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### Particle dynamics of branes

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# Appendix A

## Differential Geometry: Formulae and Conventions

In this appendix we fix the conventions used in the main text. We also give a brief introduction to the Einstein-Hilbert action.

### A.1 Conventions

We take the following metric signature convention,

$$\eta = \text{diag}(-\cdots-, +\cdots+), \quad (\text{A.1.1})$$

writing first the timelike directions and then the spacelike ones. Symmetrization of a tensor  $T_{\mu_1 \dots \mu_p}$  is given by

$$T_{(\mu_1 \dots \mu_p)} = \frac{1}{p!} \left( T_{\mu_1 \dots \mu_p} + \text{even permutations} + \text{odd permutations} \right). \quad (\text{A.1.2})$$

Whereas the anti-symmetrization of  $T_{\mu_1 \dots \mu_p}$  is given by

$$T_{[\mu_1 \dots \mu_p]} = \frac{1}{p!} \left( T_{\mu_1 \dots \mu_p} + \text{even permutations} - \text{odd permutations} \right). \quad (\text{A.1.3})$$

### A.2 General Relativity

In general relativity space-time is a  $D$ -dimensional (pseudo-)Riemannian manifold  $(M, g)$ . This means that it is a manifold  $M$  endowed with a bilinear form  $g_{\mu\nu}$  with

signature  $(-\dots-, +\dots+)$ , writing first the timelike directions and then the spacelike ones. In components we write this as

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \quad \mu, \nu = 1, \dots, D. \quad (\text{A.2.1})$$

To shorten the notation, we will omit the  $\otimes$ . For a given metric  $g_{\mu\nu}$  we use the following Levi-Civita connection

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right), \quad (\text{A.2.2})$$

from which we construct the Riemann tensor

$$\mathcal{R}_{\nu\rho\sigma}^\mu = \partial_\rho \Gamma_{\nu\sigma}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\gamma \Gamma_{\gamma\rho}^\mu - \Gamma_{\nu\rho}^\gamma \Gamma_{\gamma\sigma}^\mu. \quad (\text{A.2.3})$$

From this tensor we can define the Ricci tensors  $\mathcal{R}_{\nu\sigma}$  and the Ricci scalar  $\mathcal{R}$  via the contractions of the Riemann tensor

$$\mathcal{R}_{\nu\sigma} \equiv \mathcal{R}_{\nu\mu\sigma}^\mu, \quad \mathcal{R} \equiv \mathcal{R}_\nu^\nu. \quad (\text{A.2.4})$$

The Einstein tensor  $G_{\mu\nu}$  is defined as

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}. \quad (\text{A.2.5})$$

The Ricci tensor of a conformal re-scaled metric  $\tilde{g}_{\mu\nu} = e^{2\alpha\phi}g_{\mu\nu}$  is

$$\tilde{\mathcal{R}}(\tilde{g})_{\mu\nu} = \mathcal{R}_{\mu\nu} - (D-2)\alpha^2(\partial\phi)^2 g_{\mu\nu} + (D-2)\alpha^2\partial_\mu\phi\partial_\nu\phi - (D-2)\alpha\nabla_\mu\partial_\nu\phi - \alpha g_{\mu\nu}\square\phi. \quad (\text{A.2.6})$$

All the tensors appearing on the right-hand side are defined with respect to  $g_{\mu\nu}$ . The action of the covariant derivative  $\nabla_\eta$  on a tensor  $T_{\mu_1\dots\mu_p}^{\nu_1\dots\nu_q}$  is defined as

$$\begin{aligned} \nabla_\eta T_{\mu_1\dots\mu_p}^{\nu_1\dots\nu_q} = & \partial_\eta T_{\mu_1\dots\mu_p}^{\nu_1\dots\nu_q} - \Gamma_{\eta\mu_1}^\rho T_{\rho\mu_2\dots\mu_p}^{\nu_1\dots\nu_q} \dots - \Gamma_{\eta\mu_p}^\rho T_{\mu_1\mu_2\dots\mu_{p-1}\rho}^{\nu_1\dots\nu_q} \\ & + \Gamma_{\rho\eta}^{\nu_1} T_{\mu_1\dots\mu_p}^{\rho\nu_2\dots\nu_q} + \dots + \Gamma_{\rho\eta}^{\nu_q} T_{\mu_1\dots\mu_p}^{\nu_1\dots\nu_{q-1}\rho}. \end{aligned} \quad (\text{A.2.7})$$

Finally, the  $\square$ -operator is defined as

$$\square\phi \equiv \nabla_\mu\partial^\mu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right), \quad (\text{A.2.8})$$

where  $g$  stands for the determinant of  $g_{\mu\nu}$ .

For the cosmologies and instantons we often work in the gauge where the metric is given by

$$ds_D^2 = \epsilon f(r)^2 dr^2 + g(r)^2 g_{ij}^{D-1} dx^i dx^j. \quad (\text{A.2.9})$$

For  $\epsilon = -1$  we have a cosmology, while for  $\epsilon = +1$  this describes an instanton geometry. The function  $f$  corresponds to the gauge freedom of reparameterizing the  $r$ -coordinate. In case  $f = 1$  and  $\epsilon = -1$  we have a cosmology in the FLRW-gauge. For the metric Ansatz (A.2.9) the Ricci tensor is given by

$$\mathcal{R}_{ij} = -\epsilon \left\{ \frac{d}{dr} \left[ \frac{g\dot{g}}{f^2} \right] + \frac{g\dot{g}\dot{f}}{f^3} + (D-3) \frac{\dot{g}^2}{f^2} \right\} g_{ij}^{D-1} + \mathcal{R}_{ij}^{D-1}, \quad \mathcal{R}_{rr} = (D-1) \left\{ -\left( \frac{\ddot{g}}{g} \right) + \frac{\dot{g}\dot{f}}{gf} \right\}, \quad (\text{A.2.10})$$

where a dot refers to a derivative with respect to  $r$ .

A homeomorphism  $f : M \rightarrow M$  is an *isometry* if it preserves the metric, in components this is the statement

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p), \quad (\text{A.2.11})$$

where  $x$  and  $y$  are the coordinates of  $p$  and  $f(p)$  respectively. If a displacement  $\epsilon X$ ,  $\epsilon$  being infinitesimal, generates an isometry, the vector field  $X$  is called a *Killing vector field*. The coordinates  $x^\mu$  of a point  $p \in M$  change to  $x^\mu + \epsilon X^\mu(p)$  under this displacement. If  $f : x^\mu \rightarrow x^\mu + \epsilon X^\mu$  is an isometry, it satisfies (A.2.11). From this we can derive that  $g_{\mu\nu}$  and  $X^\mu$  satisfy the *Killing equation*

$$X^\zeta \partial_\zeta g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0. \quad (\text{A.2.12})$$

If the right-hand side is non-zero and given by  $\psi g_{\mu\nu}$  with  $\psi$  a function then  $X$  is called a conformal Killing vector field. The metric gets re-scaled by an overall factor related to  $\psi$ .

The Killing vector fields represent the direction of the symmetry of a manifold. In  $D$ -dimensional Minkowski space-time ( $D \geq 2$ ) there are  $D(D+1)/2$  Killing vector fields,  $D$  of which generate translations,  $(D-1)$  boosts and  $(D-1)(D-2)/2$  space rotations. Those space-times which admit  $D(D+1)/2$  Killing vector fields are called *maximally symmetric spaces*. One can prove that the Riemann tensor is then given by

$$\mathcal{R}_{\rho\sigma\mu\nu} = \alpha (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}), \quad (\text{A.2.13})$$

with  $\alpha$  a constant.

In the metric (A.2.9),  $g_{ij}$  often describes a Euclidean maximally symmetric space. That is we have the sphere  $S^n$  ( $k=1$ ) or the hyperboloid  $\mathbb{H}^n$  ( $k=-1$ ) or flat space  $\mathbb{E}^n$  ( $k=0$ ). The metrics read

$$ds^2 = \frac{1}{1-kr^2} dr^2 + r^2 d\Omega_{n-1}^2. \quad (\text{A.2.14})$$

Also,  $d\Omega_m^2$  is the metric on the  $S^m$  sphere

$$d\Omega_m^2 = d\theta_1^2 + \sin^2(\theta_1) d\theta_2^2 + \dots + \sin^2(\theta_1) \dots \sin^2(\theta_{m-1}) d\theta_m^2. \quad (\text{A.2.15})$$

Via the coordinate redefinition

$$\frac{1}{1 - kr^2} dr^2 = d\psi^2, \quad (\text{A.2.16})$$

we find the metric

$$\begin{aligned} k = -1 : \quad ds^2 &= d\psi^2 + \sinh^2 \psi d\Omega_{n-1}^2, \\ k = 0 \quad : \quad ds^2 &= d\psi^2 + \psi^2 d\Omega_{n-1}^2, \\ k = +1 : \quad ds^2 &= d\psi^2 + \sin^2 \psi d\Omega_{n-1}^2. \end{aligned} \quad (\text{A.2.17})$$

The convention is such that  $\alpha = k$  for these metrics. The Ricci scalar is

$$\mathcal{R}_n = kn(n-1). \quad (\text{A.2.18})$$

### A.2.1 Vielbeine

Instead of writing the metric  $g$  in terms of coordinate one-forms  $dx^\mu$ , we can use vielbein one-forms  $e^a = e^a_\mu dx^\mu$  for which the metric is

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} e^a \otimes e^b, \quad a, b = 1, \dots, D, \quad (\text{A.2.19})$$

where  $\eta$  is given by (A.1.1) and we used  $\otimes$  for clarity. We use Greek indices  $\mu, \nu, \rho \dots$  to denote space-time coordinates and Latin indices  $a, b, c \dots$  represent the so-called tangent directions, which are raised and lowered with  $\eta$ . The determinant of the vielbein is denoted by  $e$ .

We define the spin connection  $\omega^a_b = \omega^a_{\mu b} dx^\mu$  via

$$de^a = -\omega^a_b \wedge e^b, \quad \omega_{ab} = -\omega_{ba}. \quad (\text{A.2.20})$$

The spin connection can be expressed in terms of the vielbeine

$$\omega_{abc} = \omega_{\mu bc} e^\mu_a = \frac{1}{2} \left( -\Omega_{abc} + \Omega_{bca} + \Omega_{cab} \right), \quad \Omega^a_{bc} = 2\partial_{[\mu} e^a_{\nu]} e^\mu_b e^\nu_c. \quad (\text{A.2.21})$$

Here  $\Omega_{abc}$  is called the object of anholonomicity. In terms of the spin connection the two-form curvature is given by

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad (\text{A.2.22})$$

from which we find the Ricci tensor

$$\mathcal{R}_{\mu\nu} = (R^a_b)_{\rho\nu} e^b_\mu e^\rho_a. \quad (\text{A.2.23})$$

The covariant derivative with respect to local Lorentz transformations is denoted by  $\nabla_\mu$ , acting on spinors  $\chi$  as

$$\begin{aligned}\nabla_\mu\chi &= \partial_\mu\chi + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\chi, \\ \nabla_\mu\chi^\nu &= \partial_\mu\chi^\nu + \Gamma_{\mu\rho}^\nu\chi^\rho + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\chi^\nu,\end{aligned}\tag{A.2.24}$$

where in general we denote  $\Gamma_{a_1\dots a_n} = \Gamma_{[a\dots a_n]}$  and  $\Gamma_a$  is an element of the Clifford algebra, see appendix B.

### A.3 Forms

Instead of working with the index notation, it is often highly preferable to work with the language of differential forms. A differential form of order  $p$  or a  $p$ -form for short reads in components

$$A_p = \frac{1}{p!}A_{\mu_1\dots\mu_p}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p},\tag{A.3.1}$$

where the wedge product  $\wedge$  is defined by the totally anti-symmetric tensor product, for example

$$dx^{\mu_1}\wedge dx^{\mu_2} = dx^{\mu_1}\otimes dx^{\mu_2} - dx^{\mu_2}\otimes dx^{\mu_1}.\tag{A.3.2}$$

We will only give some relevant properties of differential forms, a good textbook on this subject is for example [51], see also [44].

Due to the wedge product, a  $p$ - and a  $q$ -form obey

$$A_p\wedge B_q = (-)^{pq}B_q\wedge A_p.\tag{A.3.3}$$

The action of the exterior derivative  $d$  on the  $p$ -form (A.3.1) is defined as

$$dA_p = \frac{1}{p!}\partial_{[\mu}A_{\mu_1\dots\mu_p]}dx^\mu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p},\tag{A.3.4}$$

resulting in a  $(p+1)$ -form. This exterior derivative obeys the Leibniz rule

$$d(A_p\wedge B_q) = dA_p\wedge B_q + (-)^pA_p\wedge dB_q.\tag{A.3.5}$$

In  $D$  dimensions we define the epsilon *symbol*  $\varepsilon_{\mu_1\dots\mu_D}$  via

$$\varepsilon_{1\dots D} = 1,\tag{A.3.6}$$

and it is antisymmetric in all its indices. This allows us to define the epsilon tensor  $\epsilon_{\mu_1\dots\mu_D}$  via

$$\epsilon_{\mu_1\dots\mu_D} = \sqrt{|g|}\varepsilon_{\mu_1\dots\mu_D}.\tag{A.3.7}$$

Sometimes it is also useful to define a totally antisymmetric epsilon symbol with *upstairs* indices, the components are given numerically by

$$\varepsilon^{\mu_1 \dots \mu_D} = (-)^t \varepsilon_{\mu_1 \dots \mu_D}, \quad (\text{A.3.8})$$

where  $t$  is the number of timelike coordinates. Contractions of the epsilon tensor (and symbol) obey the following relation<sup>1</sup>

$$\varepsilon_{\mu_1 \dots \mu_q \mu_{q+1} \dots \mu_D} \varepsilon^{\mu_1 \dots \mu_q \nu_{q+1} \dots \nu_D} = (-)^t q! (D-q)! \delta_{[\mu_{q+1} \dots \mu_D]}^{[\nu_{q+1} \dots \nu_D]}. \quad (\text{A.3.9})$$

The Hodge operator  $\star$  is a linear map of a  $p$ -form into a  $(D-p)$ -form, whose action on a  $p$ -form is defined by

$$\star(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = \frac{1}{(D-p)!} \varepsilon_{\nu_1 \dots \nu_{D-p}}^{\mu_1 \dots \mu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{D-p}}. \quad (\text{A.3.10})$$

As a particular case,

$$\star 1 = \varepsilon = \sqrt{|g|} d^D x, \quad (\text{A.3.11})$$

where we identify  $d^D x$  as  $dx^1 \wedge \dots \wedge dx^D$ . Three frequently used expressions for arbitrarily  $p$ -forms are

$$\begin{aligned} \star A_p \wedge B_p &= \star B_p \wedge A_p = \frac{1}{p!} A_{\mu_1 \dots \mu_p} B^{\mu_1 \dots \mu_p} \star 1, \\ \star \star A_p &= (-)^{p(D-p)+t} A_p, \end{aligned} \quad (\text{A.3.12})$$

and

$$\star d \star A_p = \frac{(-)^{p(D-p+1)-1+t} \partial_\rho \left( \sqrt{|q|} A^{\rho \mu_1 \dots \mu_{p-1}} \right)}{(p-1)! \sqrt{|q|}} g_{\mu_1 \nu_1} \dots g_{\mu_{p-1} \nu_{p-1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{p-1}}. \quad (\text{A.3.13})$$

## A.4 Euler-Lagrange Variation

To obtain the Einstein equation, we need the following relations under the variation  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, & \delta \sqrt{-g} &= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}, \\ \delta g^{\mu\nu} &= -g^{\mu\kappa} g^{\nu\rho} \delta g_{\kappa\rho}, & \delta \sqrt{-g} \mathcal{R} &= -\sqrt{-g} G^{\mu\nu} \delta g_{\mu\nu}. \end{aligned} \quad (\text{A.4.1})$$

<sup>1</sup>We define  $0! = 1$ .

The action of matter coupled to  $D$ -dimensional gravity is given by

$$S = \int d^D x \sqrt{|g|} \left( \frac{1}{2\kappa^2} \mathcal{R} + \mathcal{L} \right), \quad (\text{A.4.2})$$

where the first term on the right-hand side is called the Einstein-Hilbert action and  $\mathcal{L}$  is the matter Lagrangian density of the theory. Newton's constant  $G$  is related to  $\kappa$ , for example in four dimensions we have that  $\kappa^2 = 8\pi G$ . If the matter part of action is changed under  $\delta g_{\mu\nu}$ , the energy-momentum tensor is defined by

$$\delta S_M = \frac{1}{2} \int d^D x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu}. \quad (\text{A.4.3})$$

From demanding that the total variation of the action (A.4.2) vanishes under  $\delta g_{\mu\nu}$ , we obtain the Einstein equation

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (\text{A.4.4})$$

In particular we take for the matter Lagrangian density a  $A_{p-1}$  potential. This is described by the action

$$\mathcal{L} = * \mathcal{R} - \frac{1}{2} e^{a\phi} * dA_{p-1} \wedge dA_{p-1} - \frac{1}{2} * d\phi \wedge d\phi, \quad (\text{A.4.5})$$

where  $a$  is a real number. The equations of motion, together with the Bianchi identity, are then given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{p-1}{2(D-2)(p!)} g_{\mu\nu} e^{a\phi} F_p^2 + \frac{1}{(p-1)!2} e^{a\phi} (F_p^2)_{\mu\nu}, \quad (\text{A.4.6})$$

$$d(*e^{a\phi} F_p) = 0, \quad dF_p = 0, \quad (\text{A.4.7})$$

$$\square \phi = \frac{a}{p!2} F_p^2 e^{a\phi}, \quad (\text{A.4.8})$$

where for a  $p$ -form field strength we have used the definitions

$$\begin{aligned} F_p^2 &= F_{\mu_1 \dots \mu_p} F_{\nu_1 \dots \nu_p} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}, \\ (F_p^2)_{\mu\nu} &= F_{\mu\rho_1 \dots \rho_{p-1}} F_{\nu\nu_1 \dots \nu_{p-1}} g^{\rho_1 \nu_1} \dots g^{\rho_{p-1} \nu_{p-1}}. \end{aligned} \quad (\text{A.4.9})$$





## Appendix B

# Spinors and their Reality Properties

In this appendix, we will recall various properties of Clifford algebras and spinors. The purpose of this appendix is two-fold. On the one hand it serves to introduce our conventions and notations regarding spinors. On the other hand, the discussion on the reality conditions on spinors is also rather crucial for the results presented in chapter 6. In the first section of this appendix, we will recall some general properties of Clifford algebras in various dimensions and signatures. In the second section, we will then discuss how appropriate reality conditions can be imposed on the spinors. The latter discussion will be mainly restricted to 10 and 11 dimensions. A good review concerning the matter presented here is offered in [141], whose conventions we will mainly follow.

### B.1 Clifford Algebras in Various Dimensions and Signatures

In this section we will consider arbitrary dimensions  $d = t + s$ , where  $t$  is the number of timelike and  $s$  the number of spacelike directions. The Clifford algebra is then defined by the following anticommutation relation

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}, \quad (\text{B.1.1})$$

where  $\eta_{ab} = \text{diag}(-\dots - +\dots +)$ , writing first the timelike directions and then the spacelike ones.

We will always work with unitary representations of (B.1.1):

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1}, \quad (\text{B.1.2})$$

where we define  $A$  to be the product of all timelike  $\Gamma$ -matrices :  $A = \Gamma_1 \cdots \Gamma_t$ . In this way, timelike  $\Gamma$ -matrices are anti-hermitian, while the spacelike ones are hermitian.

In even dimensions, we will define the chirality matrix  $\Gamma_*$  as follows

$$\Gamma_* = (-i)^{d/2+t} \Gamma_1 \cdots \Gamma_d \quad \Rightarrow \quad (\Gamma_*)^2 = \mathbb{1}. \quad (\text{B.1.3})$$

When we restrict to 10 dimensions, we will also denote  $\Gamma_*$  by  $\Gamma_{11}$ . Note that in odd dimensions the product of all  $\Gamma$ -matrices is always given by a power of  $i$  times the unit matrix.

One can show that there always exists a unitary matrix  $\mathcal{C}_\eta$  such that

$$\mathcal{C}_\eta^T = -\varepsilon \mathcal{C}_\eta \quad \text{and} \quad \Gamma_a^T = -\eta \mathcal{C}_\eta \Gamma_a \mathcal{C}_\eta^{-1}, \quad (\text{B.1.4})$$

where  $\varepsilon, \eta$  can be  $\pm 1$ . In even dimensions, both signs for  $\eta$  are possible, corresponding to the fact that both  $\Gamma_a^T$  and  $-\Gamma_a^T$  are representations that are equivalent to  $\Gamma_a$ . The two possibilities for the charge conjugation matrix are then related by

$$\mathcal{C}_+ = \mathcal{C}_- \Gamma_*. \quad (\text{B.1.5})$$

In odd dimensions, due to the constraint on the product of all  $\Gamma$ -matrices, only one of the representations  $\Gamma_a^T$  or  $-\Gamma_a^T$  is equivalent to  $\Gamma_a$  and hence only one sign for  $\eta$  is possible. Once the sign of  $\eta$  is fixed, the sign of  $\varepsilon$  can be determined. The possibilities for these signs are summarized in table B.1.1.

$d \bmod 8$	0	1	2	3	4	5	6	7
$(\varepsilon, \eta)$	$(-, +)$ $(-, -)$	$(-, -)$	$(-, -)$ $(+, +)$	$(+, +)$	$(+, +)$ $(+, -)$	$(+, -)$	$(+, -)$ $(-, +)$	$(-, +)$

Table B.1.1: *The possible signs for  $\varepsilon$  and  $\eta$  for all dimensions (modulo 8).*

Defining the following matrix  $B_\eta$

$$B_\eta = -\varepsilon \eta^t \mathcal{C}_\eta A, \quad (\text{B.1.6})$$

equations (B.1.2) and (B.1.4) then imply that

$$\Gamma_a^* = (-)^{t+1} \eta B_\eta \Gamma_a B_\eta^{-1}. \quad (\text{B.1.7})$$

As for the  $\mathcal{C}_\eta$ -matrix, in even dimensions both signs of  $\eta$  are possible, while in odd dimensions only one possibility for  $\eta$  is allowed. Finally, note that the matrix  $B_\eta$  satisfies

$$B_\eta B_\eta^* = -\varepsilon \eta^t (-)^{t(t+1)/2} \mathbb{1}. \quad (\text{B.1.8})$$

## B.2 Reality Conditions for Spinors

In this thesis we define the Majorana conjugate  $\bar{\chi}$  of a spinor  $\chi$  as

$$\bar{\chi} = \chi^T \mathcal{C}_\eta, \quad (\text{B.2.1})$$

whereas the Dirac conjugate  $\bar{\chi}^D$  is given by

$$\bar{\chi}^D = \chi^\dagger A. \quad (\text{B.2.2})$$

In order to formulate reality conditions in 10 dimensions, we will work with a doublet notation, allowing us to treat type IIA and type IIB theories in a single framework. The 64-component doublets are the following

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} \text{ (type IIA),} \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \text{ (type IIB),} \quad (\text{B.2.3})$$

where  $\Gamma_* \chi^\pm = \pm \chi^\pm$ . Gamma-matrices and the charge conjugation matrix  $\mathcal{C}_\eta$  then act on the doublets by making the following replacements

$$\Gamma_a \rightarrow \Gamma_a \otimes \sigma, \quad (\text{B.2.4})$$

$$\mathcal{C}_\eta \rightarrow \mathcal{C}_\eta \otimes \sigma, \quad (\text{B.2.5})$$

where  $\sigma$  is given by  $\sigma_1$  in type IIA and by  $\mathbb{1}_2$  in type IIB. Note furthermore that  $\Gamma_*$  can be represented by  $\mathbb{1}_{32} \otimes \sigma_3$  in type IIA and by  $\mathbb{1}_{32} \otimes \mathbb{1}_2$  in type IIB. In the following and in chapter 6 we will always assume that matrices act on doublets as indicated in (B.2.4), without writing the tensor products explicitly. We use the following three Pauli matrices  $\sigma_i$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B.2.6})$$

Using this doublet notation, a general reality condition can now be denoted as follows:

$$\chi^* = -\varepsilon \eta^t \alpha_\chi \mathcal{C}_\eta A \rho \chi, \quad (\text{B.2.7})$$

where  $\alpha_\chi$  represents a phase factor. The presence of  $-\varepsilon \eta^t \mathcal{C}_\eta A$  is dictated by compatibility with Lorentz transformations. Note that the condition (B.2.7) now contains a  $2 \times 2$ -matrix  $\rho$ , that can mix the two components of the doublets (B.2.3); the action of  $\rho$  on a doublet should thus be interpreted as  $\mathbb{1}_{32} \otimes \rho$ . We will take the following possibilities for  $\rho$ :

$$\rho \in \{\mathbb{1}_2, \sigma_1, i\sigma_2, \sigma_3\}. \quad (\text{B.2.8})$$

Note that in the type IIA case the matrix  $\rho$  is required to be diagonal, since complex conjugation should preserve the chirality of the spinor. In the type IIB case, we

do not have to impose this restriction as both parts of the doublet now have the same chirality. Note that upon making a field redefinition, the reality conditions with  $\rho = \sigma_1$  and  $\rho = \sigma_3$  can be related <sup>1</sup>. We can thus restrict to  $\rho \in \mathbb{1}, i\sigma_2, \sigma_3$  without loss of generality.

The requirement that  $\chi^{**} = \chi$  leads to a non-trivial requirement:

$$(\sigma^{t+1}\rho)^2 = -\epsilon\eta^t(-)^{\frac{t(t+1)}{2}}. \quad (\text{B.2.9})$$

In the IIB case, there is moreover an extra consistency condition, due to the fact that the theory is chiral. Indeed the reality condition (B.2.7) has to respect the chirality, which in 10 dimensions is only possible when  $t$  is odd.

The different reality conditions that can be consistently imposed are then summarized in table B.2.1. In this table we always choose  $\epsilon = \eta = 1$ . This is possible as

$t \bmod 4$	0	1	2	3
IIA	*M	MW *MW	M	/
IIB	/	MW *MW	/	SMW

Table B.2.1: *This table gives all the possible ten-dimensional reality conditions of the form (B.2.7) for a doublet of chiral spinors in type IIA and IIB respectively.  $t$  denotes the number of timelike dimensions. Here M, \*M or SM respectively stand for  $\rho = \mathbb{1}_2, \sigma_3$  or  $i\sigma_2$ . The addition of W means that the reality condition respects chirality of the spinors.*

$C_- = C_+\Gamma_{11}$ , and thus (B.2.7) with the choice  $\epsilon = \eta = -1$  can always be rewritten in terms of  $C_+$  and  $\eta = \epsilon = 1$  by redefining  $\rho$  and  $\alpha_\chi$  since  $\Gamma_{11}$  can be represented as  $\sigma_3$  or  $\mathbb{1}_2$  in IIA respectively IIB.

Finally a word on notation. Note that in denoting the types of reality conditions on the fermions in table B.2.1, we reserve the \* when  $\rho = \sigma_3$  in (B.2.7). M, MW and SMW then correspond to what is known in the literature as Majorana, Majorana-Weyl and symplectic Majorana-Weyl (see for instance [141]). Although \*M suggests a Majorana condition, this is not true. For instance, what we have called \*M in Euclidean type IIA, corresponds to what in the literature is called symplectic Majorana.

<sup>1</sup>Explicitly, this redefinition is given by  $\chi'_1 = \chi_1 + \chi_2$  and  $\chi'_2 = \chi_1 - \chi_2$ . Note that this redefinition involves only real numbers. Furthermore as one can see in table 6.3.1 in the main text, this redefinition corresponds to going from IIB' to IIB\*.

# Appendix C

## Lie Group and Lie Algebra

A *Lie Group*  $G$  is a differentiable manifold which is endowed with a group structure such that the two group operations

- $\cdot : G \times G \rightarrow G, (g_1, g_2) \rightarrow g_1 \cdot g_2,$
- $^{-1} : G \rightarrow G, g \rightarrow g^{-1},$

are differentiable. A Lie group is *abelian* if  $a \cdot b = b \cdot a, \forall a, b \in G$ , else it is called non-abelian. From now on we will write  $a \cdot b$  as  $ab$ .

The Lie group  $G$  can act on a manifold  $M$ . The action of  $G$  on a point  $p$  of the manifold  $M$  is a differentiable map  $\sigma : G \times M \rightarrow M$  which satisfies the conditions

- $\sigma(e, p) = p,$
- $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p),$

where  $e$  is the identity element of  $G$ ,  $g_i \in G$  and  $p \in M$ . We call the action of  $\sigma$  *transitive* if for any  $p_1, p_2 \in M$ , there exists an element  $g \in G$  such that  $\sigma(g, p_1) = p_2$ . This means that given any point  $p \in M$ , the action of  $G$  on  $p$  allows us to go to all the points of  $M$ . Such a manifold is called *homogeneous*. For example, Lie groups act transitively on themselves via the group multiplication. The *isotropy group*  $H(p)$  of  $p \in M$  is a subgroup of  $G$  defined by

$$H(p) = \{g \in G | \sigma(g, p) = p\}. \tag{C.0.1}$$

There is a theorem that states that if a Lie group  $G$  acts on a manifold  $M$  the isotropy group  $H(p)$  for any  $p \in M$  is a Lie subgroup [51].

There are two ways to define a *Lie algebra*. First, the tangent space of a group  $G$  at the identity element  $e$  can be identified with the Lie algebra  $\mathfrak{G}$  of  $G$ . Secondly,

there is a more algebraic approach [142]. A Lie algebra is a vector space  $\mathfrak{G}$  together with a bilinear operation  $\mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$  satisfying

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z] && \text{(bilinearity),} \\ [x, y] &= -[y, x] && \text{(anticommutativity),} \\ 0 &= [x, [y, z]] + [y, [z, x]] + [z, [x, y]] && \text{(Jacobi identity),} \end{aligned} \quad (\text{C.0.2})$$

where  $x, y, z \in \mathfrak{G}$  and  $a, b \in F$  with  $F$  a field over which  $\mathfrak{G}$  is a vector space, for example  $\mathbb{R}$  or  $\mathbb{C}$ .

A Lie algebra is specified by its generators  $t_a$  and their commutation relations

$$[t_a, t_b] = f_{ab}^c t_c, \quad (\text{C.0.3})$$

here  $f_{ab}^c$  is called the structure constant. The dimension of a Lie algebra is the dimension of the underlying vector space spanned by the generators  $t_a$ .

By a *representation* of an algebra we mean a set of matrices  $T_a$  with the same commutation relations as the  $t_a$ 's given in (C.0.3). This is a mapping of the generators  $t_a$  into linear operators  $T_a$ , which act on some vector space  $V$ . When this vector space is the Lie algebra  $\mathfrak{G}$  itself we call this representation the *adjoint* representation. Let  $x \in \mathfrak{G}$  and take

$$x \rightarrow [t_a, x]. \quad (\text{C.0.4})$$

This linear transformation is called the  $\text{ad } t_a$ .

A *subalgebra*  $\mathfrak{h} \subset \mathfrak{G}$  is a subspace of  $\mathfrak{G}$  which is closed under the Lie product. An *ideal* is a special kind of subalgebra. Namely, if  $\mathfrak{h}$  is an ideal and  $x \in \mathfrak{h}$  and  $y$  is an element of  $\mathfrak{G}$  then  $[x, y] \in \mathfrak{h}$ . If  $\mathfrak{h}$  would have been a subalgebra only,  $y \in \mathfrak{h}$  instead of  $\mathfrak{G}$ . A Lie algebra which has no trivial ideals it is called *simple*. The trivial ideals are the full algebra and the ideal  $\{0\}$ . An algebra which has no abelian ideals is called *semi-simple*.

The generators of the simple Lie algebra can be chosen so that one subset of them generates a commutative Cartan subalgebra (CSA). We denote these generators by  $h_I$ , so that  $[h_I, h_J] = 0$ . The other remaining generators are eigenvectors of  $\text{ad } h$  for every  $h \in \text{CSA}$ . We call these the shift operators and denote them by  $e_\alpha$ . Here  $\alpha$  is a  $r$ -dimensional vector  $\alpha = (\alpha_1, \dots, \alpha_r)$  and  $r$  is called the rank of the algebra. The  $\alpha_I$ ,  $I = 1, \dots, r$ , are the eigenvalues of  $H_I$  in the adjoint representation, i.e.  $[H_I, E_\alpha] = \alpha_I E_\alpha$ . The  $\alpha_I$  is called a *root* which form the root vector  $\alpha$  and  $E_\alpha$  is the adjoint representation of  $e_\alpha$ .

Let  $\alpha_1, \dots, \alpha_r$  be a fixed basis of roots so that any other root  $\rho$  can be written as  $\rho = \sum_{i=1}^r c_i \alpha_i$  with  $c_i$  some coefficient. We call  $\rho$  a *positive* root if the first non-zero  $c_i > 0$ , else it is called a negative root. A *simple* root is a positive root which cannot be written as the sum of two positive roots. The number of simple roots equals the rank of the Lie algebra.

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For a general representation we indicate the basis of the CSA  $\{h_I\}$ ,  $I = 1, \dots, r$ , by matrices  $H_I$  and the step operators  $e_\alpha$  by  $E_\alpha$ . The  $H_I$  and  $E_\alpha$  act on vectors  $\phi^a$  in some space  $V$ . Since the  $H_I$  commute we take them diagonal

$$H_I \phi^a = \lambda_I^a \phi^a. \quad (\text{C.0.5})$$

The eigenvalue  $\lambda_I^a$  is called a *weight* which form the weight vector  $\lambda^a$ . We see that in case of the adjoint representation the weights are the roots.

The canonical commutation relations can be summarized by

$$[H_I, H_J] = 0, \quad [H_I, E_\alpha] = \alpha_I E_\alpha, \quad [E_\alpha, E_\beta] = N(\alpha, \beta) E_{\alpha+\beta}. \quad (\text{C.0.6})$$

The last line is to be understood as follows. If  $\alpha + \beta$  is not a root we have  $N(\alpha, \beta) = 0$ , else we have  $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$ .





# Appendix D

## Publications

- [A] E. A. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren and T. Van Riet, *Non-Extremal D-instantons and the AdS/CFT Correspondence*, JHEP **0601**, 061 (2006) [arXiv:hep-th/0510048].
- [B] J. Hartong, A. Ploegh, T. Van Riet and D. B. Westra, *Dynamics of Generalized Assisted Inflation*, Class. Quant. Grav. **23**, 4593 (2006) [arXiv:gr-qc/0602077].
- [C] W. Chemissany, A. Ploegh and T. Van Riet, *Scaling Cosmologies, Geodesic Motion and Pseudo-Susy*, [arXiv:0704.1653 [hep-th]].
- [D] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel and D. Van den Bleeken, *Pseudo-Supersymmetry and a Tale of Alternate Realities*, JHEP **0707**, 067 (2007) [arXiv:0704.3559 [hep-th]].
- [E] E. A. Bergshoeff, J. Hartong, A. Ploegh and D. Sorokin, *Q-instantons*, [arXiv:0801.4956 [hep-th]].
- [F] E. A. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet *Generating Geodesic Flows and Supergravity Solutions*, to appear.