted on p. 60), might just as well refer to the argument presented in *DGC* 1.2, rather than to the Runner-paradox; since both arguments are referred to as the 'Dichotomy', we have no way of telling. And for the existence of an argument from infinitesimal laminae the only real piece of evidence could be the cone-paradox. However, as we shall see below, the cone-paradox should not be interpreted in that way.

### *§ 2.2.2. How do atoms touch?*

In the previous section we saw that the atomists argued against the Eleatics that atoms could touch without merging into one or becoming divisible everywhere; thus they were committed to the view that an entity consisting of two atoms is at one place different from elsewhere (so that it is not dispositionally homogeneous): only there it is physically divisible, even though it may be mathematically divisible everywhere; only there the two entities touch, whereas elsewhere there is no touching in the real sense of the word. Aristotle sums it up:

> Where it is one and continous, not by contact, there it is impassive.[70]

The phenomenon of touch is very important in the physics of the atomists, for it is only by touch that the atoms interact with each other, the atoms themselves being completely inert and impenetrable.[71] Together with this goes the emphasis the atomists put on the shapes of atoms – indeed, their name for the atoms was ἰδέαι.[72] For in a world in which

70 *Physica* 8.4; 255a13

71 There are several testimonies about the impassiveness of the atoms, e.g. DK 67 A14 = Luria fr. 214; DK 68 A1 = Luria fr. 215; DK 68 A49 = Luria fr. 215

72 DK 68 A57 = Luria fr. 198

the inside of the atoms is causally inactive, only their outside, their shape, determines their behaviour.[73]

Actually, putting it thus, by drawing a contrast between the inside of an atom and its shape, may be thought misleading, because we have no explicit textual reference to the relevance of ‘shape’ in the sense of the depthless boundary of a solid body. Indeed, as also appears from the passage from Aristotle about the analysis of shapes into pyramids, with ‘shape’ or its synonym we should assume Democritus means the solid figure as a whole.

In another way, however, Democritus did refer to boundaries, quite apart from the atoms they are boundaries of, namely when he spoke of touching atoms. First of all, there are at least two related testimonies, both of them from Aristotle, in which the place of touch is referred to. The first is the continuation of the line quoted above from *Physica* 8.4:

> But where they are separated, there the one is of such a nature as to act and the other to be acted upon.[74]

The places referred to here by ‘where’ and ‘there’ are the boundaries between atoms, where they touch and where they causally interact. This is even clearer in the second passage, which we have already seen before:

> [The atoms] act and are acted upon where they happen to touch (for they are not one there).[75]

So in his physics Democritus refers to limit entities in an explanatory role. Now what kind of entity is such a boundary? Is it a line or rather plane in space, independent from the touching atoms, which just happens to be the place where they touch? Or does it belong to an atom, so that an atom cannot act or be acted upon in its middle, but only at its boundary? If one looks at the two passages, one might slightly favour the first alternative for the first passage and the second alternative for the second passage. One could, of course, conclude that the question I asked about the kind of boundary is not a very sensible one and that Democritus just did not think about the issue and never conceived of the distinction on which the question is based. As evidence that such scep-

73 Cf. the passage quoted from *De Caelo* 3.4; 303a11-14 (at the end of § 2.1). A.P.D. Mourelatos, ‘Δημόκριτος: φιλόσοφος τῆς μορφῆς’, in: L.G. Benakis (ed.), *Proceedings of the First International Congress on Democritus* (Xanthi, 1984) 109-119, at 117, has made the ingenious suggestion that the Democritean word for shape, ῥυσμός, being etymologically related to ῥέω (to run), is meant to express just that: form determining behaviour. Moreover, not only the shapes of the individual atoms, but also those of the complexes of atoms, are important in atomistic physics. The trio ῥυσμός (shape), τροπή (turning or orientation) and διαθιγή (position), for which Aristotle introduces the terms σχῆμα, θέσις and τάξις (e.g. *Metaphysica* A.4; 985b13-19), constitute the three ways in which the shape, that is, the geometrical configuration, of a complex of atoms is dependent on the shape of each individual atom, by the atoms having the shape they have, by the angle between the atoms (H and I are the same in shape, but ‘turned’ differently), and by their ordering.

74 255a14-15. By ‘being separated’ Aristotle means here being divided while touching, for otherwise he would here be ascribing to Democritus the idea that atoms can only act upon each other at a distance. Moreover, there is no specific *where* if atoms are spatially separated. Further, it is clear that in this line together with the preceding line, Aristotle wants to distinguish between two kinds of places: places where things are impassive and cannot be acted upon, namely in the middle of atoms (where they are continuous, *not by touch*), and places where they can be acted upon, namely at the boundaries of atoms (where they are continuous, but merely by touch).

75 *De Generatione et Corruptione* 1.8; 325a32-34

ticism is mistaken I will present two passages, both of which are rather difficult to understand, but which after a careful interpretation will lead us to the conclusion that Democritus did indeed draw the distinction between the two meeting boundaries of two separate atoms and the plane along which these boundaries meet.

As a first piece of evidence I want to invoke a passage from *De Generatione et Corruptione* 1.2. When it is assumed that a magnitude is divided everywhere and that there are thus only points left, a reason is provided, quite unnecessarily it seems, why all these sizeless entities together do not yield any magnitude:

> Similarly also if [a whole] consists of points, it will not be a quantity. For (i) whenever [points] touched, and there was one magnitude, and they were together, they did not make the whole any larger. For (ii) when divided into two and more, the whole is not anything smaller, nor larger (οὐδὲν ἔλαττον οὐδὲ μεῖζον), than before. Therefore (iii) even if they have all been put together, they will not produce any magnitude.[76]

This passage has caused many difficulties. Two kinds of interpretations have been offered. According to the first, the situation imagined in (i) is one in which there is one continuous magnitude in which at every place there is one point. If this magnitude is divided into two, a point is gained, as we now have two boundaries where we first had one. But since the whole length remains exactly the same, as (ii) claims, one point more or less makes no difference to the length, so, as (iii) states, no matter how many points are put together, they will not yield a magnitude.[77] This interpretation clearly does not fit the passage, for it leaves completely unexplained what is meant by the touching of points in (i). Moreover, the purpose of (i) becomes unclear, for if the picture drawn by the first interpretation were correct, (ii) would in fact have been sufficient reason to establish the conclusion; only some clarification would perhaps have been necessary, but that is not what (i) provides.

The second interpretation takes the difference between the situations of (i) and (ii) to be one between two entities touching – so that their boundaries, though remaining two, are together – and the same entities separated. As stated in (ii), there is no difference in size between the two situations.[78] Though this interpretation makes good sense of the touching of points in (i), it now again seems difficult to understand (i), because one would expect that the touching and being together of the two points would make the whole smaller rather than larger, smaller, that is, than in the situation of separation. Williams, the only one who has addressed this problem, proposes to take ‘making the whole larger’ as ‘contributing to the length of the whole’. The reasoning behind (i), then, he suggests, is that since at least one of the touching points does not contribute to the length, because the points are together, and there is no difference between the two points, neither of them contributes to the length of the whole.[79] This solution, however, seems rather contrived. Not only are the additions to the argument Williams has to

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[76] 316a30-34

[77] See White, *Continuous* 13, and M. Schramm, *Die Bedeutung der Bewegungslehre des Aristoteles für seine beiden Lösungen der zenonischen Paradoxie* (Frankfurt am Main, 1962) 249. Philoponus has the same picture in mind, but it seems to me that his comments are incoherent, since he only denies that the whole becomes *larger* when put together and *smaller* when divided; on his interpretation one would have expected that his denials would be the other way round (see *In De Generatione et Corruptione* 29.32-30.12, especially 30.6-8).

[78] See Waschkies, *Von Eudoxus* 338-341, and Williams, DGC 69-70.

[79] Williams, DGC 70

supply quite substantial, he also adopts a forced reading of ‘making the whole larger’, which in fact obliterates the clearly implied reciprocal comparison between the situations of (i) and (ii). Moreover, as Williams himself rightly notices, the emphasis in (ii) on the whole, when split into two, not being *smaller* – ‘nor larger’ clearly comes as an afterthought – does not fit his interpretation.[80]

To find a way out of these interpretational problems, we have to take a look at the context of our passage. Let me quote therefore the lines immediately preceding:

> [I]f, [after a division everywhere] there will not be any body or magnitude ..., either it will consist of points, and [there] are sizeless things of which it is composed, or [there will be] nothing at all, so that it could come to be from nothing and be composed [of nothing], and the whole would be nothing but appearance.[81]

The form of the argument is a dilemma, whose horns are both shown to be unacceptable. Now it is a presupposition common to both interpretations that the argument of our passage is the only reason given to establish that the first horn is unacceptable. For they both try to construct an argument which has as its conclusion that a single point does not contribute anything, so that all points together will also fail to make up a quantity. I have already remarked that it seems quite unnecessary to argue that sizeless entities together would never yield a magnitude – and with such a complicated argument at that! But if we look at the text, we see that the point has already been made a few lines before our passage: ‘it will consist of points, and there are sizeless things of which it is composed.’ Why take the trouble to make the same point again?

Also for that reason, I propose to understand the argument of our passage as having a much more specific purpose. It is no longer needed to establish that a point on its own does not contribute to the whole. What it is meant to do is to refute those who think that when sizeless points are in a certain relation, that is, have been put together, side by side, touching, they *do* produce length. Against them (i) says that if two points have been thus put together, touching, they do not make the whole larger than when they are not in that relation. And (ii) is on this suggestion a perspicuous reason for (i), as its emphasis on the denial of the whole of two divided magnitudes not being *smaller* is relevant for (i). For only those who assume that points related by being put together produce length, think that at every division one relation is snapped and some length disappears.

Thus the argument points to the situation envisaged by the second interpretation sketched above, that when two entities touch, they have their boundaries together, so that two distinct points coincide at one place – or at one point, to put it paradoxically. For only by letting boundaries coincide can one argue successfully against an analysis of a magnitude into successively ordered, touching points, lying side by side.

So far so good, one might say, but why should we ascribe this part of the argument of *De Generatione et Corruptione* 1.2 to Democritus? In the next few lines, two further attempts at making acceptable the dissolution of a magnitude into points (by positing something like sawdust disappearing during the division everywhere, or some separable

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[80] *Ibidem*. On this last point, however, Williams hesitates, saying that this would perhaps be pressing Aristotle’s words too much.

[81] 316a25-29

form withdrawing from the points) are briefly dismissed.[82] These seem to be Aristotelian developments and cannot possibly go back to Democritus. Moreover, elsewhere Aristotle has a view of touching entities similar to the one I claim to be intimated in our passage:

> Touching are [those things] whose extremities are together.[83]

There are a few incongruities, however, which rule out that the argument is Aristotle's own. First, though it is true that Aristotle adopts elsewhere a similar view of touching entities, he also uses the definition of touching specifically to rule out the touching of points, on the ground that points do not have extremities.[84] As far as that argument is concerned, then, he himself could not have said what is written in our passage: 'whenever points touched.' (And in the one passage where Aristotle does talk about touching points, his argument is completely different from the one we have here.[85] For as we shall see later, in Chapter Three § 4.1, Aristotle there assumes from the outset that touching involves some kind of overlap, whereas here we have an argument against the conception of a magnitude as consisting of points lying side by side, without overlap.[86]) Moreover, in other places Aristotle seems to have relinquished this view that there are two boundaries together, and to have adopted an account like the one given by the first mentioned interpretation of our passage. In *Physica* 8.8, for example, he says that when a magnitude is divided into two touching halves, the point of division is numerically one (though, as it is used twice, two in being).[87] More importantly, there are indications that in the very same chapter of *De Generatione et Corruptione*, Aristotle is working with the same conception. He says:

> [A] contact is always *one* [contact] of two things, because there is something apart from the contact and division and point.[88]

Since it is, further, so well integrated with the core of the atomistic argument (unlike the Aristotelian developments), it seems plausible that the argument of our passage goes back to Democritus himself. One implication of this is that the idea that a magnitude is composed of sizeless entities lying side by side was not only known to Democritus, but also found problematic by him. And that again is exactly what we see elsewhere, namely in the second piece of evidence I want to consider for my thesis that Democritus did

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82 316a34-b4

83 *Physica* 5.3; 226b23; cf. *DGC* 1.6; 323a3-4

84 *Physica* 6.1; 231a27-29

85 It concerns *Physica* 6.1; 231a29-31; b2-6:

> It is necessary that the points of which the continuum consists, are either continuous or in touch with each other. .. Everything touches either as a whole with a whole or as a part with a part or as a part with a whole. But since what is indivisible is partless, it must touch as a whole with a whole. But a whole touching with a whole will not be a continuum, because the continuum has different parts and divides into parts which are different in that way, that is, are separated in place.

Waschkies, *Von Eudoxus* 222-223, also mentions another passage, in *Physica* 5.3; 227a29: 'To [points] belongs touching' (ταῖς [στιγμαῖς] ὑπάρχει τὸ ἅπτεσθαι). In the context, however, this may not mean anything more than that points are involved in touching.

86 *Contra* Waschkies, *Von Eudoxus* 340.

87 *Physica* 8.8; 262a19-21 and 263a23-24.

88 316b6-8

distinguish between the boundaries belonging to atoms and the plane as the place where these boundaries meet. This piece of evidence is the cone-paradox.

The pattern of Democritus' argument is as follows (for the text, see above, p. 77). He lets a cone be cut along a plane parallel to the base. Then he considers the two segments separately, wondering whether the bottom-surface of the top-segment and the top-surface of the frustum are equal or unequal. This question introduces a dilemma. The argument for the first horn, if the surfaces are unequal, is short: the cone will become irregular, 'as it gets many step-like incisions and roughnesses.' The argument for the second horn, if the surfaces are equal, is longer: 'the segments will be equal', he says, and the cone will appear to be a cylinder 'as it is composed of equal and not of unequal circles.'

The argument of the first horn does not seem difficult to understand, for if one were to put the top of the cone back onto its frustum, there would already be one incision, as the bottom surface of the top and the top surface of the frustum are unequal. The 'many incisions', then, are brought about by adding other cuts. The second horn, however, poses more interpretational problems, for putting the top back on the frustum seems to yield a perfect cone again; there is no trace of a cylinder. Moreover, Democritus claims that the segments will be equal, but if one cuts a cone into two along a plane parallel to the base, there is in general a very clear inequality of the two segments. The solution to these problems lies in the reason Democritus provides: the cone is composed of equal circles, that is, all the circles of which the cone is composed are equal. Thus the cone will really be a cylinder. And if a cone is cut up into its constituent circles, these circles certainly deserve to be called 'segments'; thus the segments are equal.[89]

But now the main question poses itself: how does one arrive at the conclusion that all the constitutive circles are equal from the hypothesis that the surfaces of the two parts resulting from any division along a plane parallel to the base are equal? The only way to make that step is to assume a picture which allows one to argue by transitivity from the equality of any two opposing surfaces to the equality of all circles. There are two possible pictures which would do. The most commonly adopted one seems to be that with the 'circles' or 'segments' Democritus is referring to circular segments with some height, be it a very small one.[90] *Prima facie* this is quite plausible, for was not Democritus' argument in *De Generatione et Corruptione* 1.2 that a magnitude, here the total height of the cone, does not consist of sizeless entities? These scholars thus read the atomist hypothesis into the paradox. On this account Democritus would have opted for the first horn: what seems to be a smooth cone is in reality a kind of dented pyramid. It is important to notice, however, that the only kind of atoms for which the argument of the second horn works, are mathematically indivisible. Merely physically indivisible layers could have oblique sides, so that their top and bottom surfaces are unequal and the cone would be a real cone after all, even if any two opposing surfaces are equal. Only if one identifies the circles or segments and the surfaces, all one mathematical quantum thick, can one argue by transitivity from the equality of any two opposing surfaces to the equality of all circles.

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89 Cf. Hahm, 'Chrysippus' Solution' 207-209. It seems impossible that with the 'equal segments' Democritus refers to the top of the cone and the frustum, even if both of them are really cylinders, because then he should have specified that the cone be cut at a height half of the height of the whole cone.

90 See note 61 and especially Hahm, 'Chrysippus' Solution' 208.

However, it is not necessary to read the atomist hypothesis into the argument, for if one adopts a picture on which the circles or segments are still to be identified with the surfaces, but now all of them are without height, one can still arrive at the desired conclusion that what seems a cone is in fact a cylinder. For if one adds to this picture that the two opposing surfaces are in contact, but not coinciding, so that the cut occurs *between* them, then the whole cone turns out to be composed of discretely ordered circles between which one may divide it. Thus if any two such circles or surfaces will be equal, by transitivity all will be equal.[91]

This second picture seems to make for a more convincing interpretation. First, in interpreting 'circles' and 'surfaces' as referring to two-dimensional entities, it provides a much more natural reading than the first picture, on which one has to take these terms as referring to atom-thin layers. It might seem that on the second picture one also has to strain the language somewhat, by interpreting 'segments' as referring to two-dimensional circles. But once one realizes that these circles are supposed to make up the whole cone, 'segments' becomes an acceptable word for them.[92] A second and perhaps more important point is that the second picture stays closer to the dilemmatic form of the argument. On the first picture, there is no real dilemma, as the argument for the second horn makes it almost impossible not to opt for the first horn, which, nevertheless, is presented as something *prima facie* unacceptable. For because the crucial assumption is the atomistic hypothesis, it seems difficult to imagine that Democritus would want, as so often is the purpose of a dilemma, to call attention to a (hidden) assumption to be discarded. The second picture, on the other hand, gives us two completely independent lines of argument for each of the horns: a real dilemma.

We do not know from the sources how and whether Democritus proposed to solve the dilemma. The fact that Chrysippus, calling Democritus ignorant, presented his own solution, and apparently did not refer to a solution by Democritus, does not allow us to draw any conclusion in this respect.[93] But given the close parallel between the situation envisaged in the cone-paradox (on the preferred interpretation) and the position targeted by Democritus in the argument from *DGC* 1.2 against the composition of a magnitude from points, it seems plausible that Democritus wanted to call attention to the hidden assumption that surfaces, and thus circles, are ordered lying side by side, rather than coinciding. For that is the ultimate source of the problem on the second horn; if one rejects it, it is impossible to argue from the equality of the surfaces to the equality of all circles in the cone, because the required transitivity cannot be established. Moreover, there is some slight textual evidence that this is the purpose of the paradox, for, as it is set up, there is a careful distinction between the one plane by which the cone is divided and the two surfaces. And since the reference to a plane could easily have been omitted, one might think that the dividing plane is introduced on purpose. No further attention is paid to this one plane, but that is no surprise, for there is no place for it in the sequel.

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91 This analysis of the paradox is essentially the same as that of A.A. Long and D.N. Sedley, *The Hellenistic Philosophers* I *Translations of the Principal Sources with Philosophical Commentary* (Cambridge, 1987) 302, and Mendell, 'Mathematical and Physical Shapes' 6-7. Of course this analysis presupposes that it was somehow thought possible that an extended entity consists of unextended entities. But as the argument from *DGC* 1.2 dealt with above shows, this idea was around in those times.

92 As we shall see below, p. 93, the use of 'segments' for the composing circles can be explained in the context of Democritus' work in mathematics.

93 See Plutarchus' polemical report in *De communibus notitiis* 1079e-1080d.

Only if one admits that two surfaces coincide at this one plane, can one accommodate it. Now are we to believe that Democritus did not notice this incongruity in the very set-up of the paradox?

More will be said about the cone-paradox later, but it will be clear that, interpreted in this way, it argues directly for the position which is the obvious alternative for the view attacked in the argument from *DGC* 1.2: an *n*-dimensional object does not consist of *n-1*-dimensional objects lying side by side; rather the boundaries of its parts coincide. Thus we have here the distinction between the boundaries belonging to the atoms and the plane of contact, which does not belong to either atom.

Now this is a very important distinction, because in drawing it Democritus has in fact provided an argument for the distinction between physical and mathematical or conceptual divisibility. For how are we to distinguish the two coinciding boundaries? This cannot be done in mathematical terms, since they have the same position. It is only by reference to the two entities whose boundaries they are, that they can be told apart. But that requires two divided, separate entities to support the separate existence of the two coinciding boundaries. Parts which are merely conceptually divided from each other, however, will not be equal to the task, as they owe their very being to the plane which is used to distinguished them – and distinguishing them does not make them separate in the sense of creating a real discontinuity. Physically divided parts, on the other hand, can touch now this part, then that part, and on every occasion the boundary enters into a relation of coincidence with another boundary.

Perhaps this is not the way Democritus would put it, though he is committed to the idea because of his position that there is no real discontinuity within an atom. What one can say with certainty is that with these two arguments Democritus argued against the idea that one may regard every distinct physical phenomenon – the two boundaries – as distinct in mathematical terms. It is therefore noteworthy that in both arguments the language is decidedly physical. In the argument from *DGC* 1.2 the undivided situation in (i) is described in terms of contact, implying that there is no such thing in the divided situation of (ii). The cone-paradox is said to have been raised 'in a physical and lively way' (φυσικῶς καὶ ἐμψύχως). Moreover, it appeals to an understanding in terms of physical separation, because only then can one readily identify the two surfaces.

### *§ 2.2.3. Democritus limited*

As is well known, Democritus did not mind the unlimited. His universe is unlimited in extent, the number of atoms in the universe is unlimited, and the variety of atoms is unlimited.[94] In the previous sub-section it also appeared that he speaks freely about limit entities. He regularly refers to planes, lines and points, not only as independent entities, but also as boundaries, that is, limits belonging to other entities. Since he agrees with Zeno that only objects with size can be ontologically basic entities,[95] he either must have some way of answering or circumventing Zeno's arguments, especially the first paradox of plurality, or he just ignores them. And as if that is not enough, there is evidence

[94] See e.g. DK 67 A7 = Luria fr. 222; DK 67 A15 = Luria fr. 220a; DK 59 A45 = Luria fr. 145; Luria fr. 141.

[95] As appears, of course, from the existence argument analysed in § 1, but perhaps also from his famous dictum: 'In reality only atoms and void' (ἐτεῆι ἄτομα καὶ κενόν) (DK 68 B9 = Luria fr. 55; DK 68 B125 = Luria fr. 79; cf. Luria fr. 61).

which has been interpreted in such a way that Democritus is committed to there being a limit to an unlimited series (see below, p. 94). This would make the conflict with Zeno even more apparent.

Because there is not a lot of evidence on these matters, it would perhaps be wise not to say too much. On the other hand, I think there are interpretational possibilities left unexplored, which might have a bearing on them.

We have to bear in mind that the problem of the status of limits differs from another problem, namely as to how Democritus is to account for the truth of mathematical statements. Take, for example, the statement that any prism which has a triangular base may be divided into three pyramids equal to one another which have triangular bases – a statement which, as we saw, Archimedes indicates as having been shown to be true, if not proved, by Democritus.[96] Since an atomic prism is not really divisible and does not really consist of parts, what is this statement about? Does it not apply to such atoms? But then it cannot be a universally true theorem any more. In general, because Democritus does not appear to be an atomist in his mathematics, but a continuist, just as almost everybody else, there is for him not really anything his mathematics could be about. (Given the fluidity of the void, it seems unlikely that the void can come to Democritus' rescue.) So Democritus needs to explain how mathematics can apply to the real world.

I do not know how to answer on behalf of Democritus, while making use only of Democritean concepts.[97] However, the lesson we can draw from the part played by (AP) in the foundations of atomism is that mathematics does not *find* pre-existing divisions in an entity (for then the argument of *DGC* 1.2 would also apply to mathematical divisibility). Therefore it must somehow create divisions, turning in the process an entity from a unity into a plurality. This is of course a matter of make-believe: when an entity is mathematically divided, it is merely viewed as if it were a plurality. However, then it becomes clear that the problem of status of limits goes deeper than the problem about the objects of mathematics, for how could we ever view, even as a matter of make-believe, an entity as consisting of a plurality of units of which at least one is a limit?

So if Democritus is to be consistent, he should not make limits things which are to be found or created by a division procedure. Nevertheless, that he does just that, seems to be suggested by some pieces of evidence, at least in the way they are often interpreted. These are his 'proofs' of the proportion between a cone and a cylinder, and between a pyramid and a prism, and a statement by Aristotle that according to Democritus 'a sphere is a kind of angle.' I will argue, however, that rather than presupposing that a limit is what is waiting at the end of an unlimited division, these pieces of evidence show or suggest that Democritus was aware of there being a problem.

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96 Euclid, *Elementa* 12.7

97 The fullest account I know of is by K. Meakin, *Pre-Platonic Ontology of Mathematics* (unpublished Ph.D.-thesis; Cambridge, 1989). She argues on the basis of the fragments DK 68 B9 and 11 that Democritus can provide objects for his mathematics by some kind of abstraction-procedure, in accordance with which genuine cognition (γνησίη γνώμη) posits mathemathical shapes, which it has abstracted from sensory cognition (σκοτίη γνώμη), from which genuine cognition has been separated off (ἀποκεκριμένη). These mathematical shapes do then not exist in reality, but by convention (νόμῳ), just as colour and compounds taken as one, for example; mathematics can be true of them. The part about mathematical objects existing by convention is rather attractive, but needless to say there are difficulties, also because the evidence is so slender.

### § 2.2.3.1. *Proofs*

It seems unlikely that Democritus would have been credited by Archimedes with the discovery of two theorems if he had failed to supply an argument, even if it was one that fell short of a real proof, to back up his claims.[98] Several constructions are possible. If we take up first the theorem that a pyramid is a third of a corresponding prism, then the basic claim which needs to be proved is that pyramids with an equal triangle as base and an equal height are equal.[99] As we already saw in note 58, some scholars think that Democritus could have proved it by means of indivisibles: such pyramids are equal, because every pair of triangular sections parallel to the base at the same height are equal, so that all the triangular sections together are equal.[100] Another construction would be to approximate the two pyramids by inscribing them with prisms of equal height, as in fig. 1, and then to argue that for every step in the approximation every pair of inscribed prisms is equal, so that the totals of the inscribed volumes at each step are also equal. (Step 1 resulting from a division by a plane at a height half of the total height, step 2 could either result from a division by two planes, at a third and two-thirds of the total height, or from a division at the height of step 1 as well as at a quarter and three-quarter of the total height, and so on. As this latter procedure conserves the previous step, it seems easier.) From this one draws the conclusion that the two solids to be approximated in this way are also equal.[101] A third possible construction, finally, would be to follow the method of Euclid[102] and approximate the pyramids with prisms by analysing each into two equal prisms and two equal pyramids which are similar to the whole, as in fig. 2; to the two resulting pyramids the same analysis is applied again in the next step. All the resulting prisms are pairwise equal, so the pyramids are equal.[103]

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[98] As T. Heath, *The Thirteen Books of Euclid's* Elements III *Books X-XIII* (New York, 1956) 366 has pointed out, Archimedes also calls the results obtained with his own method 'without proof' just a few lines earlier (*Methodus* 428.28). We may expect that he uses this qualification in the same way when he says that Democritus stated these theorems 'without proof'.

[99] A special case of Euclid, *Elementa* 12.5. Because any polygonal pyramid and any polygonal prism can be analysed into triangular pyramids and prisms, and any triangular prism can be analysed into three pyramids with equal base and height (*Elementa* 12.7), one only needs to prove that these latter pyramids are equal.

[100] Cf. also W.R. Knorr, 'The Method of Indivisibles in Ancient Geometry', in: R. Calinger (ed.), *Vita Mathematica: Historical Research and Integration with Teaching* (Washington, DC, 1996) 67-86, at 75; Heath, *History* 180, and *Thirteen Books* III 368.

[101] Part of the reason why this method has not been proposed by anyone as the one adopted by Democritus, seems to be that at each step less than half of the remainder (the difference betweeen the pyramid and the approximating prisms) is taken away, even though with each step the ratio of what is taken away from the remainder gets closer to 1:2 (cf. Knorr, 'Method of Indivisibles' 76-77). The convergence method as adopted by Euclid, on the other hand, requires that at each step more than half of the remainder be taken away (cf. *Elementa* 10.1, used in 12.2 and 12.5). This does not make for a smooth history of geometry. (Nevertheless, W.B. Knorr,'Archimedes and the Pre-Euclidean Proportion Theory', *Archives Internationales de l'Histoire des Sciences* 28 (1978) 183-244, note 65, speculates that Eudoxus may have adopted the method I ascribe to Democritus.) The only problem, however, which should have worried Democritus if he proposed the present method, is whether every part of the pyramid will be reached eventually and there is nothing which will not be taken away. However, he may have decided that there is nothing to be worried about in this respect, starting from the fact that every two points on each of the edges of the pyramid will be divided from each other by some planar section at some stage in the approximation.

[102] *Elementa* 12.3 and 12.4

[103] Cf. T. Kouremenos, *Aristotle on Mathematical Infinity* (Stuttgart, 1995) 98, note 170.

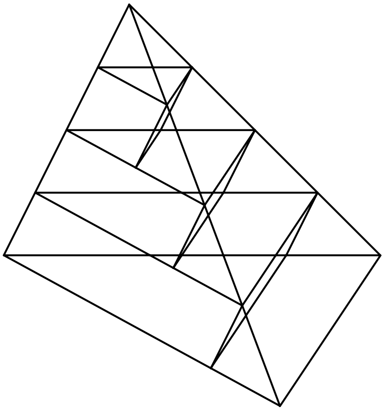

*Figure 1*

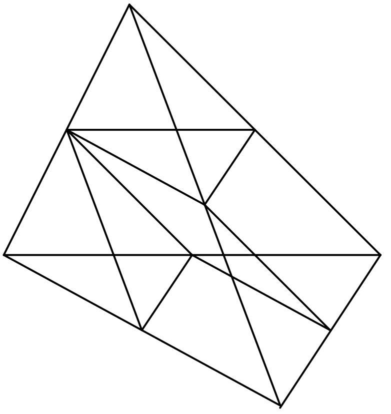

*Figure 2*

For the theorem that a cone is a third of a corresponding cylinder we already have seen the proof by way of indivisibles (see p. 77). One proof through approximation would go as follows. The circle which is the base of both the cone and the cylinder is inscribed first with a square and then further with triangles formed by bisecting the remaining arcs, thus creating polygons ever closer approximating the circle (fig. 3). On these polygons one erects pyramids with the same top as the cone and prisms equally high as the cylinder. As every polygonal pyramid is a third of its corresponding prism, the cone will be a third of the corresponding cylinder.[104] An approximation of the cone by way of inscribing cylinders, similar to the proof of the theorem about the equal pyramids, has been proposed as well, but if it is meant to calculate somehow the volume of the cone, then it seems less likely that Democritus took that road.[105] However, one may construct such an approximation, if one assumes that the chain of theorems is the same as with the demonstration by way of indivisibles. For then the approximation of the circle as above is relegated to a preparatory theorem (namely, to Hippocrates' theorem – see note 104) to be used in the final proof, which only needs to show that at every step of approximation every cylinder-segment inscribing the cone is equal to its

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104 See Heath, *History* 180. The basic idea of this argument is the approximation of the circle by way of inscribing it with a square and further triangles, together forming polygons. The same construction we know to have been employed by Antiphon, though he claimed that the circle will be reached in a limited number of steps (Simplicius, *In Physica* 55.6-8). One may doubt, however, that Antiphon was the first one to think of this method, as he employed it to such disastrous effect. (For further considerations, see W.B. Knorr, 'The Interaction of Mathematics and Philosophy', in: N. Kretzmann (ed.), *Infinity and Continuity in Ancient and Medieval Thought* (Ithaca, 1982) 112-145, at 132-133; cf. I. Mueller, 'Aristotle and the Quadrature of the Circle', *ibidem* 146-164, at 155-156; contra Heath, *History* 222.) A likely creator is Hippocrates, since he is credited by Eudemus with 'the proof that the diameters have in power the same ratio as the circles' (*apud* Simplicium, *In Physica* 61.6-9), where 'diameters being in power' is older terminology for 'diameters as they are squared.' The construction known from Antiphon strongly suggests itself (see Knorr, 'Interaction' 130-131).

105 R. Seide, 'Zum Problem' 278-279, argues that Democritus could sum the volumes of the inscribing cylindrical segments for every number of segments. This sum then approaches a third of the product of base and total height, that is, of the volume of the cylinder. As this method involves an approximation which is quite sophisticated, one should only assume it if there were no alternatives. (This method has been discussed more fully by H. Mendell, 'Mathematical and Physical Shapes' Appendix A.)

corresponding prism-segment in the pyramid with height and base equal to the height and base of the cone. As the pyramid is a third of the prism, and the prism equal to the cylinder, the cone will be a third of the cylinder.[106]

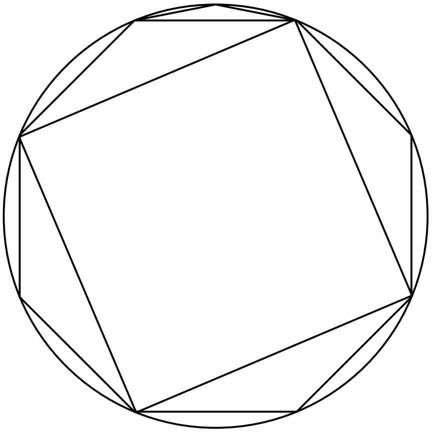

*Figure 3*

If one needs to make a choice between these possible constructions, for what construction should one opt and on what evidence? For the first theorem, scholars somehow seem to favour the method of indivisibles.[107] They always mention the cone-paradox, since there the premiss that a solid consists of plane-figures is mentioned.[108] For the second theorem, votes seem to be cast the other way round, with a greater share for the approximation by pyramids erected on the polygonals inscribing the circular base than for the method of indivisibles.[109] The only evidence mentioned for the approximation by way of erecting pyramids on a polygonal base is the passage quoted above in § 2.1 from

[106] Again the problem has been raised that this method of convergence cannot be accommodated within the Euclidean scheme (see note 101). However, there might be some evidence that it was used in ancient times, namely by Archimedes, probably in one of his earlier works; see Pappus, *Collectio* 4.2: I.234 (edition by F. Hultsch (Berlin, 1876)); for a short discussion see W.B. Knorr, *The Ancient Tradition of Geometric Problems* (Boston, 1986) 162-163; cf. 'Method of Indivisibles' 78-79. May it be the case that Archimedes drew on an earlier tradition?

[107] Heath, *History* 180, Luria, 'Infinitesimaltheorie' 143-145, Waschkies, *Von Eudoxus* 273-276. Knorr's scepticism with regard to ascribing to Democritus any use of the method of indivisibles appears in such passages as in *Ancient Tradition* 87 and 'Method of Indivisibles' 74, but a proof of the theorem by way of indivisibles as a preliminary to Euclid's treatment he seems to think quite possible ('Method of Indivisibles' 75).

[108] Mau, *Infinitesimalen* 20, ingeniously adds DK 68 B299, though without wanting to rely on it. This fragment, which some have called spurious, records a brag by Democritus:

> Nobody ever surpassed me in the composition of lines together with demonstration (γραμμέων συνθέσιος μετὰ ἀποδείξεως).

But it seems much more likely that the reference here is to mathematics in general, which in ancient times was guided in its research by problems of construction.

[109] Heath, *History* 180, Waschkies, *Von Eudoxus* 278, note 23, and Kouremenos, *Mathematical Infinity* 99, note 177, for example, are on the polygonal pyramid-side, whereas Luria, 'Infinitesimaltheorie' 142-143, is on the indivisibilist side.

Aristotle's *De Caelo*, where he ascribes to Democritus an analysis of shapes into pyramids.[110] The indivisibilists of course mention the cone again.

Now it may seem possible to take the cone-paradox as pointing to a method of indivisibles. But there is one proviso: one has to change the interpretation of both the cone-paradox and the mathematical demonstration on one point. For if one takes the crucial premiss as formulated in the cone-paradox: 'the cone is composed of circles' (συγκείμενος ἐκ κύκλων) literally, then Democritus is, because of his rejection of composition from points, committed to the denial of that premiss. It has been suggested that we read 'is composed of' in an innocuous way, in the sense that the circles 'are to be found in [the cone] at any chosen plane parallel to the base.'[111] Apart from the fact that this has an air of circularity about it (a plane-figure at any plane), there is the further problem that we should reinterpret similarly the crucial premiss occurring in the demonstration, e.g. in the case of the pyramid and the prism: if every pair of triangular sections parallel to the base at the same height are equal, then all triangular sections together are equal. But taking 'all triangular sections together' as referring to anything else than the two whole pyramids would leave a gap in the argument. Therefore it is necessary to interpret 'all .. together', in conformity with ancient mathematical practice, as summation or real composition.[112]


*Figure 4*

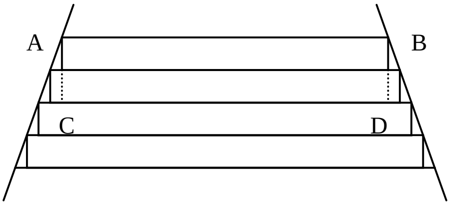


*Figure 5*

From all this it is clear that if one wants Democritus to be consistent in his rejection of composition from points, one cannot take the cone-paradox as evidence for the method of indivisibles. But if one looks closely at the cone-paradox, one will see that it does fit in any case much better the approximation method by way of inscribing prisms or cylindrical segments. The first horn reflects the situation at every step of the approximation, where the cone is inscribed by a limited number of cylindrical segments, which have unequal surfa-ces: we have a cone with many step-like incisions. The goal of the approximation, the real cone, on the other hand, could have been conceived as the situation envisaged by the second horn. For in that final situation the cone will be analysed into circles which may be thought to preserve the ordering of the segments at each step of the approximation. The reasoning behind this idea may be as follows. Take step $n$ where the cone has been divided at a certain plane (fig. 4). There are two segments meeting at this plane. Now if we assume the conservative method, where at each step to the existing divisions more divisions are added between them, the situation at step $n + 1$ will be as in fig. 5. As we see, of the two surfaces meeting along the one

110 Mendell, 'Mathematical and Physical Shapes' § 6

111 Long & Sedley, *Hellenistic Philosophers* 303

112 See Knorr, 'Method of Indivisibles' 72.

plane the 'lower' surface AB (the top-surface of the lower segment) remains the same, though the segment belonging to it has been halved. On the other hand, the 'higher' surface CD (the bottom-surface of the higher segment) has, in a very loose way of speaking, been widened together with the bisection of its segment. If one takes the goal of the approximation to be a situation in which the 'higher' surface will have become equal to the 'lower' one, and the segment belonging to it completely thinned out (just as the segment belonging to the 'lower' surface will have been completely thinned out), then one will identify the equalized surfaces with the final segments out of which the cone consists.[113] (Here we have a natural explanation for the 'equal segments' appearing in the second horn.) The fact that we can thus explain Democritus' cone-paradox as deriving from his mathematical work, is suggestive evidence for the approximation method by way of inscribing prisms and cylinders.

Can we draw from this the conclusion that it is probably not by approximation with polygonal pyramids or by the method of indivisibles that Democritus argued for his claims? As far as the method of indivisibles is concerned, we can. That is not to say, however, that Democritus has nothing to do with the ancient tradition of proofs with indivisibles. For we may speculate and suggest that the approximation with inscribing prisms and cylindrical segments gave rise to the method of indivisibles. For on Democritus' style of approximation every pair of segments below a planar section are equal if the planar sections are equal. So if all planar sections are equal, then a Democritean-style approximation method will yield the equality of the solids. In other words, the method of indivisibles may have its origin as an abbreviation for an approximation method.[114]

As far as the alternative method of approximation, by way of polygonal pyramids, is concerned, we still cannot be sure that Democritus did not use it. The analysis of rectilinear shapes into pyramids must have some context, presumably mathematical. Moreover, it is possible that Democritus used both. As we shall see below, there is evidence which *can* be interpreted in such a way that it is related to Hippocrates' approximation of the circle. And it seems anyway likely Democritus was familiar with the technique.

The most important conclusion to be drawn from this discussion, however, is that Democritus appears to have been acutely aware of a gap between the unlimited series of approximations and the limit. Approximating a solid figure with inscribing solids would seem to provide good demonstrations, but not everything which is true for each of the

113 The fault, of course, is that there are no separate surfaces CD and AB, but only the planar figure AB which has planar figure CD inside it. Again, it is the assumption that the segments at each step are really separate entities, and not the make-belief units of mathematics, which causes us to think in terms of there being two different surfaces.

114 It is the unargued assumption of Knorr, 'Method of Indivisibles', that the method of indivisibiles was for the purpose of proof converted to convergence arguments. But it seems difficult anyway to understand the composition of a solid from planar sections without an underlying approximative intuition. On the other hand, proofs as the one reconstructed by Knorr (67-68) from Theon, *Commentary on Ptolemy's* Almagest Book I (edited by A. Rome (Vatican, 1936)) 398-399, that a cube with a side of length *x* and a cylinder of the same height and with a circle having a diameter of length *x* as base, are to each other as the square of the diameter to the circle, because all the squares in the cube are in the same ratio as all the circles in the cylinder, cannot be abbreviations of an approximation method, because in such an approximation one would presuppose a theorem closely related to the one wants to prove (namely, that all rectangular segments are to all the cylindrical segments as the square of the diameter to the circle).

unlimited series of approximations is true for the limit to that series. This is shown by the cone-paradox and the passage from *DGC* 1.2 where Democritus argues against the composition from points. There is no real sense of composing a cone from circles in the way the inscribing cylindrical segments compose a whole at every step in the approximation. Nor are the limits of the cylindrical segments, the circles in the cone, ordered as those segments are ordered. Thus while on the one hand we may jump from what is true for the approximations to what is true for the limit, there are other respects in which this would be to jump too far.

### § 2.2.3.2. *Speculation*

I want now to develop a speculative interpretation of some further evidence that points to a similar awareness. The evidence is from *De Caelo* 3.8, where Aristotle discusses Democritus' idea that fire is constituted by sphere-shaped atoms. I quote the relevant lines:

> Since fire is mobile and productive of heat and combustion, some made it a sphere and others a pyramid. For these [shapes] are most mobile because they touch the smallest things (ἐλαχίστων ἅπτεσθαι) and are least stationary, and are the most productive of heat and combustion, since the [sphere] as a whole is an angle (τὸ μὲν ὅλον ἐστὶ γωνία), and the [pyramid] is the most sharp-angled, and things heat and bring to combustion by their angles, as they say.[115]

Aristotle does not like this idea of shapes determining the nature of the elements, and starts attacking the argument, by playing off the spherists against the pyramidists:

> If fire heats and brings to combustion because of its angles, all elements will be productive of heat, though the one perhaps more so than the other. For all have angles, for example the octahedron and the dodecahedron. For Democritus even the sphere, since it is a kind of angle, cuts as [something] mobile (Δημοκρίτῳ δὲ καὶ ἡ σφαῖρα ὡς γωνία τις οὖσα τέμνει ὡς εὐκίνητον).[116]

The statement ascribed to Democritus that the sphere is a kind of angle has attracted a lot of attention, because of the apparently underlying idea, reinforced by the mentioning of the octahedron and the dodecahedron, that a sphere is the limit to be approximated by a series of shapes with ever more angles, until finally there are angles everywhere.[117] That seems attractive enough, but as it would commit Democritus to a kind of composition from points, this interpretation has also its disadvantages. Perhaps there is something better?

One could of course try to explain Democritus' statement more or less away, as Simplicius seems to do, when he comments that 'what is bent is an angle and the sphere is bent all over itself.'[118] But that would turn it into a kind of flippant remark, for which it is difficult to imagine a serious context. Moreover, also Aristotle would then be guilty of gross distortion, since if read in Simplicius' way, the statement cannot fulfil the function it has in Aristotle's argument; it seems better to assume that Democritus did somehow envisage a series of figures of which the sphere was the last.

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115 306b32-307a3
116 307a13-17
117 Luria, 'Infinitesimaltheorie' 145, note 106
118 *In De Caelo* 662.10-12

On the other hand, I do not think we should put our trust in Aristotle without a critical examination of his argument, which is after all polemical. The argument seems a combination of two ideas. First there is the idea that heating occurs by the sharp angles of some shapes, for example pyramids. (In itself Democritus would accept this idea, because for him heating consists in bring about more void between atoms, which he thinks can *also* be done by sharp-angled shapes.[119]) There is also the second, Democritean, idea that fire consists of sphere-shaped atoms, which because of their shape – and small size, as we know from other sources[120] – are very mobile. It seems this mobility which is the immediate reason for the capacity of fire-atoms to cut and thus to heat and bring to combustion – so according to Democritus it need not be (only) sharp-angled shapes which cause heating.[121] These two ideas are used by Aristotle to play Democritus and the pyramidist off against each other. The base from which he does so is a generalized version of the idea that sharp-angled shapes cause heating: all angles cause heating. It is in this generalized context that the first reference to the sphere being a kind of angle comes in. And it is repeated in the second passage: all angles cause heat, so not only the sharp-angled shapes do so, but also the octahedron and the dodecahedron, and even the sphere, according to Democritus!

It is clear that Aristotle is not being fair to either side. We should not assume that Democritus and the pyramidists understand their explanation of the heating capacities of fire on the base of fire involving angles in the same way. The pyramidists have clearly sharp angles in mind; if they become obtuse, they will not cause heating any more. And if we look carefully at the Democritean side, the chain of explanations goes as follows: heating capacity is explained by mobility,[122] which on its turn is explained by being sphere-shaped, because spheres have least contact with other – straight – shapes, presumably since there is only contact at one point,[123] and because a sphere is a kind of angle.

Now should we believe there are two ultimate explanations for Democritus, that the sphere is a kind of angle and that the sphere has least contact, touching only at a point? It seems a reasonable guess that the one is a restatement of or intimately connected with the other.

But merely establishing – if the above is correct – that the 'angularity' of a sphere and the fact that a sphere touches at a point are almost the same thing is not going to enhance our understanding of Democritus' mysterious remark. Moreover, as we indicated above, we should assume that Aristotle has something to go on in Democritus

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119 See Theophrastus, *De Sensibus* 65 = DK 68 A135 (65) = Luria fr. 496 (whether or not one accepts Luria's emendation).

120 E.g. Aristotle, *De Anima* 405a5-13 = DK 68 A101 = Luria fr. 444

121 Cf. again Theophrast, *De Sensibus* 67 = DK 68 A135 (67) = Luria fr. 496:

> Sour is small and circular, as well as angular, though it has not an unequal shape. For [in its] many-angled [form] sour makes by its roughness <a body thin and hot while [in its] circular [form sour]> heats and pervades because it is small, circular and angleless. For also what is angular is like that. (τὸν δὲ δριμύν μικρόν και περιφερῆ, καὶ γωνιοειδῆ, σκαληνὸν δὲ οὐκ ἔχειν. τὸν μὲν γὰρ δριμὺν πολυγώνιον ποιεῖν τῇ τραχύτητι <ἀραιὸν καὶ θερμὸν τὸ σῶμα, τὸν δὲ δριμὺν περιφερῆ> θερμαίνειν καὶ διαχεῖν διὰ τὸ μικρὸν εἶναι καὶ περιφερῆ καὶ ἀγωνιοειδῆ· καὶ γὰρ τὸ γωνιοειδὲς εἶναι τοιοῦτον.)

For the sake of convenience I adopt the insertion proposed by Luria. I also retain, with Luria, the manuscripts' reading 'angleless' (ἀγωνιοειδῆ) and reject the emendation of Diels to 'angular' (γωνιοειδῆ). Being circular and being angular exclude each other, as appears from DK 68 A129 = Luria fr. 497.

122 Cf. the translation in *CWA* 502: '[A]nd Democritus makes even the sphere a kind of angle, which cuts because of its mobility.'

123 Cf. Simplicius, *In De Caelo* 661.32-662.1

when he suggests that the sphere comes after the octahedron and the dodecahedron, but still involves a kind of angularity. So we have to find a context in which these three elements come together: (i) the sphere touches a plane surface at one point; (ii) the sphere is a kind of angle; and (iii) the sphere is approximated with a series of solids.[124] The speculation I will indulge in now, will achieve that by positing that Democritus combined the issue whether a sphere or circle touches a straight plane or line at a point with a consideration of the approximation of a sphere or circle with inscribing solids or polygons.

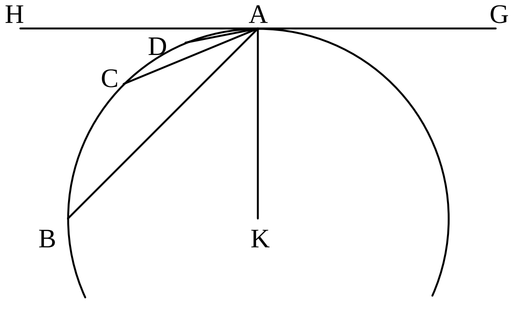


*Figure 6*

In a Hippocratean approximation of a circle, at every next step there are more points of touch between the circumference of the circle and the inscribing polygon. Now let us focus on one such point, A in fig. 6. At the first step of the approximation the circle is inscribed by a square with side AB, at the second step by an octagon with side AC, at the third by a hekkaidecagon with side AD, and so forth. At each step there is an angle enclosed by radius KA and the side of the inscribing polygon starting at A, every next step a larger angle: angle KAB, angle KAC, and so forth. Now we may wonder what the limit is to this series of angles: what is the angle between AK and the circum-ference of the circle? Well, we can prove something about that. Suppose we draw a line HAG perpendicular to AK. We can prove that HAG falls outside the circle, that is, does not have another point in common with the circumference, and that there is no other straight line through A (making an angle smaller than angle HAK) which will also be outside the circle.[125] Therefore every straight line through A not perpendicular to KA will fall inside or on the circle. But it cannot fall on the circle, as it would then have at least two points in common: such lines fall inside the circle.[126] Therefore a line through A enclosing an angle with AK which is smaller than a right angle will be a side of an approximating polygon. Therefore the limit of the series of angles, the angle between the circumference and AK is a right angle.

But is it? For HAG, which is perpendicular to AK, does not have a second point in common with the circumference, so the circumference must somehow have an incli-

[124] An element from this approximation we may, as Luria, 'Infinitesimaltheorie' 145, note 106, suggests, have encountered in the report by Aristotle that Democritus divided the sphere into eight parts (see *De Caelo* 3.4; 303b1, quoted on pp. 77-78).

[125] Euclid, *Elementa* 3.16

[126] *Elementa* 3.2

nation in relation to HAG, so that the angle formed by the circumference and AK cannot be a right angle; we can merely say that 'the angle of the semicircle contained by the straight line [KA] and the circumferenece is greater than any acute rectilineal angle.'[127]

It is this distinction that I imagine is drawn in Democritus' work with the title *On the difference of perpendicular* or *On the touching of a circle and a sphere* (Περὶ διαφορῆς γνώμονος ἢ Περὶ ψαύσιος κύκλου καὶ σφαίρης).[128] For both angles, angle HAK and the angle enclosed by the circumference and AK are greater than any acute angle and thus perpendicular. But there is also a difference, as explained above; thus we have to distinguish between two kinds of perpendicular.

The fact that the circumference, though containing together with AK an angle greater than any acute angle, still makes an angle with HAG, also explains why Democritus claims that the sphere is a kind of angle. For since there is an angle at a point, if the line going through that point is not a straight line,[129] the two angles enclosed by AK and the circumference taken together do not give us a straight line, but an angle. In this respected the circumference as punctuated at A is itself an angle.[130] But it is only a kind of angle, as it is not an angle enclosed by two sides of an approximating rectilineal polygon joined at a point.

From the same consideration, that HAG never falls inside the circle, it also follows that HAG touches the circle at one point.[131] And as HAG is the only straight line through A falling outside the circle while having at least one point in common, any straight line touching but not penetrating the circle will touch the circle at one point: the circle touches 'the smallest things.'

If these speculations are not just figments of the imagination, we can see again that Democritus points to the qualitative difference between each of the approximations and the limit of the series of approximations. Far from envisaging the sphere as consisting merely of rectilineal angles, he points out that such a composition is impossible, by

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127 *Elementa* 3.16

128 From the catalogue of works mentioned in Diogenes Laertius 9.47 = DK 68 B11l = Luria among others fr. 133. The most common form of the title, attested by three manuscripts, is *On the difference of judgement* (γνώμης), but one manuscript features γνώμωνος, obviously to be corrected to γνώμονος. For what it is worth, I think γνώμονος is the *lectio difficilior*. The main reason for opting for γνώμονος is an interpretational one. The usual account given is that Democritus argued against people like Protagoras who claimed that the mathematicians are wrong because for all we can see circles touch a straight line over some length (see Aristotle, *Metaphysica* B.2; 997b35-998a4). Democritus did so by differentiating between two kinds of judgement or cognition, one by the senses and one by reason. The senses would not be reliable witnesses on such issues; reason, that is, mathematics should tell us. On this account, however, the point made by Democritus seems rather small, at least if it is to be confined to the issue of the touch of circles and spheres. And that Democritus discussed in this work the difference between two kinds of cognition in general seems unlikely, because he had already done so in another work, titled *Standards* (Κανόνες) (see Sextus Empiricus, *AM* 7.138-139 = DK 68 B11 = Luria fr. 83), and because the present work is categorized in the catalogue of Thrasylus as mathematical (Diogenes Laertius 9.47-48). On the other hand, if Democritus had discussed the mathematical issues involved in the touching of a circle and a sphere, including proofs, it would certainly be worth a book. But then it seems difficult to connect the primarily mathematical contents with there being a difference in judgement.
The translation 'the perpendicular' for γνώμων may be somewhat adventurous, but can be justified by reference to the old word for 'perpendicular' (κάθετος): κατὰ γνώμονα, testified by Proclus, *In Elementa* 283.7-10.

129 This fits better the old, pre-Euclidean definition of angle, according to which an angle is not something formed by two meeting lines, but by one line; see Heath, *Thirteen Books* I *Introduction and Books I, II* 176-177.

130 Euclid does not know angles enclosed by two non-straight lines, but Proclus, *In Elementa* 126.17-127.16 does.

131 Cf. porism to *Elementa* 3.16.

arguing that at each of its points the circumference is a *kind of* angle, enclosing two right angles which are not of the ordinary rectilineal kind either.

### § 2.2.3.3. Concluding remarks

One can describe Democritus' position as follows: the converging series of approximations points to a limit and what is true for a limit, without yielding a limit. Limit entities are in a different league of reality from the basic entities, the atoms, and their mathematically distinguishable parts. Thus he is in open conflict with Zeno, who wants to have a conception of limits on which they are of the same kind as the basic, three-dimensional entities. We do not know whether he saw this as a problem or whether he accepted it gladly. He may have seen it as a problem for his mathematical techniques, on the ground that he could not explain how some things are true both of the approximations and the limit, while at the same time there are things which are true for the approximations and not true of the limit. We will never know – but if he did conceive of these points as problems, they were nevertheless not sufficient for him to abandon these techniques or to reject talk of limits altogether. The limits of bodies were far too important for his physics to be discounted, and he was far too proud of his mathematics.[132]

## § 3. Conclusion

The picture of Democritus which emerges from the interpretations I offered of the arguments and evidence concerning his dealings with problems of continuity and infinite divisibility, is one of a philosopher of considerable acumen. Where Zeno had suggested that there was no other way out of these problems than a complete rejection of divisibility, Democritus adopted a much more sophisticated and differentiated approach. Not only did he distinguish between divisibility in the field of physics, where he defended atomism, and divisibility in the field of mathematics, where he allowed for unlimited divisibility, but he also applied in his arguments in each field distinct metaphysical principles. With these distinctions and conceptual refinements, his philosophy represents a real step forward in the ancient debate about issues of continuity.

In the field of physics, Democritus took over from Zeno the modalized version of the principle that parts are prior to a whole, saying that what is one and whole, neither is nor can become a plurality. He used it to turn a Zenonian paradox of division into a positive argument for the existence of atoms. He agreed with Zeno that a magnitude cannot be divisible everywhere, since it would then, on the basis of this principle, already consist of sizeless parts. However, while Zeno had gone on to seal off a third way between divisibility everywhere and indivisibility by arguing that a homogeneous, bare, object cannot be divisible only at some places, Democritus showed on the basis of the same principle that this argument from homogeneity was ineffective. Since a plurality of two units cannot turn into a unity, he reasoned, an apparently homogeneous object resulting from the collision of two units is not really homogeneous, as there is still a discontinuity at the boundary between the two units. In this way he distinguished between two kinds

[132] See note 108 – I follow Mendell, 'Mathematical and Physical Shapes' Appendix C, in regarding the boast as authentic.

of homogeneity, real homogeneity, which only exists within an atom, and apparent homogeneity, which also exists when it is impossible to distinguish separate entities within an aggregate of atoms. Thus this principle, about which Democritus was so explicit that it may justifiably be called the Atomistic Principle, is the sole argumentative rock on which the whole edifice of atomism rests. There is no place any more for a conceptually independent notion of homogeneity.

In the field of mathematics, by contrast, Democritus rejected his beloved principle. He did so by allowing for a magnitude to be divisible everywhere, for only without this principle could one hold on to a magnitude being divisible everywhere without being committed to it being already divided everywhere. Thus the divisions brought about in mathematics are really newly created divisions, while the divisions in physics are in fact separations along already existing lines of division between atoms.

In another respect, however, there were for Democritus also in this field limits to divisibility, since he agreed with Zeno that a magnitude could not be composed of sizeless points; also for him limit entities could not be independent parts. It seems therefore unlikely that in his mathematics he used indivisibilist techniques. This did not stop him, however, from referring to limits in his physical theories and mathematics. In mathematics, Democritus probably used approximation methods, in which the figure about which something is to be proved is assumed to be approximately composed of a number of parts whose geometrical properties are better known. The larger this number of parts becomes, the closer the approximation is, but there is no limited number of such parts which taken together are equal to the approximated figure. This figure is therefore a kind of limit to the unlimited series of approximations. In his physics, Democritus held that all causal interaction between atoms takes place through contact. This contact, so he seems to have thought, consists in the coincidence of the limits of the two atoms involved.

The evidence suggests, I tried to argue, that Democritus was aware of the problematic status of these limits playing such an important part in his theories. By holding and even insisting that contact consists in the coincidence of two distinct limits, he must have distinguished these limits from each other by reference to the atoms to which they belong, thus making them dependent on the whole. And it is possible to interpret several pieces of evidence about Democritus' mathematics in such a way that they all indicate that Democritus wanted to point out that limits were of a very different kind from the normal parts of which bodies are composed.