# Generalized renewal measures

Kalma, Jetse Niels

CHAPTER I

# INTRODUCTION

## 1.1 Introduction and summary.

Renewal theory in its early stages concerned itself with a probabilistic model for the failure and replacement of components, such as electric light bulbs. For a general survey of the theory, and its origins, we refer to Smith ([19]*)), for proofs of the basic results see, e.g., Feller ([6]).

Consider a population of components, where each component is characterized by a nonnegative random variable $\underline{x}$, called its lifetime. We suppose these random variables $\underline{x}_1, \underline{x}_2, \ldots$ to be independent and identically distributed, and not to be degenerated at zero.

The first component is installed at the initial instant $t = 0$, say, and is replaced, instantaneously, at the time $\underline{x}_1$. The second component is, in turn, replaced at time $\underline{x}_1+\underline{x}_2$, and so on.

Thus, the time at which the $n^{\text{th}}$ renewal takes place is the $n^{\text{th}}$ partial sum of the $\{\underline{x}_i\}$, and we will denote it by

$$\underline{S}_n = \underline{x}_1+\underline{x}_2+\ldots+\underline{x}_n \qquad (n = 1,2,\ldots).$$

Let us write $\underline{N}_t$ for the largest $n$ such that $\underline{S}_n < t$, so that $\underline{N}_t$ denotes the number of renewals in $[0,t)$.

A central role in renewal theory is played by the renewal function $H(t)$, which is defined by

$$H(t) = E\{\underline{N}_t\},$$

so that $H(t)$ denotes the expected number of renewals in $[0,t)$. The simplest and oldest general result about $H(t)$ is the

*) numbers between square brackets refer to the list of references.

so-called elementary renewal theorem, which says that

$$H(t)/t \to \mu^{-1} \qquad as\ t \to \infty,$$

where $\mu$ denotes the expected lifetime $E\{\underline{x}_i\}$, and where $\mu^{-1}$ is interpreted as zero if $\mu = \infty$.

A much more fundamental - and theoretically very important - result is Blackwell's theorem, which we formulate for non-lattice distributed lifetimes. A random variable $\underline{x}$ is called nonlattice, if there exists no $d > 0$ such that the distribution of $\underline{x}$ is concentrated on the set $\{nd,\ n\ integer\}$. Blackwell's theorem then says that, for any fixed $h > 0$,

$$H(t+h)-H(t) \to h\mu^{-1} \qquad as\ t \to \infty.$$

Again, the limit is interpreted as zero if $\mu = \infty$.

Soon it became clear that the mathematically most interesting problems in the field of renewal theory - those which are concerned with the asymptotic behaviour of the renewal function - are of independent interest and they continued to be studied as such, the only trace of their origin being the fact that the resulting theorems are generally called renewal theorems.

Now the $\{\underline{x}_i\}$ no longer necessarily represent the lifetimes in some renewal process, and the assumption that the $\{\underline{x}_i\}$ are nonnegative is dropped.

The starting point is now a sequence $\underline{x}_1, \underline{x}_2, \ldots$ of independent and identically distributed random variables, with common distribution function $F$. We further define

$$\underline{S}_0 = 0; \qquad \underline{S}_n = \underline{x}_1 + \underline{x}_2 + \ldots + \underline{x}_n \qquad (n = 1,2,\ldots).$$

Such a process $\{\underline{S}_n,\ n = 0,1,\ldots\}$ is commonly called a random walk. The random variable $\underline{S}_n$ is interpreted as the position at time $n$ of a particle moving with random jumps, $\underline{x}_n$ being the jump at time $n$. The distribution function of $\underline{S}_n$ is given by $F^n$, the $n$-fold convolution of $F$.

For a Borel set $B$ and for $n \geqq 1$ the event $\{\underline{S}_n \in B\}$ is called a visit to $B$ at time $n$. The expectation of the number of visits of $\{\underline{S}_n\}$ to the set $B$ is called the renewal measure of $B$, and is denoted by $U(B)$. We have

$$U(B) = \sum_{m=1}^{\infty} F^m(B),$$

where $F^m(B) = P\{\underline{S}_m \in B\}$.

In our work we will always assume that $\int |x| F(dx) < \infty$, and that $\mu = \int xF(dx)$ is positive. This implies that $U(I) < \infty$ for any bounded interval $I$.

If the distribution function $F$ has positive, possibly infinite, first moment $\mu$ then again Blackwell's theorem is valid for the corresponding renewal measure $U$. Supposing $F$ to be nonlattice we have, for any fixed $h > 0$,

$$U[t, t+h) \to h\mu^{-1} \qquad as\ t \to \infty.$$

In this thesis we want to study measures $U_\theta$, which we will call generalized renewal measures, and which are defined, for any real $\theta$, by

$$U_\theta(B) = \sum_{m=1}^{\infty} m^\theta F^m(B)$$

for all Borel sets $B$.

What we are interested in is the asymptotic behaviour of these generalized renewal measures, e.g. the behaviour of $U_\theta[a+t, b+t)$ as $t \to \infty$.

Our investigation of the asymptotics of $U_\theta$ was motivated by the role which such generalized renewal measures and their asymptotic behaviour play in $r$-dimensional renewal theory (cf. Stam, [23]).

However, the results are perhaps also interesting from a

purely analytical point of view.
Finally, some new properties of the renewal measure $U$ itself are derived, and a few known results are obtained by new techniques.

Related problems were studied in a paper by Smith ([21]), although his primary concerns are different from ours. Smith is mainly interested in the behaviour of $U_\theta(-\infty, t)$ for $t \to \infty$, while we concentrate our attention on local asymptotic properties of $U_\theta$, namely on the behaviour of $U_\theta[a+t, b+t)$ for $t \to \infty$. Some of Smith's theorems (e.g. his corollary 5.2) are comparable with ours, but they are derived by quite different methods and in several respects they are weaker than our results.

The first four chapters of this thesis contain preparatory material. In the first chapter several, mostly well-known, renewal theorems are listed, and generalized renewal measures are introduced. We show that $U_k$, with $k = 1, 2, \ldots,$ and the $(k+1)$-fold convolution of $U$ are closely connected. In chapter II we formulate a theorem which shows that if two probability distribution functions $F$ and $G$ have suitably many moments in common, then their generalized renewal measures have the same asymptotic behaviour, up to a certain order. In chapter III the "comparison" theorem of the second chapter is used to prove some theorems on the asymptotics of renewal densities, and further some of the renewal theorems of the first chapter are sharpened. In chapter IV we consider convolutions of a certain class of signed measures; the results can be applied to investigate convolutions of the renewal measure $U$.

Our main results are derived in chapters V and VI.
In chapter V we are interested in the dominant term of $U_\theta[a+t, b+t)$, and we prove that, under appropriate conditions for $F$,

$$U_\theta[a+t,b+t) = t^\theta \mu^{-\theta-1}(b-a) + o(t^\theta), \qquad as\ t \to \infty.$$

In chapter VI we go farther. There we show that, if $F$ is absolutely continuous and has finite moment of order $\rho+1$, and if $F$ satisfies some further conditions, then we can obtain an asymptotic expansion for the density $u_\theta$ of $U_\theta$ of the form

$$u_\theta(t) = \sum_{i=0}^{[\rho]} c_i t^{\theta-i} + o(t^{\theta-\rho}), \qquad as\ t \to \infty.$$

Expressions for the coefficients $c_i$, $i = 0,1,\ldots,k$, can be obtained, and we will see that $c_i$ depends on $i$, $\theta$, $a$ and $b$, and further on the first $i+1$ moments of $F$.

An important element in our derivations is the introduction of a certain family $\{\underline{x}_t,\ t \geq t_0\}$ of integer-valued random variables. Supposing the distribution function $F$ to be non-lattice we define, for some bounded interval $I$ and for $t_0$ sufficiently large,

$$P\{\underline{x}_t = m\} = F^m(I+t)/U(I+t) \qquad (m = 1,2,\ldots).$$

Assuming that $\int_{-\infty}^{0} |x|^3 F(dx) < \infty$ we find from our expressions for the dominant term of $U_k$, with $k$ integer, that for $t \to \infty$

$$E\{\underline{x}_t\} = t\mu^{-1} + o(t); \qquad E\{\underline{x}_t^2\} = t^2\mu^{-2} + o(t^2).$$

It follows that $\underline{x}_t/t\mu^{-1}$ converges in quadratic mean, and therefore in probability, to $1$. Hence we see that the distribution of $(\underline{x}_t/t\mu^{-1})^\theta$ converges completely to a distribution which is degenerated at the point $1$. (In our work a stronger form of this weak law of large numbers will be proved, for which the assumption $\int_{-\infty}^{0} |x|^3 F(dx) < \infty$ will be seen to be superfluous.)

Now

$$U_\theta(I+t) = E\{\underline{x}_t^\theta\}U(I+t),$$

and if we may interchange limit and expectation then we find that for $t \to \infty$, denoting the length of $I$ by $|I|$,

$$U_\theta(I+t) = t^\theta \mu^{-\theta-1} |I| + o(t^\theta).$$

In this way in chapter V an expression for the dominant term of $U_\theta$, with $\theta$ real, is obtained.

In chapter VI we first show that if $F$ has exponentially vanishing tails then - by the results of chapter IV - an asymptotic expansion for $U_k(I+t)$, with $k = 1, 2, \ldots,$ can be derived, although in this expansion the coefficients are rather unmanageable.

However, if $F$ has a density with exponentially vanishing tails, then the existence of such an asymptotic expansion, together with Laplace inversion techniques, provides us with other, more satisfactory, expressions for the coefficients in this expansion, viz. in terms of moments of $F$.

Now, once more, the random variables $\{\underline{x}_t\}$ appear on the scene. Assuming $F$ to have finite second moment it turns out that, curiously enough, the $\{\underline{x}_t\}$ for $t \to \infty$ satisfy a central limit theorem. A satisfactory stochastic interpretation of this property has not yet been found.

If $F$ has a density with exponentially vanishing tails, then it is even possible to show that the moment generating function of $\underline{y}_t = (\underline{x}_t - t/\mu)/\sqrt{t/\mu}$ converges to the moment generating function of a normal law.

Furthermore

$$\begin{aligned} u_\theta(t) &= E\{\underline{x}_t^\theta\} u(t) \\ &= E\{(\underline{y}_t \sqrt{t/\mu} + t/\mu)^\theta\} u(t), \end{aligned}$$

and by expanding $(\underline{y}_t \sqrt{t/\mu} + t/\mu)^\theta$ in terms of powers of $\underline{y}_t$, and using the above-mentioned results for the moment generating function of $\underline{y}_t$, we then get an asymptotic expansion

for $u_\theta(t)$, with $\theta$ real, for the case where $F$ has a density with exponentially vanishing tails.
Finally the comparison theorem of chapter II is used to show that the finite asymptotic expansions we just obtained are valid for a more general class of probability distribution functions $F$.

## 1.2 Notations and conventions.

With $\tilde{B}$, $\tilde{B}_b$, $\tilde{B}_i$ we denote, respectively, the classes of *Borel sets*, *bounded Borel sets* and *bounded intervals* of $(-\infty,\infty)$. The *indicator function* of a set $A$ is written as $\chi_A$ or $\chi\{A\}$. The set $A+t$, for real $t$ and $A \in \tilde{B}$, denotes the *translation* of $A$ to the right over a distance $t$, so that $A+t = \{y: y-t \in A\}$.

A *signed measure* $S$ will, as usual, be defined as an extended real-valued, countably additive set function on the class of all measurable sets of a measurable space $(X,\tilde{C})$, such that $S(\emptyset) = 0$, and such that $S$ assumes at most one of the values $+\infty$ and $-\infty$. Every signed measure $S$ can be written unambiguously as the difference $S^+ - S^-$ of two measures $S^+$ and $S^-$ (of which at least one is finite) with $S^+ \perp S^-$, i.e. there exists a set $A$ such that $S^+(A) = 0$, $S^-(A^c) = 0$. The measures $S^+$ and $S^-$ are called the *positive* and *negative variation* of $S$. The sum $S^+ + S^-$ is called the *variation* of $S$, and is denoted by $|S|$. The (signed) measures we meet in the sequel are considered to be defined on $((-\infty,\infty), \tilde{B})$. A measure $R$ is called *interval-finite* if $R(I) < \infty$ for all $I \in \tilde{B}_i$.

The assumption that a signed measure $S$ cannot take both the values $+\infty$ and $-\infty$ is needed to guarantee that $S$ is defined on $((-\infty,\infty), \tilde{B})$. However, our work leads us to consider differences $R_1 - R_2$, where $R_1$ and $R_2$ are interval-finite measures