Particle dynamics of branes

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Chapter 6

Domain-Wall / Cosmology Correspondence

In the previous chapter we showed that cosmologies and domain-walls satisfy first order equations. Both type of solutions couple to a zero-form field strength given by a potential, although there is an overall minus sign difference due to the relation between the potential and superpotential. Finally, the metric Ansätze are of similar form.

Due to the first order equations (5.2.7) we know that for a given domain-wall a cosmology exist. This is called the domain-wall / cosmology correspondence [98]. In the first section we give a summary of this correspondence as given in [67, 110].

If a domain-wall can be embedded in a supergravity it can preserve (some fraction of) supersymmetry. Due to the explicit time-dependence, cosmologies break all supersymmetry. On the other hand the correspondence tells us that for a given domain-wall there is a corresponding cosmology. In section 2 we present a discussion of the correspondence in a supergravity setting. It turns out that for this to work the cosmologies need to be embedded into the star supergravities of [58]. These cosmologies then turn out to be also (fake) supersymmetric.

This work is done in collaboration with E. A. Bergshoeff, J. Hartong, J. Rosseel and D. Van Den Bleeken [111].

6.1 The Domain-Wall / Cosmology Correspondence

In 1994 it was already noticed that there is a link between domain-walls and cosmologies [66]. This has been worked out in the papers [67, 98, 110, 112–114] and is called the domain-wall / cosmology correspondence. In [110] it was noticed that the correspondence can be extended to instantons in a trivial way. For this reason we will repeat the arguments given in that paper.

For simplicity we consider the single-scalar field model of (3.5.2), that is

$$\mathcal{L} = \sqrt{\epsilon g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big)\,. \tag{6.1.1}$$

Here $\epsilon = 1$ refers to a Euclidean signature while for $\epsilon = -1$ we have a Lorentzian signature.

Since we are initially interested in domain-walls and cosmologies we require a metric Ansatz that has an one-dimensional transverse space. Furthermore, we allow for all three possible choices of $k = 0, \pm 1$. The D-dimensional metric Ansatz is given by

$$\mathrm{d}s^2 = -\epsilon\eta(\mathrm{e}^{\alpha\varphi} f)^2 \mathrm{d}z^2 + \mathrm{e}^{2\beta\varphi}\Big(-\eta\frac{\mathrm{d}r^2}{1+\eta k r^2} + r^2 \mathrm{d}\Omega_\eta^2\Big)\,. \tag{6.1.2}$$

When $\eta = -1$, $\mathrm{d}\Omega_\eta^2$ describes the SO$(D-1)$-invariant metric on the unit radius (D-2)-sphere and for $\eta = 1$ it describes the SO$(D-2,1)$-invariant metric on the unit radius $(D-2)$-hyperboloid. The functions φ and f depend only on z while α and β are given by

$$\alpha^2 = \frac{D-1}{2(D-2)}\,, \quad \beta^2 = \frac{1}{2(D-1)(D-2)}\,. \tag{6.1.3}$$

To describe a cosmology we take $\epsilon = -1$ and the choice $\eta = -1$ yields the metric of a homogeneous and isotropic cosmology, describing a universe that is closed if $k = 1$, open if $k = -1$ and flat if $k = 0$. The coordinate z is the time coordinate. For domain-walls we take $\epsilon = -1$ but now with $\eta = 1$. The worldvolume geometry of the domain-wall is anti-de Sitter if $k = -1$, de Sitter if $k = 1$ and Minkowski if $k = 0$. In this case z describes the distance from the wall, while r is the time coordinate. The instanton is described by a Euclidean metric, hence we take $\epsilon = -\eta = 1$. The scalar field ϕ can only depend on the coordinate z.

Let us now see how the correspondence comes about. Since we have maintained the re-parametrization of z due to the inclusion of f we can substitute the metric and scalar field Ansätze into the action (6.1.1). If we do this we find the effective one-dimensional Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} f^{-1}(\dot{\phi}^2 - \dot{\varphi}^2) - \epsilon\eta f \mathrm{e}^{2\alpha\varphi} V_{\text{eff}}\,. \tag{6.1.4}$$

Here a dot is a derivative with respect to z and the effective potential is given by

$$V_{\text{eff}}(\phi,\varphi) = V(\phi) - \frac{k}{2\beta^2}\mathrm{e}^{-2\beta\varphi}\,. \tag{6.1.5}$$

From (6.1.4) we see that only the product $\epsilon\eta$ appears in the effective Lagrangian. Even more, we see that (6.1.4) is invariant if we let $\epsilon\eta \to -\epsilon\eta$ together with $V \to -V$ and $k \to -k$.

The domain-wall / cosmology correspondence can be found by considering $\epsilon = -1$. We observe that for every domain-wall solution of a model with potential V there is a cosmology of the model with potential $-V$ and with opposite sign for k if it is non-zero, and *vice-versa*.

Of course, the above also follows from what we derived in section 5.2 for the case $k = 0$. There the sign difference in the potential follows from the parameter η in the relation between the potential and superpotential (5.2.5). For $\eta = 1$ we have that the potential is V, while for $\eta = -1$ we have $-V$. The analysis done in section 5.2 can be extended to include $k \neq 0$ as well.

The extension to instantons is now straightforward. For instantons we require a Euclidean metric so that $\epsilon = -\eta = 1$ or $\epsilon\eta = -1$. This is however the same condition as holds for domain-walls. We see that for a given potential V we find both a Lorentzian domain-wall and a Euclidean solution. The latter can be interpreted as an instanton, but of a model with potential $-V$ because instanton solutions of a mechanical model are precisely solutions with a flipped sign of the potential, see for example [115]. The extended correspondence of [110] can then be summarized as follows. For every domain-wall solution of a model with potential V there corresponds both a cosmology and an instanton of the model with potential $-V$ (although the latter is actually found from the effective Lagrangian with potential V).

Let us illustrate this with the four-dimensional example we worked out in section 5.4. There we showed that a cosmology coupled to a potential V indeed gives rise to a domain-wall coupled to $-V$. According to the above we should also find an instanton solution for this model with potential $-V$. Indeed we find the following Euclidean solution

$$\mathrm{d}s_4^2 = \mathrm{d}r^2 + a(r)^2\Big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big)\,, \tag{6.1.6}$$

with power-law and scalar fields given by (5.4.7). After appropriate coordinate transformations we find the uplifted solution to be

$$\mathrm{d}s_{4+n}^2 = \mathrm{d}\vec{y}_3^{\,2} + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega_n^2\,, \tag{6.1.7}$$

where $\mathrm{d}\Omega_n^2$ is the metric of a n-dimensional sphere. This metric describes $\mathbb{R}^3 \times \mathbb{R}^{n+1}$. So indeed we see that for a given domain-wall we find both a cosmology and an instanton.

The effective Lagrangian (6.1.4) gives rise to second order differential equations for ϕ and φ. On the other hand, in the previous chapter we showed that both domain-walls and cosmologies satisfy the first order equations (5.2.7). In [67] it is shown that one can introduces a function Z which depends on the scalar field ϕ such that one can derive first order equations which automatically satisfy the second order equations that follow from (6.1.4). Since the proof is rather involved, we refer to [67] for this. For $k = 0$ these first order equations agree with (5.2.7).

The analysis so far includes bosonic fields only. It is natural to ask if there is an explanation as to why cosmologies and domain-walls satisfy first order equations such as (5.2.7). Let us comment on these two issues for the case $k = 0$ [67].

The domain-wall / cosmology correspondence is based on the fact that the existence of a domain-wall solution of the effective Lagrangian (6.1.4) with potential V automatically implies the existence of a cosmological solution corresponding to $-V$. The domain-wall solutions generically are 'fake supersymmetric' [67, 98, 112]. This implies that one can write the potential V in terms of a real superpotential W. For the one scalar case this relation schematically looks like

$$V = 2\left((W')^2 - \alpha^2 W^2\right) , \tag{6.1.8}$$

where $W' = \frac{\delta W}{\delta \phi}$ and α is given in (6.1.3). This is, up to a re-scaling of W, the single scalar field version of (5.2.5).

The domain-walls allow for the existence of a Killing spinor ϵ obeying a Killing spinor equation that can be written in terms of the superpotential W as follows:

$$(D_\mu - W\Gamma_\mu)\epsilon = 0\,. \tag{6.1.9}$$

In case the Lagrangian (6.1.1) can be obtained as a truncation of a supergravity theory the equations (6.1.8, 6.1.9) can be understood as arising from the structure of the underlying supergravity theory. In particular, the Killing spinor equation could in that case be obtained by putting the supersymmetry transformations of the fermions equal to zero. In [98] it is shown that the first order-equations for domain-walls follow from (6.1.9). The authors also showed that almost all flat ($k = 0$) and AdS-sliced ($k = -1$) domain-walls preserve half of their supersymmetries. The proof of this is based on the fact that for a given domain-wall one can construct out of this solution a superpotential W such that the (fake) Killing spinor is non-zero. This superpotential W is related to the function Z we mentioned earlier. For the exact constraints we refer to [67]. In this sense the first order equations are BPS equations that guarantee the existence of a Killing spinor. For $k = 1$ we can only have a dS-foliation of either Minkowski or AdS space.

However, fake supergravity[1] is much more general and the Lagrangian (6.1.1) can be completely general and does not need to be related to any supergravity theory. The mapping between domain-walls and cosmologies implies that cosmologies also obey a property that looks very much like fake supersymmetry. In this case, it turns out that the cosmology obeys similar equations (6.1.8, 6.1.9) as its corresponding domain-wall solution, with the caveat that now the superpotential W is no longer real but is

[1] In fake supergravity one allows for a superpotential W that is not part of a genuine supergravity. This W is often called an "adapted" superpotential [100].

instead purely imaginary. Redefining $W = i\tilde{W}$, equations (6.1.8, 6.1.9) become

$$V = -2\left((\tilde{W}')^2 - \alpha^2\tilde{W}^2\right), \qquad (6.1.10)$$

$$(D_\mu - i\tilde{W}\Gamma_\mu)\epsilon = 0\,. \qquad (6.1.11)$$

Note the change of sign in (6.1.10), which indeed corresponds to $-V$ in (6.1.1). The structure (6.1.10, 6.1.11) for cosmological solutions was called *pseudo-supersymmetry* [67, 98, 112]. The structure underlying the existence of the first-order equations can be understood from Hamilton-Jacobi theory [110, 112, 114].

From a supergravity point of view, this correspondence is rather odd. Supersymmetric domain-wall solutions can be found rather generically in supergravity theories. For supersymmetric cosmological solutions this is not true. Furthermore, the correspondence involves a sign change in the potential that spoils the supersymmetry of the supergravity theory under consideration. Finally, in fake supergravity theories, one is usually not concerned with the reality properties of the (Killing) spinors and one works with arbitrary Dirac spinors. In real supergravity theories, reality conditions on the spinors have to be imposed in order to account for the correct number of degrees of freedom. In this respect, one no longer has the freedom to take W purely imaginary without upsetting the reality properties of the supersymmetry rules.

A natural question is whether one can give a meaning to pseudo-supersym- metry in a real supergravity context. The fact that the corresponding domain-wall and cosmological solutions differ in the reality properties of the superpotential suggests that, if one can give an embedding of the correspondence in supergravity, one should look for theories in which the spinors obey different reality properties. A priori, it is possible that there are two different theories in the same signature (namely $(1, 9)$) that mainly differ in the reality properties of the spinors. This can then account for a difference in reality properties of the superpotential and for the sign flip in the potential. We present an example of this is in the type II and type II* theories in signature $(1, 9)$. Starting from a supersymmetric domain-wall in type IIA, the corresponding cosmological solution then turns out to be a supersymmetric solution of the type IIA* theory. Pseudo-supersymmetry in this context corresponds to supersymmetry in a star theory.

6.2 ... in a Supergravity Setting

In the coming sections we are going to answer the question posed in the previous section, namely whether one can give a meaning to pseudo-supersymmetry in a real supergravity context. Let us begin by making two remarks.

In the coming sections we will present a complex formulation of 10- and 11-dimensional supergravity theories. One of the reasons to do this stems from the so-called "variant supergravities" in 10 and 11 dimensions, whose existence has been discussed first in [26, 59, 116]. It was argued that upon applying T-dualities along timelike directions new supergravities are found. In particular, timelike T-duality on the usual type IIA theory does not lead to the usual type IIB theory, but instead leads to a different theory, called the type IIB* theory. Similarly, the type IIA* theory is found as the timelike T-dual of the usual type IIB supergravity. Note that both type II and type II* theories share the same space-time signature $(1, 9)$. A crucial difference between type II and type II* is that in the *-theories the RR-forms are ghosts, i.e. they have wrong-sign kinetic terms. Upon applying more general dualities, one is also led to type II supergravities in different signatures. Similarly, it was argued that one should also consider eleven-dimensional supergravity in different signatures. For instance, it was shown that the type IIA* theory could be obtained by dimensional reduction over a timelike direction of 11d supergravity in signature $(2, 9)$.

In the next sections we derive the explicit actions and supersymmetry variations of these variant supergravities. For earlier work on the construction of these theories in the IIA and M-theory case, see [117, 118]. We will adopt a different approach for constructing the actions and furthermore include the IIB case. The strategy we will follow in obtaining actions and supersymmetry transformation rules for these supergravities, is based on the observations made in [119]. There, it was shown that the superalgebras underlying these variant supergravities correspond to different parameterizations of the unique real form of the superalgebra OSp(1|32). Our work can be viewed as a continuation of [119], where now we construct the complex field theory corresponding to the complex algebra presented there. More precisely, starting from the complex algebra, one can impose different reality conditions on the generators. Each choice of reality conditions gives a real superalgebra underlying one of the variant supergravities in a specific signature. Similarly we will start from a single complex action and by imposing different reality conditions obtain the different variant supergravities.

6.3 Type II Actions

In this section we will show how one can obtain supergravity actions for different signatures as different real slices of a single complex action. Sometimes this leads to different supergravity theories with the same signature.

The starting point of our construction will be a complex action that then can be reduced to different real actions. In this thesis we will not address the question of how one can in general construct sensible complex actions or investigate what a general complex action invariant under some complexified symmetry group looks like. Instead we will take a more pragmatic approach. The idea is to start from a known action in terms of some real fields[2] that is invariant under some real symmetry group. The first step is to construct a complexified version of this action that is invariant under the complexified symmetry group. We require that the real action we started from can be obtained from this complexified action by imposing certain reality conditions and similarly for the symmetries. At this point one faces the natural question: are there different real slices leading to other theories? As it will turn out, theories in different signatures are found by taking different reality conditions for a single complex action. In the case one has extended supersymmetry it can even happen that one finds multiple real theories in one signature. It is these issues that we will work out in detail for IIA and IIB supergravity in this section.

This general scheme of finding different real actions as consistent real slices of a given complex action can be applied quite generally. For the interested reader we refer to [111] for the same analysis for M-theory. One would expect the general procedure presented below to hold for all kinds of theories in various dimensions although subtleties can arise and some particular details might change from case to case.

6.3.1 The Complex Type II Action

To start we will deal with the first of the two questions posed above. We will show how one can find complex actions that can respectively be restricted to the known actions of IIA and IIB by reality conditions, and that are furthermore invariant under the complexified super Poincaré group. How the different formulations of the real 10d super Poincaré algebra can be found from the unique ten-dimensional complex OSp(1|32) algebra was described in detail in [119].

In complexifying an action it is crucial that all fields appear holomorphically in the complex action. In other words we replace fields that take values in $\mathbb{R}$ by fields that take values in $\mathbb{C}$ in such a way that no complex conjugates appear. If one does the same complexification on the symmetry transformations, the complexified action is guaranteed to be invariant under these complex transformations as checking the invariance is a pure algebraic computation that nowhere assumes reality of the involved parameters[3].

This procedure of 'holomorphic complexification' is rather straightforward and only requires some more consideration in case of the spinors. Usually spinors appear in the action through bilinears written in terms of the Dirac conjugate $\bar{\chi}^D = \chi^\dagger A$, see appendix B for our conventions and notations regarding spinors. In this form there appears a complex conjugation and as such the action is not holomorphic in

[2] By a real field, we mean a field that satisfies a reality condition, for instance a Majorana fermion.

[3] One might think that complexifying the supersymmetries in a maximal supergravity theory leads to a supergravity with 64 supercharges. This is however not the case. One should view the complexified action as a mathematical tool and not as a new theory describing new physical degrees of freedom.

the spinor χ. There is an easy way around this as using the reality condition on the spinors the original real action can equivalently be written in terms of the Majorana conjugate $\bar{\chi} = \chi^T \mathcal{C}$. In this form spinors appear holomorphically and complexification now amounts to ignoring the reality condition on the spinors.

We will now illustrate this general principle in case of the ten-dimensional type II theories. For our notations we refer to appendix B .

As a starting point we will take the actions of type IIA and type IIB as given in [120]. These actions have the following field content

$$\begin{aligned} \text{IIA} \quad &: \quad \Big\{ g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(1)}_{\mu}, C^{(3)}_{\mu\nu\rho}, \psi_\mu, \lambda \Big\} \,, \\ \text{IIB} \quad &: \quad \Big\{ g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\cdots\rho}, \psi_\mu, \lambda \Big\} \,. \end{aligned} \tag{6.3.1}$$

A combined form of the actions is given by (ignoring four fermion terms)

$$\begin{aligned} S \;=\; & -\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \, e \Big\{ \mathrm{e}^{-2\phi} \Big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2} H \cdot H \\ & -2\partial^\mu \phi \chi^{(1)}_\mu + H \cdot \chi^{(3)} + 2\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho - 2\bar{\lambda} \Gamma^\mu \nabla_\mu \lambda + 4\bar{\lambda} \Gamma^{\mu\nu} \nabla_\mu \psi_\nu \Big] \\ & + \sum_{n=0,1/2}^{3/2,2} \Big(\tfrac{1}{2} G^{(2n)} \cdot G^{(2n)} + G^{(2n)} \cdot \Psi^{(2n)} \Big) \\ & + \tfrac{1}{4} G^{(5)} \cdot G^{(5)} + \frac{1}{2} G^{(5)} \cdot \Psi^{(5)} - e^{-1} \mathcal{L}_{\mathrm{CS}} \Big\} \,. \end{aligned} \tag{6.3.2}$$

It is understood that the summation in the above action is over integers ($n = 0, 1, 2$) in the IIA case and over half-integers ($n = 1/2, 3/2$) in the IIB case. In the summation range we first write the lowest value for the IIA case, before the one for the IIB case. Remember furthermore that $G^{(5)}$ only appears in IIB and satisfies an additional self-duality constraint $G^{(5)} = \star G^{(5)}$ that does not follow from the field equations. In the IIA case, the massive theory contains an additional mass parameter $G^{(0)} = m$. The Chern-Simons terms are respectively

$$\begin{aligned} \mathcal{L}_{\mathrm{CS}} \;=\; & -\varepsilon^{\mu_1\cdots\mu_{10}} \Big(\tfrac{1}{4\cdot 24^2} \, \partial_{\mu_1} C^{(3)}_{\mu_2\mu_3\mu_4} \, \partial_{\mu_5} C^{(3)}_{\mu_6\mu_7\mu_8} \, B_{\mu_9\mu_{10}} \\ & + \tfrac{1}{2\cdot 24^2} \, G^{(0)} \, \partial_{\mu_1} C^{(3)}_{\mu_2\mu_3\mu_4} \, B^3_{\mu_5\cdots\mu_{10}} + \tfrac{1}{5\cdot 16^2} \, G^{(0)2} \, B^5_{\mu_1\cdots\mu_{10}} \Big) \ (\text{IIA}) \,, \end{aligned} \tag{6.3.3}$$

$$\mathcal{L}_{\mathrm{CS}} \;=\; -\frac{1}{3\cdot 24^2} \varepsilon^{\mu_1\cdots\mu_{10}} C^{(4)}_{\mu_1\mu_2\mu_3\mu_4} \partial_{\mu_5} C^{(2)}_{\mu_6\mu_7} \partial_{\mu_8} B_{\mu_9\mu_{10}} \qquad (\text{IIB}) \,. \tag{6.3.4}$$

The bosonic fields couple to the fermions via the bilinears $\chi^{(1,3)}$ and $\Psi^{(2n)}$, which read

$$
\begin{aligned}
\chi^{(1)}_{\mu} =& -2\bar\psi_\nu\Gamma^\nu\psi_\mu - 2\bar\lambda\Gamma^\nu\Gamma_\mu\psi_\nu\,,\\
\chi^{(3)}_{\mu\nu\rho} =& \tfrac{1}{2}\bar\psi_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\mathcal{P}\psi_\beta + \bar\lambda\Gamma_{\mu\nu\rho}{}^{\beta}\mathcal{P}\psi_\beta - \tfrac{1}{2}\bar\lambda\mathcal{P}\Gamma_{\mu\nu\rho}\lambda\,,\\
\Psi^{(2n)}_{\mu_1\cdots\mu_{2n}} =& \tfrac{1}{2}\mathrm{e}^{-\phi}\bar\psi_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}\mathcal{P}_n\psi_\beta + \tfrac{1}{2}\mathrm{e}^{-\phi}\bar\lambda\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta}\mathcal{P}_n\psi_\beta +\\
& - \tfrac{1}{4}\mathrm{e}^{-\phi}\bar\lambda\Gamma_{[\mu_1\cdots\mu_{2n-1}}\mathcal{P}_n\Gamma_{\mu_{2n}]}\lambda\,.
\end{aligned}
\tag{6.3.5}
$$

The supersymmetry rules read (here given modulo cubic fermion terms)

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^a =& \bar\epsilon\Gamma^a\psi_\mu\,,\\
\delta_\epsilon\psi_\mu =& \Big(\partial_\mu + \tfrac{1}{4}\not{\omega}_\mu + \tfrac{1}{8}\mathcal{P}\not{H}_\mu\Big)\epsilon + \tfrac{1}{8}\mathrm{e}^{\phi}\sum_{n=0,1/2}^{3/2,2}\frac{1}{(2n)!}\not{G}^{(2n)}\Gamma_\mu\mathcal{P}_n\epsilon\\
& + \tfrac{1}{16}\mathrm{e}^{\phi}\frac{1}{5!}\not{G}^{(5)}\Gamma_\mu\mathcal{P}_{5/2}\epsilon\,,\\
\delta_\epsilon B_{\mu\nu} =& -2\,\bar\epsilon\Gamma_{[\mu}\mathcal{P}\psi_{\nu]}\,,\\
\delta_\epsilon C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& -\mathrm{e}^{-\phi}\,\bar\epsilon\,\Gamma_{[\mu_1\cdots\mu_{2n-2}}\,\mathcal{P}_n\,\Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac{1}{2}\Gamma_{\mu_{2n-1}]}\lambda\Big)\\
& + (n-1)(2n-1)\,C^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\,\delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]}\,,\\
\delta_\epsilon\lambda =& \Big(\not{\partial}\phi + \tfrac{1}{12}\not{H}\mathcal{P}\Big)\epsilon + \tfrac{1}{4}\mathrm{e}^{\phi}\sum_{n=0,1/2}^{2,5/2}(-)^{2n}\frac{5-2n}{(2n)!}\not{G}^{(2n)}\mathcal{P}_n\epsilon\,,\\
\delta_\epsilon\phi =& \tfrac{1}{2}\,\bar\epsilon\lambda\,.
\end{aligned}
\tag{6.3.6}
$$

Note that for the IIB case $\Gamma_*\epsilon = \epsilon$, $\Gamma_*\psi_\mu = \psi_\mu$ and $\Gamma_*\lambda = -\lambda$, with Γ_* given by (B.1.3).

As explained in appendix B, we work both in IIA and IIB with an implicit doublet notation for the spinors. We use the conventions that symmetrization and anti-symmetrization are with weight one, slashes are short notation for $\not{H} = H^{\mu\nu\rho}\Gamma_{\mu\nu\rho}$ and $\not{H}_\mu = H_{\mu\nu\rho}\Gamma^{\nu\rho}$ and the form notations used are

$$
\begin{aligned}
A^{(p)}\cdot B^{(p)} =& \frac{1}{p!}A^{(p)}_{\mu_1\cdots\mu_p}B^{(p)\,\mu_1\cdots\mu_p}\,,\\
A^{(p)}\wedge B^{(q)} =& \frac{1}{p!q!}A^{(p)}_{\mu_1\cdots\mu_p}B^{(q)}_{\mu_{p+1}\cdots\mu_{p+q}}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{p+q}}\,,\\
A^{(p)n} =& A^{(p)}\wedge\ldots\wedge A^{(p)}\qquad (n\ \text{times})\,,
\end{aligned}
\tag{6.3.7}
$$

where the label (p) refers to the order of the p-form. The other form conventions are given in appendix A. For notational convenience we group all potentials and field strengths in the formal sums

$$\mathbf{G} = \sum_{n=0,1/2}^{2,5/2} G^{(2n)}\,, \qquad \mathbf{C} = \sum_{n=1,1/2}^{2,5/2} C^{(2n-1)}\,. \tag{6.3.8}$$

The bosonic field strengths are given by

$$H = \mathrm{d}B\,, \qquad \mathbf{G} = \mathrm{d}\mathbf{C} - \mathrm{d}B_\wedge \mathbf{C} + G^{(0)}\mathbf{e}^B\,, \tag{6.3.9}$$

where it is understood that each equation involves only one term from the formal sums (6.3.8) (only the relevant combinations are extracted). Also we will use the following abbreviation:

$$\mathbf{e}^{\pm B} \equiv \pm B + \tfrac{1}{2} B_\wedge B \pm \tfrac{1}{3!} B_\wedge B_\wedge B + \ldots \tag{6.3.10}$$

In writing down type II actions, we use the following definitions

$$\mathcal{P} \quad = \Gamma_{11} \otimes \mathbb{1}_2 = \quad \mathbb{1}_{32} \otimes \ \sigma_3 \ \text{(IIA)} \ \text{ or } \ -\mathbb{1}_{32} \otimes \sigma_3 \ \text{(IIB)}\,, \tag{6.3.11}$$

and

$$\mathcal{P}_n = (\Gamma_{11} \otimes \mathbb{1}_2)^n \ \text{ (IIA) } \ \text{or} \ \ \mathbb{1}_{32} \otimes \sigma^1 \ (\mathrm{n} + 1/2 \text{ even}), \ \mathbb{1}_{32} \otimes i\sigma^2 \ (\mathrm{n} + 1/2 \text{ odd}) \quad \text{(IIB)}\,.$$

Up till now we have just written down the action of the type IIA/B in (1,9) signature in a standard form. We will now interpret the action (6.3.2) in a different way, as a complex action. All fields are now assumed to be complex, both bosonic and fermionic. For the fermions this means that they are arbitrary Dirac spinors, as stated before they only appear holomorphically in the action through their Majorana conjugate $\bar{\chi} = \chi^T \mathcal{C}$. The gamma-matrices with flat indices remain the standard gamma-matrices of (1,9) Minkowski space. As we now allow the vielbein to be complex, the curved gamma-matrices will be part of the complexified Clifford algebra, see section 6.4 for more details. The supersymmetry transformations (6.3.6) are understood to be complex in the same way as the action (6.3.2). The complexified action remains invariant under the complexified supersymmetry transformations as basic manipulations like symmetry properties of bilinears, gamma-matrix algebra and Fierz identities are insensitive to this complexification. In the same way the complex action is invariant under the complexified Lorentz-group $\mathrm{SO}(10, \mathbb{C})$.

6.3.2 Back to Reality

Starting from the complex action and supersymmetry transformations of the previous section we will now explain how one can construct different real actions by taking different real slices. In this subsection, we will do a general analysis determining all variant supergravities. The result is summarized in table 6.3.1. In the next subsection, we will illustrate the method with some specific examples.

Let us start by explaining what we mean by taking a real slice. A reality condition on the fields cannot be chosen at will, but has to satisfy certain consistency conditions. First of all, one can only impose a limited number of reality conditions on the fermions. As is explained in appendix B this leads to the following general reality conditions on the fermionic fields (see (B.2.7))

$$\begin{aligned} \epsilon^* &= -\varepsilon\eta^t\alpha_\epsilon \mathcal{C}A\rho\epsilon\,, \\ \psi_\mu^* &= -\varepsilon\eta^t\alpha_\psi \mathcal{C}A\rho\psi_\mu\,, \\ \lambda^* &= -\varepsilon\eta^t\alpha_\lambda \mathcal{C}A\rho\lambda\,, \end{aligned} \tag{6.3.12}$$

where the α_χ represents a phase factor that can differ from field to field. On the bosonic fields, a general reality condition is given by[4]:

$$\begin{aligned} e_\mu{}^{a*} &= e_\mu{}^a\,, \\ \phi^* &= \phi\,, \\ B_{\mu\nu}^* &= \alpha_B B_{\mu\nu}\,, \\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= \alpha_n C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,, \end{aligned} \tag{6.3.13}$$

where again the α-factors represent phases. Note that we have already taken the dilaton to be real, as this is the only condition consistent with reality of the action. We also choose to work with real vielbeine. This amounts to using the flat gamma-matrices that are appropriate to a specific signature. The complex action is written in terms of fixed flat Γ-matrices in signature (1,9). In principle one could keep these fixed during the whole procedure and allow for purely imaginary vielbein components. Simultaneously redefining the vielbeine and flat gamma-matrices then brings one back to the case where the vielbeine are real and the Clifford algebra has the appropriate signature. For a more thorough and technical discussion of this point, see section 6.4. This reasoning also reveals a subtlety concerning the Chern-Simons terms. Supersymmetry of the action (6.3.2) is established thanks to the relation

$$\Gamma_{a_1\ldots a_n} = -\frac{1}{(10-n)!}\varepsilon_{a_1\ldots a_{10}}\Gamma_{11}\Gamma^{a_{10}\ldots a_{n+1}}\,. \tag{6.3.14}$$

This relation is however only valid for the Clifford algebra with signature $(1,9)$. As explained above, we choose to work with the Clifford algebra that has the same signature as space-time. For this Clifford algebra, the relation (6.3.14) is changed to

$$\Gamma_{a_1\ldots a_n} = \frac{1}{(10-n)!}\varepsilon_{a_1\ldots a_{10}}i^{t+1}\Gamma_{11}\Gamma^{a_{10}\ldots a_{n+1}}\,. \tag{6.3.15}$$

[4]To have a uniform notation the reality condition for $G^{(0)}$ is given in terms of some formal $C^{(-1)}$. This is just a shorthand implying $G^{(0)*} = \alpha_0 G^{(0)}$.

Effectively, rewriting (6.3.15) to (6.3.14) corresponds to replacing $\varepsilon_{0\dots9}$ by

$$\begin{array}{rcl}\varepsilon_{0\dots9} & \to & -(-i)^{t+1}\varepsilon_{0\dots9}\,, \\ \varepsilon^{0\dots9} & \to & -(i)^{t+1}\varepsilon^{0\dots9}\,. \end{array} \tag{6.3.16}$$

When going to a real action of a given signature, one has to replace the $\varepsilon_{0\dots9}$ in the complex Chern-Simons term via the above rule to assure invariance under supersymmetry.

The α-factors appearing in the reality conditions on the bosons and the fermions are not independent. Demanding a real action and consistency with supersymmetry relates them. The latter means that both sides of the supersymmetry rules should have the same behaviour under complex conjugation. In this way, the reality conditions on the fermions determine those of the bosons. Analyzing this in detail leads to the relations

$$\begin{array}{rcl}\alpha_\epsilon & = & \alpha_\psi\,, \\ \alpha_\lambda^2 & = & \alpha_\psi^2 = (-\eta)^{t+1}\,, \\ \alpha_\lambda & = & (-)^{t+1}\eta\rho^T\sigma\rho\sigma\,\alpha_\psi\,, \\ \alpha_H & = & \rho^T\sigma^{t+1}\sigma_3\sigma^{t+1}\rho\sigma_3\,, \\ \alpha_n & = & (-)^{(2n+1)t}(-\eta)^{(2n+1)}\rho^T\sigma^t\mathcal{P}_n\sigma^{t+1}\rho\mathcal{P}_n^{-1}\sigma\,. \end{array} \tag{6.3.17}$$

The possible solutions of these equations lead to consistent reality conditions on all fields. They are summarized in table 6.3.1. Every possible reality condition corresponds to a unique real supergravity theory that has (6.3.2) as complexified action.

Given the data in table 6.3.1, the actions and supersymmetry rules of these variant supergravities can be explicitly written down. These actions are the complex action (6.3.2), where the fields now obey the reality properties (6.3.12,6.3.13), with the α-factors the ones mentioned in table 6.3.1. One notices that in this form some fields might be purely imaginary. In this case, it is more natural to redefine the fields in terms of real fields. This leads to a change in sign of e.g. the kinetic terms of these fields. In order to write the actions in a more conventional form involving Dirac conjugates, one can use the following formula equivalent to (6.3.12) if (6.3.17) is satisfied:

$$\bar\chi = \alpha_\psi\alpha_\lambda\alpha_\chi^*\bar\chi^D\rho\,. \tag{6.3.18}$$

This allows one to rewrite Majorana conjugates appearing in (6.3.2) in terms of Dirac conjugates. As explained above in certain signatures one has to multiply the Chern-Simons term by an additional factor, this factor is given in the last row of table 6.3.1, this same factor also appears in the (anti) self-duality condition of IIB. The procedure described here will be illustrated in more detail for some specific examples in subsection 6.3.3.

	A				B			
t mod 4	0	1		2	1			3
type	*M$^+$	MW	*MW	M$^+$	MW	*MW	′MW	SMW
$\varepsilon = \eta$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
ρ	σ_3	$\mathbb{1}$	σ_3	$\mathbb{1}$	$\mathbb{1}$	σ_3	σ_1	$i\sigma_2$
$\alpha_\epsilon = \alpha_\psi$	i	1	1	i	1	1	1	1
α_λ	i	1	-1	$-i$	1	1	1	1
α_B	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$
$\alpha_0 = \alpha_2,\ \alpha_{1/2} = \alpha_{5/2}$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
$\alpha_1,\ \alpha_{3/2}$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$
$-(i)^{t+1}$	$-i$	1	1	i	1	1	1	-1

Table 6.3.1: *Possible reality conditions on the fields of type II supergravities. t is the number of timelike directions in space-time. The notation concerning the type of fermionic reality condition is explained in appendix B. Every set of reality conditions (column) corresponds to a different variant supergravity theory. The last row refers to the additional factor for the Chern-Simons terms. From this table the actions and supersymmetry transformations of all 10d variant supergravities can be constructed.*

Finally let us give a short overview of the variant theories classified by table 6.3.1. Type IIA supergravity exists in three types of signatures. Note that only in signature $t = 1$ mod 4 there are two different real theories[5]. For IIB the situation is similar. Although table 6.3.1 seems to suggest that there are three different theories in (1,9), IIB* and IIB$'$ are related by a field redefinition that can be interpreted as an S-duality. Note that IIB theories only exist in those signatures where a consistent self-duality condition can be imposed. In our conventions the five form is self-dual in signatures with $t = 1$ mod 4 and anti self-dual when $t = 3$ mod 4, this is due to the subtleties concerning the appearance of $\varepsilon_{0\ldots9}$ explained above.

6.3.3 Examples

In this subsection we will illustrate the previously discussed method of real slices for type II theories in signature (1,9). We will show how to write down the explicit form of the actions starting from table 6.3.1. To illustrate how to write down the Chern-Simons terms in case the real slice involves an additional factor multiplying $\varepsilon_{0\ldots9}$ we discuss this term in signature (0,10) in detail.

[5]The results of table 6.3.1 almost completely agree with those found in [117] for IIA, with the exception that only in signature (1,9) we find two inequivalent theories. In [117] additional IIA theories for $t = 0$ or 2 mod 4 are presented, which we are not able to reproduce in our framework.

IIA

Our first example is how one can recover the usual type IIA theory in signature (1,9). The reality conditions appropriate for this theory are summarized in the second column of table 6.3.1, leading to:

$$\begin{aligned} \epsilon^* &= -\mathcal{C}A\epsilon\,, \\ \psi_\mu^* &= -\mathcal{C}A\psi_\mu\,, \\ \lambda^* &= -\mathcal{C}A\lambda \\ B^*_{\mu\nu} &= B_{\mu\nu}\,, \\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,, \quad (n=0,1,2)\,. \end{aligned} \tag{6.3.19}$$

The real action for this theory is the complex action given above (6.3.2) but restricted to the subspace given by these reality conditions. The Majorana conditions for the spinors (6.3.19) are equivalent to

$$\begin{aligned} \bar\epsilon &= \bar\epsilon^{D}\,, \\ \bar\psi_\mu &= \bar\psi^{D}_\mu\,, \\ \bar\lambda &= \bar\lambda^{D}\,, \end{aligned} \tag{6.3.20}$$

and using these we can write the action (6.3.2) in a standard real form involving Dirac conjugates. Plugging (6.3.19 - 6.3.20) into the action (6.3.2) gives

$$\begin{aligned} S_{\rm IIA} = &-\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\Big\{\mathrm{e}^{-2\phi}\big[-\mathcal{R}\big(\omega(e)\big)-4\big(\partial\phi\big)^2+\tfrac12 H\cdot H-2\partial^\mu\phi\chi^{(1)}_\mu+H\cdot\chi^{(3)} \\ &+2\bar\psi^{D}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho-2\bar\lambda^{D}\Gamma^\mu\nabla_\mu\lambda+4\bar\lambda^{D}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\big]+\sum_{n=0,1,2}\tfrac12 G^{(2n)}\cdot G^{(2n)} \\ &+G^{(2n)}\cdot\Psi^{(2n)}+e^{-1}\varepsilon^{\mu_1\cdots\mu_{10}}\big[\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}} \\ &+\tfrac{1}{2\cdot 24^2}\,G^{(0)}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,B^3_{\mu_5\ldots\mu_{10}}+\tfrac{1}{5\cdot 16^2}\,G^{(0)2}\,B^5_{\mu_1\ldots\mu_{10}}\big]\Big\}\,, \end{aligned} \tag{6.3.21}$$

where

$$\begin{aligned} \chi^{(1)}_\mu &= -2\bar\psi^{D}_\nu\Gamma^\nu\psi_\mu-2\bar\lambda^{D}\Gamma^\nu\Gamma_\mu\psi_\nu\,, \\ \chi^{(3)}_{\mu\nu\rho} &= \tfrac12\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\Gamma_{11}\psi_\beta+\bar\lambda^{D}\Gamma_{\mu\nu\rho}{}^{\beta}\Gamma_{11}\psi_\beta-\tfrac12\bar\lambda^{D}\Gamma_{11}\Gamma_{\mu\nu\rho}\lambda\,, \\ \Psi^{(2n)}_{\mu_1\cdots\mu_{2n}} &= \tfrac12\mathrm{e}^{-\phi}\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}(\Gamma_{11})^n\psi_\beta+\tfrac12\mathrm{e}^{-\phi}\bar\lambda^{D}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta(\Gamma_{11})^n\psi_\beta \\ &\quad -\tfrac14\mathrm{e}^{-\phi}\bar\lambda^{D}\Gamma_{[\mu_1\cdots\mu_{2n-1}}(\Gamma_{11})^n\Gamma_{\mu_{2n}]}\lambda\,. \end{aligned} \tag{6.3.22}$$

The action (6.3.21) is invariant under the following supersymmetries

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^a =&\bar{\epsilon}^D \Gamma^a \psi_\mu \,, \\
\delta_\epsilon \psi_\mu =&\Big(\partial_\mu + \tfrac{1}{4} \not{\omega}_\mu + \tfrac{1}{8}\Gamma_{11} \not{H}_\mu\Big)\epsilon + \tfrac{1}{8}\mathrm{e}^\phi \sum_{n=0,1,2} \frac{1}{(2n)!} \not{G}^{(2n)}\Gamma_\mu (\Gamma_{11})^n \,\epsilon \,, \\
\delta_\epsilon B_{\mu\nu} =& -2\,\bar{\epsilon}^D \Gamma_{[\mu}\Gamma_{11}\psi_{\nu]} \,, \\
\delta_\epsilon C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& -\mathrm{e}^{-\phi}\,\bar{\epsilon}^D\,\Gamma_{[\mu_1\cdots\mu_{2n-2}}\,(\Gamma_{11})^n \Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac{1}{2}\Gamma_{\mu_{2n-1}]}\lambda\Big) \\
&+ (n-1)(2n-1)\,C^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\,\delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]} \,, \\
\delta_\epsilon \lambda =&\Big(\not{\partial}\phi + \tfrac{1}{12}\not{H}\Gamma_{11}\Big)\epsilon + \tfrac{1}{4}\mathrm{e}^\phi \sum_{n=0,1,2} \frac{5-2n}{(2n)!} \not{G}^{(2n)} (\Gamma_{11})^n \epsilon \,, \\
\delta_\epsilon \phi =&\tfrac{1}{2}\,\bar{\epsilon}^D \lambda \,.
\end{aligned}
\tag{6.3.23}
$$

As the (1,9) IIA supergravity theory was the theory we started from before complexifying, taking the real slice was rather straightforward. Things will become more interesting in case some fields are purely imaginary. We illustrate this in the following example.

IIA*

The action of the IIA* theory in (1,9) can be constructed by using the third column of table 6.3.1, which leads to the following reality conditions:

$$
\begin{aligned}
\epsilon^* &= -\mathcal{C}A\Gamma_{11}\epsilon \,, \\
\psi^*_\mu &= -\mathcal{C}A\Gamma_{11}\psi_\mu \,, \\
\lambda^* &= \mathcal{C}A\Gamma_{11}\lambda \,, \\
B^*_{\mu\nu} &= B_{\mu\nu} \,, \\
C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= -C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} \,, \quad (n=0,1,2) \,.
\end{aligned}
\tag{6.3.24}
$$

Note that now the reality condition for the Ramond-Ramond fields implies that they are purely imaginary. It is therefore natural to make a redefinition to real fields. We also prefer to have the same reality condition for all the fermionic fields. Thus we make the field redefinitions

$$
\begin{aligned}
\zeta &= -i\lambda \,, \\
A^{(2n-1)} &= -iC^{(2n-1)} \,, \\
F^{(2n)} &= -iG^{(2n)} \,.
\end{aligned}
\tag{6.3.25}
$$

In this case the relation between Majorana and Dirac conjugate of the spinors is

$$\begin{aligned}\bar\epsilon &= -\bar\epsilon^{D}\Gamma_{11}\,,\\ \bar\psi_\mu &= -\bar\psi^{D}_\mu\Gamma_{11}\,,\\ \bar\zeta &= -\bar\zeta^{D}\Gamma_{11}\,.\end{aligned}\tag{6.3.26}$$

Similarly to the IIA case one can obtain a manifestly real action, which now reads

$$\begin{aligned}S_{\mathrm{IIA}^*} =& -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\Big\{\mathrm{e}^{-2\phi}\big[-\mathcal{R}\big(\omega(e)\big)-4\big(\partial\phi\big)^2+\tfrac12 H\cdot H-2\partial^\mu\phi\xi^{(1)}_\mu+H\cdot\xi^{(3)}+\\ &-2\bar\psi^{D}_\mu\Gamma_{11}\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho-2\bar\zeta^{D}\Gamma_{11}\Gamma^\mu\nabla_\mu\zeta-4i\bar\zeta^{D}\Gamma_{11}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\big]-\sum_{n=0,1,2}\tfrac12 F^{2n}\cdot F^{2n}\\ &+F^{2n}\cdot\Delta^{(2n)}-e^{-1}\varepsilon^{\mu_1\cdots\mu_{10}}\big[\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}A^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}A^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}}\\ &+\tfrac{1}{2\cdot 24^2}\,F^{(0)}\,\partial_{\mu_1}A^{(3)}_{\mu_2\mu_3\mu_4}\,B^3_{\mu_5\cdots\mu_{10}}+\tfrac{1}{5\cdot 16^2}\,F^{(0)2}\,B^5_{\mu_1\cdots\mu_{10}}\big]\Big\}\,,\end{aligned}\tag{6.3.27}$$

where

$$\begin{aligned}\xi^{(1)}_\mu &= -2\bar\psi^{D}_\nu\Gamma^\nu\Gamma_{11}\psi_\mu+2i\bar\zeta^{D}\Gamma^\nu\Gamma_\mu\Gamma_{11}\psi_\nu\,,\\ \xi^{(3)}_{\mu\nu\rho} &= \tfrac12\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\psi_\beta-i\bar\zeta^{D}\Gamma_{\mu\nu\rho}{}^\beta\psi_\beta-\tfrac12\bar\zeta^{D}\Gamma_{\mu\nu\rho}\zeta\,,\\ \Delta^{(2n)}_{\mu_1\cdots\mu_{2n}} &= \tfrac{i}{2}\mathrm{e}^{-\phi}\bar\psi^{D}_\alpha\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}(\Gamma_{11})^{n+1}\psi_\beta+\tfrac12\mathrm{e}^{-\phi}\bar\zeta^{D}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta(\Gamma_{11})^{n+1}\psi_\beta\\ &\quad-\tfrac{i}{4}\mathrm{e}^{-\phi}\bar\zeta^{D}\Gamma_{[\mu_1\cdots\mu_{2n-1}}(\Gamma_{11})^{n+1}\Gamma_{\mu_{2n}]}\zeta\,.\end{aligned}\tag{6.3.28}$$

The action (6.3.27) is invariant under the supersymmetries

$$\begin{aligned}\delta_\epsilon e_\mu{}^a =& \bar\epsilon^{D}\Gamma^a\Gamma_{11}\psi_\mu\,,\\ \delta_\epsilon\psi_\mu =& \Big(\partial_\mu+\tfrac14\not\omega_\mu+\tfrac18\Gamma_{11}\not H_\mu\Big)\epsilon+\tfrac{i}{8}\mathrm{e}^\phi\sum_{n=0,1,2}\frac{1}{(2n)!}\not F^{(2n)}\Gamma_\mu(\Gamma_{11})^n\,\epsilon\,,\\ \delta_\epsilon B_{\mu\nu} =& -2\,\bar\epsilon^{D}\Gamma_{[\mu}\psi_{\nu]}\,,\\ \delta_\epsilon A^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& -\mathrm{e}^{-\phi}\,\bar\epsilon^{D}\,\Gamma_{[\mu_1\cdots\mu_{2n-2}}\,(\Gamma_{11})^{n+1}\,\Big(i(2n-1)\psi_{\mu_{2n-1}]}+\tfrac12\Gamma_{\mu_{2n-1}]}\zeta\Big)\\ &+(n-1)(2n-1)\,A^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\,\delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]}\,,\\ \delta_\epsilon\zeta =& -i\Big(\not\partial\phi+\tfrac{1}{12}\not H\Gamma_{11}\Big)\epsilon+\tfrac14\mathrm{e}^\phi\sum_{n=0,1,2}\frac{5-2n}{(2n)!}\not F^{(2n)}(\Gamma_{11})^n\epsilon\,,\\ \delta_\epsilon\phi =& -\tfrac{i}{2}\,\bar\epsilon^{D}\Gamma_{11}\zeta\,.\end{aligned}\tag{6.3.29}$$

Note that indeed in this real form the Ramond-Ramond fields have wrong sign kinetic terms. Furthermore, there are additional factors of i appearing in the supersymmetry transformations with respect to standard IIA. This is similar to the i's appearing in the pseudo-supersymmetry of [67, 98], we will elaborate on this in section 6.5.1. Another difference is the appearance of the chirality matrix Γ_{11} in various spinor bilinears. They appear for example in the variation of the dilaton, leading to a different transformation of this field under parity[6].

Chern-Simons terms

As explained above there are some subtleties concerning the Chern-Simons terms in certain signatures. Here we will briefly illustrate how the Chern-Simons term of IIA in (0,10) signature can be obtained, the other cases proceed analogously. Of the fields appearing in the IIA Chern-Simons term, B becomes purely imaginary while the others are real, as can be read from the first column of table 6.3.1. We thus make the redefinition

$$\tilde{B}_{\mu\nu} = -iB_{\mu\nu}\,. \tag{6.3.30}$$

Substituting this in the complex Chern-Simons term (6.3.3) and multiplying with the appropriate factor i (see table 6.3.1) gives the following real topological terms:

$$\begin{aligned} -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\varepsilon^{\mu_1\cdots\mu_{10}}\big[&\tfrac{1}{4\cdot 24^2}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,\tilde{B}_{\mu_9\mu_{10}} \\ &-\tfrac{1}{2\cdot 24^2}\,G^{(0)}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\tilde{B}^3_{\mu_5\cdots\mu_{10}} + \tfrac{1}{5\cdot 16^2}\,G^{(0)2}\,\tilde{B}^5_{\mu_1\cdots\mu_{10}}\big]\,. \end{aligned} \tag{6.3.31}$$

Note that apart from the changes in the Chern-Simons term also the relation between the real potentials and field strengths gets modified, e.g.

$$G^{(4)} = \mathrm{d}C^{(3)} + \mathrm{d}\tilde{B}\wedge A^{(1)} - G^{(0)}\tilde{B}^2\,, \tag{6.3.32}$$

instead of the standard relation (6.3.9).

Similar to this example one can find the Chern-Simons terms and field strengths in other signatures.

IIB*

As our final example we derive the action and supersymmetry equations of IIB*, the alternate real IIB theory in signature (1,9). The reality conditions are:

$$\begin{aligned} \epsilon^* &= \mathcal{C}A\mathcal{P}\epsilon\,, \\ \psi_\mu^* &= \mathcal{C}A\mathcal{P}\psi_\mu\,, \\ \lambda^* &= \mathcal{C}A\mathcal{P}\lambda\,, \\ B^*_{\mu\nu} &= B_{\mu\nu}\,, \\ C^{(2n-1)*}_{\mu_1\cdots\mu_{2n-1}} &= -C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\,, \quad (n = 1/2, 3/2, 5/2)\,. \end{aligned} \tag{6.3.33}$$

[6] Under parity we understand the transformation $x^i \to -x^i$ that reverses the sign of all 9 spacelike directions.

We redefine the imaginary fields in term of real fields as follows:

$$\begin{aligned} A^{(2n-1)} &= -iC^{(2n-1)}\,, \\ F^{(2n)} &= -iG^{(2n)}\,. \end{aligned} \tag{6.3.34}$$

The reality conditions for the spinors are equivalent to the conditions

$$\begin{aligned} \bar{\epsilon} &= \bar{\epsilon}^{D}\mathcal{P}\,, \\ \bar{\psi}_\mu &= \bar{\psi}_\mu^{D}\mathcal{P}\,, \\ \bar{\lambda} &= \bar{\lambda}^{D}\mathcal{P}\,. \end{aligned} \tag{6.3.35}$$

Substituting this into the complex IIB action (6.3.2) leads to

$$\begin{aligned} S_{IIB*} =& -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x e\Big\{e^{-2\phi}\Big[-\mathcal{R}\big(\omega(e)\big) - 4\big(\partial\phi\big)^2 + \tfrac{1}{2}H\cdot H + \\ & -2\partial^\mu\phi\zeta_\mu^{(1)} + H\cdot\zeta^{(3)} + 2\bar{\psi}_\mu^D\mathcal{P}\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho - 2\bar{\lambda}^D\mathcal{P}\Gamma^\mu\nabla_\mu\lambda + 4\bar{\lambda}^D\mathcal{P}\Gamma^{\mu\nu}\nabla_\mu\psi_\nu\Big] \\ & -\sum_{n=1/2}^{3/2}\left(\frac{1}{2}F^{(2n)}\cdot F^{(2n)} + F^{(2n)}\cdot\Delta^{(2n)}\right) - \frac{1}{4}F^{(5)}\cdot F^{(5)} - \frac{1}{2}F^{(5)}\cdot\Delta^{(5)} \\ & - e^{-1}\frac{1}{3\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}A^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_5}A^{(2)}_{\mu_6\mu_7}\partial_{\mu_8}B_{\mu_9\mu_{10}}\Big\}\,. \end{aligned} \tag{6.3.36}$$

This action needs to be supplemented with the usual self-duality for the RR-five form $F^{(5)}$. The bosonic fields couple to the fermions via the bilinears

$$\begin{aligned} \zeta_\mu^{(1)} =& -2\bar{\psi}_\nu^D\mathcal{P}\Gamma^\nu\psi_\mu - 2\bar{\lambda}^D\mathcal{P}\Gamma^\nu\Gamma_\mu\psi_\nu\,, \\ \zeta_{\mu\nu\rho}^{(3)} =& \tfrac{1}{2}\bar{\psi}_\alpha^D\Gamma^{[\alpha}\Gamma_{\mu\nu\rho}\Gamma^{\beta]}\psi_\beta + \bar{\lambda}^D\,\Gamma_{\mu\nu\rho}{}^\beta\psi_\beta - \tfrac{1}{2}\bar{\lambda}^D\Gamma_{\mu\nu\rho}\lambda\,, \\ \Delta^{(2n)}_{\mu_1\cdots\mu_{2n}} =& -i\Big(\tfrac{1}{2}\mathrm{e}^{-\phi}\bar{\psi}_\alpha^D\Gamma^{[\alpha}\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^{\beta]}\mathcal{P}\mathcal{P}_n\psi_\beta + \tfrac{1}{2}\mathrm{e}^{-\phi}\bar{\lambda}^D\Gamma_{\mu_1\cdots\mu_{2n}}\Gamma^\beta\mathcal{P}\mathcal{P}_n\psi_\beta \\ & - \tfrac{1}{4}\mathrm{e}^{-\phi}\bar{\lambda}^D\Gamma_{[\mu_1\cdots\mu_{2n-1}}\mathcal{P}\mathcal{P}_n\Gamma_{\mu_{2n}]}\lambda\Big)\,. \end{aligned} \tag{6.3.37}$$

The supersymmetry rules are

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^a =& \bar\epsilon^D \mathcal{P} \Gamma^a \psi_\mu \,, \\
\delta_\epsilon \psi_\mu =& \Big(\partial_\mu + \tfrac14 \not\omega_\mu + \tfrac18 \mathcal{P} \not H_\mu\Big)\epsilon + \tfrac{i}{8} e^\phi \sum_{n=1/2}^{3/2} \frac{1}{(2n)!} \not F^{(2n)} \Gamma_\mu \mathcal{P}_n \epsilon \\
& + \tfrac{i}{16} e^\phi \frac{1}{5!} \not F^{(5)} \Gamma_\mu \mathcal{P}_{5/2} \epsilon \,, \\
\delta_\epsilon B_{\mu\nu} =& -2\, \bar\epsilon^D \Gamma_{[\mu} \psi_{\nu]} \,, \\
\delta_\epsilon A^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}} =& i e^{-\phi}\, \bar\epsilon^D\, \Gamma_{[\mu_1\cdots\mu_{2n-2}}\, \mathcal{P}\mathcal{P}_n \Big((2n-1)\psi_{\mu_{2n-1}]} - \tfrac12 \Gamma_{\mu_{2n-1}]}\lambda\Big) \\
& + (n-1)(2n-1)\, A^{(2n-3)}_{[\mu_1\cdots\mu_{2n-3}}\, \delta_\epsilon B_{\mu_{2n-2}\mu_{2n-1}]} \,, \\
\delta_\epsilon \lambda =& \Big(\not\partial\phi + \tfrac{1}{12} \not H \mathcal{P}\Big)\epsilon + \tfrac{i}{4} e^\phi \sum_{n=1/2}^{5/2} (-)^{2n} \frac{5-2n}{(2n)!} \not F^{(2n)} \mathcal{P}_n \epsilon \,, \\
\delta_\epsilon \phi =& \tfrac12\, \bar\epsilon^D \mathcal{P} \lambda \,.
\end{aligned}
\tag{6.3.38}
$$

Note that in contrast to the standard IIB action the IIB* action is no longer invariant under the full S-duality group, but gets mapped to the IIB$'$ theory. Another viewpoint is thus that IIB$'$ is nothing else than a field redefinition of IIB*. As such we will not construct its action and supersymmetry transformations here. They can be obtained either from performing an S-duality or taking a real slice with the appropriate reality conditions in table 6.3.1.

6.3.4 Extended vs. Unextended Supersymmetry

It might be remarkable that in certain signatures different real slices exist while in others only one real theory is consistent. This is related to the number of independent supersymmetries. Although we always discussed theories with 32 real supercharges, this does not necessarily mean that their supersymmetry is extended. Depending from signature to signature the dimension of a real irreducible spinor is 16 or 32. Only in the signatures in which it is 16, and thus the 32 supercharges imply extended supersymmetry, different real slices can occur. This can be understood as in this case different reality conditions can be imposed on the two independent 16-dimensional spinors.

This suggests that in any signature only one real slice of the complex 10d $\mathcal{N}=1$ supergravities exists. These $\mathcal{N}=1$ supergravities can be seen as truncations of the type II theories by a $\mathbb{Z}_2$ truncation. Thus one would expect both the standard theories and their star versions to truncate to the same theory. We will now show that this is indeed the case in IIA. The truncation is made by only keeping the fields invariant under the following fermion number symmetry [120]:

$$\begin{aligned}\{\phi, g_{\mu\nu}, B_{\mu\nu}\} &\to \{\phi, g_{\mu\nu}, B_{\mu\nu}\}\,,\\ \{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\} &\to -\{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\}\,,\\ \{\psi_\mu, \lambda, \epsilon\} &\to \Gamma_{11}\{\psi_\mu, -\lambda, \epsilon\}\,.\end{aligned} \tag{6.3.39}$$

One can see that both IIA and IIA* project to the same theory under identification by this symmetry as this identification is equivalent to demanding the reality conditions (6.3.19) and (6.3.24) to be identical. The other IIA truncation that is given in [120] is no longer consistent.

The situation is similar in IIB. For IIB in (1,9) signature both truncations given in [120] lead to the same result, for IIB* only one truncation is consistent with the reality properties of the spinors while the other identification is the only consistent one for IIB$'$. In the end all possible truncations lead to the same $\mathcal{N}=1$ theory.

6.4 Reality of the Vielbeine

6.4.1 Imaginary Vielbeine and Signature Change

In this section we will give some more details on the equivalence between choosing to work with on the one hand fixed flat gamma-matrices of signature (1,9) and possibly imaginary vielbein or on the other hand gamma-matrices of the appropriate signature and a real vielbein.

It is important to stress that the flat gamma-matrices appearing in the complex action (6.3.2) are elements of the Clifford algebra of signature (1,9) obeying the standard reality condition[7]

$$\Gamma^{a*} = -\mathcal{C}\Gamma_0\Gamma^a\Gamma_0\mathcal{C}^{-1}. \tag{6.4.1}$$

The curved gamma-matrices $\Gamma_\mu = \Gamma_a e_\mu{}^a$ no longer obey a reality condition as $e_\mu{}^a$ (and the other fields) are complex. Because the vielbein, and thus the metric as well, is complex there is no longer a concept of space-time signature. Note that the complex metric is defined as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta^{(1,9)}_{ab}$, where $\eta^{(1,9)}_{ab} = \mathrm{diag}(- + \ldots +)$.

When we impose reality conditions on the fields appearing in the action we recover a real theory in a signature that can differ from the (1,9) signature we started from. This can happen as some components of the vielbein can be purely imaginary such that $g_{\mu\nu}$ is real but has a signature different from that of $\eta^{(1,9)}_{ab}$.

As explained a choice of reality conditions for the fermions determines the reality properties of all the bosonic fields as well. For the vielbein this happens through the supersymmetry transformation

$$\delta_\epsilon e_\mu{}^a = \bar{\epsilon}\Gamma^a\psi_\mu\,. \tag{6.4.2}$$

[7] In this section we will make the choice $\epsilon = \eta = 1$.

As explained in appendix B in the reality conditions for the spinors (B.2.7) the operator A appears. This A is the product of the timelike gamma-matrices. So by choosing A in the reality conditions for the complex fermions one decides in which space-time signature the real fermions will be consistent. As we will see below consistency of the above supersymmetry variation (6.4.2) implies that also the real metric given by these reality conditions has that signature. If one for example makes a real slice to a theory in signature (t,s), $t+s=10$, fermions satisfy the following reality conditions

$$\begin{aligned} \epsilon^* &= -\varepsilon\eta^t\alpha_\epsilon \mathcal{C}A\rho\epsilon\,, \\ \psi_\mu^* &= -\varepsilon\eta^t\alpha_\psi \mathcal{C}A\rho\psi_\mu\,, \end{aligned} \tag{6.4.3}$$

with

$$A = (\Gamma_0)(i\Gamma_1)\dots(i\Gamma_{t-1})\,, \tag{6.4.4}$$

where Γ_a are elements of the (1,9) Clifford algebra, i.e. those appearing in the complex action and (6.4.2). We propose the following reality conditions for the vielbeine:

$$(e_\mu{}^a)^* = \alpha_\mu^a e_\mu{}^a\,. \tag{6.4.5}$$

Using this definition and (6.4.3), one can calculate that[8]

$$\alpha_\mu^a = (-)^t A^{-1}\Gamma_0\Gamma^a\Gamma_0 A(\Gamma^a)^{-1}\,, \tag{6.4.6}$$

by taking the complex conjugate of (6.4.2). In the case we take A of the form (6.4.4) and divide the index a as $i=1\dots t-1$, $j=t\dots 9$ this implies

$$\alpha_\mu^0 = 1\,,\quad \alpha_\mu^i = -1\,,\quad \alpha_\mu^j = 1\,. \tag{6.4.7}$$

So parts of the vielbein are imaginary and indeed this exactly implies the metric $g_{\mu\nu}$ now has the signature (t,s).

Although everything works perfectly in this way it is rather odd to work with vielbeine that have imaginary components. This is why in the previous section we preferred to work in a formulation where the vielbein is always completely real. This can be accomplished by simultaneously redefining the appropriate components $e_\mu^{\ i} = i\tilde{e}_\mu^{\ i}$ and $\Gamma^i = i\tilde{\Gamma}^i$. It is clear that this redefinition changes the signature of the flat metric η_{ab} as the Clifford algebra now has signature (t,s). Furthermore, in all supersymmetry transformations and the action the vielbeine and Γ's appear in pairs of the form $e^\mu{}_a\Gamma^a$ or $e_\mu{}^a\Gamma_a$ and as such always in a combination where one of the redefined variables appears through its inverse. This means that we can put tildes everywhere without changing the form of the expressions or having to add i's or minus signs. One should read the previous section with this redefinition in mind although we did not explicitly write the tildes, i.e. in section 6.3 flat gamma-matrices appearing in real actions and supersymmetry transformations are always elements of the Clifford algebra that has the same signature as space-time and all vielbeine are real.

[8] One has to use that $\alpha_\epsilon\alpha_\psi = (-)^{t+1}$, which follows from analysing the other supersymmetry variations.

6.4.2 Imaginary Vielbeine without Signature Change

The discussion above brings about another point. One can also take some of the components of the vielbein imaginary and still obtain signature (1,9) for the curved metric. This can be achieved by taking for instance the following matrix A:

$$A = i\Gamma_9\,. \tag{6.4.8}$$

This still leads to a consistent reality condition for the fermions. As explained before A determines what is space and what is time in the real slice. The choice (6.4.8) corresponds to

$$\alpha^0_\mu = -1\,,\quad \alpha^i_\mu = 1\,,\quad \alpha^9_\mu = -1\,,\qquad (i = 1\ldots 8)\,. \tag{6.4.9}$$

The naturally redefined $\tilde{\eta}^{(1,9)}_{ab}$ now has $\tilde{\eta}^{(1,9)}_{00} = \tilde{\eta}^{(1,9)}_{ii} = 1$ and $\tilde{\eta}^{(1,9)}_{99} = -1$ while the original $\eta^{(1,9)}_{ab}$ from the complex theory had $\eta^{(1,9)}_{00} = -1$ and $\eta^{(1,9)}_{ii} = 1 = \eta^{(1,9)}_{99}$.

This choice for the vielbein does not lead to new real actions. Changing the role of different coordinates from timelike to spacelike and vice versa, but keeping the signature fixed, amounts to no more than a relabelling of the coordinates. The action and supersymmetry variations are not affected by this permutation of coordinates.

This is not true however for solutions of its equations of motion. A generic solution is not invariant under exchange of a timelike and spacelike coordinate. For a complexified version of such a solution, interchanging coordinates again is equivalent to a relabelling that does not lead to a different complex solution, as there is no notion of space or time anymore. So given a real solution, if we complexify it and then go back to a real form by imposing different reality conditions it can happen that two coordinates interchange their space- and timelike character. To keep track of this effect when taking real slices of a complex solution it is most practical to work with imaginary vielbeine. In this way one can see explicitly which coordinates will be timelike and which spacelike in a different real form. One can see this explicitly at work in section 6.5.1.

6.5 Domain-Walls and Cosmologies

In this section we will apply the previously discussed method of complex actions and real slices to construct and relate different real solutions. We discuss two examples. Our first example is in massive IIA, where we find a realisation of the domain-wall/cosmology correspondence of [67, 98, 112] in a supersymmetric theory. After that we look at 9d gauged maximal supergravity. This as an example to show that our method works in different dimensions. Note that our examples are all in theories with extended supersymmetry, as this seems necessary to be able to take different real slices in the same signature with our method.

In this section we work in the Einstein frame (E), related to the string frame (S) via $g^{(S)}_{\mu\nu} = \mathrm{e}^{\phi/2} g^{(E)}_{\mu\nu}$.

6.5.1 10d Massive IIA/A*

We truncate the complex massive IIA theory (6.3.2) to the following action

$$S_{\mathrm{mIIA}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x\, e \Big(\mathcal{R} - \frac{1}{2}\big(\partial\phi\big)^2 - \tfrac{1}{2}\mathrm{e}^{5\phi/2} m^2 \Big) , \tag{6.5.1}$$

where $m = G^{(0)}$ is the Romans' mass parameter. The fermionic part of the truncated supersymmetry transformations is

$$\begin{aligned} \delta\psi_\mu &= \Big(\nabla_\mu - \frac{1}{32} W \Gamma_\mu \Big) \epsilon , \\ \delta_\epsilon \lambda &= \Big(\partial\!\!\!/\phi + \frac{\delta W}{\delta\phi} \Big) \epsilon . \end{aligned} \tag{6.5.2}$$

For our theory (6.5.1) the scalar potential and superpotential are respectively

$$V = \frac{1}{2}\Big(\frac{\delta W}{\delta\phi}\Big)^2 - \frac{9}{32} W^2 = \frac{1}{2}\mathrm{e}^{5\phi/2} m^2 , \quad W = \mathrm{e}^{5\phi/4} m . \tag{6.5.3}$$

The complex equations of motion are given by

$$\begin{aligned} 0 &= \frac{1}{e}\partial_\mu \Big(e g^{\mu\nu} \partial_\nu \phi \Big) - \frac{5}{4}\mathrm{e}^{5\phi/2} m^2 , \\ G_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} (g^{\eta\rho}\partial_\eta\phi\partial_\rho\phi) - \frac{1}{2} g_{\mu\nu} \mathrm{e}^{5\phi/2} m^2 \Big) . \end{aligned} \tag{6.5.4}$$

We propose the following complex Ansatz for a supersymmetric solution. As we will show, it can be seen as the complexification of both a domain-wall and a cosmology:

$$\begin{aligned} e_\mu^0 &= a_0 H^{1/16} \delta_\mu^0 , \\ e_\mu^i &= a_i H^{1/16} \delta_\mu^i \qquad (i = 1 \ldots 8) , \\ e_\mu^9 &= a_9 H^{9/16} \delta_\mu^9 , \\ \phi &= -\frac{5}{4} \log H , \end{aligned} \tag{6.5.5}$$

here a_a are some constant complex numbers and H is a complex function depending only on the coordinate x^9. The complex metric is given by $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta^{(1,9)}_{ab}$, as in the previous section 6.3. For this Ansatz the equations of motion (6.5.4) and the supersymmetry condition from (6.5.2) reduce to:

$$\partial_9 H = a_9 m . \tag{6.5.6}$$

So we find the following complex solution to the complexified massive theory:

$$\begin{aligned} \mathrm{d}s^2 &= H^{1/8}\Big(-a_0^2(\mathrm{d}x^0)^2 + (a_i)^2(\mathrm{d}x^i)^2\Big) + H^{9/8}a_9^2(\mathrm{d}x^9)^2\,, \\ \mathrm{e}^\phi &= H^{-5/4}\,, \qquad \text{with } \ H = 1 + a_9 m x^9\,. \end{aligned} \tag{6.5.7}$$

It is invariant under the following complex supersymmetries:

$$\Gamma_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,, \tag{6.5.8}$$

where ϵ_0 is a constant Dirac spinor. In section 6.3 we explained how the complex action (6.5.1) can give rise to several different real theories by taking different real slices. If we now apply these reality conditions on the bosonic fields to our complex solution we will find different real solutions. The different inequivalent reality properties consistent with section 6.3 and (6.5.6) are given for some signatures in table 6.5.1, note that here we allow for imaginary vielbeine. Let us illustrate how the complex Ansatz reduces to a supersymmetric domain-wall in massive IIA (mIIA) and a supersymmetric cosmology in mIIA* by imposing the reality conditions.

t	0	1		2
type	mIIA*	mIIA	mIIA*	mIIA
α_m	$+$	$+$	$-$	$-$
α_ϕ	$+$	$+$	$+$	$+$
α_μ^0	$-$	$+$	$-$	$+$
α_μ^1	$+$	$+$	$+$	$+$
α_μ^i	$+$	$+$	$+$	$+$
α_μ^9	$+$	$+$	$-$	$-$
A	$\mathbb{1}$	Γ_0	$i\Gamma_9$	$i\Gamma_0\Gamma_9$

Table 6.5.1: *Possible reality conditions on the fields of the truncated massive IIA supergravity (mIIA), consistent with the equations of motion. t is the number of timelike directions in space-time. When dealing with solutions it is preferable to allow for imaginary vielbeine as explained in section 6.4. A is the product of all Γ's that are timelike in the real theory and it appears in e.g. the reality condition for the fermions.*

Domain-wall in mIIA The standard reality conditions lead to mIIA and can be found in the second column of table 6.5.1. As all the fields are real, the action coincides with (6.5.1). The complex solution becomes the well known domain-wall or D8-brane of mIIA:

$$\begin{aligned} \mathrm{d}s^2 &= H^{1/8}\Big(-(\mathrm{d}x^0)^2 + (\mathrm{d}\vec{x})^2\Big) + H^{9/8}(\mathrm{d}x^9)^2\,, \\ \mathrm{e}^{\phi} &= H^{-5/4}\,, \quad \text{ with } \; H = 1 + mx^9\,. \end{aligned} \tag{6.5.9}$$

The complex supersymmetry variations (6.5.2) become

$$\begin{aligned} \delta\psi_\mu &= \Big(\nabla_\mu - \frac{1}{32} W \Gamma_\mu\Big)\epsilon\,, \\ \delta_\epsilon\lambda &= \Big(\partial\!\!\!/\phi + \frac{\delta W}{\delta\phi}\Big)\epsilon\,, \end{aligned} \tag{6.5.10}$$

where W is given by

$$W = \mathrm{e}^{5\phi/4} m\,, \quad \text{with } m \in \mathbb{R}\,. \tag{6.5.11}$$

It is not difficult to verify that the domain-wall (6.5.9) has the following unbroken real supersymmetries

$$\Gamma_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,, \tag{6.5.12}$$

where ϵ_0 now is a constant Majorana spinor.

Cosmology in mIIA* Alternatively, we can apply the reality conditions of mIIA* to the complex solution (6.5.7). As can be read from table 6.5.1 in this case m is purely imaginary, as are two components of the vielbein: ${e_\mu}^0$ and ${e_\mu}^9$. This implies that in this case $a_9 = a_0 = i$. We redefine $m = i\tilde{m}$, ${e_\mu}^0 = i{\tilde{e}_\mu}{}^0$, ${e_\mu}^9 = i{\tilde{e}_\mu}{}^9$, $\Gamma^0 = i\tilde{\Gamma}^0$ and $\Gamma^9 = i\tilde{\Gamma}^9$. Substituting all this in the complex solution (6.5.7) gives us a supersymmetric cosmological solution of mIIA*:

$$\begin{aligned} \mathrm{d}s^2 &= H^{1/8}\Big((\mathrm{d}x^0)^2 + (\mathrm{d}\vec{x})^2\Big) - H^{9/8}(\mathrm{d}x^9)^2\,, \\ \mathrm{e}^{\phi} &= H^{-5/4}\,, \quad \text{ with } \; H = 1 - \tilde{m}x^9\,, \end{aligned} \tag{6.5.13}$$

where $\tilde{m} = F^{(0)}$. Note that this is the E9-brane of [26]. Note that also the real action of mIIA* is different than that of mIIA. It is given in terms of real fields by

$$S_{\text{mIIA}^*} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\, e\,\Big(\mathcal{R} - \frac{1}{2}\big(\partial\phi\big)^2 + \tfrac{1}{2}\mathrm{e}^{5\phi/2}\tilde{m}^2\Big)\,, \tag{6.5.14}$$

with the corresponding supersymmetry variations

$$\begin{aligned} \delta\psi_\mu &= \Big(\nabla_\mu - \frac{i}{32}\tilde{W}\Gamma_\mu\Big)\epsilon\,, \\ \delta\zeta &= \Big(-i\partial\!\!\!/\phi + \frac{\delta\tilde{W}}{\delta\phi}\Big)\epsilon\,. \end{aligned} \tag{6.5.15}$$

The superpotential $\tilde{W}$ is real and given by

$$\tilde{W} = e^{5\phi/4}\tilde{m}\,, \quad \text{with } \tilde{m} \in \mathbb{R}\,. \tag{6.5.16}$$

As was the case for the domain-wall it is easy to check that the cosmology (6.5.13) preserves the following supersymmetries

$$i\tilde{\Gamma}_9\epsilon = \epsilon\,, \quad \epsilon = H^{1/32}\epsilon_0\,. \tag{6.5.17}$$

Note that now ϵ is not a standard Majorana spinor but satisfies a *MW reality condition instead.

The domain-wall and cosmology presented above are a particular example of the domain-wall / cosmology correspondence of [67, 98, 112], where an embedding in extended supergravity is possible. Indeed the truncated mIIA theory is exactly of the gravity-scalar form as proposed there, as is its domain-wall solution. Furthermore the truncated mIIA* theory is equal to the truncated mIIA theory up to a relative sign in front of the scalar potential. This example places the domain-wall / cosmology correspondence in a supersymmetric context. This means that the Killing spinor of the solution generates a supersymmetry of the theory. Furthermore we see that what was called a pseudo-Killing spinor in [67, 98, 112] now is a generator of a genuine supersymmetry, but in a star theory. In this example pseudo-supersymmetry in an extended supersymmetric theory coincides with the supersymmetry of a superalgebra obeying star reality conditions.

Instanton in Euclidean mIIA* In section 6.1 we noticed that for every domain-wall of a model with potential V there corresponds, besides the cosmology with $-V$, an instanton [113]. The first column of table 6.5.1 agrees with this. The α_m is the same for Euclidean mIIA* in (0,10) and mIIA in (1,9) space-time dimensions, hence they have the same potential. Also do we see that $\alpha^0_\mu = -1$ and hence the corresponding vielbein is imaginary. As a result we have a Euclidean theory.

6.5.2 Maximal Gauged Supergravity in 9d

In the previous example the only scalar field that played a role was the dilaton. In subsection 5.3.2 pseudo-supersymmetry was studied in systems with explicit multiple scalar fields. As mentioned there, in [99, 103] it was shown that for the domain-wall / cosmology correspondence subtleties can appear when axions are included. In light of this we consider a more general example with multiple scalar fields including an axion.

The theory we will be working with is given by the following truncated $\mathcal{N} = 2$, d=9 gauged supergravity Lagrangian

$$\mathcal{L}_{9\text{d}} = \tfrac{1}{2}\, e\left[\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}(\partial l)^2 - \tfrac{1}{2}(\partial\varphi)^2 - V(\phi, l, \varphi)\right], \tag{6.5.18}$$

with V given by

$$V = \frac{1}{2}\mathrm{e}^{-2\phi+4\varphi/\sqrt{7}}\Big(q_1^2 + 2\mathrm{e}^{2\phi}q_1(-q_2+q_1l^2) + \mathrm{e}^{4\phi}(q_2+q_1l^2)^2\Big)\,. \qquad (6.5.19)$$

The details of the reduction from IIB are given in [121, 122]. The scalar fields are given by ϕ, φ and l. The constants q_1 and q_2 specify the gauging. We group the 9d, 16-component $\mathcal{N}=2$ spinors χ_i in doublets

$$\chi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}\,. \qquad (6.5.20)$$

From table B.1.1 we see that $\epsilon=\eta=-1$. In this section we will use $\mathcal{C}=\gamma_0$. The (1,8) gamma-matrices γ_μ are then purely imaginary. In this notation the supersymmetry transformations of the fermions are

$$\begin{aligned}
\delta_\epsilon\psi_\mu &= \big[\partial_\mu + \omega_\mu - (\tfrac{1}{4}\mathrm{e}^{\phi}\partial_\mu l - \tfrac{i}{28}\gamma_\mu W)i\sigma_2\big]\,\epsilon\,,\\
\delta_\epsilon\lambda &= (i\,\partial\!\!\!/\phi - \mathrm{e}^{-\phi}\delta_l W)\sigma_3\,\epsilon + \mathrm{e}^{\phi}(i\,\partial\!\!\!/ l + \mathrm{e}^{-\phi}\delta_\phi W)\sigma_1\,\epsilon\,,\\
\delta_\epsilon\tilde\lambda &= i\,\partial\!\!\!/\varphi\sigma_3\epsilon + \delta_\varphi W\sigma_1\,\epsilon\,.
\end{aligned} \qquad (6.5.21)$$

The superpotential W is given by

$$W = \mathrm{e}^{2\varphi/\sqrt{7}}\Big(\mathrm{e}^{-\phi}q_1 + \mathrm{e}^{\phi}(q_2+q_1l^2)\Big)\,. \qquad (6.5.22)$$

The above action and supersymmetry rules can be made complex via the method of section 2. It turns out that there are two real slices for signature (1,8), see table 6.5.2. We denote the star version of the truncated $\mathcal{N}$=2, d=9 theory by 9d*. Inspired by

	9d	9d*
$\varepsilon=\eta$	$-$	$-$
ρ	$\mathbb{1}$	σ_3
α_l	$+$	$-$
$\alpha_\epsilon=\alpha_\lambda=\alpha_{\tilde\lambda}$	$+$	$+$
$q_1=q_2$	$+$	$-$

Table 6.5.2: *The two sets of reality conditions appearing in the truncated $\mathcal{N}$=2, d=9 gauged supergravity Lagrangian leading to signature (1,8).*

the domain-walls of [122], we propose the following complex Ansätze:

$$
\begin{aligned}
e_\mu^0 &= a_0 h^{1/28}\delta_\mu^0\,,\\
e_\mu^i &= h^{1/28}\delta_\mu^i \qquad (i=1\ldots 7)\,,\\
e_\mu^8 &= a_8 h^{-3/14}\delta_\mu^8\,,\\
\mathrm{e}^{\phi} &= h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\quad l=c_1{h_1}^{-1}\,,\\
h &= h_1h_2-c_1^2\,,
\end{aligned}
\tag{6.5.23}
$$

where a_a and c_1^2 are some arbitrary complex constants and h_1 and h_2 are functions of x^8 only. This Ansatz is a supersymmetric complex solution if

$$
\begin{aligned}
\partial_8 h_1 &= 2a_8q_1\,,\\
\partial_8 h_2 &= 2a_8q_2\,.
\end{aligned}
\tag{6.5.24}
$$

In this case it has a complex Killing spinor of the form

$$
\epsilon = h^{1/56}(\cos f\,\mathbb{1}_2 - i\sin f\,\sigma_2)\epsilon_0\,, \tag{6.5.25}
$$

with

$$
f=\frac{1}{4}\arctan\left(\frac{2c_1q_1h^{1/2}}{q_2h_1^2-q_1h+q_1c_1^2}\right)\,, \tag{6.5.26}
$$

and ϵ_0 is a doublet of constant Dirac spinors that satisfies

$$
\gamma^8\sigma_2\epsilon_0=\epsilon_0\,. \tag{6.5.27}
$$

As in the previous subsection one can now take two different real slices leading to a pair of real solutions in signature (1,8). Taking all the fields in (6.5.23) real leads back to the familiar domain-walls of [122]:

$$
\begin{aligned}
&ds^2 = h^{1/14}(-(\mathrm{d}x^0)^2+\mathrm{d}\vec{x}^2)+h^{-3/7}(\mathrm{d}x^8)^2\,,\\
&\mathrm{e}^{\phi}=h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\qquad l=c_1{h_1}^{-1}\,,\\
&h_1=2q_1x^8+k_1^2,\ h_2=2q_2x^8+k_2^2 \quad\text{and}\quad h=h_1h_2-c_1^2\,,
\end{aligned}
\tag{6.5.28}
$$

where k_i are integration constants. As noted above there is another set of consistent reality conditions. This second real slice of (6.5.23) will give a cosmological solution. From table 6.5.2 one can see which fields become purely imaginary. To write everything in terms of real fields we redefine $q_i=i\tilde{q}_i$, $c_1=i\tilde{c}_1$ and $l=i\tilde{l}=i\tilde{c}_1h_1^{-1}$. Consistency with the equations of motion also requires ${e_\mu}^0$ and ${e_\mu}^8$ to be imaginary, i.e. $a_8=a_0=i$. The Ansatz (6.5.23) written in terms of these real fields gives the following cosmological solution:

$$
\begin{aligned}
&ds^2 = h^{1/14}((\mathrm{d}x^0)^2+\mathrm{d}\vec{x}^2)-h^{-3/7}(\mathrm{d}x^8)^2\,,\\
&\mathrm{e}^{\phi}=h^{-1/2}h_1\,,\quad \mathrm{e}^{\sqrt{7}\varphi}=h^{-1}\,,\qquad \tilde{l}=\tilde{c}_1{h_1}^{-1}\,,
\end{aligned}
\tag{6.5.29}
$$

where

$$h = h_1 h_2 + \tilde{c}_1^2 \,, \qquad h_1 = 2\tilde{q}_1 x^8 + k_1^2 \,, \qquad h_2 = 2\tilde{q}_2 x^8 + k_2^2 \,. \tag{6.5.30}$$

This is a real cosmological solution of the star version of the 9d theory, of which the action can easily be constructed along the lines of section 6.3.

Again we find a natural relation between domain-wall solutions and cosmological solutions as different real slices of a single complex solution. In this case the relation between the two real theories is slightly more involved than just reversing the overall sign of the scalar potential. It is not difficult to see that the scalar potential in the 9d* theory is now

$$V = -\frac{1}{2} \mathrm{e}^{-2\phi + 4\varphi/\sqrt{7}} \Big(\tilde{q}_1^2 - 2\mathrm{e}^{2\phi} \tilde{q}_1 (\tilde{q}_2 + \tilde{q}_1 \tilde{l}^2) + \mathrm{e}^{4\phi} (\tilde{q}_2 - \tilde{q}_1 \tilde{l}^2)^2 \Big) \,. \tag{6.5.31}$$

So apart from an overall change in sign with respect to (6.5.19) there are also relative sign changes between the different terms in the potential. This goes together with a signature change of the scalar manifold in the 9d* case as the axion $\tilde{l}$ has wrong sign kinetic term. Note that if the theory is truncated by setting the axion to zero we again find an example where one can embed the correspondence of [67, 98, 112] in a supersymmetric theory. Also in this case the pseudo-supersymmetry of the cosmology can be interpreted as the vanishing of the fermionic supersymmetry transformations of a theory with star reality conditions.

6.5.3 E-branes

In the previous sections we have seen that star supergravities have a non-Riemannian scalar coset. In section 6.5.1 we gave an example of a time-dependent half supersymmetric solution of mIIA*. This solution is called an E8-brane or Ep-brane in general. The E stands for the Euclidean worldvolume and the p reflects that we have a (p+1)-dimensional worldvolume[9]. Note that Sp-branes also have a (p+1)-dimensional worldvolume but they are brane solutions of standard type II theories instead of type II* theories.

In [26, 59] the full BPS-brane analysis for the star supergravities was done. Just like Sp-branes they are time-dependent. The difference is that E-branes are BPS-solutions of star supergravities, while Sp-branes are not supersymmetric solutions of standard type II supergravities. At the end of section 2.4 we mentioned that if we demand that an extremal S-brane satisfies the extremality condition (2.4.20) the field strength turns out to be imaginary. Such a solution can be embedded in a star supergravity. This shows that in a star supergravity we can have solutions satisfying (2.4.20).

[9] In the original papers an Ep-brane means a brane with a p-dimensional Euclidean worldvolume. We however prefer to use the same notation as that for p- and Sp-branes.

Let us show this explicitly for the S(−1)-brane given in section 2.4. We need to find the extremal version of this solution. The only deformation parameter of the metric (2.4.32) is $||v||$, we expect that the limit of $||v|| \to 0$ should give us the extremal solution. For this we need to restrict to a $k = -1$ slicing since only this describes Minkowski space-time in Milne coordinates. However, from (2.4.33) we see that the limit $||v|| \to 0$ gives us real and constant scalar fields. We therefore have to re-scale the constants in a specific way.

Inspired by the extremal limit given in [123] we consider the following series of limits

$$c_2 \to c_2 + \frac{1}{2}\pi i\,, \quad c_1 \to \frac{c_1}{i||v||}\,, \quad c_2 \to \frac{g_s||v||}{c_1}\,, \quad c_3 \to ic_3\,, \quad ||v|| \to 0\,. \tag{6.5.32}$$

A direct calculation shows that this leads to the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^2\mathrm{d}\mathbb{H}^2_{D-1}\,, \tag{6.5.33}$$

and the axion and dilaton are given by

$$\mathrm{e}^{\phi} = h\,, \quad \chi = i(\pm h^{-1} + c_3)\,. \tag{6.5.34}$$

Note that the axion is imaginary as required. The harmonic function h is given by

$$h = g_s + \frac{c_1}{(D-2)t^{D-2}}\,. \tag{6.5.35}$$

This is the solution as given in [41]. We see that the extremal limit of the S(−1)-brane indeed leads to an imaginary solution which can be embedded in IIB* supergravity. We have derived the extremal E(−1)-brane as given in [26].

6.6 Discussion

In the first section we gave an overview of the domain-wall / cosmology correspondence. A natural question that arose is whether one can give a meaning to pseudo-supersymmetry in a real supergravity context.

To achieve this we complexified the type II supergravities and their supersymmetry rules. These complex actions do not describe physical theories but are a useful mathematical tool that allows to write down the actions for all variant supergravities as real slices of the complex action. We illustrated the method in detail for the standard type II theories and their corresponding star versions in signature (1,9). Although we restricted our analysis to 10 dimensions one can generalize it to lower dimensions, for some related results see [124–132]. In this chapter we gave an additional example for $\mathcal{N} = 2$ in 9 dimensions.

In the last part of this chapter, we have looked at solutions of these complex theories and shown that one can obtain solutions of the different real theories by taking real slices. In particular, we have seen that in this way supersymmetric domain-walls and (pseudo-) supersymmetric cosmologies can arise as different slices of one complex solution. The domain-walls are solutions in an ordinary supergravity, while the cosmologies arise as solutions of the star version. In this sense the pseudo-supersymmetry of cosmologies corresponds to supersymmetry in the star theory. We presented a ten-dimensional example where the domain-wall / cosmology correspondence of [67, 98, 112] can be embedded into an extended supergravity context. We also noticed that it can be extended to include instantons as well [110]. In another example in 9 dimensions we again constructed a domain-wall and corresponding cosmology. A noteworthy feature of this last example is that the potential no longer gets an overall sign flip, but only certain terms in the potential change sign. Furthermore the scalar manifold changes signature. This might hint that also in a fake supergravity context more general changes in the potential could appear under the map of a domain-wall to a cosmology.

In the last subsection we pointed out a relation between extremal Sp- and Ep-branes.