Particle dynamics of branes

Ploegh, André René

Chapter 3

Dimensional Reduction of Branes

In the previous chapter we mentioned that the superstring requires a ten-dimensional space-time. To make a connection to our four-dimensional universe we introduce in this chapter dimensional reduction to link a higher-dimensional theory to a lower-dimensional one. We will restrict to torus reductions and reductions on maximally symmetric Einstein spaces.

In the last section we show that via dimensional reduction over the worldvolume of a brane we obtain a link between p-branes and instantons, and similarly between Sp-branes and S(-1)-branes. If on the other hand we reduce over the maximally symmetric transverse space of a brane, we generate a potential. This way we have a relation between branes and domain-walls and cosmologies. These observations form the basis for the rest of the thesis.

Some useful references about Kaluza–Klein reductions are [44, 45].

3.1 Dimensional Reduction

Consider a free scalar field $\hat{\phi}$ in $\hat{D} = D + 1$ dimensions, depending on the coordinates $x^{\hat{\mu}} = (x^{\mu}, y)$. We put hats on the fields when they are in $D + 1$ dimensions. What happens if we reduce the theory to D dimensions via compactifying the coordinate y on a circle S^1 of radius R? The first thing we can do is expand the scalar field $\hat{\phi}$ in a Fourier series. Due to the circle we have to impose the following boundary condition on the scalar field

$$\hat{\phi}(x^{\mu}, 0) = \hat{\phi}(x, 2\pi R)\,. \tag{3.1.1}$$

We expand $\hat{\phi}$ as

$$\hat{\phi}(x^\mu, y) = \sum_n \phi_n(x) \mathrm{e}^{iny/R}\,. \tag{3.1.2}$$

We obtain a discrete spectrum of fields $\phi_n(x)$ with quantized momentum $k = n/R$, see (2.3.18). The equation of motion for $\hat{\phi}$ is

$$\hat{\Box}\hat{\phi} = (\Box + \partial_y \partial^y)\hat{\phi}\,. \tag{3.1.3}$$

Using (3.1.2) we see that the lower-dimensional scalar fields $\phi_n(x)$ obey

$$\Box\phi_n(x) - \frac{n^2}{R^2}\phi_n(x) = 0\,. \tag{3.1.4}$$

We thus see that $\phi_0(x)$ is a massless field in D dimensions and that the other modes are massive fields with masses $m^2 = n^2/R^2$. The usual Kaluza–Klein approach is to assume that the radius R of S^1 is very small, in which case the masses of the modes with $n \neq 0$ will be enormous. This holds for general compact internal spaces and fields $\hat{\phi}$. This is the physical reason as to why we can choose to work with fields independent of y, since when we take the radius R of the extra dimension to be small so that we do not observe it, the massive fields ϕ_n become extremely massive and will not play a role in the effective D-dimensional theory. On the level of the equations of motion (3.1.4) this means that we truncate the fields $\phi_{n>0}$. These massive fields are knows as Kaluza–Klein states and when they are truncated we see from (3.1.2) that $\hat{\phi}$ is independent of the extra dimension y.

Let us therefore assume that all higher-dimensional fields are independent of the coordinate y. For a scalar field we just have that $\phi(x) = \hat{\phi}(x)$, but for fields that transform non-trivially under coordinate transformations we need to do something more. A vector will give $\hat{A}_{\hat{\mu}} = (A_\mu(x), \chi(x))$, i.e. it gives a D-dimensional vector $A_\mu(x)$ and a scalar field $\chi(x)$. The metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ will give rise to a D-dimensional metric $g_{\mu\nu}$, a vector $A_\mu(x)$ and a scalar field $\varphi(x)$, since we can write it as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \varphi \end{pmatrix}\,. \tag{3.1.5}$$

If we would start in $D+1$ dimensions with a scalar field, a vector and a metric, we would have in D dimensions three scalar fields, two vectors and one metric.

Not all compact manifolds are allowed, since a reduction must be consistent. This means that a solution of the lower-dimensional theory is also a solution of the higher-dimensional theory via tracing back the steps of the reduction. This procedure is called uplifting or oxidation.

An example of an inconsistency is choosing the scalar field φ to be a constant in the reduction (3.1.5). This follows from the equation of motion for φ, i.e. the details of the interactions between the various lower-dimensional fields prevent the truncation of the scalar φ. A different inconsistency can appear from the truncation of the massive modes in the expansion (3.1.2) and similarly for the metric and other fields. It can be that turning off the massive modes is not allowed, although one can show that this is not a problem for our S^1 reduction. For example, suppose we had kept all the modes in the Fourier expansion. It could have been that from the resulting equations of motion for these higher modes we would have found that it is not allowed to take the modes with $\phi_{n>0}$ to be zero. In more complicated Kaluza–Klein reductions, the issue of the consistency of the truncation to the massless sector is a tricky one. For these reasons it is best to reduce the equations of motion instead of the action. However in the examples we will consider this consistency is know to be in order.

If the massive sector of a certain reduction cannot be consistently truncated this does not automatically mean that the reduction is of no use. Assuming that the massive modes are very heavy, it is probable that the massive modes have negligible interactions with the massless sector because they are so heavy. So even in the case that the massive modes cannot be consistently truncated, leaving them out of the theory at low energies might still be a good approximation.

We will also use dimensional reduction for a different reason. We will show that reducing a theory over some of its dimensions leads to a theory which is easier to solve than the original one. Via uplifting the solution back to $\hat{D}$ dimensions we have generated a solution of the higher-dimensional system. This way we have constructed a solution-generating technique. It is clear that for this to work, the reduction must be consistent to be sure that it leads to a solution of the higher-dimensional theory.

So far we have focussed on a compact internal manifold such as S^1. Later we will see examples where the internal manifold is not compact. The only thing we require is that we can consistently truncate to the massless sector, although there is no physical motivation as to why we should do this truncation. For the solution-generating technique this turns out to be useful. Such a reduction is called a non-compactification, in contrast to a compactification on a compact manifold.

3.2 Torus Reduction of Gravity

As a first example we will work out the sphere reduction we introduced in the previous section. From (3.1.5) we see that the metric of the higher-dimensional theory is given by $\mathrm{d}s^2_{D+1} = \mathrm{d}s^2_D + \mathrm{d}\phi^2 + 2A_\mu \mathrm{d}x^\mu \mathrm{d}y$. We could use this Ansatz and plug it in the higher-dimensional Einstein-Hilbert action

$$S = \int \mathrm{d}^{D+1}x \sqrt{-\hat{g}}\,\hat{\mathcal{R}}\,. \tag{3.2.1}$$

The resulting action would not look familiar, for example we will not find the standard Einstein-Hilbert term. We would have to introduce some field redefinitions to fix this. To avoid this we will use the following $(D+1)$-dimensional metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\phi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\phi}(\mathrm{d}y + A_\mu \mathrm{d}x^\mu)^2\,, \tag{3.2.2}$$

with

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}\,, \quad \beta = -(D-2)\alpha\,. \tag{3.2.3}$$

These constants are chosen such that the lower-dimensional action immediately contains the Einstein-Hilbert term. The conventions we use for the Christoffel symbol are given in appendix A.2. Plugging (3.2.2) in (3.2.1) leads to the following D-dimensional Lagrangian

$$\mathcal{L} = *\mathcal{R}_D - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{-2(D-1)\alpha\phi} * \mathrm{d}A \wedge \mathrm{d}A\,. \tag{3.2.4}$$

Due to the coupling between ϕ and A we cannot take ϕ to be zero.

Let us now consider the reduction over a n-torus $\mathbb{T}^n$, i.e. $\mathbb{T}^n = S^1 \times \ldots \times S^1$ (n times). This reduction can be obtained from further reducing pure gravity on a series of circles. For each step the reduction of the metric generates a Kaluza–Klein vector A^i and a Kaluza–Klein scalar ϕ^i. The vectors that are already present from an earlier reduction give rise to a lower-dimensional vector and a scalar called axion. After reduction on an $n-$torus we count n vector fields A^m, n dilaton scalars ϕ^m that correspond to the radii of the circles and $n(n-1)/2$ axions χ^α. The dilatons and axions together parameterize the *coset* $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n) = \mathbb{R}\times\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$. The concept of a coset will be discussed in more detail in section 3.4.

Instead of doing a circle by circle reduction, it is easier to do the torus reduction in one step. Similar to the case of S^1 (3.1.5) we now write

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu^n \\ A_\nu^m & \varphi_{mn} \end{pmatrix}\,, \tag{3.2.5}$$

with φ_{mn} a symmetric strictly positive definite matrix of scalars, it contains the $n(n+1)/2$ axions and dilatons. It is positive definite since we are reducing over a Euclidean torus. The indices μ,ν (m,n) run from $1,\ldots,D$ $(D+1,\ldots,D+n)$. To avoid having to introduce field redefinitions in the lower-dimensional theory, we define $\varphi_{mn} = \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}$, with $\mathrm{e}^{2\beta\varphi}$ the determinant of φ_{mn}. This means that φ determines the volume of the torus. It is therefore called the volume modulus or the *breathing mode*. The $(D+n)$-dimensional Ansatz that gives us the Einstein-Hilbert action in D dimensions is

$$\mathrm{d}s_{D+n}^2 = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s_D^2 + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}(\mathrm{d}y^m + A^m) \otimes (\mathrm{d}y^n + A^n)\,, \tag{3.2.6}$$

with

$$\alpha^2 = \frac{n}{2(D+n-2)(D-2)}\,, \quad \beta = -\frac{(D-2)\alpha}{n}\,. \tag{3.2.7}$$

In doing this reduction we need the inverse of the metric Ansatz. This can best be achieved by using the vielbeine $\hat{e}_{\hat{\mu}}{}^{\underline{\hat{a}}}$ related to the metric as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\underline{\hat{a}}}\hat{e}_{\hat{\nu}}{}^{\underline{\hat{b}}}\hat{\eta}_{\underline{\hat{a}\hat{b}}}\,, \quad \hat{\eta}_{\underline{\hat{a}\hat{b}}} = \mathrm{diag}(-1,1,\ldots,1)\,, \tag{3.2.8}$$

see appendix A.2.1. Indices raised and lowered with $\hat{\eta}$ are underlined for clarity here. For the metric Ansatz (3.2.6) we derive that

$$\hat{e}_{\hat{\mu}}{}^{\underline{\hat{a}}} = \begin{pmatrix} \mathrm{e}^{\alpha\varphi} e_\mu{}^{\underline{\nu}} & \mathrm{e}^{\beta\varphi} A^p_\mu L_p{}^{\underline{m}} \\ 0 & \mathrm{e}^{\beta\varphi} L_n{}^{\underline{m}} \end{pmatrix}\,. \tag{3.2.9}$$

Here L is the vielbein of the torus

$$L_m{}^{\underline{k}} L_n{}^{\underline{l}} \hat{\eta}_{\underline{kl}} = \mathcal{M}_{mn}\,, \tag{3.2.10}$$

and $\underline{k},\underline{l}$ run from $D+1,\ldots,D+n$. Since the vielbein (3.2.9) is upper triangular it can easily be inverted. From this we find the inverse metric

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \hat{e}_{\underline{\hat{a}}}{}^{\hat{\mu}} \hat{e}_{\underline{\hat{b}}}{}^{\hat{\nu}} \hat{\eta}^{\underline{\hat{a}\hat{b}}}\,, \tag{3.2.11}$$

where $\hat{e}_{\underline{\hat{a}}}{}^{\hat{\mu}}$ is the inverse vielbein. The inverse metric can be written in terms of $\mathcal{M}_{ab}$ again.

If we plug (3.2.6) in the $(D+n)$-dimensional Einstein-Hilbert action we find the Lagrangian

$$\mathcal{L} = *\mathcal{R}_D - \frac{1}{2}*\mathrm{d}\varphi\wedge\mathrm{d}\varphi + \frac{1}{4}*\mathrm{d}\mathcal{M}_{mn}\wedge\mathrm{d}\mathcal{M}^{mn} - \frac{1}{2}\mathrm{e}^{2(\beta-\alpha)\varphi}\mathcal{M}_{mn}*\mathrm{d}A^m\wedge\mathrm{d}A^n\,. \tag{3.2.12}$$

Here $\mathcal{M}^{mn}$ means the inverse, i.e. $\mathcal{M}^{mn} = (\mathcal{M}^{-1})_{mn}$. It is consistent to put A^m to zero.

As mentioned, the scalar field φ is called the breathing mode since it describes the overall volume of the torus. The scalars in $\mathcal{M}$ can be interpreted as shape-moduli of the torus. These scalars parameterize the coset $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$. Together with the breathing mode φ we have the coset $\mathrm{GL}(n,\mathbb{R})/\,\mathrm{SO}(n) = \mathbb{R}\times\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$.

3.2.1 Torus Reduction over Time

Above we considered the Euclidean torus reduction of pure gravity. If we want to include time in the dimensional reduction as well we need to reduce over a n-torus with a Lorentzian signature $\mathbb{T}^{n-1,1}$. The Ansatz (3.2.6) is still valid, but with the difference that now Det $\mathcal{M} = -1$ due to the Lorentzian signature of the torus. To take care of this we replace $\mathcal{M}$ (3.2.10) by

$$\mathcal{M} = L\eta L^T\,, \quad \eta = \text{diag}(-1, 1, \ldots, 1)\,. \tag{3.2.13}$$

The reduction leads to the Lagrangian (3.2.12) but now with $\mathcal{M}$ given by (3.2.13). The scalar coset parameterizes $\text{GL}(n, \mathbb{R})/\,\text{SO}(n-1, 1)$. We have a non-compact version of the $\text{SO}(n)$-subgroup due to the reduction over the Lorentzian torus.

Something special happens if we reduce down to three dimensions. Due to Hodge duality (A.3.10) we can dualize *all* gauge potentials A^m to scalars. Schematically this goes as

$$\partial_\mu \tilde{\phi} \propto \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}\,, \tag{3.2.14}$$

where $F = \text{d}A^m$ and $\epsilon_{\mu\nu\rho}$ the three-dimensional epsilon tensor see (A.3.7). The three-dimensional gauge potential A^m can be described by the new scalar field $\tilde{\phi}^m$. If we do this for all the gauge potentials this leads to extra scalar fields in three dimensions. As a result, there is a symmetry enhancement since it can be shown that the extra scalars combine with the existing scalars into the coset $\text{SL}(n+1, \mathbb{R})/\,\text{SO}(n-1, 2)$. There is no decoupled $\mathbb{R}$ in this case.

For the reduction over a Euclidean torus from $3+n$ to three dimensions we have the coset $\text{SL}(n+1, \mathbb{R})/\,\text{SO}(n+1)$.

Due to the non-compactness of the subgroup, such as $\text{SO}(n-1, 1)$, the theory will contain ghosts. A ghost is an axion field with the opposite sign for the kinetic term in the Lagrangian. For future use let us discuss the ghost content for the theory with scalar coset $\text{GL}(p+q)/\,\text{SO}(p, q)$.

For a general coset $\text{GL}(p+q)/\,\text{SO}(p, q)$ the number of ghosts is pq. For the Kaluza–Klein moduli spaces this can be seen as follows. When one considers a reduction over time then there are two possible origins for ghosts. Ghost fields χ^Λ appear as the time-component of a one-form $\hat{A}^\Lambda$ in the higher dimension, that is, $\hat{A}^\Lambda = \chi^\Lambda \text{d}t + A^\Lambda$. Alternatively, extra ghost fields appear in three dimensions upon dualisation of the one-forms. The extra minus sign is due to the fact that the three-dimensional theory is Euclidean. Therefore, imagine we reduce Einstein gravity in $D+n$ dimensions to $D+1$ dimensions over a spacelike torus and then perform a subsequent reduction over a timelike circle, then the $n-1$ Kaluza–Klein vectors in $D+1$ dimensions give $n-1$ ghostlike axions. This fits with the fact that the scalar coset is $\text{GL}(n)/\,\text{SO}(n-1, 1)$. If $D = 3$ then we can further dualise those $n-1$ descendants of the Kaluza–Klein vectors to $n-1$ ghostlike axions, thereby doubling the number of ghosts. The Kaluza–Klein vector that appears from the last timelike reduction does not dualise to a ghost but to a normal axion since that vector appeared with a wrong sign in three dimensions. This indeed explains why there are $2(n-1)$ ghosts in $\text{SL}(n+1)/\,\text{SO}(n-1, 2)$.

3.3 Maximally Symmetric Compactification

The torus reduction we considered so far did not generate a potential V in the lower-dimensional theory. In this section we want to show an example where this does happen.

For this let us consider gravity with the following metric Ansatz

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\varphi} g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\varphi} g_{mn}(y)\mathrm{d}y^m \mathrm{d}y^n\,, \tag{3.3.1}$$

with the same α and β as before but note that g_{mn} depends on the coordinates y^m. The only scalar field is the breathing mode $\varphi(x)$. The indices μ, ν run from $1, \ldots, D$ and m, n run from $D+1, \ldots, D+n$. To see what kind of condition g_{mn} has to satisfy we work out the higher-dimensional Ricci tensor

$$\begin{aligned}
\hat{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{\mu\nu} + a(\partial\varphi)^2 g_{\mu\nu} + b\partial_\mu\varphi\partial_\nu\varphi + c\nabla_\mu\partial_\nu\varphi - \alpha g_{\mu\nu}\Box\varphi\,,\\
\hat{\mathcal{R}}_{mn} &= \mathcal{R}_{mn} - \beta \mathrm{e}^{2(\beta-\alpha)\varphi} g_{mn}\Box\varphi\,,\\
\hat{\mathcal{R}}_{\mu m} &= \hat{\mathcal{R}}_{m\mu} = 0\,.
\end{aligned} \tag{3.3.2}$$

The coefficients a, b and c are given by

$$a = -(D-2)\alpha^2 - n\beta\alpha\,, \quad b = (D-2)\alpha^2 + 2n\alpha\beta - \beta^2 n\,, \quad c = -(D-2)\alpha - n\beta\,. \tag{3.3.3}$$

Let us now focus on the internal manifold metric g_{mn}. We assume that g_{mn} is a n-dimensional *Einstein space*. Such a space is defined by

$$\mathcal{R}_{mn} = d\, g_{mn} \to \mathcal{R}_n = d\, n\,, \tag{3.3.4}$$

with d a constant and $\mathcal{R}_n$ the Ricci scalar of the n-dimensional internal manifold. From (A.2.18) we see that if we have a sphere S^n ($k=+1$), a hyperboloid $\mathbb{H}^n$ ($k=-1$) or flat space $\mathbb{E}^n$ ($k=0$) that $d=(n-1)k$. From the Einstein equation $\hat{\mathcal{R}}_{mn} = 0$ (3.3.2) we see that, if g_{mn} is one of these three Einstein spaces that this becomes an equation of motion for φ coupled to some potential V.

To be precise, the n-dimensional field equations (3.3.2) can be derived from reducing the $(D+n)$-dimensional Einstein-Hilbert action

$$\int \sqrt{-\hat{g}}\hat{\mathcal{R}} = \mathrm{Vol}(\mathcal{M}_n) \int \sqrt{-g_D}\Big(\mathcal{R}_D - \frac{1}{2}(\partial\varphi)^2 + \mathrm{e}^{2(\alpha-\beta)\varphi}\mathcal{R}_n\Big)\,, \tag{3.3.5}$$

where $\mathrm{Vol}\,\mathcal{M}_n$ is the volume of the internal manifold and we ignore the total derivative $\Box\varphi$. When g_{mn} belongs to one of the three Einstein spaces discussed above $\mathcal{R}_n$ simplifies to $d\,n$. This means that we can identify the potential V as [46]

$$V(\varphi) = -kn(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi}\,. \tag{3.3.6}$$

In case we restrict to positive potentials, we see from (3.3.6) that $k = -1$. However, $\mathbb{H}^n$ is not a compact space and $\mathrm{Vol}(\mathcal{M}_n)$ is not finite. To resolve this we mention that $\mathbb{H}^n$ can be seen as the coset $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, just as the sphere S^n is the coset $\mathrm{SO}(n+1)/\mathrm{SO}(n)$. To make $\mathbb{H}^n$ compact, we can mod out with a discrete non-compact symmetry. Since the metric is local, it does not care about topological issues such as discrete identifications.

To make contact with S- and p-branes, we add to the higher-dimensional theory a $(p-1)$-form gauge potential and a dilaton ϕ

$$\mathcal{L} = *\hat{\mathcal{R}} - \frac{1}{2} * \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{a\phi} * \mathrm{d}A_{p-1} \wedge \mathrm{d}A_{p-1}\,. \tag{3.3.7}$$

The equations of motion and Bianchi identity are

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{p-1}{2(D-2)(p!)}g_{\mu\nu}\mathrm{e}^{a\phi}F_p^2 + \frac{1}{(p-1)!2}\mathrm{e}^{a\phi}(F_p^2)_{\mu\nu}\,, \tag{3.3.8}$$

$$\mathrm{d}\Big(*\mathrm{e}^{a\phi}F_p\Big) = 0\,, \quad \mathrm{d}F_p = 0\,, \tag{3.3.9}$$

$$\Box\phi = \frac{a}{p!2}F_p^2\mathrm{e}^{a\phi}\,. \tag{3.3.10}$$

The Ansatz for the metric is again (3.3.1), but now we have that $D = p$. For the field strength we use a Freund-Rubin [47] like Ansatz

$$F_p = f\mathrm{e}^{(D\alpha - n\beta)\varphi - a\phi}\epsilon_D(x^\mu)\,, \quad \phi = \phi(x^\mu)\,. \tag{3.3.11}$$

Here f is a constant and ϵ_D is the D-dimensional epsilon tensor (A.3.7). This expression for the field strength is in agreement with the equation of motion for F_p and its Bianchi identity (3.3.9). One can now again reduce the higher-dimensional equations of motion (3.3.8-3.3.10) by using the Ansätze for the field strength and metric. As it turns out, these lower-dimensional equations can be derived from the following Lagrangian

$$\mathcal{L}_D = \sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - V(\phi,\varphi)\Big)\,, \tag{3.3.12}$$

where the scalar potential V now gets a positive contribution from the p-form flux

$$V(\phi,\varphi) = \frac{f^2}{2}\mathrm{e}^{2(D-1)\alpha\varphi - a\phi} - kn(n-1)\mathrm{e}^{2(\alpha-\beta)\varphi}\,. \tag{3.3.13}$$

Here the first term comes from the flux and the second term we already found in (3.3.6). We see that the flux adds a positive contribution to the potential. In case $k = 0$ and if the scalar fields can be fixed this leads to a positive cosmological constant Λ. Due to the charge quantization condition (2.4.18) Λ would be quantized.

It is important to mention that in the above reduction we reduced the equations of motion and not the action. Had we reduced the latter we would have found the wrong sign in front of the flux contribution. This is because filling in an Ansatz means filling in on-shell information and that can lead to problems. An Euler-Lagrange variation with respect to the remaining (unfixed) degrees of freedom can be inconsistent. For this reason it is better to reduce the equations of motion and then see what kind of action leads to these equations of motion. For the torus reduction discussed in the previous section such a problem does not appear.

An interesting generalization was considered in the papers [48–50]. We will use the result of [45]. Consider the internal manifold as a product of M different spaces $\mathcal{M}_i$

$$\mathcal{M}_{\text{int}} = \Pi_{i=1}^{M} \times \mathcal{M}_i \,. \tag{3.3.14}$$

The dimensions of each internal spaces is $\mathcal{M}_i = n_i$, obeying the sum $\sum_i n_i = n$. Each space $\mathcal{M}_i$ is assumed to be an Einstein space

$$(\mathcal{R}_i)_{A_i\,B_i} = k_i(n_i - 1)(g_i)_{A_i\,B_i} \,. \tag{3.3.15}$$

The generalization of the metric Ansatz (3.3.1) is

$$\mathrm{d}s^2_{D+n} = \mathrm{e}^{2\alpha\varphi(x)} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\varphi(x)} \sum_{i=1}^{M} X_i(x)(g_i)_{A_i B_i} \mathrm{d}y^{A_i} \mathrm{d}y^{B_i} \,. \tag{3.3.16}$$

To consider φ as the field that determines the overall volume of the internal manifold we have to require that

$$\Pi_{i=1}^{M} X_i^{n_i} = 1 \,. \tag{3.3.17}$$

With this condition we see from (3.3.16) that the only x-dependence of the determinant of the internal manifold is given by $\varphi(x)$. Due to (3.3.17) we have only $M-1$ independent X^i. It is therefore convenient to write

$$X_i = \mathrm{e}^{-\vec{\beta}_i \cdot \vec{\phi}} \,, \tag{3.3.18}$$

where $\vec{\beta}_i \cdot \vec{\phi} = \sum_{I=1}^{M-1} \beta_{iI} \phi_I$, with $\vec{\beta}_i$ a constant $(M-1)$-dimensional vector. In this case we see that we have M scalar fields in total. From (3.3.17) we find that the vectors $\vec{\beta}_i$ satisfy

$$\sum_{i=1}^{M} n_i \beta_{iI} = 0 \,. \tag{3.3.19}$$

The reduced Einstein equations for the metric Ansatz (3.3.16) can be derived from the action

$$S = \int *\mathcal{R} - \frac{1}{2} * \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} \sum_{I=1}^{M-1} *\mathrm{d}\phi^I \wedge \mathrm{d}\phi^I - V(\varphi, \vec{\phi}) \,, \tag{3.3.20}$$

with the multi-exponential potential

$$V(\varphi, \vec{\phi}) = -\mathrm{e}^{2(\alpha-\beta)\varphi} \sum_{i=1}^{M} k_i n_i (n_i - 1) \mathrm{e}^{\vec{\beta}_i \cdot \vec{\phi}} \,. \tag{3.3.21}$$

3.4 Coset Geometry

When we discussed the torus reduction in section 3.2 we found out that the scalar fields parameterize the Riemannian coset $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{SO}(n)$. In this section we want to give a general discussion about cosets G/H with H the maximal compact subgroup of G. In particular we will show how we can obtain for such cosets a metric G_{ij}. We comment later on the case when H is non-compact due to a reduction over a Lorentzian torus.

Let us begin by defining what we mean with a coset G/H. Let G be a group with a subgroup H. The coset space G/H is the set of elements $[g]$ of G with the equivalence relation

$$[g] = [g'] \quad \text{if} \quad g' = g\, h \,, \tag{3.4.1}$$

where h is an element of H. As an example, let G be a Lie group and H any subgroup of G. We can form the coset G/H. This coset space admits a differentiable structure and G/H becomes a manifold M with dim G/H=dim G-dim H.

Assume now that G is a Lie group which acts transitively on a manifold M. That means that given any point $p \in M$, the action of G on p allows us to go to all the points of M. Such a manifold is called *homogeneous*. For example, let $H(p)$ be an isotropy group of $p \in M$, then G/H is a homogenous space. In fact, if G, $H(p)$ and M satisfy certain requirements it can be shown that $G/H(p)$ is diffeomorphic to M [51]. As an example consider the Lie group $SO(3)$ acting transitively on S^2. The isotropy group H is $SO(2)$ and as a result we have that $SO(3)/SO(2) \cong S^2$.

Let us focus on coset manifolds of the form G/H with H the maximal compact subgroup. We want to define a metric G_{ij} on G/H, which fixes the kinetic term for the scalar fields

$$e^{-1}\mathcal{L} \propto G(\phi)_{ij} \partial\phi^i \partial\phi^j \,. \tag{3.4.2}$$

This metric will not be unique, but we make the demands that the isometry group must be G and that we have invariance under local H-transformations. We mention [44, 52–54] as a few examples about the use of cosets in supergravity.

If we succeed in parameterizing G/H with some coordinates y^i, then a coset representative $L(y)$ is a representation of G/H with the extra condition that if $y \neq y'$ then there cannot exist an element h of H for which $L(y) = L(y')h$. This is because of the coset requirement (3.4.1). On the other hand, for a given y there exist multiple $L(y)$ since for all $h \in H$, $L(y)h$ is an equivalent representation of the same coset element.

It is not difficult to construct a coset representative using the Lie algebras $\mathfrak{G}$ and $\mathfrak{H}$ of G and H respectively. Since H is a subgroup of G we have the decomposition $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{F}$, with $\mathfrak{F}$ the complement of $\mathfrak{H}$ in $\mathfrak{G}$. For a given representation of the algebra $\mathfrak{G}$ we define a coset representative via

$$L(y) = \exp(y^i \mathbf{f_i}) \,, \qquad (3.4.3)$$

where the $\mathbf{f_i}$ form a basis of $\mathfrak{F}$ in some representation of $\mathfrak{G}$. This defines correctly a representative since if we assume $L(y) = L(y')h$ we find that $y = y'$ as is required for a representative.

We will be interested in a decomposition $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{F}$ which can be done in such a way that

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H} \,, \quad [\mathfrak{F}, \mathfrak{H}] \subset \mathfrak{F} \,, \quad [\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{H} \,. \qquad (3.4.4)$$

Such a coset is called a *symmetric space*.

To derive the metric we define a Lie algebra valued one-form from the coset representative $L(y)$ via

$$L^{-1}\mathrm{d}L \equiv E + \Omega \,, \qquad (3.4.5)$$

where E takes values in $\mathfrak{F}$ and Ω in $\mathfrak{H}$. We notice that $L^{-1}dL$ is invariant under left multiplication with an y-independent element $g \in G$. Multiplying L from the right with local elements $h \in H$ results in

$$E \to h^{-1}\, E\, h \,, \qquad \Omega \to h^{-1}\, \Omega\, h + h^{-1}\mathrm{d}h \,. \qquad (3.4.6)$$

In supergravity the parameters y^i are scalar fields that depend on the space-time coordinates $y^i = \phi^i(x)$. The one-form $L^{-1}\mathrm{d}L$ can be written out in terms of coset-coordinate one-forms $\mathrm{d}\phi^i$ which themselves can be pulled back to space-time coordinate one-forms $\mathrm{d}\phi^i = \partial_\mu \phi^i \mathrm{d}x^\mu$. Now we can write

$$L^{-1}\mathrm{d}L = E_\mu \mathrm{d}x^\mu + \Omega_\mu \mathrm{d}x^\mu \,. \qquad (3.4.7)$$

Under the ϕ-dependent H-transformations $h(\phi(x))$ we have that $\Omega_\mu \to h^{-1}\Omega_\mu h + h^{-1}\partial_\mu h$ and $E_\mu \to h^{-1}E_\mu h$. We see that E_μ is covariant under local H-transformations and Ω_μ transforms like a connection. Using this connection Ω_μ we can make the following H-covariant derivative on L and L^{-1}

$$D_\mu L = \partial_\mu L - L\Omega_\mu \,, \qquad D_\mu L^{-1} = \partial_\mu L^{-1} + \Omega_\mu L^{-1} \,. \qquad (3.4.8)$$

To find a kinetic term for the scalars we notice that the object

$$\mathrm{Tr}[D_\mu L D^\mu L^{-1}] = -\mathrm{Tr}[E_\mu E^\mu] \,, \qquad (3.4.9)$$

has all the right properties as it contains single derivatives on the scalars, it is a space-time scalar, it is invariant under rigid G transformations and under local H-transformations. Thus,

$$e^{-1}\mathcal{L}_{\mathrm{scalar}} = -\mathrm{Tr}[E_\mu E^\mu] \equiv -\tfrac{1}{2} G(\phi)_{ij} \partial_\mu \phi^i \partial^\mu \phi^j \,. \qquad (3.4.10)$$

So far we have been completely general, that is we did not specify the coordinates ϕ^i nor the representation. Let us therefore make a connection to section 3.2, i.e. we focus on a coset where H is the maximal compact subgroup SO(n) of G in the fundamental representation. The Lie algebra of SO(n) is the vector space of antisymmetric matrices and we have the split

$$E = \frac{L^{-1}\mathrm{d}L + (L^{-1}\mathrm{d}L)^T}{2}\,, \qquad \Omega = \frac{L^{-1}\mathrm{d}L - (L^{-1}\mathrm{d}L)^T}{2}\,, \tag{3.4.11}$$

and a calculation shows that

$$e^{-1}\mathcal{L}_{\text{scalar}} = -\mathrm{Tr}[E^2] = +\tfrac{1}{4}\mathrm{Tr}[\partial\mathcal{M}\partial\mathcal{M}^{-1}] = -\tfrac{1}{2}G(\phi)_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{3.4.12}$$

Here $\mathcal{M}$ is the SO(n)-invariant matrix

$$\mathcal{M} = LL^T\,. \tag{3.4.13}$$

Under the global isometry group G it transforms as

$$\mathcal{M} \to \mathcal{M}' = g\mathcal{M}g^T\,, \quad g \in G\,. \tag{3.4.14}$$

To find the metric $G(\phi)_{ij}$ from (3.4.12) we still need to use an explicit realization of the Lie algebra. This means a choice for the coordinate frame on G/H. However we found that (3.4.12) still has local H-invariance. We use these h-transformations to bring L in a 'nice' form for computations, i.e. we make a gauge choice.

Let us look for the 'nice' gauge in case H is the maximal compact subgroup of G. The gauge we will be using is due to the Iwasawa decomposition [44, 55]. This states that every element g in the Lie group G can be obtained by exponentiating the lie algebra $\mathfrak{G}$ as follows

$$g = g_N g_C g_H\,, \tag{3.4.15}$$

where g_N is the exponentiation of the positive-root part of the algebra $\mathfrak{G}$, $g_\mathcal{C}$ the exponentiation of the Cartan subalgebra and g_H is the maximal compact subgroup H in G. In appendix C we present a short overview of Lie algebras and Lie groups.

For the algebra of G we denote the Cartan generators by H_I with $I = 1, \ldots, r$ and r is the rank of the algebra. All the positive root generators are denoted by E_α. The commutation relations read

$$[H_I, H_J] = 0\,, \quad [H_I, E_\alpha] = \alpha_I E_\alpha\,, \quad [E_\alpha, E_\beta] = N(\alpha,\beta)E_{\alpha+\beta}\,. \tag{3.4.16}$$

The last line is to be understood as follows. If $\alpha+\beta$ is not a root we have $N(\alpha,\beta) = 0$, else we have $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$. We call the algebra formed by H_I and the positive root generators E_α the *Borel subalgebra*. For the Borel Lie algebra the matrix representation can be chosen such that all elements of it are upper-triangular [55]. We can then parameterize the coset elements in this gauge as

$$L = \exp[\mathfrak{s}]\,, \tag{3.4.17}$$

with $\mathfrak{s} = \mathcal{C} \oplus \sum E_\alpha$, the sum is over all the positive roots α. To be precise, as a representative L we take

$$L = \Pi_\alpha \exp[\chi^\alpha E_\alpha] \Pi_I \exp[\frac{1}{2}\phi^I H_I]\,, \tag{3.4.18}$$

where the ϕ^I are called the dilatons and χ^α the axions. The number of dilatons is given by the rank of $\mathfrak{g}$ and the number of axions is given by the number of positive roots.

In terms of the Iwasawa decomposition (3.4.15) we see that our coset representative is written in terms of $L = g_N g_{\mathcal{C}}$. If we now multiply this representative from the left with an element $g \in G$ and make use of the Iwasawa decomposition we see that we must be able to write $gL(y)$ as

$$gL(y) = L(y')g_H\,, \tag{3.4.19}$$

where $L(y') = g'_N g'_{\mathcal{C}}$. We can now use a local $h \in H$ to remove g_H such that we are back in the Borel gauge. This is the gauge obtained via exponentiating the Borel subalgebra.

What we have done here is only valid in case G is so-called *maximally non-compact*. A group G is maximally non-compact if the Iwasawa decomposition allows the representative to be given by *all* the Cartan generators.

In general the Iwasawa decomposition ensures the existence of a solvable Lie algebra[1] *Solv*, that is a real semisimple Lie algebra $\mathfrak{G}$ of a group G can be written as $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a solvable Lie algebra consisting out of the non-compact part of the Cartan generators and a subset of the positive root generators [53, 55]. One can then similarly use this solvable algebra as the basis for the representative. For the case we have a maximally non-compact G the solvable gauge is called the Borel gauge. In case not all Cartan generators are in $\mathfrak{s}$ we call G non-maximally non-compact. See [52] for a discussion of this in the case of dimensionally reduced heterotic supergravity.

3.4.1 The Coset $\mathrm{SL}(n, \mathbb{R})/\,\mathrm{SO}(n)$

Now we specify to $G = \mathrm{SL}(n, \mathbb{R})$, which has rank $n-1$ and $\mathrm{SO}(n)$ as its maximal compact subgroup. The number of positive roots is $n(n-1)/2$. There are therefore $n-1$ dilatons ϕ^I and $n(n-1)/2$ axions χ^α. The Cartan generators are given in terms of the weights $\vec{\beta}$ of $\mathrm{SL}(n, \mathbb{R})$ in the fundamental representation

$$(\vec{H})_{ij} = (\vec{\beta}_i)\delta_{ij}\,. \tag{3.4.20}$$

[1] A solvable Lie algebra is defined as follows. Let $\mathfrak{G}^0 = \mathfrak{G}$ and for $k > 0$ we define $\mathfrak{G}^{k+1} = [\mathfrak{G}^k, \mathfrak{G}^k]$. If for finite n this series terminates, i.e. $\mathfrak{G}^n = 0$, then we call the Lie algebra $\mathfrak{G}$ solvable.

The weights can be taken to obey the following algebra

$$\sum_i \beta_{iI} = 0\,, \quad \sum_i \beta_{iI}\beta_{iJ} = 2\delta_{IJ}\,, \quad \vec{\beta}_i \cdot \vec{\beta}_j = 2\delta_{ij} - \frac{2}{n}\,. \tag{3.4.21}$$

The first of these identities holds in all bases since it follows from the tracelessness of the SL generators. The second and third identity can be seen as convenient normalizations of the generators. The positive step operators E_{ij} are all upper triangular and a handy basis is that they have only one non-zero entry $[E_{ij}]_{ij} = 1$. The negative step operators are the transpose of the positive. The SO(n) algebra is spanned by the following combinations

$$\frac{1}{\sqrt{2}}(E_\beta - E_{-\beta})\,. \tag{3.4.22}$$

The action will generically look complicated but when all axions are set to zero L is diagonal $L = \mathrm{diag}[\,\exp(-\frac{1}{2}\vec{\beta}_i \cdot \vec{\phi})]$ and the action becomes

$$+\tfrac{1}{4}\mathrm{Tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} = -\tfrac{1}{4}(\sum_i \beta_{iJ}\beta_{iI})\partial\phi^I\partial\phi^J = -\tfrac{1}{2}\delta_{IJ}\partial\phi^I\partial\phi^J\,. \tag{3.4.23}$$

This action describes $n-1$ dilatons that parameterize the flat scalar manifold $\mathbb{R}^{n-1}$. Above we have set all the axions χ^α to zero, this should be consistent with the equations of motion that follow from (3.2.12)

$$\partial_\mu(\sqrt{-g}\mathcal{M}^{-1}\partial^\mu\mathcal{M}) = 0\,. \tag{3.4.24}$$

In general one can truncate a set of scalar fields that parameterize a scalar manifold M to a smaller set of scalar fields that parameterize a submanifold $M' \subset M$ if M' is a totally geodesically submanifold [56]. This means that any geodesic in M' is also a geodesic in M.

The simplest example is SL$(2,\mathbb{R})/$ SO(2). The algebra is given by

$$[H, E_2] = 2E_2\,, \quad [H, E_{-2}] = -2E_{-2}\,, \quad [E_2, E_{-2}] = H\,. \tag{3.4.25}$$

The two-dimensional fundamental representation is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.4.26}$$

From which we find the coset representative

$$L = \exp[\chi E_2]\exp[\frac{1}{2}\phi H] = \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2}\chi \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix}, \tag{3.4.27}$$

which leads to the kinetic term

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,. \tag{3.4.28}$$

This is the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$-coset of type IIB supergravity[2]. In section 2.3.3 we found that the $\mathrm{SL}(2,\mathbb{R})$ extends to the whole Lagrangian, leading to S-duality. We refer to [45] for a realization of the somewhat less trivial example $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$ with five scalar fields.

From the above we can construct the Lorentzian version of this coset, that is $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. This gives rise to the D(-1)-instanton of type IIB. As explained in section 3.2.1 we can use the same expression for L, but need to modify $\mathcal{M}$ to $L\eta L^T$ with $\eta = (-1,1)$. We find the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,. \tag{3.4.29}$$

Indeed we see that the metric is no longer positive definite.

It is important to mention that for maximally non-compact G the Borel gauge only covers the whole manifold if the subgroup H is its maximal compact subgroup. As we saw from the torus reduction this is no longer true when we reduce over time. The Borel gauge does not cover the whole manifold any more as shown in [57].

This can also be seen explicitly for the coset (3.4.29). We can rewrite it as

$$\mathcal{L} = -\frac{1}{2}\mathrm{e}^{-2\phi}(\partial\mathrm{e}^{\phi})^2 + \frac{1}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2 \,, \tag{3.4.30}$$

which is the metric on two-dimensional anti-de Sitter space (AdS_2) in terms of the coordinates (e^{ϕ},χ). It is known that these coordinates do not cover the whole manifold. Whereas we can rewrite (3.4.28) as Euclidean AdS_2, which does cover the whole manifold.

It is therefore better not to rely on the Borel gauge for this kind of computation at all, but it seems that a general good gauge choice (a gauge that can always be imposed) is unavailable [57]. To avoid this problem with the Borel gauge, we will work in chapter 7 on the level of $\mathcal{M}$ directly when we discuss the generating solution for instantons.

3.4.2 Maximally Extended Supergravities

Let us now see what happens if we consider the dimensional reduction of type IIA and type IIB on a n-torus and eleven-dimensional supergravity on a $(n+1)$-torus. As it turns out, we find the unique maximal supergravities in $D \leq 10$. That is D-dimensional supergravities, whose supersymmetry is the maximal allowed by the

[2]The minus sign in the exponent can be removed via the field redefinition $\phi \to -\phi$.

	Minkowskian	Euclidean
$D=10$	SO(1,1)	SO(1,1)
$D=9$	$\frac{\mathrm{GL}(2,\mathrm{I\!R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{GL}(2,\mathrm{I\!R})}{\mathrm{SO}(1,1)}$
$D=8$	$\frac{\mathrm{SL}(3,\mathrm{I\!R})}{\mathrm{SO}(3)} \times \frac{\mathrm{SL}(2,\mathrm{I\!R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{SL}(3,\mathrm{I\!R})}{\mathrm{SO}(2,1)} \times \frac{\mathrm{SL}(2,\mathrm{I\!R})}{\mathrm{SO}(1,1)}$
$D=7$	$\frac{SL(5,\mathrm{I\!R})}{\mathrm{SO}(5)}$	$\frac{\mathrm{SL}(5,\mathrm{I\!R})}{\mathrm{SO}(3,2)}$
$D=6$	$\frac{\mathrm{SO}(5,5)}{\mathrm{S}[\mathrm{O}(5)\times\mathrm{O}(5)]}$	$\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5,\mathbb{C})}$
$D=5$	$\frac{\mathrm{E}_{6(+6)}}{USp(8)}$	$\frac{\mathrm{E}_{6(+6)}}{USp(4,4)}$
$D=4$	$\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}(8)}$	$\frac{E_{7(+7)}}{\mathrm{SU}^*(8)}$
$D=3$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}(16)}$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}^*(16)}$

Table 3.4.1: *Cosets for maximal supergravities in Minkowskian and Euclidean signatures.*

space-time dimension D. The reason that we obtain maximal supergravities is that a torus reduction does not break supersymmetry.

One can classify these (maximal) supergravities by the scalar field interactions in the Lagrangian. Just as for example the $\mathrm{SL}(2,\mathrm{I\!R})/\,\mathrm{SO}(2)$ coset specifies the Lagrangian (3.4.28). The scalar fields parameterize a Riemannian manifold whose geometry fixes the interactions terms in the supergravity Lagrangian. We summarize the scalar manifolds of the maximally extended supergravities that appear after dimensional reduction of 11-dimensional supergravity on a torus in table 3.4.1 [58]. For future use we show it both for Minkowskian and Euclidean maximal supergravities[3]. The cosets G/H in the left column are all maximally non-compact since G is the maximal non-compact real slice of a semi-simple algebra and H is the maximal compact subgroup. Since H is compact the metric is strictly positive definite and the coset is Riemannian. The cosets G/H' in the right column only differ in the isotropy group H' which is some non-compact version of H and as a consequence G/H' is not Riemannian.

There is a third class of maximally extended supergravities which we did not put in the table. Namely the so-called *star supergravities* [58, 59]. These are Lorentzian theories, but do have a non-compact isotropy group H. We will meet these theories in

[3]The relation between supergravity theories and geometries is not always completely one-to-one. For example, in section 3.3 we have seen that a reduction can also generate a potential V. This potential is in general not determined by the geometry.

chapter 6 and show how they are related to the Minkowski theories in the left column of table 3.4.1 for the case $D = 10$.

3.5 From Branes to "Particles"

As we discussed in section 2.4, many supergravity solutions have the structure of p-branes. That is, they are charged electrically under a $(p+1)$-form gauge potential A_{p+1} or magnetically under a $(D-p-3)$-form gauge potential A_{D-p-3}, where D is the space-time dimension of the supergravity theory. Another characteristic of brane solutions is that the brane geometry has a flat $(p+1)$-dimensional worldvolume, see for example (2.4.12).

In general two different kinds of brane solutions are considered; timelike p-branes that are related to the string theory D-branes [60] (or M-branes) or spacelike p-branes (known as S-branes) who are conjectured to describe time-dependent phenomena in string theory [8]. Timelike p-branes have a Lorentzian worldvolume and are stationary solutions whereas Sp-branes have a Euclidean worldvolume and are explicitly time-dependent. The metrics are given by[4]

$$\begin{aligned}
&\text{timelike brane:} \qquad \mathrm{d}s_D^2 = \mathrm{e}^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2B(r)}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2}),\\
&\text{spacelike brane:} \qquad \mathrm{d}s_D^2 = \mathrm{e}^{2A(t)}\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2B(t)}(-\mathrm{d}t^2 + t^2\mathrm{d}\mathbb{H}^2_{D-p-2})\,, \qquad (3.5.1)
\end{aligned}$$

where A, B are arbitrary functions and δ, η are respectively the Euclidean and the Lorentzian metric. There exist less symmetric solutions that break the worldvolume symmetries ($\mathrm{ISO}(p,1)$ and $\mathrm{ISO}(p+1)$) and the transversal symmetries ($\mathrm{SO}(D-p-1)$ and $\mathrm{SO}(D-p,1)$). There are two standard ways to achieve this. First there are extra functions multiplying the $\mathrm{d}x\mathrm{d}x$-terms on the worldvolume. Secondly, there are off-diagonal terms that mix worldvolume directions with transversal directions ($\mathrm{d}x\mathrm{d}\theta$), like for rotating timelike branes or twisted spacelike branes [39].

Solutions that are carried by a metric and scalars alone have a simpler mathematical structure then those solutions that are carried by non-trivial p-form potentials. At first sight, there is only a restricted class of brane solutions that can be found as solutions of a scalar-metric Lagrangian of the type

$$\mathcal{L} = \sqrt{|g|}\Big(\mathcal{R} - \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j - V(\phi)\Big)\,, \qquad (3.5.2)$$

where G_{ij} is the metric on moduli space and $V(\phi)$ is a scalar potential. If we regard a scalar potential V as a 0-form "field strength" then it can couple magnetically to $(D-2)$-branes, thus domain-walls (timelike) and cosmologies (spacelike) based on the reasoning explained in section 2.4.

[4]We choose $A = C$ in (2.4.22).

On the other hand $\partial\phi$ is a 1-form field strength and can therefore couple magnetically to $(D-3)$-branes and electrically to (-1)-branes. Note that for timelike (-1)-branes the worldvolume is zero-dimensional and the transverse space covers the whole space. Timelike (-1)-branes are solutions of Euclidean supergravity, i.e. they are instantons[5].

Apart from the $(D-3)$-branes (like the IIB 7-branes) the scalars only depend on one coordinate and the Ansatz is given by

$$\mathrm{d}s_D^2 = \epsilon f(r)^2 \mathrm{d}r^2 + g(r)^2 g_{ab}^{D-1} \mathrm{d}x^a \mathrm{d}x^b, \quad \phi^i = \phi^i(r)\,. \tag{3.5.3}$$

The function f corresponds to the gauge freedom of re-parameterizing the r-coordinate. For $\epsilon = -1$ the radial coordinate corresponds to time ($r = t$) and g_{ab} is a metric on a Euclidean maximally symmetric space (the three possible FLRW geometries). When $\epsilon = +1$ (3.5.3) this describes an instanton geometry with r the direction of the tunneling process. For $\epsilon = +1$ and g_{ab}^{D-1} a Lorentzian maximally symmetric space (AdS, Minkowski or dS) (3.5.3) is a domain-wall geometry with r the transversal distance from the wall. The difficulty with $(D-3)$-branes is that these solutions depend on one complex coordinate rather that on one real coordinate. For this reason we do not consider $(D-3)$-branes in this thesis.

Let us now explain that also the other p-brane solutions can be related to the Lagrangian (3.5.2).

The worldvolume of a p-brane corresponds to Killing directions of space-time, and for that to be valid the matter fields do not depend on the worldvolume coordinates. This implies that one can “dimensionally reduce” a p-brane over its worldvolume[6]. In the dimensionally reduced theory the p-brane then corresponds to a (-1)-brane, since the worldvolume is zero-dimensional. These reductions are the torus reductions we described in section 3.2. Comparing (3.5.1) with (3.2.6) we see that we have to turn off the Kaluza–Klein vectors. Furthermore, the worldvolume of the theory is identified with $\mathcal{M}_{mn}$. We see that in general $\mathcal{M}_{mn}$ breaks the worldvolume symmetries ($\mathrm{ISO}(p,1)$ and $\mathrm{ISO}(p+1)$), since we will obtain extra terms multiplying the $\mathrm{d}x\mathrm{d}x$-terms on the worldvolume. If we reduce to $D = 3$ we dualize all Kaluza–Klein vectors to scalars, see (3.2.14). These Kaluza–Klein vectors will lead to off-diagonal terms that mix worldvolume directions with transversal directions $\mathrm{d}x\mathrm{d}\theta$.

We see that p- and Sp-branes reduced over their worldvolume lead to a system containing gravity and scalar fields only! In the case of timelike branes this is probably best known for the correspondence between four-dimensional black holes (0-branes) and three-dimensional instantons [56, 61]. We refer to [62] for a similar discussion in the case of spacelike p-brane solutions in maximal supergravity.

[5] In chapter 6 and 7 we consider a few examples where we have a Euclidean theory *with* a potential. We call these solutions instantons as well.

[6] We put dimensionally reduce between “” since the worldvolume is not compact.

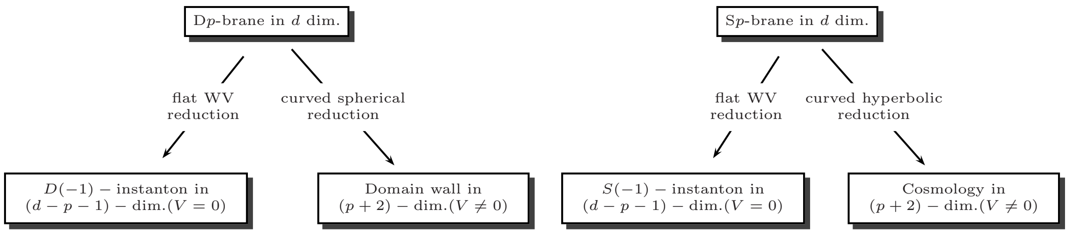

Figure 3.5.1: Starting from a brane in a Lorentzian d-dimensional space-time we show four possible reductions. The worldvolume reduction of a Dp-brane leads to an instanton in the lower-dimensional theory. Reducing over its transverse space gives a domain-wall. Starting from an Sp-brane, the worldvolume reduction leads to an S(-1)-brane, while the reduction over its transverse space leads to a cosmology.

If we instead compare the metric Ansätze (3.5.1) with that of (3.3.1) we see that it should also be possible to reduce a brane solution over its transversal space; a $(D-p-2)$-sphere ($\mathrm{d}\Omega^2_{D-p-2}$) for timelike branes and a $(D-p-2)$-hyperboloid ($\mathrm{d}\Sigma^2_{D-p-2}$) in case of a spacelike brane.

As we showed in section 3.3, after such a reduction the reduced brane is a $(D-2)$-brane that couples to a non-zero scalar potential $V(\phi)$. In case of timelike p-branes this is the known procedure to obtain brane solutions via uplifting domain-wall solutions of gauged supergravities [63]. In case of spacelike branes this is known from the fact that some (accelerating) cosmological quintessence-like solution obtained from hyperbolic reductions lift up to S-branes as shown in many papers (see for instance [64, 65]).

So if we start with gravity alone in the higher-dimensional space-time we see that the branes (3.5.1), when reduced over their worldvolume or over their transverse space, lead to the general Lagrangian (3.5.2). Furthermore, the Ansätze for the metric and scalar fields are given by (3.5.3). Everything depends only on one parameter r. This is like the physics of a "particle"! This explains the title of the thesis, 'Particle Dynamics of Branes'. In the next chapter we will see that the inclusion of a higher-dimensional $(p+2)$-form field strength leads to extra axions in the lower-dimensional theory. So again we have only scalar fields. In figure 3.5.1 we have summarized the various reductions.

As said, an action containing a metric and scalar fields alone have a simpler mathematical structure than those solutions that are carried by non-trivial p-form potentials. When we have solved the lower-dimensional (scalar) equations of motion we can lift up the solution to the original theory. This way we have a solution carried by a non-trivial p-form potential as well. Let us discuss the situation with and without a potential separately.

(−1)-branes

In case $V = 0$ we have (-1)-branes. The metric Ansatz is given by (3.5.3). The trick in solving the equations of motion is that the scalar part of (3.5.2) describes geodesic motion on the scalar manifold. To see this we re-parameterize the coordinate r as the harmonic function $h(r)$ via

$$\mathrm{d}h(r) = g^{1-D} f \mathrm{d}r\,, \tag{3.5.4}$$

then the scalar part of the action becomes

$$S = \int G_{ij}(\phi)\partial_h\phi^i\partial_h\phi^j \mathrm{d}h\,. \tag{3.5.5}$$

From this it follows that the solution describes geodesic motion on the moduli space with h as an affine parameter. Namely, in section 2.1 we have seen that the variation of the action $S \propto \int \sqrt{-g_{\mu\nu}x'^{\mu}x'^{\nu}}\mathrm{d}s$ leads to the geodesic equation (2.1.6), where a prime means a derivative with respect to the geodesic length s. One can show that the action

$$S \propto \int g_{\mu\nu}x'^{\mu}x'^{\nu}\mathrm{d}s\,, \tag{3.5.6}$$

leads to the same geodesic equation. Comparing this with (3.5.5) we see that $G_{ij}(\phi)$ takes on the role of $g_{\mu\nu}$, the scalar fields that of x^μ and h replaces the affine parameter s. We see that in case there is no potential, the scalar fields trace out geodesics on the scalar manifold. From this we know that the affine velocity $||v||^2$ defined by

$$||v||^2 = G_{ij}\partial_h\phi^i\partial_h\phi^j\,, \tag{3.5.7}$$

is a constant.

The Einstein equation for a (-1)-brane is given by

$$\mathcal{R}_{rr} = \tfrac{1}{2}G_{ij}\partial_r\phi^i\partial_r\phi^j\,, \qquad \mathcal{R}_{ab} = 0\,. \tag{3.5.8}$$

For the metric (3.5.3) we derive that the Ricci tensor is given by

$$\begin{aligned}\mathcal{R}_{ab} &= -\epsilon\Big\{\frac{\mathrm{d}}{\mathrm{d}r}[\frac{g\dot{g}}{f^2}] + \frac{g\dot{g}\dot{f}}{f^3} + (D-3)\frac{\dot{g}^2}{f^2}\Big\}g^{D-1}_{ab} + \mathcal{R}^{D-1}_{ab}\,,\\ \mathcal{R}_{rr} &= (D-1)\Big\{-(\frac{\ddot{g}}{g}) + \frac{\dot{g}\dot{f}}{gf}\Big\}\,,\end{aligned} \tag{3.5.9}$$

where a dot refers to a derivative with respect to r. Combining the Einstein equations together with (3.5.7) we deduce the following first-order equation

$$\dot{g}^2 = \frac{||v||^2}{2(D-2)(D-1)}f^2 g^{4-2D} + \epsilon k f^2\,. \tag{3.5.10}$$

A solution exists when the right-hand side remains positive. There is no equation of motion for f since it corresponds to the re-parametrization freedom for r. We thus see that the metric can be solved without having to know anything about the scalar field solutions!

We now have to make a difference between the reduction of space- and timelike branes. The former gives rise to a coset with a compact isotropy group. This means that the metric G_{ij} will be positive definite and hence $||v||^2 > 0$. So we have only spacelike geodesics. Via uplifting the S(-1)-brane we obtain a (fluxless) S-brane. *We thus have an S(-1)-brane / Sp-brane map.* This will be the subject of the next chapter.

For the timelike branes on the other hand the isotropy group is non-compact, for example SO$(n-1,1)$ instead of SO(n). Because of this G_{ij} will not be a positive definite metric on the scalar manifold and $||v||^2$ can be zero, positive or negative. We call these respectively lightlike, spacelike or timelike geodesics. We will discuss these (-1)-branes in chapter 7. *This way we obtain a (-1)-brane / p-brane map.*

Domain-walls and cosmologies

Let us now discuss what happens if $V \neq 0$. We know that the presence of the potential leads to domain-walls (timelike) and cosmologies (spacelike). Due to the potential there is a priori no reason to assume that these solutions are still geodesics of the scalar manifold. Under certain conditions however this turns out to be the case. Namely when we have a so-called scaling solution, see subsection 5.1.3. Let us illustrate this. A scaling solution has the property that if we calculate the *on-shell* potential V and kinetic energy $T = \frac{1}{2}G_{ij}\partial_r\phi^i\partial_r\phi^j$ we find that they have the same r-dependence. Effectively, we can consider $T + V$ as some new T only. From what we discussed above, we know that this means geodesic motion. Of course, filling in on-shell information is rarely a consistent procedure. We analyze this in chapter 5. There we will also show that scaling solutions are important since they correspond to the so-called critical points of autonomous differential equations governing the evolution of cosmologies. The critical points say a lot about the general evolution of a cosmology.

Besides that both type of solutions couple to a potential, their Lorentzian Ansätze also look very much the same. In [66] it was first noted that for a given domain-wall one also finds a cosmology. This was worked out in detail in [67] and is called the domain-wall / cosmology correspondence. We give a summary of this correspondence in chapter 6.