Particle dynamics of branes

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Chapter 4

Massless Time-Dependent Solutions

In this chapter we are going to look for time-dependent solutions of the Lagrangian (3.5.2) without a potential V. We are then discussing S(-1)-branes belonging to the action

$$S = \int \mathrm{d}^D x \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2} G_{ij} \partial\phi^i \partial\phi^j\Big) . \tag{4.0.1}$$

We will restrict to scalar manifold metrics G_{ij} which belong to maximally non-compact cosets G/H with H its maximal compact subgroup. If we consider solutions that depend only on the time, $\vec{\phi}$ is a geodesic on the scalar manifold as we explained in section 3.5. To find the most general geodesic we are going to construct a solution-generating technique.

From section 3.5 we know that the above action can be obtained from reducing gravity together with a dilaton and a p-form over a Euclidean torus. If we oxidize the time-dependent geodesic solution back to the original higher-dimensional theory we will obtain a (fluxless) Sp-brane. This leads to the S(-1)-brane / Sp-brane map.

The work in this section is done together with E. A. Bergshoeff, W. Chemissany, T. Van Riet and M. Trigiante [68, 69].

4.1 S(-1)-brane Geometries

We want to look for solutions belonging to the action (4.0.1) which only depend on the time coordinate t. The Ansätze for the time-dependent S(-1)-brane is given by (3.5.3)

$$\mathrm{d}s_D^2 = -f^2(t)\mathrm{d}t^2 + g^2(t) g_{ab}^{D-1} \mathrm{d}x^a \mathrm{d}x^b , \qquad \phi^i = \phi^i(t) . \tag{4.1.1}$$

In section 3.5 we showed that the scalar part of (4.0.1) leads to a geodesic on the scalar manifold with affine parameter the harmonic function h. In terms of this affine parameter the velocity $||v||$ is a strictly positive constant

$$||v||^2 = G_{ij}\partial_h\phi^i\partial_h\phi^j > 0\,. \tag{4.1.2}$$

Via combining the scalar field equations and the Einstein equations we deduced that the metric can be found from solving (3.5.10). If we choose to work in the gauge where $g^2 = t^2$ we find that the Einstein equations (3.5.10) give the following D-dimensional metric

$$\mathrm{d}s_D^2 = -\frac{\mathrm{d}t^2}{a\,t^{-2(D-2)} - k} + t^2\mathrm{d}\Sigma_k^2\,, \qquad a = \frac{||v||^2}{2(D-1)(D-2)}\,, \tag{4.1.3}$$

while the scalar fields trace out geodesic curves with the harmonic function $h(t)$ as affine parameter. The harmonic function is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)}\log\Big|\sqrt{a}t^{2-D} + \sqrt{at^{2(2-D)} - k}\Big| + c\,. \tag{4.1.4}$$

We take $c = 0$ in what follows since it just corresponds to a shift in the affine parameter h.

Now that we have solved the metric, we proceed by explaining how one can find the scalar field geodesics.

4.2 A Solution-Generating Technique

To discuss geodesic curves it is useful to introduce coordinates (scalar fields) on the moduli space. As explained in section 3.4, we use the solvable gauge which for maximally non-compact manifolds G/H coincides with the Borel gauge. In the Borel gauge the scalar fields are divided in dilatons ϕ^I and axions χ^α. This is done by choosing the coset element as follows

$$L = \Pi_\alpha \exp\,[\chi^\alpha E_\alpha]\Pi_I \exp\,[\tfrac{1}{2}\Phi^I H_I]\,, \tag{4.2.1}$$

where H_I are the Cartan generators of the Lie algebra of G and the E_α are the positive root operators. The number of Cartan generators is the rank r of the Lie algebra of G and for the cosets listed in table 3.4.1 the rank is $r = 11 - D$. The number of axions equals the dimension of the isotropy group H for maximally non-compact cosets since the Lie algebra of H is spanned by the combinations $E_\alpha - E_{-\alpha}$.

Our approach to understand all the geodesic curves is by constructing *the generating solution*. By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group

G generates all other geodesics from the generating solution. Below we will explain that for cosets G/H where G is a maximally non-compact real slice of a complex semi-simple group and H is the maximal compact subgroup, the generating solution can be taken to be the straight line through the origin carried by the dilaton fields

$$\boxed{\phi^I(h) = v^I\,h\,, \qquad \chi^\alpha = 0\,, \quad I = 1,\ldots r\,.} \tag{4.2.2}$$

Here h is the harmonic function. This solution contains r arbitrary integration constants v^I, with r the rank of G. This theorem applies to all the cosets in the left column of table 3.4.1.

Since the straight line solution is the generating solution, by definition G-transformations on this solution generate all the other geodesic curves. The number of independent constants in G is the dimensions of G which is $r + 2\dim H$. In total this gives us $2\,r + 2\dim H$ arbitrary (integration) constants as expected since there are $r + \dim H$ scalars (coordinates) for which we have to specify the initial place and velocity. The number of dilatons is given by r, the number of axions by $\dim H$. However this counting exercise is no proof since it might be that the action of G does not create independent integration constants or if the solutions lie in disconnected areas. The latter is the case for the cosets in the right column of table 3.4.1. There the straight line solution is not generating since the affine velocity is positive

$$||v||^2 = \sum(v^I)^2 > 0\,. \tag{4.2.3}$$

The affine velocity is invariant under G-transformations and by transforming the straight line we only generate spacelike geodesics. But cosets with non-compact isotropy H have metrics with indefinite signature and therefore allow for spacelike, lightlike and timelike geodesics. In chapter 7 we derive the generating solutions for cosets with non-compact isotropy group $\mathrm{SO}(p,q)$.

Let us repeat the proof of (4.2.2) as given in [68][1]. In the Borel gauge the geodesic equation is

$$\ddot\phi^I + \Gamma^I_{JK}\dot\phi^J\dot\phi^K + \Gamma^I_{\alpha J}\dot\chi^\alpha\dot\phi^J + \Gamma^I_{\alpha\beta}\dot\chi^\alpha\dot\chi^\beta = 0\,, \tag{4.2.4}$$

$$\ddot\chi^\alpha + \Gamma^\alpha_{JK}\dot\phi^J\dot\phi^K + \Gamma^\alpha_{\beta J}\dot\chi^\beta\dot\phi^J + \Gamma^\alpha_{\beta\gamma}\dot\chi^\beta\dot\chi^\gamma = 0\,. \tag{4.2.5}$$

Since $\Gamma^I_{JK} = 0$ and $\Gamma^\alpha_{JK} = 0$ at points for which $\chi^\alpha = 0$ a trivial solution is given by

$$\phi^I = v^I\,t\,, \qquad \chi^\alpha = 0\,, \tag{4.2.6}$$

for some parameter t. How many other solutions are there? A first thing we notice is that every global G-transformation $\Phi \to \tilde\Phi$ brings us from one solution to another

[1] See also the appendix of [70] for earlier remarks.

solution. Since G generically mixes dilatons and axions we can construct solutions with non-trivial axions in this way. We now prove that in this way *all* geodesics are obtained and this depends on the fact that G is maximally non-compact with H the maximal compact subgroup of G.

Consider an arbitrary geodesic curve $\Phi(t)$ on G/H. The point $\Phi(0)$ can be mapped to the origin $L = \mathbb{1}$ using a G-transformation, since we can identify $\Phi(0)$ with an element of G and then we multiply the geodesic curve $\Phi(t)$ with $\Phi(0)^{-1}$, generating a new geodesic curve $\Phi_2(t) = \Phi(0)^{-1}\Phi(t)$ that goes through the origin. The origin is invariant under H-rotations but the tangent space at the origin transforms under the adjoint of H. One can prove that there always exists an element $k \in H$, such that $\text{Adj}_k \dot{\Phi}_2(0) \in \text{CSA}$ [71]. Therefore $\dot{\chi}_2^\alpha = 0$ and this solution must be a straight line. So we started out with a general curve $\Phi(t)$ and proved that the curve $\Phi_3(t) = k\Phi(0)^{-1}\Phi(t)$ is a straight line. If we take $t = h$ it follows that the scalar fields are given by (4.2.2).

4.3 Spacelike Branes

In this section we consider the time-dependent (-1)-brane solutions in D dimensions and their uplift to general Sp-branes in $D + p + 1$ dimensions.

Sp-branes are solutions of the following action

$$\mathcal{L} = \sqrt{-g}\Big(\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2(p+2)!}\mathrm{e}^{b\phi}F_{p+2}^2\Big)\,, \tag{4.3.1}$$

with b the dilaton coupling constant. The reduction Ansatz for the metric is as in section 3.2

$$\mathrm{d}s^2_{D+p+1} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_D + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}\mathrm{d}z^n \otimes \mathrm{d}z^m\,, \tag{4.3.2}$$

where

$$\alpha^2 = \frac{p+1}{2(D+p-1)(D-2)}\,, \qquad \beta = -\frac{(D-2)\alpha}{p+1}\,. \tag{4.3.3}$$

The matrix $\mathcal{M}$ and the scalar φ are the moduli of the $(p+1)$-torus and depend on the D-dimensional coordinates. In particular $\mathcal{M}$ is a positive-definite symmetric $(p+1)\times(p+1)$ matrix with unit determinant and the modulus φ controls the overall volume. For a dimensional reduction over a Euclidean torus the scalars parameterize $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$ where φ belongs to the decoupled $\mathbb{R}$-part and $\mathcal{M}$ is the $\mathrm{SL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$ part. More precisely $\mathcal{M} = LL^T$ where L is the vielbein matrix of the internal torus and it also plays the role of the coset representative of $\mathrm{SL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$.

The reduction of a $(p+1)$-form A^{p+1} over a $(p+1)$-torus gives a scalar χ and various other forms of lower degree in D dimensions since

$$A^{p+1} = \sum_{i=0}^{p+1} A^{(i)}(x)\mathrm{d}z^{i+1} \wedge \mathrm{d}z^{i+2} \wedge \ldots \wedge \mathrm{d}z^{p+1}\,. \tag{4.3.4}$$

Here the $A^{(i)}$ are the gauge potentials of rank i. If one of the non-trivial forms in the series is a $(D-2)$-form, it can be dualized to a scalar field χ_2 in the lower dimension. Since we start from an electric Ansatz, we can have magnetic flux only if we have a dyonic solution in $D+p+1$ dimensions. This gives the constraint $p+2 = D-1$. Non-zero values for χ and χ_2 imply then respectively non-zero electric and magnetic flux. The reduction Ansatz for electrical solutions is $\hat{A} = \chi(x)\mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^{p+1}$.

The reduced D-dimensional Lagrangian is

$$\mathcal{L} = \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} - \tfrac{1}{2}\mathrm{e}^{b\phi+2(D-2)\alpha\varphi}(\partial\chi)^2\Big\}\,. \tag{4.3.5}$$

If the scalar fields in $\mathcal{M}$ are non-trivial then $\mathcal{M} \neq \mathbb{1}$ and the $\mathrm{ISO}(p+1)$ worldvolume symmetries of the brane becomes smaller. The fact that we are able to write down the most general solution with a deformed worldvolume illustrates the power of our approach.

After an appropriate SO(2)-rotation of the two dilatons φ and ϕ we get the more familiar Lagrangian for the scalars that parameterize $\mathbb{R} \times \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$

$$\mathcal{L} = \sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}(\partial\varphi')^2 - \tfrac{1}{2}(\partial\phi')^2 - \tfrac{1}{2}\mathrm{e}^{c\phi'}(\partial\chi)^2 + \tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1}\Big\}\,, \tag{4.3.6}$$

where the $'$ denotes that the scalars are rotated versions of the original scalars and where the radius of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ part is given by

$$c = \sqrt{b^2 + 2\tfrac{(D-2)(p+1)}{D+p-1}}\,. \tag{4.3.7}$$

The $\mathrm{SL}(2,\mathbb{R})$ transformations Ω work in a non-linear fashion on ϕ' and χ, but on the level of the scalar matrix

$$\mathcal{M}_2 = \mathrm{e}^{\frac{c}{2}\phi'}\begin{pmatrix} \frac{c^2}{4}\chi^2 + \mathrm{e}^{-c\phi'} & \frac{c}{2}\chi \\ \frac{c}{2}\chi & 1 \end{pmatrix}\,, \tag{4.3.8}$$

the transformation is $\mathcal{M}_2 \to \Omega\mathcal{M}_2\Omega^T$.

4.3.1 Pure Gravity

We start by considering Sp-brane solutions of pure gravity. The corresponding S(-1)-brane is given by geodesics on $\mathrm{GL}(p+1,\mathbb{R})/\,\mathrm{SO}(p+1)$. In section 4.2 we showed that the most general geodesic solution is given by the most general $\mathrm{SL}(p+1,\mathbb{R})$-transformation of the "straight line" through the origin (which is therefore the generating solution):

$$\varphi = v^\varphi h + c^\varphi\,, \qquad \phi^I = v^I h\,, \tag{4.3.9}$$

where I runs from 1 to p and $v^\varphi\,, c^\varphi$ and v^I are integration constants and h is given by (4.1.4). The ϕ^I are the dilaton scalars of $\mathrm{SL}(p+1,\mathbb{R})/\,\mathrm{SO}(p+1)$. The dilatons are related to the diagonal components of the metric $\mathcal{M}$ on the internal space via $\mathcal{M} = LL^T$ with L given by (4.2.1).

In case all axions are truncated we have that

$$\mathcal{M} = \mathrm{diag}(\exp[\beta_{iI}\phi^I])\,, \tag{4.3.10}$$

where the $\vec{\beta}_i$ are the weights of $\mathrm{SL}(p+1,\mathbb{R})$ in a suitable basis where they obey (3.4.21). The affine velocity follows from (4.2.3) and is given by $||v^2|| = (v^\varphi)^2 + \sum_I (v^I)^2$.

Uplifts

Since the scalar field matrix transforms as $\mathcal{M} \to \Omega\mathcal{M}\Omega^T$ with $\Omega \in \mathrm{SL}(p+1,\mathbb{R})$ we notice that we only need to uplift the straight line geodesic since all other geodesics are just Ω-transformations which can be absorbed by redefining the torus coordinates $\mathrm{d}\vec{z}' = \Omega \mathrm{d}\vec{z}$. The higher-dimensional geometries we find depend on the curvature k of the lower-dimensional FLRW-space.

- For flat FLRW-spaces ($k = 0$) the uplift becomes the Kasner solution [68]

$$\mathrm{d}s^2 = -\tau^{2p_0}\mathrm{d}\tau^2 + \sum_{a=1}^{D-1} \tau^{2p_a}(\mathrm{d}x^a)^2 + \sum_{b=1}^{p+1} \tau^{2p_b}(\mathrm{d}z^b)^2\,, \tag{4.3.11}$$

where

$$\begin{aligned} p_0 &= \frac{\alpha v^\varphi}{\sqrt{a}} + (D-2)\,, \\ p_a &= \frac{\alpha v^\varphi}{\sqrt{a}} + 1\,, \\ p_b &= \frac{\beta v^\varphi}{\sqrt{a}} + \frac{\beta_{bI} v^I}{2\sqrt{a}}\,, \end{aligned} \tag{4.3.12}$$

and a is given by (4.1.3). These numbers p obey the constraints

$$p_0 + 1 = \sum_{i=1}^{D+p} p_i\,, \quad (p_0+1)^2 = \sum_{i=1}^{D+p} p_i^2\,. \tag{4.3.13}$$

• If we consider a curved FLRW-space ($k \neq 0$) in the lower dimension then the uplift gives a vacuum solution with a bit more complicated metric. The uplift gives

$$\mathrm{d}s^2_{D+p+1} = W^u \left[-\frac{\mathrm{d}t^2}{\mathrm{e}^{2\omega} t^{-2(D-2)} - k} + t^2\,\mathrm{d}\Sigma_k^2 \right] + \sum_{b=1}^{p+1} W^{w_b} (\mathrm{d}z^b)^2\,, \qquad (4.3.14)$$

with

$$W = \mathrm{e}^{\omega} t^{2-D} + \sqrt{\mathrm{e}^{2\omega} t^{2(2-D)} - k}\,, \qquad (4.3.15)$$

$$u = \tfrac{-2v^{\varphi}}{(D-2)||v||} \sqrt{\tfrac{(p+1)(D-1)}{D+p-1}}\,, \qquad (4.3.16)$$

$$w_b = \frac{2v^{\varphi}}{||v||} \sqrt{\tfrac{D-1}{(D+p-1)(p+1)}} - \sqrt{\frac{2(D-1)}{||v||(D-2)}} \beta_{bI} v^I\,, \qquad (4.3.17)$$

and ω is left arbitrary.

If we choose $v^I = 0$ and $k = -1$ the solution is the fluxless S-brane of [8, 65, 72]. For $k = 0, +1$ the solution is strictly not called an S-brane since there is no Lorentzian symmetry group $\mathrm{SO}(D-p-2, 1)$.

4.3.2 Dilaton-Gravity

Now we complicate matters by considering a non-zero dilaton ϕ. In D-dimensions the solution is

$$\varphi = v^{\varphi} h + c^{\varphi}\,, \qquad \phi = v^{\phi} h + c^{\phi}\,, \qquad \phi^I = v^I h\,. \qquad (4.3.18)$$

The uplift to a fluxless Sp-brane gives a metric of the form (4.3.14) but now there is a non-constant dilaton $\phi(t)$ and $||v||$ gets an extra contribution

$$\phi(t) = -\tfrac{v^{\phi}}{||v||} \sqrt{\tfrac{2(D-1)}{(D-2)}} \log W + c^{\phi}\,, \qquad (4.3.19)$$

$$||v||^2 = (v^{\varphi})^2 + (v^{\phi})^2 + \sum_{I=1}^{p} (v^I)^2\,. \qquad (4.3.20)$$

When we put the dilaton to constant via $v^{\phi} = 0$ we end up with the pure gravitational solution (4.3.14-4.3.17).

4.3.3 ... with Non-Trivial Flux

The uplift of the general solution with χ possibly non-zero requires the uplift of all $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ geodesics. But as explained before they can be obtained by an

SL(2, $\mathbb{R}$)-transformation on the straight line solution trough the origin $\phi = v^\phi h$. Thus we can obtain the general solution by transforming the fluxless solution (4.3.19) with $c^\phi = 0$. The metric reads

$$\mathrm{d}s^2_{D+p+1} = W^u \Big(\zeta'^2 W^z + \eta^2\Big)^x \Big[-\frac{\mathrm{d}t^2}{\mathrm{e}^{2\omega} t^{-2(D-2)} - k} + t^2\,\mathrm{d}\Sigma_k^2\Big] + \tag{4.3.21}$$

$$\sum_{b=1}^{p+1} W^{w_b} \Big(\zeta'^2 W^z + \eta^2\Big)^y (\mathrm{d}z^b)^2\,, \tag{4.3.22}$$

with

$$x = \tfrac{4(p+1)}{b^2(D+p-1)+2(D-2)(p+1)}\,, \quad y = -\tfrac{D-2}{p+1} x\,, \tag{4.3.23}$$

$$u = \frac{-2v^\varphi}{||v||(D-2)}\sqrt{\frac{(p+1)(D-1)}{D+p-1}}\,, \tag{4.3.24}$$

$$z = \frac{bv^\phi + 2(D-2)\alpha v^\varphi}{||v||}\sqrt{\frac{2(D-1)}{(D-2)}}\,. \tag{4.3.25}$$

The function W is defined in (4.3.15), w_b is given by (4.3.17) and $||v||$ by (4.3.20). The dilaton and the form field strength are given by

$$\phi(t) = \tfrac{2(D+p-1)b}{b^2(D+p-1)+2(D-2)(p+1)} \log\Big(\zeta'^2 W^z + \eta^2\Big) - \tfrac{v^\phi}{||v||}\sqrt{\tfrac{2(D-1)}{(D-2)}} \log W\,,$$

$$F_{ti_1\ldots i_{p+1}} = \Big(2(D-2)\sqrt{a}\,\zeta\eta\psi\Big)\times \tag{4.3.26}$$

$$\Big(\frac{t^{1-D}\sqrt{\mathrm{e}^{2\omega}t^{2(2-D)} - k} + \mathrm{e}^\omega t^{3-2D}}{W^{c\psi+1}[\zeta'^2 W^{-c\psi} + \eta^2]^2\sqrt{\mathrm{e}^{2\omega}t^{2(2-D)} - k}}\Big)\,\varepsilon_{i_1\ldots i_{p+1}}\,. \tag{4.3.27}$$

Let us explain the various integration constants. As before ω is left arbitrary and a and c are defined as before. The parameters ζ' and ψ are given by

$$\zeta' = \zeta\frac{\sqrt{a}}{e^\omega}\,, \tag{4.3.28}$$

$$\psi = -\tfrac{1}{c||v||}\sqrt{\tfrac{2(D-1)}{D-2}}\Big(b\,v^\phi + 2(D-1)\sqrt{\tfrac{p+1}{2(D+p-1)(D-2)}}v^\varphi\Big)\,, \tag{4.3.29}$$

where ζ, η come from the SL(2, $\mathbb{R}$)-transformation

$$\Omega = \begin{pmatrix} \gamma & \delta \\ \zeta & \eta \end{pmatrix}\,, \qquad \gamma\eta - \delta\zeta = 1\,. \tag{4.3.30}$$

The numbers v^φ and v^ϕ are the "velocities" of respectively φ and ϕ in the fluxless solution. One readily checks that the choice $\Omega = \mathbb{1}$ indeed reproduces the fluxless solution given in subsection 4.3.2 with $c^\phi = 0$. If we restrict to $D = 10$, $b = 2$, $p = -1$ and $v^\varphi = 0$ we have the S(−1)-brane of type IIB in a different coordinate frame as discussed in section 2.4.3.

In this section the have written down the most general Sp-brane with a deformed worldvolume.

S0-brane

As an illustration we consider the four-dimensional S0-brane considered in [8]. We do this for three reasons. First of all to show that we indeed reproduce known S-branes. Secondly, the parameters labelling the solutions do not yet have a physical meaning and finally to show that from a higher-dimensional point of view not all parameters are independent.

The four-dimensional non-dilatonic S0-brane belongs to the action (4.3.1) if we take $D = 4$, $p = 0$ and $k = -1$

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big(\mathcal{R} - \tfrac{1}{4}F_2^2\Big)\,. \tag{4.3.31}$$

There is no dilaton present, so we need to take $b = v^\phi = 0$. We then have four remaining parameters ω, v^φ, ζ and η.

The S0-brane follows from (4.3.21) if we require

$$(v^\varphi)^2 \to \frac{2Q\tau_0}{\zeta^2}\,, \quad \omega \to \frac{1}{2}\log(\tau_0^2)\,, \quad \eta^2 \to \frac{Q}{2\tau_0}\,, \tag{4.3.32}$$

together with the new time coordinate τ defined as

$$t^2 = \tau^2 - \tau_0^2\,. \tag{4.3.33}$$

From these relations we derive the metric

$$\mathrm{d}s^2 = -\frac{Q^2}{\tau_0^2}\frac{\tau^2}{\tau^2-\tau_0^2}\mathrm{d}\tau^2 + \frac{\tau_0^2}{Q^2}\frac{\tau^2-\tau_0^2}{\tau^2}\mathrm{d}z^2 + \frac{Q^2}{\tau_0^2}\tau^2\mathrm{d}\mathbb{H}_2^2\,. \tag{4.3.34}$$

This is the S0-brane as given in [8].

Not all parameters we started from are independent parameters from the four-dimensional view point. From the explicit metric and field strength expressions we can see that v^φ only appears in the field strength via a. Furthermore, ζ always appears in the combination $\zeta\sqrt{a}$. So ζ and v^φ are *not* independent variables. This agrees with the first relation in (4.3.32). We can choose $\zeta = 1$ without loss of generality. Similarly one can show that there is one relation between the other three parameters.

Actually, we can also remove the parameter τ_0 from (4.3.34) via re-scaling the coordinates as follows

$$t = \frac{Q}{\tau_0}\tau\,, \quad r = \frac{\tau_0}{Q}z\,. \tag{4.3.35}$$

If we do this we end up with

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{1-\frac{Q^2}{t^2}} + (1-\frac{Q^2}{t^2})\mathrm{d}r^2 + t^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,, \tag{4.3.36}$$

from which it follows that Q is related to the electric charge. The symmetry of the metric (4.3.34) is $\mathrm{SO}(2,1)\times\mathbb{R}$. Here the $\mathrm{SO}(2,1)$ is the symmetry transverse to the brane worldvolume and is referred to as the R-symmetry.

The S0-brane (4.3.34) is a singular solution [8], this follows for example from considering the invariant $\mathcal{R}_{\mu\nu\rho\eta}\mathcal{R}^{\mu\nu\rho\eta}$ where $\mathcal{R}_{\mu\nu\rho\eta}$ is the Riemann tensor. There are two ways that the singularity might disappear in the full theory. Namely the singularity might be smoothed out by stringy effects which are non-perturbative in α' or g_s [8]. All the original isotropic S-branes have singularities similar to the S0-brane.

In 2004 non-singular S-branes were found via a different way. Let us illustrate this with the examples given in [39, 73]. The original S-branes are homogenous, isotropic and time dependent. These solutions can be derived from known isotropic p-brane solutions via analytic continuations. For example, the S0-brane we described above follows from an analytically continuated Reissner-Nordström black hole with mass m and charge Q. To be specific, if we apply the following set of analytic continuations [73]

$$t \to ir\,, \quad r \to it\,, \quad \theta \to i\theta\,, \quad m \to im\,, \tag{4.3.37}$$

to the Reissner-Nordström metric we find

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{1-\frac{2m}{t}-\frac{Q^2}{t^2}} + (1-\frac{2m}{t}-\frac{Q^2}{t^2})\mathrm{d}r^2 + t^2(\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2)\,. \tag{4.3.38}$$

If we consider the massless limit this becomes (4.3.36) [74]. Similarly, the other S-branes can be related to analytic continuations of isotropic p-branes.

A new class of S-branes can be found by deforming the $\mathrm{SO}(D-p-2,1)$ R-symmetry of the S-branes. The easiest way is to consider the analytic continuation of known non-isotropic branes. In [75] is was shown that an analytic continuation of *rotating* p-branes leads to non-singular S-branes. In general, the rotating p-branes have singularities for large angular momenta. However, after the analytic continuation the resulting S-branes are regular everywhere. These branes are called twisted S-branes. Reducing the R-symmetry is thus one way to cure the singularity problem of the original S-branes.

4.4 Discussion

In this chapter we first introduced the concept of a generating solution. A generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of the isometry group G generates all other geodesics from the generating solution. We then presented the theorem that for maximally non-compact cosets G/H, with H its maximal compact subgroup, the generating solution can be constructed from the Cartan subalgebra only.

We illustrated this technique for Sp-branes. That is we studied a Lagrangian containing gravity, a dilaton and a $(p+1)$-form potential. Reducing this over the worldvolume of the Sp-brane gives the coset $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. Using the above mentioned theorem we presented the generating solution for the S(-1)-brane belonging to this coset. Acting with the $\mathrm{SL}(2,\mathbb{R})$-group on this solution and oxidizing to the higher-dimensional theory we obtained the most general Sp-brane solution with a deformed worldvolume. This is the S(-1)-/Sp-brane map.

However, we made various simplifications. First we did not dualize any forms to scalars in the reduced theory which would add magnetic flux to the above Sp-brane solutions. Secondly we did not consider intersections of S-branes which are carried by multiple forms with different degrees. Nonetheless with our approach they can be found with some extra effort. These extensions would just add extra axions to the lower-dimensional Lagrangian which extend the coset to the cosets in the left column of table 3.4.1. All the geodesics on these cosets must correspond to specific time-dependent S-brane solutions. Since the generating geodesics for the cosets on the left column of table 3.4.1 are the dilatonic straight lines, it must be that all S-brane type solutions can be rotated to pure gravitational solutions in 11 dimensions or to dilaton–Einstein solutions in type II supergravity. For example, if we take $\zeta = 0$ and $\eta = 1$ in (4.3.27) the flux becomes zero. In this way we see that the $SL(2,\mathbb{R})$-subgroup mixes physically distinct solutions in the higher-dimensional theory.

If we reduce to three dimensions a symmetry-enhancement of the coset takes place. The dualisation of the three-dimensional Kaluza–Klein vectors generate the coset $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+2)$ instead of the expected $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. However the generating solution of the $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+2)$-coset has only non-trivial dilatons and is therefore the same as the generating solution of $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. Nonetheless, there is an important difference with the time-dependent solutions from $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. In that case a solution-generating transformation $\in \mathrm{GL}(p+1,\mathbb{R})$ can be interpreted as a coordinate transformation in $D+p+1$ dimensions and therefore maps the vacuum solution to the same vacuum solution in different coordinates. In the case of symmetry enhancement to $\mathrm{SL}(p+2,\mathbb{R})$ a solution-generating transformation is not necessarily a coordinate transformation in $D+p+1$ dimensions. Instead the time-dependent vacuum solution transforms into a "twisted" vacuum solution. Where the twist indicates off-diagonal terms that cannot be redefined away. Such twisted solutions with $k = -1$ have received considerable interest since they can be regular [38, 39].

The solution-generating technique presented here should be considered complementary to the "compensator method" developed by Fré et al in [62]. There the straight line also serves as a generating solution but instead of rigid G-transformations one uses local H-transformations that preserve the solvable gauge to generate new non-trivial solutions. This technique is an illustration of the integrability of the second–order geodesic equations of motion [76].