Particle dynamics of branes

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Chapter 5

Massive Time-Dependent Solutions

In this chapter we extend the time-dependent analysis to Lagrangians with a potential

$$\mathcal{L} = \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j - V(\phi)\Big)\,. \tag{5.0.1}$$

We mentioned in section 3.5 that we can regard the scalar potential V as a 0-form field strength. This can couple magnetically to $(D-2)$-branes, i.e. domain-walls (timelike) and cosmologies (spacelike). In this chapter we will focus mainly on the latter, although many results apply to domain-walls as well.

We begin this chapter with a brief introduction to cosmologies. We will focus on a specific cosmological model, namely the generalization of the multi-exponential potential we introduced in section 3.3. We will not look for the most general solution, but restrict ourself to a critical point analysis. It turns out that these critical points are so-called scaling solutions. The surprising thing is that these scaling solutions are *still* geodesics of the scalar manifold, the presence of the potential does not upset this. In this chapter we are going to state the condition when for a potential V the scaling solution is still a geodesic of the scalar manifold. For a general discussion about cosmologies we refer to e.g. [77, 78].

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5.1 Cosmologies

Due to cosmological observations we know now that our universe is both homogeneous and isotropic on the large scale. This means that our place in the universe is not special (homogeneous) and that the universe looks the same in all directions (isotropic). For this reason cosmological space-times are described by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 g_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{5.1.1}$$

where g_{ij} is the three-dimensional spatial metric and $a(\tau)$ is called the scale factor. Due to the observation that our universe is both homogeneous and isotropic g_{ij} can written as

$$\mathrm{d}s^2 = \frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{n-1} \,. \tag{5.1.2}$$

The Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G\, T_{\mu\nu} \,, \tag{5.1.3}$$

relates the space-time metric to the matter distribution in space-time. The latter is encoded in the energy-momentum tensor $T_{\mu\nu}$. For the FLRW-metric (5.1.1) we derive that this must have the form

$$T_{\mu\nu} = \mathrm{diag}(\rho, p g_{ij}) \,. \tag{5.1.4}$$

Here $\rho(\tau)$ is the energy density and $p(\tau)$ the pressure. The matter distribution is called the cosmological fluid. After rewriting the Einstein equations we derive the Friedmann equations

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \,, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(3p+\rho) \,. \end{aligned} \tag{5.1.5}$$

Here the function $H = \dot{a}/a$ is called the Hubble parameter and the dot is with respect to τ. Due to the conservation of energy we have for the cosmological fluid the continuity equations

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} + 3H(\rho + p) = 0 \,. \tag{5.1.6}$$

Finally, the relation between the energy density and the pressure is given by the equation of state parameter ω

$$p = \omega\rho \,. \tag{5.1.7}$$

For 'ordinary' matter such as radiation or dust $-1/3 < \omega < 1$. All cosmological fluids can be grouped in two different classes. Namely, those that respect the strong energy condition (SEC) and those that violated it, see e.g. [80]. The SEC is a specific condition on the energy momentum tensor. For the cosmological fluid this means that the matter has to obey $\omega > -1/3$.

For a flat universe we derive from the Friedmann equations (5.1.5) that they imply

$$\ddot{a} > 0 \longleftrightarrow (3\omega + 1)\rho < 0\,. \tag{5.1.8}$$

In other words, if matter does not obey the SEC we have that $\omega < -1/3$ and we see that we have accelerated expansion ($\rho > 0$).

When ω is constant we find from the (5.1.6) that

$$\rho \propto \frac{1}{a^{3(1+\omega)}}\,. \tag{5.1.9}$$

Using this together with the Friedmann equations we can solve for the scale factor, in case $k = 0$ we find

$$a(t) \propto \tau^{\frac{2}{3(\omega+1)}}\,. \tag{5.1.10}$$

Such a scale factor is called a power-law. When $k \neq 0$ the scale factor can also be solved but is more complicated.

5.1.1 Multi-Field Cosmology

Let us assume that we have a system consisting of N scalar fields ϕ^i with scalar manifold metric $G_{ij} = \delta_{ij}$ and a potential $V(\vec{\phi})$

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\Big[\frac{1}{2\kappa^2}\mathcal{R} - \tfrac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} - V(\vec{\phi})\Big]\,, \tag{5.1.11}$$

where $\kappa^2 = 8\pi G$ with G Newton's constant and

$$\partial\vec{\phi}\cdot\partial\vec{\phi} = \sum_{i=1}^{N}\delta_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{5.1.12}$$

The Ansatz for the metric is that of a flat ($k = 0$) FLRW-universe (5.1.1, 5.1.2) and accordingly the scalars only depend on cosmic time τ. The equations of motion are

$$H^2 = \tfrac{\kappa^2}{3}\Big[T + V(\vec{\phi})\Big]\,, \tag{5.1.13}$$

$$\dot{H} = -\kappa^2 T\,, \tag{5.1.14}$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \partial_i V(\vec{\phi}) = 0\,, \tag{5.1.15}$$

where T stands for the kinetic energy

$$T = \frac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi}. \tag{5.1.16}$$

Equation (5.1.15) is referred to as the Klein–Gordon equation. In terms of the fluid language we have

$$\rho = T + V\,, \quad p = T - V\,, \quad \omega = \frac{T-V}{T+V}\,. \tag{5.1.17}$$

The above equations of motion are coupled second order differential equations. Therefore the most general solution is hard to find. Instead we will focus on the late-time behaviour of the cosmological solution. For this we need to find the asymptotic behaviour of a solution. It often happens that this behaviour is determined by simpler equations that the ones above. To see why this is so we show that we can rewrite the equations of motion as an autonomous system.

Autonomous systems

Assume that we have a set of variables $x^i(t)$ that obey a first order equation that can be cast into the following form

$$\dot{x}^i(t) = f^i(x)\,. \tag{5.1.18}$$

Here $f^i(x)$ depends only on the variables x and do not contain the evolution parameter t explicitly. We see that we can consider $\vec{f}$ as the velocity field belonging to the curve $\vec{x}$. Assume now that this vector field $\vec{f}$ has a zero at some point $\vec{x}_0$. This simplifies (5.1.18) to

$$\vec{f}(\vec{x}_0) = \vec{0} \Longrightarrow \dot{x}^i(t) = 0\,. \tag{5.1.19}$$

Such a point $\vec{x}_0$ is called a fixed or critical point. Such a critical point is a trivial solutions since we can easily integrate (5.1.19).

What has this to do with late-time cosmology? A critical point can either be a stable or an unstable solution. The stability follows from checking whether a perturbation δ^i of a fixed point x_0^i vanishes or not. For this we have to plug the perturbation $x_0^i + \delta^i$ into the equations of motion (5.1.18) and only keep terms linear in δ^i. This leads to the set of first order equations

$$\dot{\delta}^i = (\partial_j f^i)|_{x=x_0}\delta^j\,. \tag{5.1.20}$$

The general solutions for the perturbations δ^i are given by

$$\delta^i = \sum_j C^i_j e^{\lambda_j t}\,, \tag{5.1.21}$$

where λ_j are the eigenvalues of the matrix $(\partial_j f^i)|_{x=x_0}$ and C^i_j are real constants. If it happens that all λ_j are negative, we see from (5.1.21) that the perturbations decay exponentially. Such a critical point is called an attractor or sink.

If on the other hand some of the λ_j are positive these perturbations grow exponentially. Such a critical point is called a saddle point. When all eigenvalues are positive the point is called a repeller or source.

Let us now answer the question we posed above. Although we cannot solve for a general solution, it will generically be a curve in phase space interpolating between two critical points. For example, the curve can start at a repeller and will asymptotically reach the attractor. Thus we see that critical points, determined by the properties of the vector field $\vec{f}$, are important in understanding the general interpolating solution. In particular, the late-time cosmology is determined by the attractor critical point. Due to the exponential behaviour the attractor will not be reached in finite time.

We will now illustrate this with a specific potential.

5.1.2 Generalized Assisted Inflation

In section 3.3 we found the multi-exponential potential (3.3.21). Such exponential potentials also arise in models motivated by string theory such as supergravities obtained from dimensional reduction (see for instance [46,81–84] and references therein), descriptions of brane interactions [85–87], nonperturbative effects and the effective description of string gas cosmology (see for instance [88]). Also, these models allow to find exact solutions, which correspond to critical points in an autonomous system[1].

Let us generalize (3.3.21) to a sum of M exponential terms

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa \langle \alpha_a \, , \, \phi \rangle], \tag{5.1.22}$$

where $\langle \alpha_a \, , \, \phi \rangle = \sum_{i=1}^{N} \alpha_{ai}\phi_i$. There are M vectors α_a with N components α_{ai}. The indices $i, j, \ldots$ run from 1 to N and denote the components of the vectors ϕ and α_a. The indices $a, b, \ldots$ run from 1 to M and label the different vectors α_a and the constants Λ_a.

Let us now make use of linear field redefinitions. If the scalars transform linearly as $\phi \longrightarrow \phi' = S\phi$, where S is an element of $\mathrm{GL}(\mathbb{R}, N)$ then the vectors α_a transform in the dual representation $\alpha_a \longrightarrow \alpha'_a = S^{-T}\alpha_a$. This can be seen from the definition of α'_a

$$\langle \alpha_a \, , \, \phi \rangle \equiv \langle \alpha'_a \, , \, \phi' \rangle \, . \tag{5.1.23}$$

Field redefinitions that shift the scalar fields leave the α_a invariant, but change the Λ_a.

From the action we can deduce some properties of this system by looking at transformations in scalar space. The kinetic term is invariant under constant shifts and $O(N)$-rotations of the scalars. These transformations map the multiple exponential

[1] For a review on dynamical systems in cosmology see [89].

potential to another multiple exponential potential but with different Λ_a and α_a. Such redefinitions do not alter the physics, they just rewrite the equations. Therefore qualitative features only depend on $O(N)$-invariant combinations of the α_a-vectors (for example $\langle\alpha_a\,,\,\alpha_b\rangle$). By shifting the scalars we can always re-scale R of the Λ_a to be ±1, where R is the number of independent α-vectors

$$R = \mathrm{Rank}[\alpha_{ai}]\,. \tag{5.1.24}$$

The number of linearly independent vectors α_a is denoted by R. If $R < N$ one can rotate the scalars such that $\phi_{R+1}, \ldots, \phi_N$ no longer appear in the potential ($\alpha_{ai} = 0$ for $i > R$). These scalars are then said to be decoupled or free.

Let us illustrate this for the single exponential potential

$$V = \Lambda \mathrm{e}^{\vec{\alpha}\cdot\vec{\phi}}\,. \tag{5.1.25}$$

According to the above we should be able to remove all but one scalar field from the potential (5.1.25). To achieve this consider the orthogonal field redefinition $\vec{\phi} \to \vec{\phi}'$

$$\phi'_1 = \frac{1}{||\vec{\alpha}||}\vec{\alpha}\cdot\vec{\phi}\,, \tag{5.1.26}$$

and the ϕ'_i $(i > 1)$ are constructed orthogonal to this direction via a Gramm-Schmidt procedure. This procedure is guaranteed to preserve the kinetic term, but the potential now contains only one scalar field as claimed

$$V = \Lambda \mathrm{e}^{||\vec{\alpha}||\phi'_1}\,. \tag{5.1.27}$$

We end up with one massive field $\vec{\phi}'_1$ and $N-1$ massless fields ϕ'_i $(i > 1)$. Although (5.1.25) looks like an interaction term, we find that it can be removed via field redefinitions. The resulting theory has only one self-interaction term and $N-1$ free fields.

One can rewrite equations (5.1.13-5.1.15) as an autonomous system. For this we first note that (5.1.14) is not an independent equation and can therefore be ignored. Secondly, we define the following dimensionless variables

$$X_i = \frac{\kappa\dot{\phi}_i}{\sqrt{6}\,H}, \quad Y_a = \frac{\kappa^2}{3\,H^2}\,\Lambda_a \exp[-\kappa\,\langle\alpha_a\,,\,\phi\rangle]\,. \tag{5.1.28}$$

If we write $X^2 = \sum_i X_i^2$ and $Y = \sum_a Y_a$ the equations of motion become

$$X^2 + Y - 1 = 0\,, \tag{5.1.29}$$

$$X'_i = 3X_i\Big(-1 + X^2\Big) + \sqrt{\tfrac{3}{2}}\sum_a \alpha_{ai} Y_a\,, \tag{5.1.30}$$

$$Y'_a = Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 6X^2\Big)\,, \tag{5.1.31}$$

where the prime denotes differentiation with respect to $\log(a)$ [2]. It can be shown that if $X^2+Y-1=0$ initially then equations (5.1.30-5.1.31) guarantee that it is satisfied at all times. Hence, given the correct initial conditions the dynamics is described by equations (5.1.30-5.1.31).

In the next section we will construct all critical point solutions of the autonomous system (5.1.30-5.1.31). For that purpose it is useful to consider the matrix A

$$A_{ab} = \langle \alpha_a \, , \, \alpha_b \rangle \, . \tag{5.1.32}$$

The models are divided into two classes. The first class is defined by $R = M$ and the second class by $R < M$. Algebraically the two differ in the following way

$$1.\ R = M \quad \Longleftrightarrow \quad \mathrm{Det}A > 0\, , \tag{5.1.33}$$
$$2.\ R < M \quad \Longleftrightarrow \quad \mathrm{Det}A = 0. \tag{5.1.34}$$

The first possibility, where A is invertible, is called generalized assisted inflation. We will not discuss the second class, for this we refer to [79].

Assisted inflation

Assisted inflation is the subclass where A is diagonal. This implies that in assisted inflation the α_a are perpendicular to each other and that one can choose an orthonormal basis in which $\alpha_{ai} = \alpha_a \, \delta_{ai}$. In that basis the potential becomes

$$V(\phi) = \sum_{a=1}^{M} \Lambda_a \exp[-\kappa \, \alpha_a \, \phi_a] \, . \tag{5.1.35}$$

It is this particular form of the potential that is referred to as assisted inflation in the literature. We want to emphasize that the latter definition is basis-dependent. Potentials different from (5.1.35) but with a diagonal matrix A can be brought to the form (5.1.35) through an $O(N)$-rotation of the scalar fields. An $O(N)$-invariant definition of assisted inflation is that A is a diagonal matrix.

In any system with multiple fields but a single exponential such as studied in [91], the matrix A is trivially diagonal. One can perform a rotation on the scalars such that only one scalar appears in the potential and all the others are decoupled. In order to have a system whose scalars are mutually interacting one needs at least two exponential terms both containing more than one scalar.

[2] The use of $\log(a)$ as a time coordinate fails when $\dot{a} = 0$. This is no problem for studying critical points and their stability because in the neighborhood around a critical point there always exists a region where the coordinate is well defined. In reference [90] an explicit example is given where $\dot{a}$ becomes zero at some point.

5.1.3 The Critical Points

Critical points are defined as solutions of the autonomous system for which $X_i' = Y_a' = 0$. From the acceleration equation,

$$\dot{H} = -\tfrac{\kappa^2}{2}\partial\vec{\phi}\cdot\partial\vec{\phi}\,, \tag{5.1.36}$$

it follows that in a critical point $\dot{H}/H^2$ is constant. If the constant differs from zero we put $\dot{H}/H^2 \equiv -1/p$ and then the scale factor becomes

$$a(\tau) = a_0\,\big(\frac{\tau}{\tau_0}\big)^p\,. \tag{5.1.37}$$

In terms of the dimensionless variables p can be expressed as

$$p = \frac{1}{3X^2}\,. \tag{5.1.38}$$

When $\dot{H}/H^2 = 0$ the scale factor is

$$a(\tau) = a_0\,\mathrm{e}^{H\tau}\,, \tag{5.1.39}$$

and space-time is de Sitter.

The requirement that $Y_a' = 0$ can be satisfied in two ways as can be seen from

$$Y_a' = Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 6X^2\Big)\,. \tag{5.1.40}$$

Either $Y_a = 0$ or the second factor on the right-hand side equals zero. If we put $Y_a = 0$ by hand and then solve for the X_i and the remaining Y_a, the critical point is called a *nonproper critical point.* If we put the second factor to zero by hand and then solve for X_i and Y_a, the critical point is called a *proper critical point.* The name nonproper is given since a critical point with some $Y_a = 0$ has ∞-valued scalar fields. Therefore these critical points are no proper solutions to the equations of motion, they are asymptotic descriptions of solutions. The proper critical points generically have non-zero Y_a and therefore are proper solutions to the equations of motion. But in some cases one finds that $Y_a = 0$ for proper critical points, although one did not put those Y_a to zero by hand. We will not discuss the non-proper solutions, see [79].

Regardless of whether critical points are proper solutions to the equations of motion, they are all equally important in providing information about the orbits. That is, they are either repellers, attractors or saddle points.

Proper critical points

To construct the proper critical points we demand that $X'_i = Y'_a = 0$, we have the algebraic constraints

$$\boxed{\begin{aligned} 3X_i\Big(-1+X^2\Big) + \sqrt{\tfrac{3}{2}}\sum_a \alpha_{ai}Y_a = 0\,, \\ Y_a\Big(-\sqrt{6}\,\langle\alpha_a\,,\,X\rangle + 6X^2\Big) = 0\,, \end{aligned}} \qquad (5.1.41)$$

To solve these constraints we multiply the first equation above with α_{bi} and sum over i and use (5.1.38, 5.1.41) to find

$$Y_a = 2\frac{3p-1}{3p^2}\sum_b [A^{-1}]_{ab}\,, \quad X_i = \frac{1}{p}\sqrt{\tfrac{2}{3}}\sum_{ab}\alpha_{ai}[A^{-1}]_{ab}\,. \qquad (5.1.42)$$

The value of p is found by combining equations (5.1.29, 5.1.38, 5.1.42)

$$p = 2\sum_{ab}[A^{-1}]_{ab}\,. \qquad (5.1.43)$$

In terms of the scalar fields the solutions read

$$\boxed{\phi_i = \frac{\sqrt{6}X_i\,p}{\kappa}\log|\tau| + c_i\,.} \qquad (5.1.44)$$

The critical point constructed above does not always exist. It is clear from the definition of the Y_a-variables that they must have the same sign as the Λ_a, i.e. [3]

$$(\frac{1}{p} - 3)\sum_b [A^{-1}]_{ab} \gtrless 0 \quad \text{for} \quad \Lambda_a \gtrless 0\,. \qquad (5.1.45)$$

Let us briefly discuss what happens if we couple the system we described so far to a barotropic fluid ρ which represents the matter in our universe and allow the curvature to be non-zero ($k = \pm 1$) [79]. It turns out that we can classify the solutions as *curvature*, *matter-scaling* or *scalar-dominated* scaling solutions. The curvature scaling solutions have the property that $k \neq 0$ while $\rho = 0$ and $p = 1$ in (5.1.38). For the matter-scaling solutions we have that $k = 0$ while $\rho \neq 0$ and finally for scalar dominated solutions we have $k = \rho = 0$. Especially the matter-scaling solution are of interest to cosmologists, since these solutions have the property that they have a non-zero constant ratio between the energy density of the scalar fields and that of

[3] In [82] it is shown that critical points that violate the existence conditions can still play a role in understanding the late time behaviour of general solutions.

the barotropic fluid. If the matter-scaling solution are attractors they could explain why today we see the same order of energy densities for matter and dark energy. The dark energy is related to the scalar fields. In [79] it is assumed that there is no direct coupling between the scalar fields and the barotropic fluid. The only interaction is via gravity. In such cases one can derive that the power-law is given by $p = \frac{2}{3(\omega+1)}$. Here ω is the relation between the energy density and pressure of the barotropic fluid (5.1.7). Surprisingly, this is the same power-law as that of only a barotropic fluid. In this way we see that the scalar field mimics the barotropic fluid.

It was noticed in [70, 79, 92] that for the scaling solutions that we discussed above there exist a field redefinitions $\phi^i \to \varphi^i$ such that the potential can be written as

$$V(\varphi) = \mathrm{e}^{c\varphi_1} U(\varphi_2, \ldots, \varphi_N) \,. \tag{5.1.46}$$

To prove this [70] we first note that if the $\vec{\alpha}_a$ are linearly independent there exist a vector $\vec{E}$ such that

$$\vec{\alpha}_a \cdot \vec{E} = c \,, \tag{5.1.47}$$

with c a number. The above can be proven by noting that the $R \times R$ matrix

$$B_{ij} = \sum_{a=1}^{M} \alpha_{ai}\alpha_{aj} \,, \tag{5.1.48}$$

is invertible since the $\vec{\alpha}_a$ are linearly independent. If we now multiply (5.1.47) with α_{aj} and summing over a we see that

$$\sum_i B_{ij} E^i = c \sum_a \alpha_{aj} \,. \tag{5.1.49}$$

Due to the existence of the inverse of B we can find E^i. The above mentioned field redefinition is given by

$$\vec{\phi} = \varphi_1 \vec{E} + \vec{\varphi}_\perp \,. \tag{5.1.50}$$

If we substitute this in (5.1.22) we see that $\alpha_{a1} = c$ and hence we have derive that (5.1.46) holds.

Scaling solutions

The solution (5.1.44) is called a *scaling solution*. The name can be understood as follows. If we calculate the Hubble factor and the kinetic and potential energy we note that they have the same scaling behaviour

$$H^2 \propto V \propto T \,. \tag{5.1.51}$$

We take (5.1.51) as the definition of a scaling solution. These relations imply that the scale factor is a power-law. This follows from the Friedmann equation (5.1.14)

$$\dot{H} \propto H^2 \,. \tag{5.1.52}$$

From this we derive that for scaling solutions we have $H^2 \propto \tau^{-2}$ and as a result $a(\tau)$ must be a power-law. For the scaling solution of the previous section we see from (5.1.37) that this is indeed the case[4].

Interestingly, scaling solutions correspond to the FLRW-geometries that possess a timelike conformal vector field ξ coming from the transformation

$$\tau \rightarrow \mathrm{e}^{\lambda}\tau \,, \qquad x^a \rightarrow \mathrm{e}^{(1-p)\lambda} x^a \,, \tag{5.1.53}$$

where x^a are the spacelike Cartesian coordinates[5]. In the forthcoming we reserve the indices $a, b, \ldots$ to denote spacelike coordinates when we consider cosmological space-times.

Scaling cosmologies have received a great deal of attention in the dark-energy literature, see [93] for a review and references. Apart from the intriguing cosmological properties of scaling solutions they are also interesting for understanding the dynamics of a general cosmological solution since scaling solutions are often critical points of an autonomous system of differential equations as we explained in subsection 5.1.1. Scaling cosmologies often appear in supergravity theories (see for instance [70, 94]), but remarkably they also appear by spatially averaging inhomogeneous cosmologies in classical general relativity [95].

Let us finish this section by making one surprising observation. When we look at the solution for the scalar fields (5.1.44) we see that this is still a geodesic of the scalar manifold since $G_{ij} = \delta_{ij}$! Apparently the presence of the (complicated) potential does not upset this. A natural question is under what condition does a solution remain a geodesic of the scalar manifold in the presence of a potential. This will be the subject of the next section.

5.2 First Order Formalism

In what follows we consider scalar fields ϕ^i that parameterize a Riemannian manifold with metric G_{ij} coupled to gravity through the action

$$S = \int \mathrm{d}^D x \sqrt{-g} \Big\{ \mathcal{R} - \tfrac{1}{2} G_{ij} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - V(\phi) \Big\} \,. \tag{5.2.1}$$

[4]In the case of curved FLRW-universes we also demand that $H \sim k/a^2$, which is only possible for $p = 1$. But in what follows we will not consider the case $k \neq 0$.

[5]For curved FLRW-space-times the spacelike coordinates are invariant.

We restrict to solutions with the following D-dimensional space-time metric

$$\mathrm{d}s_D^2 = g(r)^2 \mathrm{d}s_{D-1}^2 + \eta f(r)^2 \mathrm{d}r^2\,, \quad \mathrm{d}s_{D-1}^2 = (\delta_\eta)_{ab} \mathrm{d}x^a \mathrm{d}x^b\,, \tag{5.2.2}$$

where $\eta = \pm 1$ and $\delta_\eta = \mathrm{diag}(-\eta, 1, \dots, 1)$. The case $\eta = -1$ describes a flat FLRW-space-time and $\eta = +1$ a Minkowski-sliced domain-wall (DW) space-time. The scalar fields that source these space-times can only depend on the r-coordinate $\phi^i = \phi^i(r)$. The function f corresponds to the gauge freedom of re-parameterizing the r-coordinate.

We will use two coordinate frames to describe scaling cosmologies

$$\tau - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p}\, \mathrm{d}s_{D-1}^2\,, \tag{5.2.3}$$

$$t - \text{frame}: \qquad \mathrm{d}s^2 = -\mathrm{e}^{2t}\, \mathrm{d}t^2 + \mathrm{e}^{2pt} \mathrm{d}s_{D-1}^2\,. \tag{5.2.4}$$

The first is the usual FLRW-coordinate system and the second can be obtained via the substitution $t = \log \tau$.

If the scalar potential $V(\phi)$ can be written in terms of another function $W(\phi)$ as follows

$$V = \eta \left\{ \tfrac{1}{2} G^{ij} \partial_i W \partial_j W - \tfrac{D-1}{4(D-2)} W^2 \right\}, \tag{5.2.5}$$

then the action can be written as "a sum of squares" plus a boundary term when reduced to one dimension:

$$\begin{aligned} S =& \eta \int \mathrm{d}r\, f g^{D-1} \Big\{ \tfrac{(D-1)}{4(D-2)} \Big[W - 2(D-2) \frac{\dot{g}}{fg} \Big]^2 - \tfrac{1}{2} || \frac{\dot{\phi}^i}{f} + G^{ij} \partial_j W ||^2 \Big\} \\ & + \eta \int \mathrm{d} \Big\{ g^{D-1} W - 2(D-1) \dot{g} g^{D-2} f^{-1} \Big\}\,, \end{aligned} \tag{5.2.6}$$

where a dot denotes a derivative w.r.t. r. The term $||\dot{\phi}^i/f + G^{ij}\partial_j W||^2$ is a shorthand notation and the square involves a contraction with the field metric G_{ij}. It is clear that the action is stationary under variations if the terms within brackets are zero[6], leading to the following *first-order* equations of motion

$$\boxed{W = 2(D-2) \frac{\dot{g}}{fg}\,, \qquad \frac{\dot{\phi}^i}{f} + G^{ij} \partial_j W = 0\,.} \tag{5.2.7}$$

For $\eta = +1$ these equations are the standard Bogomol'nyi-Prasad-Sommerfield (BPS) equations for domain-walls that arise from demanding the supersymmetry-variation (susy) of the fermions to vanish, which guarantees that the domain-wall preserves a fraction of the total supersymmetry of the theory. The function W is then the

[6]For completeness we should have added the Gibbons-Hawking term [96] in the action which deletes that part of the above boundary term that contains $\dot{g}$.

superpotential that appears in the susy-variation rules and equation (5.2.5) with $\eta = +1$ is natural for supergravity theories. It is clear that for every W that obeys (5.2.5) we can find a corresponding domain-wall solution, and if W is not related to the susy-variations we call the solutions fake supersymmetric [97].

For $\eta = -1$ these equations are the generalization to cosmologies for arbitrary space-time dimension D and field metric G_{ij}. We refer to these first-order equations as pseudo-BPS equations and W is named the pseudo-superpotential because of the immediate analogy with BPS domain-walls in supergravity [67, 98]. For the case of cosmologies there is no natural choice for W as cosmologies cannot be found by demanding vanishing susy-variations of the fermions. The cosmological solutions is therefore called *pseudo-supersymmetric*.

If we can solve (5.2.7) for a domain-wall we immediately have a cosmological solution by construction. This is called the domain-wall / cosmology correspondence [67,98]. In the next chapter we will discuss this correspondence in more detail.

In [98] it is proven that for all single-scalar cosmologies (and domain-walls) a pseudo-superpotential W exists such that the cosmology is pseudo-BPS and that one can give a fermionic interpretation of the pseudo-BPS flow in terms of so-called pseudo-Killing spinors. This does not necessarily carry over to multi-scalar solutions as was shown in [99]. Nonetheless, a multi-field solution can locally be seen as a single-field solution [100] because locally we can redefine the scalar coordinates such that the curve $\phi(r)$ is aligned with a scalar axis and all other scalars are constant on this solution. A necessary condition for the single-field pseudo-BPS flow to carry over (locally) to the multi-field system is that the truncation down to a single scalar is consistent (this means that apart from the solution one can put the other scalars always to zero) [99].

5.3 Multi-Field Scaling Cosmologies

Let us turn to scaling solutions in the framework of these first order equations and see how the geodesic motion arises that we found at the end of section 5.1. First we consider the rather trivial case with vanishing scalar potential V and after that we add a scalar potential V. Pseudo-supersymmetry is only discussed in the case of non-vanishing V.

5.3.1 Pure Kinetic Solutions

If there is no scalar potential the solutions trace out geodesics as we learned in section 3.5. The affine velocity $G_{ij}\partial_h\phi^i\partial_h\phi^j = ||v||^2$ is positive and for the metric Ansatz (5.2.2) we derive the Einstein equations

$$\mathcal{R}_{rr} = \tfrac{1}{2}G_{ij}\dot{\phi}^i\dot{\phi}^j = \frac{||v||^2}{2}g^{2-2D}f^2\,, \qquad \mathcal{R}_{ab} = 0\,. \tag{5.3.1}$$

In the gauge $f = 1$ the solution is given by $g = e^{C_2}(r + C_1)^{\frac{1}{D-1}}$, with C_1 and C_2 arbitrary integration constants, but with a shift of r we can always put $C_1 = 0$ and C_2 can always be put to zero by re-scaling the spacelike coordinates. In the case of a four-dimensional cosmology the geometry is a power-law FLRW-solution with $p = 1/3$.

5.3.2 Potential-Kinetic Scaling Solutions

In a recent paper of Tolley and Wesley an interesting interpretation was given to scaling solutions [101], which we repeat here. The finite transformation (5.1.53) leaves the equations of motion invariant if the action S scales with a constant factor, which is exactly what happens for scaling solutions since all terms in the Lagrangian scale like τ^{-2}. Under (5.1.53) the metric scales like $e^{2\lambda}g_{\mu\nu}$ and in order for the action to scale as a whole we must have

$$V \to e^{-2\lambda}V\,, \qquad T = \tfrac{1}{2}g^{\tau\tau}G_{ij}\dot{\phi}^i\dot{\phi}^j \to e^{-2\lambda}T\,. \tag{5.3.2}$$

Equations (5.3.2) imply that $G_{ij}\dot{\phi}^i\dot{\phi}^j$ remains invariant from which one deduces that $\frac{d\phi^i}{d\lambda} = \xi^i$ must be a Killing vector. The curve that describes a scaling solution follows an isometry of the scalar manifold. It depends on the parametrization whether the tangent vector $\dot{\phi}$ itself is Killing. This happens for the parametrization in terms of $t = \log\tau$ since

$$\xi^i = \frac{\mathrm{d}\phi^i}{\mathrm{d}\lambda} = \lim_{\lambda\to 0}\frac{\phi^i(e^\lambda\tau) - \phi^i(\tau)}{\lambda} = \frac{\mathrm{d}\phi^i}{\mathrm{d}\log\tau}\,. \tag{5.3.3}$$

Thus a scaling solution is associated with an invariance of the equations of motion for a re-scaling of cosmic time and is therefore associated with a conformal Killing vector on space-time and a Killing vector on the scalar manifold.

Pseudo-supersymmetry comes into play when we check the geodesic equation of motion

$$\nabla_{\dot{\phi}}\dot{\phi}_i = \dot{\phi}^j\nabla_j\dot{\phi}_i = \dot{\phi}^j\left\{\nabla_{(j}\dot{\phi}_{i)} + \nabla_{[j}\dot{\phi}_{i]}\right\}, \tag{5.3.4}$$

where we denote $\dot{\phi}_i = G_{ik}\dot{\phi}^k$. Now we have that the symmetric part is zero if we parameterize the curve with $t = \log\tau$ since scaling makes $\dot{\phi}$ a Killing vector. We also have that $\nabla_{[j}\dot{\phi}_{i]} = 0$ since the pseudo-BPS condition makes $\dot{\phi}$ a curl-free flow $\dot{\phi}_i = -f\partial_i W$. To check that the curl is indeed zero (when $f \neq 1$) one has to notice that in the parametrization of the curve in terms of $t = \log\tau$ the gauge is such that $\dot{g}/g$ is constant and that $f \sim W^{-1}$. Since the curl is also zero we notice that the curve is a geodesic with $\log\tau$ as affine parametrization[7]

$$\nabla_{\dot\phi}\dot\phi^i = 0 = \ddot\phi^i + \Gamma^i_{jk}\dot\phi^j\dot\phi^k\,. \qquad (5.3.5)$$

The link between scaling and geodesics was discovered by Karthauser and Saffin in [102], but no conditions on the Lagrangian were given in [102] such that the relation scaling-geodesic holds. An example of a scaling solution that is not a geodesic was given by Sonner and Townsend in [103].

A more intuitive understanding of the origin of the geodesic motion for some scaling cosmologies comes from the on-shell substitution $V = (3p-1)\,T$ in the Lagrangian to get a new Lagrangian describing seemingly massless fields. Although this is rarely a consistent procedure we believe that this is nonetheless related to the existence of geodesic scaling solutions.

Single field

For single-field models the potential must be exponential $V = \Lambda e^{\alpha\phi}$ in order to have scaling solutions. The simplest pseudo-superpotential belonging to an exponential potential is itself exponential

$$W = \pm\sqrt{\tfrac{8\Lambda}{3-\alpha^2}}\; e^{\frac{\alpha\phi}{2}}\,. \qquad (5.3.6)$$

If we choose the plus sign the solution to the pseudo-BPS equation is

$$\phi(\tau) = -\tfrac{2}{\alpha}\log\tau + \tfrac{1}{\alpha}\log[\tfrac{6-2\alpha^2}{\alpha^4\Lambda}]\,, \qquad g(\tau)\sim\tau^{\frac{1}{\alpha^2}}\,. \qquad (5.3.7)$$

The minus sign corresponds to the time reversed solution.

Multiple fields

For a general multi-field model a scaling solution with power-law scale factor τ^p obeys $V = (3p-1)T$ from which we derive the **on-shell** relation

$$G^{ij}\partial_i W\partial_j W = \frac{W^2}{4p} \quad\Rightarrow\quad W = \pm\sqrt{\frac{8\,p\,V}{3p-1}}\,. \qquad (5.3.8)$$

In general the above expression for the superpotential $W\sim\sqrt{V}$ does not hold off-shell, unless the potential is a function of a specific kind:

$$\frac{1}{p} = \frac{G^{ij}\partial_i V\partial_j V}{V^2}\,. \qquad (5.3.9)$$

[7]One could wonder whether the results works in two ways. Imagine that a scaling solution is a geodesic. This then implies that $\nabla_{[j}\dot\phi_{i]} = 0$ and therefore the flow is locally a gradient flow $\dot\phi_i = \partial_i \log W \sim f\partial_i W$.

Scalar potentials that obey (5.3.9) with the extra condition that $p \gtrless \frac{1}{3} \leftrightarrow V \gtrless 0$ allow for multi-field scaling solutions. For a given scalar potential that obeys (5.3.9) there probably exist many pseudo-superpotentials W compatible with V but if we make the specific choice $W = \sqrt{8\,p\,V/(3p-1)}$ then all pseudo-BPS solutions must be scaling and hence geodesic. As a consistency check we substitute the first-order pseudo-BPS equations into the right-hand side of the following second-order equations of motion

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^k\dot{\phi}^j = -f^2 G^{ij}\partial_j V - \Big[3(\dot{\log g}) - (\dot{\log f})\Big]\dot{\phi}^i\,, \tag{5.3.10}$$

and choose a gauge for which

$$\frac{\dot{f}}{f^2} = \frac{1}{4p}W\,, \tag{5.3.11}$$

then we indeed find an affine geodesic motion since the right-hand side of (5.3.10) vanishes.

For some systems one first needs to perform a truncation in order to find the above relation (5.3.9). A good example is the multi-field potential appearing in Assisted Inflation [104]

$$V(\phi^1,\ldots,\phi^n) = \sum_i^n \Lambda_i\,\mathrm{e}^{\alpha_i\phi^i}\,, \qquad G_{ij} = \delta_{ij}\,. \tag{5.3.12}$$

The scaling solution of this system was proven to be the same as the single-exponential scaling [92]. The reason is that we can perform the orthogonal transformation in field space that we discussed below (5.1.46). As a result, the form of the kinetic term is preserved but the scalar potential is given by

$$V = \mathrm{e}^{\alpha\varphi}\,U(\phi^1,\ldots,\phi^{n-1})\,, \quad \frac{1}{\alpha^2} = \sum_i \frac{1}{\alpha_i^2}\,. \tag{5.3.13}$$

The scaling solution is such that $\phi_1,\ldots,\phi_{n-1}$ are frozen in a stationary point of U. This follows from the Klein–Gordon equation for the new fields and making use of the fact that this is a critical point solution. Therefore the system is truncated to a single-field system that obeys (5.3.9). The same proof holds for generalized assisted inflation discussed in section 5.1.2 [79]. As shown in (5.1.44) the scaling solution reads $\phi^i = A^i\log\tau + B^i$, which is clearly a straight line and thus a geodesic.

The scaling solutions of [99,103] were constructed for an axion-dilaton system with an exponential potential for the dilaton

$$S = \int \mathrm{d}^D x\sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mathrm{e}^{\mu\phi}(\partial\chi)^2 - \Lambda\mathrm{e}^{\alpha\phi}\Big\}\,. \tag{5.3.14}$$

Clearly this two-field system obeys (5.3.9) and (one of) the pseudo-superpotential(s) is given by (5.3.6). The pseudo-BPS scaling solution therefore has constant axion and is effectively described by the dilaton in an exponential potential. Note that this solution indeed describes a geodesic on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ with $\log\tau$ as affine parameter. All examples of scaling solutions in the literature seem to occur for exponential potentials, however by performing a $\mathrm{SL}(2,\mathbb{R})$-transformation on the Lagrangian (5.3.14) the kinetic term is unchanged and the potential becomes a more complicated function of the axion and the dilaton. The same scaling solution then trivially still exists (and (5.3.9) still holds) but the axion is not constant in the new frame and instead the solution follows a more complicated geodesic on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

However another scaling solution is given in [103] that is not geodesic and with varying axion in the frame of the above action (5.3.14). This is an illustration of the above, since the solution is not geodesic we know that there does not exists any other pseudo-superpotential for which the varying axion solution is pseudo-BPS, consistent with what is shown in [99] for that particular solution.

5.4 Uplifts

In this section we illustrate with an example that the cosmologies we worked out in subsection 5.1.3 can be uplifted over their transverse space by using the reduction explained in section 3.3 [45, 64, 70]. We only consider the simple example of a single scalar field and we illustrate that for a given cosmology we find a domain-wall belonging to a model with minus the potential.

In section 3.3 we showed that the reduction of a fluxless brane over its maximally symmetric transverse space with $k = -1$ leads to a Lagrangian with exponential potential (3.3.6)

$$V(\varphi) = n(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi}\,. \tag{5.4.1}$$

Here n is the dimension of the internal manifold and α, β are given in (3.2.7). The four-dimensional $k = 0$ solution with $\kappa^2 = 1/2$ is found from (5.1.43–5.1.44) to be

$$a(\tau) = \tau^{\frac{n}{n+2}}\,, \quad \varphi(\tau) = -\frac{2}{c}\log(\tau) - \frac{2}{c}\log(\frac{n+2}{2})\,, \tag{5.4.2}$$

with $c^2 = (2(\alpha-\beta))^2 = 1 + 2/n$. The constant c_1 in (5.1.44) gets fixed due to the Friedmann equations.

To uplift we plug these solutions in (3.3.1) and derive

$$\mathrm{d}s^2_{4+n} = -\Big(\frac{2}{2+n}\Big)^{\frac{2n}{2+n}}\tau^{-\frac{2n}{2+n}}\mathrm{d}\tau^2 + \Big(\frac{2}{2+n}\Big)^{\frac{2n}{2+n}}\mathrm{d}\vec{x}_3^2 + \Big(\frac{2}{2+n}\Big)^{-\frac{4}{2+n}}\tau^{\frac{4}{2+n}}\mathrm{d}\mathbb{H}_n^2\,. \tag{5.4.3}$$

To identify the solution we apply the following coordinate transformations

$$t = \Big(\frac{2}{n+2}\Big)^{-\frac{2}{2+n}}\tau^{\frac{2}{2+n}}\,, \quad y^i = \Big(\frac{2}{2+n}\Big)^{\frac{n}{2+n}}x^i\,, \tag{5.4.4}$$

and we find

$$\mathrm{d}s_{4+n}^2 = \mathrm{d}\vec{y}_3^2 - \mathrm{d}t^2 + t^2\mathrm{d}\mathbb{H}_n^2\,. \tag{5.4.5}$$

This metric describes $\mathbb{R}^3 \times \mathrm{Milne}_{n+1}$, the latter is a patch of Minkowski space-time in unconventional coordinates. The uplifted solution describes thus a flat space solution.

The extension to the multi-exponential potential given in (3.3.21) does not lead to any qualitative chance. The reason is that the attractor solution is such that only one scalar field is turned on [45]. To find genuine S-branes we need to take flux into consideration. This requires the uplift of a solution belonging to (3.3.12) [45, 105].

According to (5.2.7) the four-dimensional cosmology should give rise to a domain-wall with minus the potential as given in (5.4.1). From (5.2.2) and $\eta = 1$ we indeed find the solution

$$\mathrm{d}s_4^2 = \mathrm{d}r^2 + a(r)^2\Big(-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2\Big)\,, \tag{5.4.6}$$

with power-law and φ given by

$$a(r) = r^{\frac{n}{n+2}}\,, \quad \varphi(r) = -\frac{2}{c}\log(r) - \frac{2}{c}\log(\frac{n+2}{2})\,. \tag{5.4.7}$$

Due to the minus sign in the potential this domain-wall lifts up to a spherical transverse space and we find after appropriate coordinate transformations

$$\mathrm{d}s_{4+n}^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{y}_2^2 + \mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega_n^2\,, \tag{5.4.8}$$

where $\mathrm{d}\Omega_n^2$ is the metric of a n-dimensional sphere. This metric describes three-dimensional Minkowski space-time $\times\, \mathbb{R}^{n+1}$.

5.5 Discussion

In the first part of this chapter we gave a brief introduction to cosmology and focussed on generalized assisted inflation models. These models have the characteristic that they all have a multi-exponential potential. The analysis was restricted to critical points via re-writing the equations of motion as an autonomous system. These critical points turn out to be scaling solutions and we noted that they are still geodesics of the scalar manifold.

In the second half of this chapter we explained under what condition we have geodesic motion in the presence of a potential. For this we have studied multi-field scaling solutions using a first-order formalism for scalar cosmologies. We derived these first-order equations via a Bogomolnyi-like method that was known to work for domain-wall solutions as was first shown in [106–108] and we showed that it trivially extends to cosmological solutions. This first-order formalism allows a better understanding of the geodesic motion that comes with a specific class of scaling solutions. One of the main results of this chapter is a proof that shows that *all pseudo-BPS* *cosmologies that are scaling solutions must be geodesic.* This complements to the discussion in [99] where the first example of a non-geodesic scaling cosmology was shown to be non-pseudo-BPS. Moreover we gave constraints on multi-field Lagrangians for which the pseudo-BPS cosmologies are geodesic scaling solutions.

By now the first order formalism has been extended to branes of arbitrary dimensions, both space- and timelike. This has been initiated by [109] where it was shown that the *non-extremal* Reissner-Nordström black hole solution of Einstein-Maxwell theory can be found from first-order equations by rewriting the action as a sum of squares à la Bogomol'nyi. In the recent paper [74] this construction of BPS-type equations is extended to branes of arbitrary dimension and to time-dependent solutions. The authors presented the fake- and pseudo-BPS equations for all stationary branes (timelike branes) and all time-dependent branes (spacelike branes) of an Einstein-dilaton-p-form system in arbitrary dimensions[8]. As mentioned before, the word fake refers to time-independent solutions where the superpotential W used in the derivation of the first order equations has no relation to the superpotential appearing in the supersymmetry transformation. In case of time-dependent cosmological solutions the word pseudo-BPS is used for the first order equations governing the dynamics of cosmologies.

[8]They did not included branes with co-dimensions less than three. When the co-dimension is one, the stationary branes are domain walls and the time-dependent branes are cosmologies. The case of branes with co-dimension two is not included as these solutions depend on one complex coordinate rather than on one real coordinate.