Particle dynamics of branes

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Chapter 2

String Theory

In this chapter we begin with introducing the relativistic point particle and the free bosonic string. We then move to the superstring and focus on its low energy limit, obtaining supergravities. This allows us to introduce Dp- and Sp-brane solutions, which will play an important role in the coming chapters. We will not give many details, for this we refer to books and lecture notes such as [3–7].

2.1 Classical String Theory

Before discussing string theory, it is interesting to remind ourself how to describe a free relativistic particle of mass $m > 0$ in a Minkowski space-time, given by the D-dimensional flat metric $\eta_{\mu\nu} = (-1, 1, \ldots, 1)$ with line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \sum_{i=1}^{D-1}(\mathrm{d}x^i)^2\,, \tag{2.1.1}$$

where c is the speed of light, t is the time coordinate and x^i are spatial coordinates. Since we are dealing with a free particle, we expect it to trace out a straight line in space-time. The action for such a particle is given by the shortest path

$$S = -mc\int_{\mathcal{P}}\sqrt{-\mathrm{d}s^2} = -mc\int_{\lambda_i}^{\lambda_f}\sqrt{-\eta_{\mu\nu}x'^\mu x'^\nu}\mathrm{d}\lambda\,, \tag{2.1.2}$$

where c is the speed of light, $x'^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for some parameter λ describing the curve $x^\mu(\lambda)$ and λ_i and λ_f are the values of λ at the initial and final points of the world-line $\mathcal{P}$. The presence of mc is dictated by requiring the right units for an action S. To see that this is the action we are after, we note that with the help of (2.1.1) and using $\lambda = t$ we can rewrite it as

$$S = -mc \int_{t_i}^{t_f} c\sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t\,, \tag{2.1.3}$$

where the velocity squared is given by $v^2 = \sum_{i=1}^{D-1}(\mathrm{d}x^i/\mathrm{d}t)^2$. This clearly shows that v is bounded by the speed of light c, in agreement with the special theory of relativity. Also note that a Taylor expansion around small v/c leads to the Lagrangian

$$\mathcal{L} = -mc^2 + \frac{1}{2}mv^2\,, \tag{2.1.4}$$

which is the expected expression for a non-relativistic free particle. From now on we take $c = 1$.

The action (2.1.2) can be extended to a curved space-time described by the metric $g_{\mu\nu}(x)$ via replacing the line element (2.1.1) with

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{2.1.5}$$

By extremizing the corresponding action and using for λ an affine parameter[1], we derive the equations of motion

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0\,, \tag{2.1.6}$$

with the Christoffel symbol $\Gamma^\mu_{\rho\nu}$ given by (A.2.2). This equation is called the geodesic equation. If we use $g_{\mu\nu} = \eta_{\mu\nu}$ we find the equations of motion for a free particle in Minkowski space-time. We can interpreted the action (2.1.2) as a map from the parameter space λ to an embedding in a D-dimensional space-time described by x^μ.

A string is the two-dimensional extension of this. Instead of the world-line we have a two-dimensional surface called the world-sheet Σ of the string. It is common to describe this world-sheet by the parameters (τ, σ). We then consider the mapping from the (τ, σ) world-sheet to the D-dimensional space-time described by X^μ [2].

What is the action describing a string in a Minkowski space-time? The particle is parameterized by λ, the "metric" induced on the one-dimensional world-line is given by

$$g_{\lambda\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\eta_{\mu\nu}\,, \tag{2.1.7}$$

[1] An affine parameter means that λ is related to $s = \int_{\mathcal{P}} \sqrt{-\mathrm{d}s^2}$ via $\lambda = as + b$ with $a, b \in \mathbb{R}$. This means that we parameterize the curve by the distance along the curve, $x^\mu = x^\mu(s)$.

[2] In string theory it is conventional to use capital letters for the embedding coordinates, i.e. X^μ instead of x^μ.

and we note that the determinant (Det) g of $g_{\lambda\lambda}$ appears in the integrand in (2.1.2). For the string we have the natural extension to the so-called Nambu-Goto action

$$S = -T \int d^2\zeta \sqrt{-g}\,, \quad \text{with } g_{ij} = \frac{dX^\mu}{d\zeta^i} \frac{dX^\nu}{d\zeta^j} \eta_{\mu\nu}\,, \tag{2.1.8}$$

where $\zeta^i = (\tau, \sigma)$ and $g = \text{Det}(g_{ij})$. The fields X^μ are the coordinates of the string in the Minkowski space-time described by the flat metric $\eta_{\mu\nu}$. The tension T has units of force. It is the force which tries to pull the string together to a point and is therefore called the string tension. It is often rewritten as $T = 1/2\pi\alpha' = 1/l_s^2$. The parameter l_s is called the string length and introduces a fundamental scale in the theory. Instead of working with (2.1.8) it is more convenient to work with the alternative action

$$S = -\frac{T}{2} \int_\Sigma d^2\zeta \sqrt{-h} h^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\,. \tag{2.1.9}$$

This is called the Howe-Tucker-Polyakov action [3]. Here h_{ij} is an independent metric on the worldvolume, independent of the induced metric g_{ij}. From the equation of motion for h_{ij} we obtain

$$g_{ij} = \frac{1}{2} h_{ij} (h^{ab} g_{ab})\,. \tag{2.1.10}$$

This can be used to show that (2.1.9) is classically equivalent to (2.1.8). From (2.1.9) it follows that h_{ij} allows for a conformal re-scaling symmetry

$$h'_{ij} = f(\zeta) h_{ij}\,, \tag{2.1.11}$$

with $f(\zeta)$ an arbitrary function of the world-sheet coordinates. This is called the Weyl re-scaling. The action (2.1.9) has two more symmetries. Namely general coordinate transformations on the world-sheet and global Poincaré transformations in D-dimensional space-time

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu\,, \tag{2.1.12}$$

where Λ^μ_ν is an SO(1,$D-1$) matrix obeying $\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \eta_{\mu\nu} = \eta_{\kappa\lambda}$ and a^μ is a constant vector.

It is a well known result of two-dimensional geometry that a coordinate re-parametrization allows an arbitrary metric h_{ij} to be cast locally in a conformal flat metric

$$h_{ij} = \rho^2(\zeta) \eta_{ij}\,, \tag{2.1.13}$$

where $\eta = \text{diag}(-1, 1)$ and ρ is called the conformal factor. The action (2.1.9) in this *conformal gauge* reduces to

$$S = -\frac{T}{2} \int_\Sigma d^2\zeta \eta^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\,. \tag{2.1.14}$$

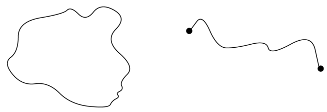

Figure 2.1.1: The figure on the left (right) represents a closed (open) string.

and (2.1.10) becomes in this gauge the constraint

$$\Big(\partial_\tau X^\mu \pm \partial_\sigma X^\mu\Big)^2 = 0\,, \tag{2.1.15}$$

where the square means a contraction with $\eta_{\mu\nu}$.

Now we are going to derive the equations of motion for X^μ. Given an initial and a final condition at τ_i and τ_f, we need to vary (2.1.14) with respect to X^μ, i.e. $\delta X^\mu(\tau_i,\sigma) = \delta X^\mu(\tau_f,\sigma) = 0$. We now have to make a difference between *open* and *closed* strings, see figure 2.1.1. As the name suggests, an open string has two end-points labeled by $\sigma = 0$ and $\sigma = l$. The variation of (2.1.14) leads to

$$\delta S = T\int_\Sigma \mathrm{d}^2\zeta \Box X^\mu \delta X_\mu + T\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \partial_\sigma X^\mu \delta X_\mu|_{\sigma=0}^{\sigma=l} = 0\,. \tag{2.1.16}$$

The first term on the right-hand side of the expression above leads to the well known wave equation

$$\Box X^\mu(\tau,\sigma) = \Big(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\Big) X^\mu(\tau,\sigma) = 0\,, \tag{2.1.17}$$

with the general solution

$$X^\mu(\tau,\sigma) = X^\mu_-(\tau-\sigma) + X^\mu_+(\tau+\sigma)\,. \tag{2.1.18}$$

The subscript $-$ $(+)$ stand for right (left) moving modes on the string. There are two ways to make the second term on the right-hand side of (2.1.16) zero.

First we can choose to work with so-called *Neumann* boundary conditions. These are specified by

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,l) = 0\,. \tag{2.1.19}$$

The end-points of the strings can move freely. Alternatively we can choose to keep (some of) the end-points of the string fixed. For this we have to restrict the variations to

$$\delta X^\mu(\tau,0) = \delta X^\mu(\tau,l) = 0 \rightarrow X^\mu(\tau, 0 \text{ or } l) = \text{constant}\,. \tag{2.1.20}$$

These are called *Dirichlet* boundary conditions. Because the string is fixed in the directions where these Dirichlet conditions are applied, momentum cannot be conserved in these directions. Therefore these boundary conditions imply that the open string has to couple to a dynamical object which is called a D-brane. The name comes from the Dirichlet boundary condition and the word brane generalizes the concept of a membrane. D-branes are an important class of extended objects in string theory and have played a crucial role in understanding the non-perturbative structure of string theory.

Extended objects are in general called p-branes. Here p stands for the number of spatial directions of the extended object. The free relativistic particle we discussed before is in this language a 0-brane and the string a 1-brane. An open string that has both endpoints confined to the same Dp-brane satisfies Neumann conditions in the $(p+1)$ directions which make up the worldvolume of the brane. Note that time is considered part of this worldvolume. The $(D-p-1)$ Dirichlet conditions are transverse to this plane. An exception is the D(-1)-brane or D-instanton. This brane lives in a Euclidean background, the time coordinate has been replaced by a spatial direction and all these spatial directions are transverse to the brane. If we instead take for the time coordinate a Dirichlet condition the brane is called an Sp-brane, which has by definition a Euclidean $(p+1)$-dimensional worldvolume and time is one of the transversal coordinates [8]. This means that the time coordinate X^0 obeys a Dirichlet condition. The S stands for the spacelike worldvolume. In section 2.4 we will discuss both p- and Sp-branes.

Finally for the *closed* strings the same variation as given in (2.1.16) holds. The difference is that the end-points do not exists and we have to demand a periodic boundary condition specified by

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+l) \quad \forall \quad \mu\,. \tag{2.1.21}$$

2.2 Quantization of the Bosonic String and Curved Backgrounds

So far we have been working with classical bosonic strings in a Minkowski background $g_{\mu\nu} = \eta_{\mu\nu}$. In this background the theory can be quantized exactly. The first noteworthy feature is that to regain Lorentz covariance the space-time dimension needs to be $D = 26$. The oscillation modes for the open string lead to the following mass spectrum

- the vacuum with mass squared $M^2 = -\hbar/\alpha'$, corresponding to the tachyonic scalar T,
- the first excited state with $M^2 = 0$, corresponding to a massless vector A_μ,
- an infinite tower of massive modes.

The tachyonic particle has $M^2 < 0$, which leads to an instability of the theory[3]. The mass gap between each subsequent mass level is $\hbar/\alpha'$.

The first two closed string spectrum levels are

- the vacuum with mass squared $M^2 = -4\hbar/\alpha'$, corresponding to a tachyonic scalar T ,
- the first excited state with $M^2 = 0$, consisting out of a symmetric traceless field $g_{\mu\nu}$, an anti-symmetric 2-form field $B_{\mu\nu}$ and a scalar field ϕ called the dilaton .

The field mediating the gravitational force is identified with the symmetric traceless tensor $g_{\mu\nu}$. This identification follows from the fact that the degrees of freedom of a classical D-dimensional gravitational field is carried by a symmetric, traceless tensor field with number of independent components $1/2\ D(D-3)$. The closed string spectrum shows that gravity is part of the quantized closed bosonic string. For this reason it is believed that string theory could form the basis of a theory of quantum gravity.

So far we have only considered non-interacting strings, moving in a flat Minkowski background. Just as we did for the particle, we now want to extend this to a more general background $g_{\mu\nu}(X)$. A possible starting point is the Howe-Tucker-Polyakov action (2.1.9), but now with $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$. We can think of this string moving in a coherent background of gravitons [3]. We have seen that the graviton is itself an excited state of the string, so we can generalize this by also turning on backgrounds for the other two massless fields appearing in the closed string spectrum. Therefore we consider the closed string in a background consisting out of the massless states ϕ, $g_{\mu\nu}$ and $B_{\mu\nu}$. The action can be obtained in the following way. We assume at most two world-sheet derivatives and we extend the symmetries of the action (2.1.9) to this case. That means that we require general covariance on the world-sheet and in the target space, as well as local Weyl invariance. It turns out that the following action

[3]The instability can be understood as follows. Consider a scalar field ϕ with mass M^2 that depends only on the time t. The equation of motion of such a particle with potential $V = \frac{1}{2}M^2\phi^2$ is given by

$$\frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} + M^2\phi(t) = 0\,. \tag{2.2.1}$$

In case $M^2 > 0$ we see that the solution for ϕ is given by $\phi(t) = \phi_0 \sin(Mt + \alpha)$ with ϕ_0, α two integration constants. The scalar field can "sit" at $\phi = 0$ forever since it is a stable point. If on the other hand $M^2 < 0$ we see that the general solution is given by $\phi(t) = A\cosh(mt) + B\sinh(mt)$ with m given by $M^2 = -m^2$ and A, B two constants of integration. It is clear that for $A = 0$ and $|t| \to \infty$ the scalar field $|\phi|$ blows up. This time $\phi = 0$ is a maximum of the theory. So if the scalar field sits at $\phi = 0$, a small perturbation $\delta\phi$ will cause the field to start rolling down the potential [6].

will do[4]

$$S = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\zeta\sqrt{-h}\Big(h^{ij}g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu - \epsilon^{ij}B_{\mu\nu}\partial_i X^\mu \partial_j X^\nu - \alpha'\phi\mathcal{R}(h)\Big)\,, \quad (2.2.2)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the world-sheet metric h_{ij}. This action is an example of a non-linear σ-model. A non-linear σ-model is a scalar field field theory in which the scalar fields take on values in some non-trivial manifold M. The last term in (2.2.2) plays a special role. Assume that we have a constant mode of the dilaton ϕ_0. The Gauss-Bonnet theorem [9] states that

$$\chi = \frac{1}{4\pi}\int_{\Sigma} \mathrm{d}^2\zeta\sqrt{-h}\mathcal{R}(h) = 2(1-g)\,, \quad (2.2.3)$$

where the genus g is the number of handles of the world-sheet. One can now calculate scattering amplitudes between different string modes via the string path integral based on the action (2.2.2). From (2.2.3) we see that the amplitude of a string diagram of genus g is multiplied by $(\mathrm{e}^{\phi_0})^{2g-2}$. As a consequence every interaction will have an associated string coupling constant g_s given by the expectation value of e^{ϕ_0}. A world-sheet with genus g can therefore be seen as the g-th loop correction to string theory.

The scattering amplitude for the massless modes can be summarized by an effective action. For the closed bosonic string it is given by [10]

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\sqrt{-g}\,\mathrm{e}^{-2\phi}\Big(\mathcal{R}(g) + 4(\partial\phi)^2 - \frac{1}{2(3!)}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\,, \quad (2.2.4)$$

where κ_0 is related to the 26-dimensional Newton's constant G_{26} via $\kappa = \kappa_0 g_s = \sqrt{8\pi G_{26}}$ and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$. Note that this is not the standard Einstein-Hilbert action as given in appendix A.4. The Ricci scalar is coupled to the dilaton in a specific way. This defines the so-called string frame $g^{(S)}_{\mu\nu}$. Via the conformal mapping $g^{(S)}_{\mu\nu} = \mathrm{e}^{\phi/2}g^{(E)}_{\mu\nu}$ we find the action in the Einstein frame (E)

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{26}x\sqrt{g^{(E)}}\Big(\mathcal{R}(g^E) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(3!)}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\,. \quad (2.2.5)$$

We thus see that the bosonic string action leads to the effective 26-dimensional action (2.2.5). In the next section we are going to include world-sheet fermions and obtain effective actions via the method mentioned here.

[4] At the quantum level Weyl invariance is broken by an anomaly and the last term in the action (2.2.2) is needed to restore the symmetry.

2.3 Superstrings and Supergravities

The free bosonic string of the previous section has two drawbacks. First there are the tachyons signaling an instability. Secondly, there are no fermions present in the theory. To fix this we add fermions to the world-sheet and we will see that this also solves the tachyon problem. In this section we will work in the conformal gauge.

The standard way to proceed is by adding D world-sheet fermions $\psi^\mu = (\psi_+^\mu,\, \psi_-^\mu)$ to the Howe-Tucker-Polyakov action (2.1.9) via the term

$$S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\zeta \Big(\eta^{ij}\partial_i X^\mu \partial_j X_\mu + i\bar{\psi}^\mu\gamma^j\partial_j\psi_\mu\Big)\,, \tag{2.3.1}$$

where γ^j are the Γ-matrices in two dimensions

$$\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\,, \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \tag{2.3.2}$$

satisfying $\{\gamma_i, \gamma_j\} = \eta_{ij}$, see appendix B for more details. The action (2.3.1) is invariant under the following world-sheet *supersymmetry* transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu\,, \;\; \delta\psi^\mu = -i(\partial\!\!\!/ X^\mu)\epsilon\,, \tag{2.3.3}$$

where $\epsilon = (\epsilon_+\,, \epsilon_-)$ is a constant spinor. For the free theory we are considering in this section we see that the bosonic and fermionic sector decouple. We have the same bosonic solution (2.1.18). A variation with respect to $\bar{\psi}^\mu$ gives the following equations of motion

$$(\partial_\tau \mp \partial_\sigma)\psi_\pm^\mu = 0\,, \tag{2.3.4}$$

together with the boundary condition

$$\Big(\psi_+^\mu \delta\,\psi_{+\mu} - \psi_-^\mu\delta\,\psi_{-\mu}\Big)\Big|_{\sigma=0}^{\sigma=l} = 0\,. \tag{2.3.5}$$

From (2.3.4) we see that the most general solution is given by $\psi_\pm^\mu(\tau\pm\sigma)$, where ψ_+ (ψ_-) is called the left (right) mover.

Let us first focus on the open strings. Then (2.3.5) is satisfied if $\psi_+^\mu = \pm\psi_-^\mu$ and $\delta\psi_+^\mu = \pm\delta\psi_-^\mu$. Since an overall sign in the boundary conditions is irrelevant, we can set $\psi_+^\mu(\sigma=0) = \psi_+^\mu(\sigma=l)$. We find the following two possibilities for an open string

$$\begin{aligned} \text{Ramond (R)}:\ & \psi_+^\mu(l,\tau) = \psi_-^\mu(l,\tau)\,, \\ \text{Neveu-Schwarz (NS)}:\ & \psi_+^\mu(l,\tau) = -\psi_-^\mu(l,\tau)\,. \end{aligned} \tag{2.3.6}$$

For the closed string we have the periodic identification for σ. This means we can impose (anti)-periodicity for the left- and right-moving component $\psi_\pm^\mu$ separately

$$\begin{aligned} \text{Ramond (R)}:\ & \psi_\pm^\mu(0,\tau) = \psi_\pm^\mu(l,\tau)\,, \\ \text{Neveu-Schwarz (NS)}:\ & \psi_\pm^\mu(0,\tau) = -\psi_\pm^\mu(l,\tau)\,. \end{aligned} \tag{2.3.7}$$

As a result we get four differen sectors for the closed string: R-R, NS-NS, R-NS and NS-R.

We can now quantize the free superstring. To regain Lorentz covariance we find that we need to require $D = 10$. To have space-time supersymmetry we apply the Gliozzi-Scherk-Olive (GSO) projection. This basically truncates the states that do not have a counterpart in the other sector. This projection also eliminates the tachyonic ground state from the spectrum. Due to the space-time supersymmetry we refer to this theory as the superstring. Choosing several combinations for the boundary conditions in the open and closed string case leads to five differen supersymmetric string theories. Namely type IIA, type IIB, Type I and heterotic $E_8 \times E_8$ and heterotic SO(32). The type IIA and type IIB are closed string theories, containing $\mathcal{N} = 2$ space-time supersymmetry. In type IIB the supersymmetry parameters have the same chirality, in type IIA they are opposite. Type I is the only open string theory and has $\mathcal{N} = 1$ supersymmetry. The heterotic theories also have $\mathcal{N} = 1$ supersymmetry, they differ in their gauge group.

2.3.1 Supergravities

In section 2.2 we mentioned that we can obtain a 26-dimensional effective action for the free bosonic string. This method can similarly be applied to the five superstring theories. As it turns out these theories are *supergravities*. This means that the symmetries of such a theory combine general coordinate transformations and local supersymmetry. That is the spinor ϵ depends on the space-time coordinates. In this section we will only write down the bosonic sector (in the string frame). In chapter 6 we will make use of the fact that these are local supersymmetric theories.

Type IIA

The action is given by the following expression

$$\begin{aligned} S_{\mathrm{IIA}} &= -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\Big\{\mathrm{e}^{-2\phi}\Big[-\mathcal{R}-4\big(\partial\phi\big)^2+\tfrac{1}{2}H\cdot H\Big] \\ &\quad +\tfrac{1}{2}\sum_{n=1}^{2}G^{(2n)}\cdot G^{(2n)}-\frac{1}{\sqrt{|g|}}\mathcal{L}_{\mathrm{CS}}\Big\}\,, \end{aligned} \tag{2.3.8}$$

where

$$\mathcal{L}_{\mathrm{CS}} = -\tfrac{1}{4\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}\,\partial_{\mu_1}C^{(3)}_{\mu_2\mu_3\mu_4}\,\partial_{\mu_5}C^{(3)}_{\mu_6\mu_7\mu_8}\,B_{\mu_9\mu_{10}}\,, \tag{2.3.9}$$

and we have the following expressions for the field strengths

$$H = \mathrm{d}B\,,\quad G^{(2)} = \mathrm{d}C^{(1)}\,,\quad G^{(4)} = \mathrm{d}C^{(3)} - H^{(3)}\wedge C^{(1)}\,. \tag{2.3.10}$$

Here κ_0^2 is related to the physical coupling κ^2 via

$$\frac{1}{2\kappa_0^2} = \frac{\mathrm{e}^{2\phi_0}}{2\kappa^2} \,. \tag{2.3.11}$$

Type IIB

The action is given by the following expression

$$\begin{aligned} S_{\mathrm{IIB}} \;&=\; -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{|g|}\Big\{\mathrm{e}^{-2\phi}\Big[-\mathcal{R}-4\big(\partial\phi\big)^2+\tfrac{1}{2}H\cdot H\Big] \\ &\quad +\tfrac{1}{2}\sum_{n=1/2}^{3/2} G^{(2n)}\cdot G^{(2n)} + \frac{1}{4}G^{(5)}\cdot G^{(5)} - \frac{1}{\sqrt{|g|}}\mathcal{L}_{\mathrm{CS}}\Big\}\,, \end{aligned} \tag{2.3.12}$$

where

$$\mathcal{L}_{\mathrm{CS}} \;=\; -\frac{1}{3\cdot 24^2}\varepsilon^{\mu_1\cdots\mu_{10}}C^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\mu_5}C^{(2)}_{\mu_6\mu_7}\partial_{\mu_8}B_{\mu_9\mu_{10}}\,. \tag{2.3.13}$$

The scalar $C^{(0)}$ is called the axion and we have the following expressions for the field strengths

$$G^{(1)} = \mathrm{d}C^{(0)}\,,\quad G^{(3)} = \mathrm{d}C^{(2)} - H^{(3)}C^{(0)}\,. \tag{2.3.14}$$

To get the right number of degrees of freedom, we must impose that the five-form field strength is self-dual

$$G^{(5)} = *G^{(5)}\,. \tag{2.3.15}$$

This constraint is added to the equations of motion. We will not write down the other three $\mathcal{N}=1$ supergravities resulting from type I and the two heterotic string theories since we will not make use of them. They can be found in e.g. [11].

11d Supergravity

Although Lorentz covariance requires superstrings to live in $D=10$, there does exist a supergravity in $D=11$ [12]. Its bosonic action is given by

$$\begin{aligned} S = &-\frac{1}{4\kappa_{11}^2}\int \mathrm{d}^{11}x\,\sqrt{|g|}\Big[-\mathcal{R}+\frac{1}{2}G^{(4)}\cdot G^{(4)}\Big] \\ &-\frac{1}{4\kappa_{11}^2}\int \mathrm{d}^{11}x\frac{1}{144^2}\,\varepsilon^{\alpha_1\ldots\alpha_4\beta_1\ldots\beta_4\mu\nu\rho}G_{\alpha_1\ldots\alpha_4}G_{\beta_1\ldots\beta_4}C_{\mu\nu\rho}\,, \end{aligned} \tag{2.3.16}$$

we see that it has a 3-form gauge potential related to $G_{\mu\nu\rho\sigma} = 4\,\partial_{[\mu}\,C_{\nu\rho\sigma]}$. It is a theory with $\mathcal{N}=1$ supersymmetry.

2.3.2 T-duality

Let us go back to the bosonic string in 26 dimensions. We assume that the 25th coordinate X^{25} has the topology of a circle S^1 with radius R. This can be achieved by imposing that all points along this direction are identified if they differ by $2\pi R$.

For the closed string we have to modify the boundary condition (2.1.21) as

$$X^{25}(\tau, \sigma + l) = X^{25}(\tau, \sigma) + 2\pi R m \,, \tag{2.3.17}$$

where the integer m now indicates how many times the closed string is wrapped around the circle. This leads to quantized momentum along this direction

$$p^{25} = \frac{k}{R} \,, \tag{2.3.18}$$

with k an integer. This follows from $\mathrm{e}^{ip^{25}X^{25}}$ together with the boundary condition (2.3.17). The mass spectrum is given by

$$M^2 \propto \left(\frac{k^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} \right) . \tag{2.3.19}$$

It is invariant under the inversion of the radius with a simultaneous interchange of k with m

$$R \to \frac{\alpha'}{R} \,, \quad k \to m \,. \tag{2.3.20}$$

This transformation is called T-duality.

The surprising thing is that if we apply T-duality to type IIA string theory we will end up with type IIB string theory. This can be shown for example by noting that when both type II supergravities are reduced to nine dimensions the same action appears. This winding is a stringy effect, in field theories particles cannot wrap around a compact dimension.

2.3.3 S-duality

This type of duality is a strong-weak duality. Let us show this for type IIB supergravity. We combine the dilaton ϕ and axion $C^{(0)}$ in a complex scalar τ via

$$\tau = C^{(0)} + ie^{-\phi} \,. \tag{2.3.21}$$

The scalar part of (2.3.12) containing ϕ and $C^{(0)}$ can be written as

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{2\phi}(\partial C^{(0)})^2 = -\frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\tau_2^2} \,, \tag{2.3.22}$$

where τ_2 is the imaginary part of τ. Note that we are working in the Einstein frame. Define now the following fractional linear transformation on τ

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}\,, \quad \text{with } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\,. \tag{2.3.23}$$

We can group these numbers in a SL(2, $\mathbb{R}$) transformation Λ

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.3.24}$$

Under this SL(2, $\mathbb{R}$) transformation the full type IIB supergravity is invariant if the two 2-form potentials transform as a doublet

$$\begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix}, \tag{2.3.25}$$

while the 4-form transforms as a singlet. Assume that we have a background in which $C^{(0)}$ vanishes. An S-duality transformation is the specific SL(2, $\mathbb{R}$) transformation with $a = d = 0$ and $b = -c = 1$ such that

$$\phi \rightarrow -\phi\,, \quad C^{(2)} \rightarrow B^{(2)}\,, \quad B^{(2)} \rightarrow -C^{(2)}\,. \tag{2.3.26}$$

This means that the string coupling g_s goes to $1/g_s$. If g_s is small initially, (2.3.26) maps the theory from weak to strong coupling.

The five string theories we mentioned above turn out to be related to one another by dualities such as T- and S-duality, see for example [11]. This suggest that these five theories represent various limits of one single fundamental theory, called M-theory. The idea is that the 11d supergravity (2.3.16) is one of the low energy approximation of M-theory.

2.4 Brane Solutions

In this section we will discuss two different types of brane solutions belonging to type II supergravities. First we discuss time-independent p-branes, after that we will look at time-dependent Sp-branes.

2.4.1 p-branes

In section 2.1 we have introduced D-branes as objects arising due to the Dirichlet boundary conditions applied to open strings. We will now focus on the type II supergravity actions and show that the Dp-branes are a special class of p-branes.

In the previous section we have seen that both type II theories and 11d supergravity contain higher rank gauge fields $C^{(n+1)}$

$$\begin{aligned} \text{IIA} \quad &: \quad \left\{C^{(1)}_{\mu}, C^{(3)}_{\mu_1\mu_2\mu_3}\right\}, \\ \text{IIB} \quad &: \quad \left\{C^{(0)}, C^{(2)}_{\mu_1\mu_2}, C^{(4)}_{\mu_1\cdots\mu_4}\right\}, \\ \text{11d} \quad &: \quad \left\{C^{(3)}_{\mu_1\mu_2\mu_3}\right\}. \end{aligned} \tag{2.4.1}$$

For a charged 0-brane (or a point particle) we know that the coupling to a one-form gauge field $A^{(1)}$ is of the form

$$S = -mc\int_{\mathcal{P}} \mathrm{d}s + q\int_{\mathcal{P}} A^{(1)} - \frac{1}{2}\int_{\mathcal{M}} *\mathrm{d}A^{(1)} \wedge \mathrm{d}A^{(1)}\,, \tag{2.4.2}$$

where q is the electric charge of the 0-brane, $A^{(1)} = A_\mu \mathrm{d}x^\mu$ the one-form gauge field, $\mathcal{M}$ the space-time manifold and $\mathcal{P}$ the world-line. We see that the 0-brane couples naturally to a one-form gauge field $A^{(1)}$.

The existence of the higher rank $C^{(n+1)}$ gauge field suggests a coupling to a higher-dimensional object, namely a $(p = n)$-brane instead of a 0-brane. In section 2.1 we mentioned that a p-brane is a $(p+1)$-dimensional object in space-time. The coupling of a $(n + 1)$-rank gauge field to a $(p = n)$-brane generalizes to

$$T\int_{\Sigma} \mathrm{d}^{n+1}\zeta \frac{1}{(n+1)!}\partial_{a_1}X^{\mu_1}\ldots\partial_{a_{n+1}}X^{\mu_{n+1}}A_{\mu_1\ldots\mu_{n+1}}\varepsilon^{a_1\ldots a_{n+1}}\,. \tag{2.4.3}$$

Here T is called the brane tension and Σ is the $(n + 1)$-dimensional worldvolume of the brane. The expression (2.4.3) is called the Wess-Zumino (WZ) term[5].

From the type II supergravities (in the Einstein frame) it is clear that there are in general couplings between the field strengths and the dilaton. The action for a p-branes is given by

$$S = \frac{1}{2\kappa^2}\int\Big(*\mathcal{R} - \frac{1}{2}*\mathrm{d}\phi\wedge\mathrm{d}\phi - \frac{1}{2}\mathrm{e}^{a\phi}*\mathrm{d}A_{n+1}\wedge\mathrm{d}A_{n+1}\Big)\,, \tag{2.4.4}$$

where we allow for an arbitrary dilaton coupling parameter a. The equations of motion together with the Bianchi identity are given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{n+1}{2(D-2)((n+2)!)}g_{\mu\nu}\mathrm{e}^{a\phi}F^2_{n+2} + \frac{1}{(n+1)!2}\mathrm{e}^{a\phi}(F^2_{n+2})_{\mu\nu}\,, \tag{2.4.5}$$

$$\mathrm{d}\Big(*\mathrm{e}^{a\phi}F_{n+2}\Big) = (-)^{(D-n)}2\kappa^2 * J^{(n+1)}\,, \quad \mathrm{d}F_{n+2} = 0\,, \tag{2.4.6}$$

[5]Besides the Wess-Zumino term, we can also add the D-brane low energy effective worldvolume action. This term is called the Dirac Born-Infeld action [13].

$$\Box\phi = \frac{a}{(n+2)!2} F^2_{n+2} e^{a\phi} \,, \tag{2.4.7}$$

where $J^{(n+1)}$ follows from the variation of the WZ-term (2.4.3), $\Box$ is given by (A.2.8) and F^2_{n+2} and $(F^2_{n+2})_{\mu\nu}$ are given by (A.4.9).

From (2.4.6) we see that $\mathrm{d} * J^{(n+1)} = 0$, so we define the electric charge Q_e as

$$Q_e = \frac{1}{\sqrt{2\kappa^2}} \int_{S^{D-n-2}} * e^{a\phi} F^{(n+2)} \,, \tag{2.4.8}$$

where S^{D-n-2} is the higher-dimensional sphere surrounding the brane.

We will ignore the WZ-term and focus on the bulk action (2.4.4). This action has an electric/magnetic duality. To see this we define the dual field strength $\tilde{F}^{(D-n-2)}$ via Hodge duality $\tilde{F}^{(D-n-2)} = * e^{a\phi} F^{(n+2)}$. The equations of motion (2.4.5-2.4.7) are invariant under the following "duality transformations"

$$a\phi \to -a\phi \,, \quad (n+2) \to (D-n-2) \,, \quad F^{(n+2)} \to \tilde{F}^{(D-n-2)} \,. \tag{2.4.9}$$

Under this duality the Bianchi identity and equation of motion for $F^{(n+2)}$ swap their role. This means that there exists also a magnetic solution with charge

$$Q_m = \frac{1}{\sqrt{2\kappa^2}} \int_{S^{n+2}} F^{(n+2)} \,. \tag{2.4.10}$$

The action (2.4.3) therefore has both an electric solution with a $(n+1)$-dimensional worldvolume and a magnetic solution with a $(D-n-3)$-dimensional worldvolume. Let us present these solutions in some detail.

We denote the worldvolume coordinates of an arbitrary p-brane by x^i with $i = 0, 1 \ldots p$ and the coordinates of the space transverse to the brane by y^a with $a = p+1, \ldots D-1$. We assume that the worldvolume has Poincaré symmetry $\mathrm{ISO}(1,p)$ and the transverse space $\mathrm{SO}(D-p-1)$. The following Ansätze will do

$$\mathrm{d}s^2 = e^{2A(r)} \mathrm{d}x^i \mathrm{d}x^j \eta_{ij} + e^{2B(r)} \mathrm{d}y^a \mathrm{d}y^b \delta_{ab} \,, \quad \phi(r) \,, \tag{2.4.11}$$

with $r = \sqrt{y^a y^b \delta_{ab}}$ and A, B are arbitrary functions. As discussed above, there are now two different solutions, namely an electric $(p = n)$-brane or a magnetic $(p = (D-n-4))$-brane. Solving the equations of motion gives the following metric

$$\mathrm{d}s^2 = h^{\frac{-4(D-p-3)}{(D-2)\Delta}} \eta_{ij} \mathrm{d}x^i \mathrm{d}x^j + h^{\frac{4(p+1)}{(D-2)\Delta}} \delta_{ab} \mathrm{d}y^a \mathrm{d}y^b \,, \tag{2.4.12}$$

where the parameter Δ is given by [14]

$$\Delta = a^2 + 2\frac{(p+1)(D-p-3)}{D-2} \,. \tag{2.4.13}$$

The electric ($p = n$)-brane is given by [14, 15]

$$e^{\phi} = h^{2a/\Delta}\,, \quad F^{(n+2)} = \frac{2}{\sqrt{\Delta}} dx^0 \wedge \ldots \wedge dx^n \wedge dh(r)^{-1}\,. \qquad (2.4.14)$$

For the magnetic solution we prefer to work with transversal coordinates given by

$$\delta_{ab} dy^a dy^b = dr^2 + r^2 d\Omega^2_{D-p-2}\,. \qquad (2.4.15)$$

Here $d\Omega^2_m$ is the metric on the S^m sphere see (A.2.15). The magnetic ($p = (D-n-4)$)-brane is described by

$$e^{\phi} = h^{-2a/\Delta}\,, \quad F^{(n+2)} = Q\frac{2}{\sqrt{\Delta}}\sqrt{g_{S^{(n+2)}}} d\theta^1 \wedge \ldots \wedge d\theta^{n+2}\,, \qquad (2.4.16)$$

where $g_{S^{(n+2)}}$ is the determinant of the metric on $S^{(n+2)}$. The function $h(r)$ is the harmonic function of the transverse space. For $p < D - 3$ we have

$$h = 1 + \frac{Q}{(D-p-3)r^{D-p-3}}\,. \qquad (2.4.17)$$

In both cases Q is related to either the electric or magnetic charge of the brane. Note that the metric approaches Minkowski space-time when r goes to infinity. For the special case that $p = (D - 3)$ we have a logarithmic harmonic function, while if $p = (D - 2)$ we have a linear harmonic function.

It can be shown that the electric Q_e and magnetic charge Q_m satisfy a Dirac quantization condition [11, 16, 17]

$$Q_e Q_m = 2\pi n\,, \quad n = \text{integer}\,. \qquad (2.4.18)$$

This is the generalization of Dirac's quantization for electric and magnetic monopoles. The $SL(2, \mathbb{R})$ symmetry we mentioned earlier (2.3.23), is broken down to $SL(2, \mathbb{Z})$ in the quantum theory due to this charge quantization.

2.4.2 Dp-branes

So far we have discussed general p-branes. To make contact to the type II supergravities we choose $D = 10$ and $a = (3 - p)/2$ [6]. We defined Dp-branes in section 2.1 as hyperplanes on which open strings can end. They are a special class of p-brane solutions with a coupling to the RR-potentials and satisfy Dirichlet boundary conditions along their transversal spacelike coordinates[7]. The above discussion of p-branes

[6]Note that this is a consistent truncation of the type II supergravities, with this we mean that any solution of the truncated theory is also a solution of the full theory.

[7]A different Dp-brane picture is given in [18]. Here one considers a stable Dp-brane as a tachyonic kink solution on the worldvolume of an unstable D($p + 1$)-brane.

showed us that a $(n+1)$-gauge potential leads to a $(p = n)$- or a $(p = (D - n - 4))$-brane. From (2.3.8) we see that in type IIA we have odd-form gauge potentials. These give rise to D0-, D2-, D4- and D6-branes. In type IIB there are D1-, D3-, D5- and D7-branes coupled to even-form potentials. The D3-brane is special in that its dual is the D3-brane itself. It is a dyonic solution, which means that it carries both electric and magnetic charge. The field strength should be self-dual, see (2.3.15)[8].

There are two special branes that play an important role in this thesis. First there are the $p = (D - 2)$-branes. These are called domain-walls, since the brane has only one transverse direction, separating space-time into two regions. The corresponding field strength is a zero form. Such a term can for example be added to type IIA supergravity, obtaining massive IIA [19].

From the field content of type IIB we see that there is also an axion present, this leads to a $D(p = -1)$-brane or the $D(-1)$-instanton. As the name suggest, we have a zero-dimensional worldvolume and all the transversal directions are spatial. This branes lives in a Euclidean background and is dual to the D7-brane.

Finally note that in both type II theories there is also the NS-NS 2-form $B_{\mu\nu}$. This couples to the fundamental F1-string and its dual the NS5-brane.

One of the successes of the D-branes is the Maldacena conjecture [20–22] or AdS/CFT correspondence. This correspondence arises from considering the near horizon limit of a N D3-branes in which we consider the region close to $r = 0$, where the metric has geometry $\mathrm{AdS}_5 \times \mathrm{S}^5$. Here AdS_5 is a five-dimensional Anti-de Sitter space and S^5 a five sphere. On the other hand, far away from the D3-brane we have free bulk supergravity. From the D3-brane action perspective, the dynamics far away from the brane is also free bulk supergravity. However near the brane we have a supersymmetric $\mathrm{SU}(N)$ gauge theory. The conjecture is that $\mathcal{N} = 4$ $\mathrm{SU}(N)$ super-Yang-Mills theory in 3+1 dimensions is the same as (or dual to) type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [22].

All these branes can be shown to preserve half of their supersymmetries. Such solutions have the property that they fulfill some first-order differential equations which arise from demanding that the fermion supersymmetry transformations vanish. These first-order equations are now referred to as Bogomol'nyi or BPS equations, named after Bogomol'nyi [23] and Prasad and Sommerfield [24]. Witten and Olive showed in [25] the relation to preserved supersymmetry of solitons in supersymmetric theories. Nowadays the term BPS–equation is used for first order equations of motion that are found by rewriting the action as a sum of squares. In general supersymmetric solutions belong to this class. Stationary non-extremal (see below) and time-dependent solutions cannot preserve supersymmetry in ordinary supergravity theories. Nonetheless, we will later see that these solutions often can be found from first–order equations. Even more, we will see that a time-dependent solution sometimes does preserve supersymmetry if we embed it in a so-called star supergravity [26].

[8]This can be obtained from our solution if we replace $F_5 \rightarrow \frac{1}{2}(F_5 + *F_5)$.

Such a theory is closely linked to the supergravity theories of subsection 2.3.1.

In the literature these supersymmetric branes are also called extremal. The word extremal comes from the fact that the branes obey a relation between the mass and the charge of the D-branes. To be precise, when the mass equals the charge a brane is called extremal [27,28][9]. When this is not the case the brane is called non-extremal.

Non-extremal p-branes

There are two standard types of deformations of the extremal p-brane. In the literature these are called type 1 and type 2 non-extremal p-branes [30].

For type 1 deformations the form of the D-dimensional metric Ansatz remains the same as in the extremal case, namely

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\mathrm{d}x^i\mathrm{d}x^j\eta_{ij} + \mathrm{e}^{2B}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2})\,, \tag{2.4.19}$$

where A and B are functions of r only. For the extremal case we discussed above these two functions are related via

$$X = (p+1)A + (D-p-3)B = 0\,. \tag{2.4.20}$$

For the type II supergravity D-branes this relation follows from the requirement that the brane solutions preserve some unbroken supersymmetry see e.g. [15, 27]. For the non-extremal type 1 deformations we have $X \neq 0$.

The type 2 deformation begins from a modified form for the metric Ansatz [31,32], namely

$$\mathrm{d}s^2 = \mathrm{e}^{2A}(-\mathrm{e}^{2f}\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^j\delta_{ij}) + \mathrm{e}^{2B}(\mathrm{e}^{-2f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-p-2})\,. \tag{2.4.21}$$

Here f is a function of r only and in this case the relation $X = 0$ still holds.

Although both type 1 and type 2 deformations introduce an additional function, namely X or f, the way in which they enter the metric Ansatz is quite different. The two become equivalent when $p = 0$.

2.4.3 Sp-branes

In the previous section we discussed p-branes. A natural question is what happens if we choose a Dirichlet condition for the time-coordinate [8]. Since time is then no longer part of the worldvolume, we have a brane with a Euclidean worldvolume. Such branes are called *spacelike* branes or S-branes for short.

[9]However, an extremal brane is not necessarily supersymmetric, see for example [29].

Just as for a p-brane, an S-brane is carried by a metric, a dilaton and a $(p+2)$-form field strength. An Sp-brane will have a $(p+1)$-dimensional Euclidean worldvolume and $D-p-1$ transversal coordinates of which one is the time coordinate.

In [8] the S-branes were first introduced. The metric of an Sp-brane describes a time-dependent geometry which schematically looks like

$$\mathrm{d}s^2 = -\mathrm{e}^{2A(t)}\mathrm{d}t^2 + \mathrm{e}^{2B(t)}\delta_{ab}\mathrm{d}y^a\mathrm{d}y^b + \mathrm{e}^{2C(t)}\mathrm{d}\mathbb{H}^2_{D-p-2}\,. \tag{2.4.22}$$

The δ_{ab} makes sure that we have a Euclidean worldvolume with symmetry ISO$(p+1)$. The transverse space consists out of time and a $(D-p-2)$-dimensional hyperbolic space, see appendix A.2. This gives the symmetry SO$(D-p-2,1)$. In [8] it was argued that this is the required symmetry for an S-brane Ansatz. In the gauge where $A=C$ the Ansatz looks like that of the metric of a p-brane (2.4.11). In [33–39] many different S-brane solutions are given.

By now this has been extended to many different time-dependent Ansätze. Such as Ansätze where the hyperbolic space has been replaced by a compact sphere. It is unlikely that this has anything to do with the original Sp-branes of [8]. For example, when t goes to infinity these solutions do not even approach flat Minkowski space any more. Nonetheless, in this thesis we will define Sp-branes in this generalized sense. That is, we call a time-dependent solution carried by a metric, possibly a field strength and a scalar field an Sp-brane.

Due to the time-dependence the S-branes belonging to type II supergravities are not supersymmetric. As a result the solutions are more complicated to write down. We will not discuss general Sp-branes as we did for the p-branes. Instead we will focus on S(-1)-branes. In this section we will consider the S(-1)-brane belonging to type IIB supergravity. This brane can be considered as the time-dependent 'twin' of the Euclidean D(-1)-instanton.

The action for the S(-1)-brane follows from the truncation of type IIB supergravity to its scalar sector (for $D=10$)

$$S = \int \mathrm{d}^D x\sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2\Big)\,, \tag{2.4.23}$$

where ϕ is the dilaton and we denote the axion with χ instead of $C^{(0)}$. Observe that we can introduce a metric G_{ij} on the scalar manifold. With this we mean that we can write the scalar part of (2.4.23) as follows

$$\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mathrm{e}^{2\phi}(\partial\chi)^2 \equiv \frac{1}{2}G_{ij}\partial\phi^i\partial\phi^j\,. \tag{2.4.24}$$

If we consider ϕ and χ as coordinates, we can read of what G_{ij} has to be. In general it depends on the scalar fields. To show that G_{ij} indeed describes a metric on the scalar manifold, consider the following action with N scalar fields ϕ^i

$$S = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g}\,G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,. \tag{2.4.25}$$

The scalar fields ϕ^i define a local map from a space-time parameterized by the coordinate x^μ to a N-dimensional Riemannian manifold parameterized by ϕ^i

$$\phi : \mathcal{M}_{\text{space-time}} \to \mathcal{M}_{\text{scalar}} : \ \ x^\mu \to \phi^i(x)\,. \tag{2.4.26}$$

Let us show that G_{ij} indeed transforms as a metric under coordinate transformations on $\mathcal{M}_{\text{scalar}}$

$$G_{ij}\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x^\nu} = G_{ij}\frac{\partial\phi^i(\phi')}{\partial\phi'^k}\frac{\partial\phi^j(\phi')}{\partial\phi'^l}\frac{\partial\phi'^k}{\partial x^\mu}\frac{\partial\phi'^l}{\partial x^\nu} \equiv G'_{kl}\frac{\partial\phi'^k}{\partial x^\mu}\frac{\partial\phi'^l}{\partial x^\nu}\,. \tag{2.4.27}$$

We see that G_{ij} transforms as a bilinear.

The equations of motion that follow from (2.4.23) are given by

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\mathrm{e}^{2\phi}\partial_\mu\chi\partial_\nu\chi\,, \tag{2.4.28}$$

$$\partial_\mu\Big(\mathrm{e}^{2\phi}\sqrt{-g}g^{\mu\nu}\partial_\nu\chi\Big) = 0\,, \tag{2.4.29}$$

$$\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) - \sqrt{-g}\mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi = 0\,. \tag{2.4.30}$$

When $p = -1$ all space is transverse so the part containing B is not present in the Ansatz (2.4.22). We choose the gauge where $\mathrm{e}^{2C} = t^2$ and we generalize the Ansatz to

$$\mathrm{d}s^2 = -f(t)^2\mathrm{d}t^2 + t^2\Big(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big)\,, \tag{2.4.31}$$

such that it is valid for all k. The part between brackets describes for $k = 0$ flat space, for $k = +1$ a sphere and for $k = -1$ a hyperboloid, see appendix A.2. Only for $k = -1$ there is an expected string theory interpretation. This follows from the fact that when t goes to infinity the metric describes a flat Minkowski space-time only for $k = -1$ (if f approaches one), see for example (2.4.32). The two scalars depend only on t.

In section 3.5 we will show a way to solve the equations of motion (2.4.28-2.4.30) in a structured way. Basically this comes down to realizing that the scalar fields trace out a geodesic on the scalar manifold described by the metric G_{ij}. For now let us just give the metric solution

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{\frac{||v||^2}{2(D-1)(D-2)}t^{-2(D-2)} - k} + t^2\Big(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big)\,. \tag{2.4.32}$$

Here $||v||^2$ is a strictly positive number. This constant will turn out to be the affine velocity labelling the geodesic. The scalar fields are given by

$$\begin{aligned}\phi(t) &= \log\Big(c_1\cosh(||v||h + c_2)\Big)\,,\\ \chi(t) &= \pm\frac{1}{c_1}\tanh(||v||h + c_2) + c_3\,.\end{aligned} \tag{2.4.33}$$

Here c_i are constants of integrations and the harmonic function h is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)} \log\left|\sqrt{a}t^{2-D} + \sqrt{at^{2(2-D)} - k}\right| + c\,, \qquad a = \frac{||v||^2}{2(D-1)(D-2)}\,, \tag{2.4.34}$$

with c a constant.

The link between the two scalar fields and the geodesic is due to the following relation

$$(\partial_h \phi)^2 + \mathrm{e}^{2\phi}(\partial_h \chi)^2 = ||v||^2\,. \tag{2.4.35}$$

We give an explanation of this in section 3.5.

For $k = -1$ we have the S(-1)-brane of type IIB supergravity [33]. For $k = 0$ the brane describes a so-called power-law universe in FLRW-coordinates. With this we mean that after a coordinate transformation we find that the metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2\Big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\Big)\,. \tag{2.4.36}$$

Here $a(\tau)$ is called the scale factor for an obvious reason. For the S(-1)-brane solution we discuss here we find that $a(t) = \tau^p$ with $p = 1/(D-1)$. Such a scale factor is called a power-law.

In case $k = 1$ we cannot really consider it as an S(-1)-brane, actually the metric (2.4.36) describes a transition from a Big Bang to a Big Crunch for a closed universe [40]. For example, in three dimensions we easily derive that $a(\tau)^2 \propto (||v||^2/4 - \tau^2) > 0$. At $\tau = \pm||v||/2$ the Ricci scalar blows up and hence we see that this describes a Big Bang to a Big Crunch universe. A different reason to use the FLRW metric instead of (2.4.32) is that the latter has a coordinate singularity for some finite t. In the FLRW frame the metric covers the whole manifold, but finding explicit expressions for $h(\tau)$ and $a(\tau)$ is difficult in general.

In chapter 4 we will use the relation between geodesics and the scalar fields to write down the most general Sp-brane with a deformed worldvolume.

A supersymmetric brane obeys the extremality condition (2.4.20), i.e. $X = 0$. In [41] it was shown that if one demands that the extremality condition also holds for an Sp-brane with $k = -1$ one finds that the resulting field strength is *imaginary*. This shows that the extremality condition cannot be satisfied for real S-brane solutions. However, in chapter 6 we will see that there is a different interpretation possible for the imaginary solution.

In subsection 2.4.2 we mentioned the AdS/CFT correspondence. For S-branes this leads to a proposed dS/CFT correspondence [42, 43]. Since the worldvolume of an S-brane is Euclidean, the field theory will be a Euclidean conformal field theory. Unlike the AdS/CFT correspondence there is not much support yet for the dS/CFT one.