Particle dynamics of branes

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Chapter 7

Instantons

So far we have considered Riemannian scalar cosets. In the previous chapter we showed that the domain-wall / cosmology correspondence can be extended to include instantons as well. It is therefore not surprising that the analysis we did for Riemannian scalar cosets can be extended to non-Riemannian scalar cosets. Actually, we know already from section 3.5 that if there is no potential present the instantons also describe geodesic motion on the scalar manifold, with the complication that the affine velocity $||v||^2$ is no longer strictly positive due to the fact that we now have to deal with a non-Riemannian scalar manifold. This leads to different classes of instantons, labeled by the sign of $||v||^2$.

The prototype Lorentzian scalar coset is $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ of Euclidean type IIB supergravity. The extremal solution belonging to this coset is the D(-1)-instanton [133, 134]. In our language this is a lightlike geodesic. The extension to non-extremal D-instantons was considered in [123, 135]. These are related to space- and timelike geodesics. In this chapter we are going to consider two extensions of the scalar coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$.

First we begin with considering more general non-Riemannian cosets. In section 3.2.1 we noticed that reducing pure gravity over a Lorentzian torus gives the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$. In the next section we will derive the generating solution for this coset and extend it to $\mathrm{GL}(p+q,\mathbb{R})/\mathrm{SO}(p,q)$.

The second extension will be the inclusion of a potential. This extends the results of section 5.2 to non-Riemannian cosets. After that, we restrict ourselves to a potential given by a cosmological constant Λ. We will see that this potential is special in that it never upsets the geodesic motion of the scalar fields.

The work in this chapter is done together with E. A. Bergshoeff, W. Chemissany, A. Collinucci, T. Van Riet, M. Trigiante and S. Vandoren [40, 69].

7.1 Instanton Geometries

From section 3.5 we know that the geometry of the (-1)-brane or instanton entirely depends on the character of the geodesic curve (spacelike, lightlike or timelike), independently of the scalar manifold coset. The metric is given by (3.5.3) where g can be found by solving (3.5.10) with $\epsilon = +1$

$$\dot{g}^2 = \frac{||v||^2}{2(D-2)(D-1)} f^2 g^{4-2D} + k f^2 \,. \tag{7.1.1}$$

Some of these solutions have appeared in the literature before [40, 56, 123, 135, 136].

- $||v||^2 > 0$

  For this class of instantons we will be using the gauge $f = g$. In the table below we present the conformal factor f that determines the metric and the radial harmonic function h. Note that for all three values of k the solutions have singularities.

$k=-1$	$f(r) = (\frac{\vert\vert v\vert\vert^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \cos^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{8(D-1)}{(D-2)\vert\vert v\vert\vert^2}} \,\mathrm{arctanh}[\tan(\frac{D-2}{2} r)] + b$
$k=0$	$f(r) = \left(\sqrt{\frac{(D-2)\vert\vert v\vert\vert^2}{2(D-1)}}\, r\right)^{\frac{1}{D-2}}$ $h(r) = \sqrt{\frac{2(D-1)}{(D-2)\vert\vert v\vert\vert^2}} \log r + b$
$k=+1$	$f(r) = (\frac{\vert\vert v\vert\vert^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \sinh^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{2(D-1)}{\vert\vert v\vert\vert^2(D-2)}} \log[\tanh(\frac{D-2}{2} r)] + b$

Table 7.1.1: *The Euclidean geometries with* $||v^2|| > 0$ *in the gauge* $f = g$. *The real number* b *is an integration constant.*

- $||v||^2 = 0$

  We will be using the Euclidean "FLRW gauge" for which $f = 1$. It is clear from (3.5.10) that for $k = -1$ we do not find a solution and that for $k = 0$ we find flat space in Cartesian coordinates ($g = 1$) and for $k = +1$ we find flat space in spherical coordinates ($g = r$). This makes sense since a lightlike geodesic motion comes with zero "energy-momentum"[1]. The harmonic function is

$$\begin{aligned} k = 0: \qquad & h(r) = a\,r + b\,, \\ k = 1: \qquad & h(r) = \frac{a}{r^{D-2}} + b\,. \end{aligned} \tag{7.1.2}$$

In Euclidean IIB supergravity the axion-dilaton parameterize $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$ and for $||v||^2 = 0$ and $k = 1$ we have the standard half-supersymmetric D-instanton [133, 134].

- $||v||^2 < 0$

  We will present the solutions in the conformal gauge $g = fr$, which has the advantage that the coordinates cover the whole space. For $k = 0$ and $k = -1$ we clearly have no solutions since the right-hand side of (3.5.10) is always negative. For $k = +1$ a solution does exist, and in the conformal gauge it is given by

$$f(r) = \Big(1 - \tfrac{||v||^2}{8(D-1)(D-2)} r^{-2(D-2)}\Big)^{\frac{1}{D-2}}\,, \tag{7.1.3}$$

  where indeed only $||v||^2 < 0$ is valid. This geometry is smooth everywhere and describes a wormhole since there is an inversion-symmetry which interchanges the two asymptotically flat regions, see figure 7.1.1. This symmetry acts as follows [123]

$$r^{D-2} \rightarrow \tfrac{-||v||^2}{8(D-1)(D-2)} r^{-(D-2)}\,. \tag{7.1.4}$$

  The neck of the wormhole is located at the self-dual radius defined by $r_s^{2(D-2)} = \frac{-||v||^2}{8(D-1)(D-2)}$. The two asymptotic regions are connected via a neck with a minimal physical radius at the self-dual radius r_s. This physical radius can be calculated in the FLRW-gauge

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + a(\rho)^2 \mathrm{d}\Omega_{D-1}^2\,. \tag{7.1.5}$$

  From this we find that the physical radius is given by $a(\rho_s)^{D-2} = r^{D-2} f(r_s)$.

  The harmonic function is given by

$$h(r) = \sqrt{-\tfrac{8(D-1)}{(D-2)||v||^2}}\,\arctan\Big(\sqrt{\tfrac{-||v||^2}{8(D-1)(D-2)}}\; r^{-(D-2)}\Big) + b\,. \tag{7.1.6}$$

[1] The fact that the $k = -1$ solution does not exists reflects that there does not exist a hyperbolic slicing of the Euclidean plane.

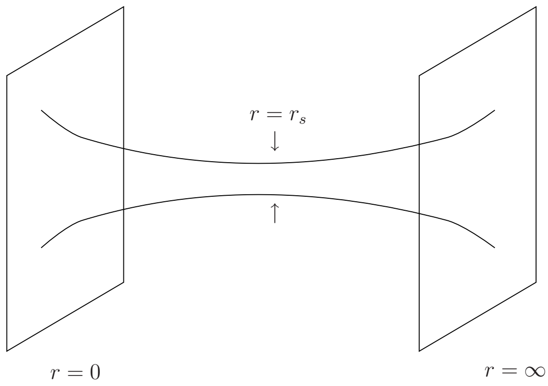

Figure 7.1.1: For the class $||v||^2 < 0$ the space is a wormhole with the neck of the wormhole at the self-dual radius r_s. The two asymptotically flat regions at $r = 0$ and $r = \infty$ are connected via a neck with a minimal physical radius at the self-dual radius r_s.

7.2 Solutions of Kaluza–Klein Theory

We know that the scalar fields are geodesics, but these are not described by the Cartan subalgebra only since $||v||^2$ can have any sign. Let us therefore focus on the geodesic motion that comes about. The approach that we will follow allows us to re-derive the geodesics of $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{SO}(n)$ coset but also allows for a generalization to $\mathrm{GL}(p + q = n, \mathbb{R})/\,\mathrm{SO}(p, q)$.

7.2.1 The Geodesic Curves

In the following we will not make use of a coordinate system on the coset, instead we will work directly on the level of $\mathcal{M}$, see section 3.4. So that we do not need to be bothered with subtleties regarding the Borel gauge.

The action for the geodesic curves can be compactly written in terms of the symmetric coset matrix $\hat{\mathcal{M}}$

$$S = \int \mathrm{Tr}[\partial\hat{\mathcal{M}}\partial\hat{\mathcal{M}}^{-1}]\,, \qquad \hat{\mathcal{M}} = \hat{L}\eta\hat{L}^T\,, \quad \eta = (-\mathbb{1}_p, \mathbb{1}_q)\,. \tag{7.2.1}$$

Here we have included the breathing mode φ in $\mathcal{M}$, see (3.2.12), to make $\hat{\mathcal{M}}$. Here $\hat{L}$ is a representative of $\mathrm{GL}(p + q = n, \mathbb{R})/\,\mathrm{SO}(p, q)$. The relation between $\hat{\mathcal{M}}$ and the moduli φ and $\mathcal{M}$ is as follows

$$\hat{\mathcal{M}} = (|\mathrm{det}\hat{\mathcal{M}}|)^{\frac{1}{n}}\mathcal{M}\,, \qquad |\mathrm{det}\hat{\mathcal{M}}| = \exp\sqrt{2n}\varphi\,. \tag{7.2.2}$$

The equations of motion can compactly be written as

$$[\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}']' = 0\,, \tag{7.2.3}$$

where a prime is a derivative with respect to an affine parameter. This implies that $\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}' = K$ with K a constant matrix, which can be seen as the matrix of Noether charges. The affine velocity squared of the geodesic curve is $||v||^2 = \frac{1}{2}\mathrm{Tr}[K^2]$. Since $\hat{\mathcal{M}}^{-1}\hat{\mathcal{M}}' = K$ the problem is integrable and a general solution is given by

$$\hat{\mathcal{M}}(h) = \hat{\mathcal{M}}(0)\mathrm{e}^{Kh}\,. \tag{7.2.4}$$

The isometry group $\mathrm{GL}(n,\mathrm{I\!R})$ has a transitive action on the coset space which implies that we can restrict ourselves to geodesics that go trough the origin. Since we have the freedom of affine re-parameterization of h we can assume that $\hat{\mathcal{M}}(0) = \eta$. The matrix of Noether charges K is not completely arbitrary and the only constraint it fulfills can be derived from the properties of $\hat{\mathcal{M}}$

$$\eta K = K^T\eta\,, \tag{7.2.5}$$

where the signature of η depends on the isotropy group $\mathrm{SO}(p,q)$ we are considering, see (7.2.1).

K is an element of the Lie algebra of $\mathrm{GL}(n,\mathrm{I\!R})$ and accordingly it transforms in the adjoint of $\mathrm{GL}(n,\mathrm{I\!R})$

$$K \rightarrow \Omega\, K\, \Omega^{-1}\,, \tag{7.2.6}$$

under which the n Casimirs $\mathrm{Tr}K$, $\mathrm{Tr}K^2$,..., $\mathrm{Tr}K^n$ are invariant. The constraints given in (7.2.5) are not invariant under the total isometry group $\mathrm{GL}(n,\mathrm{I\!R})$ but only under the smaller isotropy group $\mathrm{SO}(p,q)$.

In the following sections we derive the generating geodesics for the three possible cases $\mathrm{GL}(n,\mathrm{I\!R})/\,\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathrm{I\!R})/\,\mathrm{SO}(n-1,1)$ and $\mathrm{GL}(n+1,\mathrm{I\!R})/\,\mathrm{SO}(n-1,2)$, although it can easily be extended to $\mathrm{GL}(p+q,\mathrm{I\!R})/\,\mathrm{SO}(p,q)$. As explained in subsection 4.3.1, for pure Kaluza–Klein theory in $D>3$ all geodesics that are related through a $\mathrm{SL}(n)$-transformation lift up to exactly the same vacuum solution in $D+n$ dimensions since the $\mathrm{SL}(n)$ corresponds to rigid coordinate transformations from a $(D+n)$-dimensional point of view. Here the $\mathrm{SL}(n,\mathrm{I\!R})$ follows from $\mathrm{GL}(n,\mathrm{I\!R}) = \mathbb{R}\times\mathrm{SL}(n,\mathrm{I\!R})$ with $\mathbb{R}$ related to the breathing mode. So, in this sense it is absolutely necessary to understand the generating geodesic since it classifies higher-dimensional solutions modulo coordinate transformations. Of course, this is not true for $D=3$ where $\mathrm{SL}(n+1)$ maps higher-dimensional solutions to each other that are not necessarily related by coordinate transformations.

7.2.2 Normal Form of $\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$

Consider $K\in\mathfrak{gl}(p+q)/\mathfrak{so}(p,q)$. By definition K obeys (7.2.5)

$$\eta K = K^T\eta\,,\quad \text{with}\quad \eta = (-\mathbb{1}_p, +\mathbb{1}_q)\,. \tag{7.2.7}$$

Two eigenvectors of K, $\mathbf{v_1}$ and $\mathbf{v_2}$, that belong to different eigenvalues λ_1 and λ_2 are necessarily orthogonal with respect to the inner product (,) defined with the bilinear form η, because $(\mathbf{v_2}, K\mathbf{v_1}) = (K\mathbf{v_2}, \mathbf{v_1})$ and thus $\lambda_1(\mathbf{v_1}, \mathbf{v_2}) = \lambda_2(\mathbf{v_1}, \mathbf{v_2})$. Now if $\lambda_1 \neq \lambda_2$ then this is only consistent if $(\mathbf{v_1}, \mathbf{v_2}) = 0$. If two eigenvectors have the same eigenvalue we can always perform a pseudo Gramm–Schmidt procedure such that they become orthogonal with respect to η, which we refer to as pseudo-orthogonal.

Assume we have a complex eigenvalue $\lambda \neq \bar{\lambda}$ with corresponding eigenvectors $\mathbf{v}$ and $\bar{\mathbf{v}}$. If we write $\mathbf{v} = \mathbf{v_1} + \mathbf{i}\mathbf{v_2}$ and $\lambda = \lambda_1 + i\lambda_2$ then this means that

$$K\mathbf{v_1} = \lambda_1\mathbf{v_1} - \lambda_2\mathbf{v_2}\,, \qquad K\mathbf{v_2} = \lambda_2\mathbf{v_1} + \lambda_1\mathbf{v_2}\,, \tag{7.2.8}$$

(pseudo)-orthogonality between $\mathbf{v}$ and $\bar{\mathbf{v}}$ implies $(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2})$.

In what follows we will construct a normal form for K using the eigenvectors as a basis.

$\mathbf{min}(p, q) = 0$

In this case we see from (7.2.7) that K is a symmetric matrix. With the help of SO(n) we can diagonalize K to a real matrix, written in terms of its orthogonal basis of eigenvectors as

$$K_N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}. \tag{7.2.9}$$

Note that this result is compatible with the result given in (4.2.2). The fact that K_N is diagonal reflects that the generating solution can be rotated to the Cartan subalgebra only.

If we instead consider SO(p, q) we will have in general complex eigenvalues (and its conjugates). Since $\hat{\mathcal{M}}$, see eq. (7.2.4), contains the scalar fields it should always be real. One complex eigenvalue and its conjugate can always be written as a 2×2 real block. So for each complex eigenvalue the matrix K_N will have a 2×2 block. Therefore knowing the maximal number of complex eigenvalues leads us to the normal form. In the following we will derive the maximal number of complex eigenvalues one can have for the coset $\mathrm{GL}(p+q, \mathbb{R})/\,\mathrm{SO}(p, q)$. We will show that this number is given by $\min(p, q)$.

$\mathbf{min}(p, q) = 1$

Assume there are at least two complex eigenvalues λ, σ, that correspond to respectively $\mathbf{v} = \mathbf{v_1} + \mathbf{i}\mathbf{v_2}$ and $\mathbf{w} = \mathbf{w_1} + \mathbf{i}\mathbf{w_2}$, and that they are not related through conjugation ($\lambda \neq \bar{\sigma}$). Then the four vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}$ obey (from pseudo-orthogonality constraints between the vectors themselves and the vectors with the conjugated vectors)

$$(\mathbf{v_1},\mathbf{v_1}) = -(\mathbf{v_2},\mathbf{v_2})\,, \qquad (\mathbf{w_1},\mathbf{w_1}) = -(\mathbf{w_2},\mathbf{w_2})\,, \qquad (\mathbf{w_i},\mathbf{v_j}) = 0\,, \quad \forall\, i,j = 1,2. \tag{7.2.10}$$

From here on we write a vector $\mathbf{v}$ as $\mathbf{v} = (v, \vec{V})$. Here v is the time component and $\vec{V}$ are the spatial components of $\mathbf{v}$. Now assume $\mathbf{v}_i$ as $\mathbf{w}_j$ are all lightlike. Lightlike vectors then obey $v^2 = \vec{V}\cdot\vec{V}$. Pseudo-orthogonality between the lightlike vectors $\mathbf{v_1}$ and $\mathbf{w_1}$ implies $v_1 w_1 = \vec{V_1}\cdot\vec{W_1}$. Therefore $(\vec{V_1}\cdot\vec{W_1})^2 = v_1^2 w_1^2 = (\vec{V_1}\cdot\vec{V_1})(\vec{W_1}\cdot\vec{W_1})$, which, according to Cauchy–Schwarz, is only possible when $\vec{V_1}$ and $\vec{W_1}$ are parallel. But in that case we find $\mathbf{v_1} \sim \mathbf{w_1}$. Similarly we find that $\mathbf{v_1} \sim \mathbf{w_2}$ and $\mathbf{v_2} \sim \mathbf{w_2}$ which implies $\mathbf{v_1} \sim \mathbf{v_2}$ which is impossible for complex eigenvectors. Therefore our assumption was wrong.

Assume that one couple of vectors is lightlike, say $\mathbf{v_1},\mathbf{v_2}$. Then $\mathbf{w_1}$ is spacelike and $\mathbf{w_2}$ timelike (or vice versa). We can always find a frame in which $\mathbf{w_2}$ is given by $(w_2, \vec{0})$. Now it is straightforward that there cannot exist a lightlike vector (like $\mathbf{v_1}$) pseudo-orthogonal to $\mathbf{w_2}$. Therefore our assumption was wrong.

Assume that no couple is lightlike. Take $\mathbf{v_2}$ and $\mathbf{w_2}$ timelike. In a frame in which $\mathbf{w_2}$ is given by $(w_2, \vec{0})$ it is clear that $\mathbf{v_2}$ cannot exist as there does not exist a timelike vector pseudo-orthogonal to $\mathbf{w_2}$.

Therefore, having at least two different complex eigenvalues, not related through complex conjugation is a false assumption and we conclude that there can be at maximum one complex eigenvalue (and its conjugated one).

From this we can find the normal form K_N. For that we write the normal form in terms of a pseudo-orthogonal basis made from the (real and imaginary) parts of the eigenvectors.

Assume $\mathbf{v_1}$ is timelike and normalized, $(\mathbf{v_1},\mathbf{v_1}) = -1$. Because $\mathbf{v_1}$ is timelike all vectors orthogonal to it are spacelike. In the following we assume that the $\mathbf{v_i}$ are normalized $(\mathbf{v_i},\mathbf{v_i}) = +1$ for $i > 1$. We define the orthonormal basis $(\mathbf{u_i}, i = 1,\dots n)$ where the unit vectors $\mathbf{u_i}$, $i > 2$ are orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$ via the Gramm-Schmidt procedure

$$\mathbf{u_1} = \mathbf{v_1}\,, \tag{7.2.11}$$

$$\mathbf{u_2} = \sin\alpha\,\mathbf{v_1} + \cos\alpha\,\mathbf{v_2}\,, \tag{7.2.12}$$

where $\tan\alpha = (\mathbf{v_1},\mathbf{v_2})$. Using (7.2.8) we find that

$$K\mathbf{u_1} = (\lambda_1 + \lambda_2\tan\alpha)\mathbf{u_1} - \lambda_2\cos^{-1}\alpha\,\mathbf{u_2}\,,\ K\mathbf{u_2} = \lambda_2\cos^{-1}\alpha\,\mathbf{u_1} + (-\lambda_2\tan\alpha + \lambda_1)\mathbf{u_2}\,. \tag{7.2.13}$$

From this we easily read off the components of K_N in the new basis

$$K_N = \begin{pmatrix} \lambda_1 + \lambda_2 \tan\alpha & -\lambda_2 \cos^{-1}\alpha & 0 & \dots & 0 \\ \lambda_2 \cos^{-1}\alpha & -\lambda_2 \tan\alpha + \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & \dots & \dots & \lambda_n \end{pmatrix} . \tag{7.2.14}$$

If $\mathbf{v_1}$ is spacelike, then the above normal form is still valid if we interchange $\mathbf{v_1}$ with $\mathbf{v_2}$ in the definition of the orthonormal basis.

Assume on the other hand that $\mathbf{v_1}$ and $\mathbf{v_2}$ are lightlike. Like before it is easy to understand that all other eigenvectors $\mathbf{v_{i>2}}$ must be spacelike $(\mathbf{v_{i>2}}, \mathbf{v_{i>2}}) = +1$. We define an orthonormal basis $(\mathbf{u_i}, i = 1, \dots n)$, again the unit vectors $\mathbf{u_i}$, $i > 2$ are orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$ via the Gramm-Schmidt procedure

$$\mathbf{u_1} = \frac{\mathbf{v_1} \pm \mathbf{v_2}}{\sqrt{2|(\mathbf{v_1}, \mathbf{v_2})|}} , \tag{7.2.15}$$

$$\mathbf{u_2} = \frac{\mathbf{v_1} \mp \mathbf{v_2}}{\sqrt{2|(\mathbf{v_1}, \mathbf{v_2})|}} . \tag{7.2.16}$$

Here the upper sign must be chosen when $(\mathbf{v_1}, \mathbf{v_2}) < 0$ and vice versa. The normal form is given by

$$K_N = \begin{pmatrix} \lambda_1 & \pm\lambda_2 & 0 & \dots & 0 \\ \mp\lambda_2 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & \dots & \dots & \lambda_n \end{pmatrix} . \tag{7.2.17}$$

Note that the 2×2 block in both (7.2.14) and (7.2.17) correspond to at most one complex eigenvalue (and its conjugated one).

$\mathbf{min}(p, q) = 2$

Now, assume there exist at least three complex eigenvalues (which is possible if $p+q \geq 6$). This implies the existence of six vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}, \mathbf{x_1}, \mathbf{x_2}$ that obey

$$(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2}) , \quad (\mathbf{w_1}, \mathbf{w_1}) = -(\mathbf{w_2}, \mathbf{w_2}) , \quad (\mathbf{x_1}, \mathbf{x_1}) = -(\mathbf{x_2}, \mathbf{x_2}) , \tag{7.2.18}$$

$$(\mathbf{v_i}, \mathbf{w_j}) = (\mathbf{v_i}, \mathbf{x_j}) = (\mathbf{w_i}, \mathbf{x_j}) = \mathbf{0} , \qquad \forall \quad \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2} . \tag{7.2.19}$$

Let us first assume all vectors are lightlike. There always exists a frame for which $\mathbf{v_1} = (v_1, 0, \vec{V_1})$, $\mathbf{v_2} = (0, v_2, \vec{V_2})$. Then write $\mathbf{w_1} = (a, b, \vec{W_1})$. We have that $a^2 + b^2 = \vec{W_1} \cdot \vec{W_1}$. But orthogonality implies $a = ||\vec{V_1}||^{-1} \vec{V_1} \cdot \vec{W_1}$ and $b = ||\vec{V_2}||^{-1} \vec{V_2} \cdot \vec{W_1}$. Also

$\vec{W}_1$ is fixed and it equals $a\vec{V}_1 + b\vec{V}_2$. Therefore all other lightlike vectors are the same and thus parallel, which is not what we want.

So, at maximum two couples can be lightlike, say $(\mathbf{v_1}, \mathbf{v_2})$ and $(\mathbf{w_1}, \mathbf{w_2})$. Then say $\mathbf{x_1}$ is timelike and thus $\mathbf{x_2}$ spacelike. We can find a frame for which $\mathbf{x_1} = (x, 0, \vec{0})$. So all lightlike vectors perpendicular to these must have the form $\mathbf{L_i} = (0, l_i, \vec{L}_i)$. And if we want the lightlike vectors to be mutually perpendicular (with respect to η) we find again that they are all parallel which gives rise to contradictions.

Now, maximally one couple of the vectors can be lightlike, say the couple $(\mathbf{v_1}, \mathbf{v_2})$ and the rest not lightlike. There must exist two timelike vectors, say $\mathbf{w_1}$ and $\mathbf{x_1}$. As before we can always fix a frame in which $\mathbf{w_1} = (0, w, \vec{0})$ and $\mathbf{x_1} = (x, 0, \vec{0})$. But clearly we cannot find a lightlike vector orthogonal to them.

Finally assume that none of them are lightlike. Then we have three spacelike and three lightlike vectors. This however is impossible because they must be mutually orthogonal with respect to η. To show this assume $\mathbf{v_1}, \mathbf{w_1}, \mathbf{x_1}$ are timelike. After a boost there always exists a frame in which $\mathbf{v_1} = (v, 0, \vec{0})$. There is still a $\mathrm{SO}(1, n)$-boost $\in \mathrm{SO}(2, n)$ to bring $\mathbf{w_1}$ to the form $\mathbf{w_1} = (0, w, \vec{0})$. Clearly there does not exist a $\mathbf{x_1} = (x, y, \vec{X}_1)$ since the orthogonality condition implies $x = y = 0$ and thus $\mathbf{x_1}$ is not timelike, contrary to the assumption.

Therefore, having at least three different complex eigenvalues is a false assumption and we conclude that there can be at most two complex eigenvalues (and the conjugated ones). Similar to eq. (7.2.17), we now have in general two 2×2 blocks in K_N and $n - 4$ number of diagonal elements.

$\mathbf{min}(p, q) > 2$

In case $(p, q) > 2$ a similar analysis applies. Now we need at least $\min(p, q) + 1$ complex eigenvalues to find a contradiction. We can therefore have at most $\min(p, q)$ complex eigenvalues. The normal form of K_N will in general have $\min(p, q)$ number of 2×2 blocks and $n - 2\min(p, q)$ number of diagonal elements.

7.2.3 Uplift to Vacuum Solutions

In order to uplift the solutions from $D > 3$ dimensions to $D + n$ dimensions one uses the Kaluza–Klein Ansatz (4.3.2) with Kaluza–Klein vectors put to zero

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}\mathrm{d}z^m\mathrm{d}z^n \,. \tag{7.2.20}$$

Consider the symmetric coset matrix $\hat{\mathcal{M}}(h) = \eta \exp K_N h$ with K_N the normal form of K that generates all other geodesics and h the harmonic function given in section 7.1. The relation between $\hat{\mathcal{M}}$ and the moduli φ and $\mathcal{M}$ is given in (7.2.2).

Time-dependent solutions from $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n)$

Although the method we use here is different from that we used in chapter 4, the vacuum solutions are of course the same. We refer to [69] to see the vacuum solutions in this approach explicitly.

Stationary solutions from $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n-1,1)$

For K_N we use the normal form $K_N =$

$$\begin{pmatrix} \lambda_1 & \omega & 0 & \dots & 0 \\ -\omega & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_a & \omega & 0 & \dots & 0 \\ -\omega & -\lambda_a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_b & 0 & 0 & \dots & 0 \\ 0 & \lambda_b & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} . \tag{7.2.21}$$

We exponentiate this to

$$\hat{\mathcal{M}}(h(r)) = \eta\, \mathrm{e}^{K_N h(r)} = \begin{pmatrix} a(r) & b(r) & 0 & \dots & 0 \\ b(r) & c(r) & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 h} & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{e}^{\lambda_n h} \end{pmatrix} , \tag{7.2.22}$$

with

$$\begin{aligned} a(r) &= -\mathrm{e}^{\lambda_b h(r)}\Big(\cosh(\Lambda h(r)) + \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda}\Big) , \\ b(r) &= -\omega \mathrm{e}^{\lambda_b h(r)} \frac{\sinh(\Lambda h(r))}{\Lambda} , \\ c(r) &= \mathrm{e}^{\lambda_b h(r)}\Big(\cosh(\Lambda h(r)) - \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda}\Big) , \end{aligned} \tag{7.2.23}$$

and we define the $\mathrm{SO}(1,1)$ invariant quantity Λ as

$$\Lambda = \sqrt{\lambda_a^2 - \omega^2}\, . \tag{7.2.24}$$

There exist three distinctive cases depending on the character of Λ. If Λ is real the above expression does not need rewriting but we can put ω to zero using a $\mathrm{SO}(1,1)$-boost and then the generating solution is just the straight line solution. If $\Lambda = i\tilde{\Lambda}$ with $\tilde{\Lambda}$ real then the terms with $\cosh(\Lambda h)$ become $\cos\tilde{\Lambda}$ and $\Lambda^{-1}\sinh\Lambda h$ become $\tilde{\Lambda}^{-1}\sin\tilde{\Lambda}h$. Finally, if $\Lambda = 0$ then the term $\Lambda^{-1}\sinh\Lambda h$ becomes just h and the term with $\cosh\Lambda h$ becomes equal to one.

To discuss the zoo of solutions one should make a classification in terms of the different signs for k, $||v||^2$ and Λ^2. The solutions with spherical symmetry have the more interesting properties that they lift up to vacuum solutions that can be asymptotically flat. Moreover, the brane solutions in supergravity always have $k = +1$. Let us briefly discuss them.

- $||v||^2 > 0$: There are three metric solutions, depending on the sign of Λ^2. Only in case $\Lambda^2 > 0$ we can diagonalize it to a straight line via a SO(1,1) boost. In the other two cases there will be a cross-term.
- $||v||^2 = 0$: There are only two solutions, namely $\Lambda^2 = 0$ and $\Lambda^2 < 0$. Both will have a cross-term.
- $||v||^2 < 0$: This implies that $\Lambda^2 < 0$ and hence there will be a cross term.

As an example consider $||v||^2 < 0\,, \Lambda^2 < 0$. The metric solution is given by

$$\begin{aligned} \mathrm{d}s^2 =& \mathrm{e}^{p_0 h(r)} f(r)^2 \Big(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{D-1}^2\Big) \\ &+ \mathrm{e}^{p_1 h(r)} \Big(-\tilde{a}(r)\mathrm{d}t^2 + 2\tilde{b}(r)\mathrm{d}t\mathrm{d}x + \tilde{c}(r)\mathrm{d}x^2\Big) + \mathrm{e}^{2p_{i-1}}\mathrm{d}z^i\mathrm{d}z_i\,, \end{aligned} \tag{7.2.25}$$

where $f(r)$ can be found in (7.1.3) and $h(r)$ is defined in equation (7.1.6). The numbers p_i are

$$p_0 = (2\lambda_b + \sum_{i=3}^{n} \lambda_i)\sqrt{\tfrac{1}{(D+n-2)(D-2)}}\,, \tag{7.2.26}$$

$$p_1 = -\frac{(D-2)p_0 - \lambda_b(n-2) + \sum_{i=3}^n \lambda_i}{n}\,, \tag{7.2.27}$$

$$p_{i-1} = -\frac{(D-2)p_0 + 2\lambda_b + \sum_{j=3}^n \lambda_j - \lambda_i}{n}\,, \tag{7.2.28}$$

and the functions $\tilde{a}(r), \tilde{b}(r), \tilde{c}(r)$ are given by

$$\tilde{a}(r) = \cos(|\Lambda| h(r)) + \lambda_a \frac{\sin(|\Lambda| h(r))}{|\Lambda|}\,, \tag{7.2.29}$$

$$\tilde{b}(r) = \sqrt{\lambda_a^2 - \Lambda^2}\frac{\sin(|\Lambda h(r)|)}{|\Lambda|}\,, \tag{7.2.30}$$

$$\tilde{c}(r) = \cos(|\Lambda| h(r)) - \lambda_a \frac{\sin(|\lambda| h(r))}{|\Lambda|}\,. \tag{7.2.31}$$

In [69] all the different $k = +1$ metrics are given.

7.3 Massive Instantons

Just as we did for time-dependent solutions, we now investigate the effect of adding a potential to our non-Riemannian scalar coset. The analysis we did in section 5.2 can be extended to the instanton case as well, we only consider the case $k = 0$.

Consider the following action

$$S = \int \mathrm{d}^D x \sqrt{\epsilon g} \Big\{ \mathcal{R} - \tfrac{1}{2} G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) \Big\} . \tag{7.3.1}$$

For $\epsilon = 1$ we have a Euclidean theory, while for $\epsilon = -1$ we have a Lorentzian theory. The metric Ansatz is

$$\mathrm{d}s_D^2 = -\eta\epsilon f(r)^2 \mathrm{d}r^2 + g(r)^2(-\eta\, \mathrm{d}\rho^2 + \mathrm{d}\vec{x}_{D-2}^2) \,, \tag{7.3.2}$$

and we extend (5.2.5) to

$$V = -\eta\epsilon \left\{ \tfrac{1}{2} G^{ij} \partial_i W \partial_j W - \tfrac{D-1}{4(D-2)} W^2 \right\} . \tag{7.3.3}$$

If $\epsilon = -1$ we rediscover the analysis of section 5.2. That is for $\eta = 1$ we have a domain-wall and for $\eta = -1$ a cosmology. When $\epsilon = 1$ and $\eta = -1$ we have a Euclidean metric. We see that both the domain-wall and instanton have $\epsilon\eta = -1$, just as we found in section 6.1. For instantons the same first order equations (5.2.7) hold, since we derive the same effective action as (5.2.6). The only difference is that the overall factor η has been replaced with $-\eta\epsilon$. For the Euclidean case the scalar metric G_{ij} is in general non-compact as we saw in the previous section.

There is therefore one complicating issue. The above analysis shows that for a given domain-wall we find a Euclidean solution belonging to the *same* scalar manifold metric G_{ij}. We know however from the previous section that in general a Euclidean theory contains ghost fields, while domain-walls are considered to belong to theories without ghosts. In case we have only a dilaton this problem does not occur, see for example the instanton of massive IIA* we discussed in subsection 6.5.1. However, let us consider the multi-scalar case (5.3.14) with $\alpha = 0$ for simplicity. Below we show that the Euclidean version of this action requires the axion χ to become a ghost. So we have a different scalar coset. That means that for a given domain-wall we do not have an instanton belonging to the Euclidean version of (5.3.14), instead the correspondence gives us a Euclidean solution of the same scalar coset without a ghost.

Note that we found a similar thing in section 6.5.2 when we discussed the 9d $\mathcal{N} = 2$ domain-wall / cosmology example. When we include the ghost axions the supergravity embedding of the correspondence does not simply require V replaced by $-V$.

Cosmological constant

We now restrict to the case that the potential is a cosmological constant

$$S = \int \mathrm{d}^D x \sqrt{g}\Big(\mathcal{R} + \frac{1}{4}\mathrm{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) - \Lambda\Big)\,. \tag{7.3.4}$$

In chapter 5 we have seen that the solution of a theory with a potential is only under certain conditions still a geodesic of the scalar manifold, namely when the cosmology is pseudo-BPS. The potential that we consider here is a negative cosmological constant. There is no coupling of the scalar fields in $\mathcal{M}$ to Λ. The equations of motion for $\mathcal{M}$ are still given by

$$\partial_\mu \Big(\sqrt{g}\mathcal{M}^{-1}\partial^\mu \mathcal{M}\Big) = 0\,. \tag{7.3.5}$$

That means that if we introduce the harmonic function h as the new parameter they can still be geodesics. Indeed if we define the Noether charge as $\mathcal{M}^{-1}\partial_h \mathcal{M} = K$ and the affine velocity as $||v||^2 = 1/2\mathrm{Tr}(K^2)$, the metric can be solved again independently. The presence of the cosmological constant will only change the shape of the metric. This is no longer true if we allow for a direct coupling of the scalars to Λ.

Let us now turn to the solutions. For all the cosets we considered in the previous sections $\mathcal{M}$ is unchanged, since we still have geodesics on the scalar manifold. The metric can be solved independently of $\mathcal{M}$, a similar analysis as for the case $\Lambda = 0$ gives

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{\frac{||v||^2}{2(D-2)(D-1)} r^{-2(D-2)} - \frac{\Lambda}{(D-2)(D-1)} r^2 + k} + r^2 \Big(\frac{\mathrm{d}\rho^2}{1-k\rho^2} + \rho^2 \mathrm{d}\Omega^2_{D-2}\Big)\,. \tag{7.3.6}$$

As a concrete example consider the Euclidean version of the action (5.3.14) with $\alpha = 0$ and $\mu = 2$

$$S = \int \mathrm{d}^D x \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2 - \Lambda\Big\}\,. \tag{7.3.7}$$

For a proper derivation as to why we need to replace in the Euclidean theory $(\partial\chi)^2$ by $-(\partial\chi)^2$ we refer to e.g. [137] and references therein. The scalar field solutions are given in table 7.3.1 [2].

The harmonic function h satisfies the differential equation

$$\partial_r h(r) = \frac{1}{\sqrt{\Big(\frac{||v||^2}{2(D-2)(D-1)} r^{-2(D-2)} - \frac{\Lambda}{(D-2)(D-1)} r^2 + k\Big) r^{2(D-1)}}}\,. \tag{7.3.8}$$

[2] The $||v||^2 > 0$ geodesic is related to $||v||^2 < 0$ via the continuations $||v|| \to i||\tilde{v}||\,,\quad c_2 \to ic_2\,,\quad c_1 \to \frac{c_1}{i}$. Similarly the $||v||^2 = 0$ geodesics follows from $||v||^2 > 0$ if we apply $c_1 \to \frac{c_1}{||v||}\,,\quad c_2 \to \frac{g_s||v||}{c_1}$.

$\|\|v\|\|^2 > 0$	$\phi = -\log\|c_1 \sinh(\|\|v\|\|h + c_2)\|$	$\chi = \pm\frac{1}{c_1}\coth(\|\|v\|\|h + c_2) + c_3$
$\|\|v\|\|^2 = 0$	$\phi = -\log\|c_1 h + g_s\|$	$\chi = \pm(c_1 h + g_S)^{-1} + c_3$
$\|\|v\|\|^2 < 0$	$\phi = -\log\|c_1 \sin(\|\|\tilde{v}\|\|h + c_2)\|$	$\chi = \pm\frac{1}{c_1}\cot(\|\|\tilde{v}\|\|h + c_2) + c_3$

Table 7.3.1: The scalar fields belonging to the action (7.3.7) for each sign of $||v||^2$. For clarity we have defined $v = i\tilde{v}$ such that $||\tilde{v}||^2 > 0$.

In general the harmonic function h can no longer be solved explicitly with the exception $D = 3$ [40]. In case $||v|| = 0$ we can solve (7.3.8), but since the solution is rather involved we do not write it down.

There is a close connection to the S(-1)-brane of subsection 2.4.3 and the E(-1)-brane we discussed in subsection 6.5.3. For simplicity we consider only those solutions that are related to type IIB supergravities and have a string theory interpretation.

Let us first consider the link to the type IIB $k = -1$ S(-1)-brane. For this we consider the $||v||^2 > 0$ instanton with $k = 1$ and $\Lambda = 0$. Consider the analytically continuation $r \to it$. We see from the g_{rr}-component of (7.3.6) that $||v||^2 r^{-2(D-2)} \to ||v||^2(-1)^{D-2}t^{-2(D-2)}$. The latter should be positive for S(-1)-branes, so in general we have to make a difference between even and odd dimensions. If we restrict to $D = 10$ and apply the following analytical continuations on the instanton

$$r \to it\,, \quad \rho \to i\tilde{\rho}\,, \quad c_2 \to c_2 + i\frac{\pi}{2}\,, \quad c_1 \to -ic_1\,, \quad c_3 \to ic_3\,, \tag{7.3.9}$$

we find the S(-1)-brane solution (2.4.32–2.4.34) with $k = -1$.

To find the non-extremal version of the type IIB* E(-1)-brane it is sufficient to consider the Wick rotations $r \to it$ and $\rho \to i\tilde{\rho}$ applied to the $||v||^2 > 0$ and $k = 1$ instanton. We derive the following non-extremal type IIB* E(-1)-brane

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\mathrm{d}t^2}{\frac{||v||^2}{144}t^{-16} + 1} + t^2\Big(\frac{\mathrm{d}\tilde{\rho}^2}{1+\tilde{\rho}^2} + \tilde{\rho}^2\mathrm{d}\Omega_8^2\Big)\,, \\ \phi &= \log\Big[c_1\sinh\Big(||v||h + c_2\Big)\Big]\,, \\ \chi &= \pm\frac{1}{c_1}\coth\Big(||v||h + c_2\Big) + c_3\,, \\ h(t) &= \frac{3}{2||v||}\mathrm{arctanh}\Big(\frac{||v||}{\sqrt{144t^{16} + v^2}}\Big)\,, \end{aligned} \tag{7.3.10}$$

which after appropriate re-scalings can be linked to (6.5.34). Note that the spatial part of the metric describes a nine-dimensional hyperboloid as expected and that we can extend both (-1)-branes to theories with $\Lambda \neq 0$.

Let us finish by relating our general solution to other known type IIB solutions. For this we restrict to $k = +1$. For $D = 10$ and $\Lambda = 0$ the solutions are the extremal [133,134] and non-extremal D-instanton [123,135] of type IIB for respectively $||v||^2 = 0$ and $||v||^2 \neq 0$. For $2 < D < 10$ and $\Lambda = 0$ it can be considered as a consistent truncation of type IIB reduced over a torus. In case we have $D = 5$ and consider a negative cosmological constant, the action is a compactification of type IIB over S^5. This is the natural setting for the AdS/CFT correspondence [40, 138–140].

7.4 Discussion

In this chapter we have extended the analysis of the generating solution to non-Riemannian cosets. In particular we focussed on $\mathrm{GL}(p+q, \mathbb{R})/\,\mathrm{SO}(p,q)$ and showed that the number of complex eigenvalues is at most $\min(p,q)$. For $\min(p,q) = 1$ we discussed the oxidation of the various generating solutions to vacuum solutions.

In the last section we looked at what happens if we consider massive instantons. We showed that we can find similar first order equations as we derived for domain-walls and cosmologies in section 5.2.7. The main difference is that the scalar metric is no longer positive definite. We then focused on the special case that the potential is a cosmological constant. The scalar fields still describe geodesic motion. At the end of the last section we showed a link between non-extremal D(-1)-instantons, S(-1)-branes and non-extremal E(-1)-branes.