Nature-inspired microfluidic propulsion using magnetic artificial cilia

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Chapter 2

Magneto-mechanical model

2.1 Introduction

In this chapter we derive the numerical model and dimensionless parameters used in this thesis for two-dimensional simulations of fluid flow due to magnetically-driven artificial cilia. The physical quantities to be solved for are the magnetic field, cilia motion and fluid flow. The magnetic field and magnetization of the cilia are calculated by solving the Maxwell's equations using a boundary element approach at every instant in the deformed configuration. From the magnetization and the applied magnetic field, the magnetic forces acting on the cilia are calculated, which are given as input to the solid dynamics model. The solid dynamics model considers the cilia to be an assemblage of Euler-Bernoulli beam elements, taking into consideration inertia and geometric nonlinearity within an updated Lagrangian framework. The Navier-Stokes equation, which describes the fluid behaviour, is solved using an Eulerian finite element approach. The solid and fluid domains are coupled using the no-slip boundary condition on the cilia within a fictitious domain framework (van Loon *et al.*, 2006).

In the magneto-mechanical model a range of forces can be identified: the elastic and inertia forces of the cilia, the viscous and inertia forces of the fluid and the magnetic forces acting on the cilia. By introducing relevant length and time scales into the principle of virtual work, we derive how these forces scale with the different parameters involved. From the dimensionless form of the virtual work equation, we identify five dimensionless parameters: the magnetic number - the ratio of magnetic forces to elastic forces, the fluid number - the ratio of fluid viscous forces to elastic forces, the inertia number - the ratio of inertial to elastic forces of the cilia, the flapping Reynolds number - the ratio of inertia forces to viscous forces in the fluid and the diffusion Reynolds number - the ratio of momentum diffusion time to ciliary cycle time. These five dimensionless parameters completely capture the physical behaviour of the cilia and fluid flow, in addition to the geometric parameters such as cilia spacing and channel height.

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2.2 Equations of motion

2.2.1 Solid dynamics model

As a starting point for the Euler-Bernoulli beam element formulation we use the principle of virtual work (Malvern, 1977); i.e., the virtual work of the external forces ($\delta W_{\text{ext}}^{t+\Delta t}$) is equal to the internal virtual work ($\delta W_{\text{int}}^{t+\Delta t}$),

$$\delta W_{\text{int}}^{t+\Delta t} = \delta W_{\text{ext}}^{t+\Delta t}, \tag{2.1}$$

with

$$\delta W_{\text{int}}^{t+\Delta t} = \int_V \left(\sigma \delta \epsilon + \rho(\ddot{u}\delta u + \ddot{v}\delta v)\right) dV, \tag{2.2}$$

where u and v are the axial and transverse displacements of a point along the beam and ρ is the mass density. Furthermore, σ is the axial stress and ϵ is the corresponding strain, given by

$$\epsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - y\frac{\partial^2 v}{\partial x^2}.$$

The external virtual work is

$$\begin{aligned}\delta W_{\text{ext}}^{t+\Delta t} = &\int \left(f_x \delta u + f_y \delta v + N_z \frac{\partial \delta v}{\partial x}\right) A dx \\ &+ \int \left(t_x \delta u + t_y \delta v\right) b dx,\end{aligned} \tag{2.3}$$

where f_x and f_y are the magnetic body forces in axial and transverse directions, N_z is the magnetic body couple in the out-of-plane direction, t_x and t_y are the surface tractions and b is the out-of-plane thickness of the film.

It is assumed that the solid material is isotropic and linear elastic, specified by the elastic modulus E and Poisson's ration ν. We follow the approach used by Annabattula *et al.* (2010) to linearise and discretise the principal of virtual work to arrive at the discretised form of the virtual work equation at time $t + \Delta t$ (see appendix A),

$$\delta \boldsymbol{p}^T \left(\boldsymbol{K}\Delta\boldsymbol{p} + \boldsymbol{M}\ddot{\boldsymbol{p}}^{\text{t}+\Delta\text{t}} - \boldsymbol{F}_{\text{ext}}^{\text{t}+\Delta\text{t}} + \boldsymbol{F}_{\text{int}}^{\text{t}}\right) = 0, \tag{2.4}$$

where $\boldsymbol{K}$ is the stiffness matrix that combines both material and geometric contributions, $\boldsymbol{M}$ is the mass matrix which can be found in (Cook *et al.*, 2001), $\boldsymbol{F}_{\text{ext}}^{\text{t}+\Delta\text{t}}$ is the external force vector, $\boldsymbol{F}_{\text{int}}^{\text{t}}$ is the internal force vector, $\Delta\boldsymbol{p}$ is the nodal displacement increment vector and $\ddot{\boldsymbol{p}}$ is the nodal acceleration vector.

2.2.2 Fluid dynamics model

The principle of virtual work in the rate form for the fluid problem is (Bathe, 1996)

$$\int_V \sigma_{ij}\delta D_{ij} dV + \rho^f \int_V \frac{du_i}{dt}\delta u_i dV + \int_V \delta p \frac{\partial u_i}{\partial x_i} dV = 0, \tag{2.5}$$

where the first and the second terms represent the work due to the internal stresses and inertia forces in the fluid, respectively, while the third term imposes the incompressibility condition. In Eqn. 2.5, σ_{ij} represent the components of stress tensor in the fluid, D_{ij} represent the components of the deformation rate tensor in the fluid, ρ^f is the fluid density, u_i[1] represents the components of fluid velocity in the i^{th} direction, $\frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$ represents the total derivative, p is the pressure and $dV = bdxdy$. The constitutive relation for the fluid is $\sigma_{ij} = -p\delta_{ij} + 2\mu D_{ij}$, where μ is the fluid viscosity. It is to be noted that x represents a spatial point in the fluid domain (Eulerian point of view), while it represents the position occupied by a material point on the beam in the solid domain (Lagrangian point of view). The following shape functions are used to interpolate velocity and pressure:

$$\begin{aligned} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} &= \left[\begin{array}{ccccccc} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \dots \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & \dots \end{array} \right] \left\{ \begin{array}{ccc} u_1^1, & u_2^1, & \dots \end{array} \right\}^T \\ &= \phi_{Ii} U_I = \boldsymbol{\phi}^T \boldsymbol{U}, \\ p &= \left[\begin{array}{cccc} \psi_1 & \psi_2 & \psi_3 & \dots \end{array} \right] \left\{ \begin{array}{cccc} p^1, & p^2, & p^3 & \dots \end{array} \right\}^T = \psi_I P_I = \boldsymbol{\psi}^T \boldsymbol{P}, \end{aligned} \tag{2.6}$$

where, u_i^J represents the components of fluid velocity in the i^{th} direction at the J^{th} node, p^J represents the magnitude of pressure at the J^{th} node and ϕ_J and ψ_J are the shape functions used to interpolate the nodal velocities and pressure, respectively. The shape functions are chosen so that the resulting element satisfies the Inf-Sup condition. In this work we use the Taylor-Hood Q_2Q_1 element which interpolates the velocities quadratically and the pressures linearly (Bathe, 1996). Using Eqn. 2.6 the various terms in the virtual work equation are evaluated (see appendix B):

$$\int_V \sigma_{ij} \delta D_{ij} dV = \int_V \left(-p\delta_{ij} + 2\mu D_{ij}\right) \delta D_{ij} = \delta \boldsymbol{U}^T \left(\boldsymbol{K^{UU}} \boldsymbol{U} + \boldsymbol{K^{UP}} \boldsymbol{P} \right), \tag{2.7}$$

$$\int_V \delta p \frac{\partial u_i}{\partial x_i} dV = \int_V \delta p_I \psi_I \frac{\partial \phi_{Ji}}{\partial x_i} U_J dV = \delta p_I \int_V \psi_I \frac{\partial \phi_{Ji}}{\partial x_i} dV U_J = \delta \boldsymbol{P}^T \left(\boldsymbol{K^{UP}} \right)^{\boldsymbol{T}} \boldsymbol{U}. \tag{2.8}$$

Assuming that the solution at time t is known, the convective term is linearized with respect to the known solution at time t and the time derivative of velocity is discretised using an implicit Euler scheme. Neglecting the higher order terms we get (see appendix B):

$$\begin{aligned} \int_V \frac{du_i^{t+\Delta t}}{dt} \delta u_i dV &= \rho^f \int_V \left(\delta u_i \frac{\partial u_i^{t+\Delta t}}{\partial t} + \delta u_i \frac{\partial u_i^{t+\Delta t}}{\partial x_j} u_j^{t+\Delta t} \right) dV \\ &= \delta \boldsymbol{U}^T \hat{\boldsymbol{M}} \boldsymbol{U} + \delta \boldsymbol{U}^T \boldsymbol{K}^1 \boldsymbol{U} + \delta \boldsymbol{U}^T \boldsymbol{K}^2 \boldsymbol{U} - \delta \boldsymbol{U}^T \boldsymbol{F^f}. \end{aligned}$$

Substituting the above expressions in Eqn. 2.5 yields its discretised form

$$\delta \boldsymbol{U}^T \boldsymbol{K^{UP}} \boldsymbol{P} + \delta \boldsymbol{U}^T \hat{\boldsymbol{K}}_{\boldsymbol{f}} \boldsymbol{U} - \delta \boldsymbol{U}^T \boldsymbol{F^f} + \delta \boldsymbol{P}^T (\boldsymbol{K^{UP}})^{\boldsymbol{T}} \boldsymbol{U} = 0, \tag{2.9}$$

where $\hat{\boldsymbol{K}}_{\boldsymbol{f}} = \boldsymbol{K^{UU}} + \hat{\boldsymbol{M}} + \boldsymbol{K}^1 + \boldsymbol{K}^2$.

[1] u (without a subscript) represents the axial displacement of a point on the beam, while u_i (with a subscript) represents the fluid velocity.

2.2.3 Fluid-structure interaction

We now couple the Lagrangian beam element formulation of section 2.2.1 to the Eulerian finite element fluid model of section 2.2.2. The solid beam is considered as an internal boundary to the fluid domain. The solid and fluid models are coupled using the constraint that the velocity (or displacement) of the solid is equal to the velocity (or displacement) of the fluid. This coupling is established using the method of Lagrange multipliers (Lanczos, 1952). In what follows, we apply the constraint to the fluid dynamics model, then add the virtual work done by the constraint force to the solid dynamics model and finally couple the equations so that the solid and the fluid equations of motion can be solved implicitly.

We first add the variation of the Lagrange multiplier times the constraint to the fluid dynamics model,

$$\begin{aligned}\delta\boldsymbol{U}^T\boldsymbol{K^{UP}P}+\delta\boldsymbol{U}^T\hat{\boldsymbol{K}}_{\boldsymbol{f}}\boldsymbol{U}-\delta\boldsymbol{U}^T\boldsymbol{F^f}+\delta\boldsymbol{P}^T(\boldsymbol{K^{UP}})^{\boldsymbol{T}}\boldsymbol{U}\\ +\,\delta(\lambda_i^J(u_i^{Jf}-\dot{p}_i^J))=0,\end{aligned} \tag{2.10}$$

where, u_i^{Jf} and $\dot{p}_i^J$ represent the components of the fluid and solid beam velocity, respectively, in the i^{th} direction at the location where the J^{th} node of the solid beam is present. Discretising the constraint equation (see appendix B) yields,

$$\begin{aligned}\delta\boldsymbol{U}^T\boldsymbol{K^{UP}P}+\delta\boldsymbol{U}^T\hat{\boldsymbol{K}}_{\boldsymbol{f}}\boldsymbol{U}-\delta\boldsymbol{U}^T\boldsymbol{F^f}+\delta\boldsymbol{P}^T(\boldsymbol{K^{UP}})^{\boldsymbol{T}}\boldsymbol{U}+\delta\boldsymbol{U}^T\boldsymbol{\phi}^J\boldsymbol{\lambda}^J\\ +\delta\boldsymbol{\lambda}^{JT}(\boldsymbol{\phi^{JT}U}-\boldsymbol{A}\dot{\boldsymbol{p}}^J)=0,\end{aligned}$$

where $\boldsymbol{A}$ is a matrix that eliminates the rotational degrees of freedom from $\dot{\boldsymbol{p}}$. It can be noted that the term $\delta\boldsymbol{U}^T\boldsymbol{\phi}^J\boldsymbol{\lambda}^J$ in the above equation represents the virtual work done by the Lagrange multiplier, i.e., the Lagrange multiplier acts as a force due to the constraint. Hence, the components of the Lagrange multiplier are to be interpreted as fluid drag forces. Invoking the arbitrary nature of the virtual fields and performing the standard finite element assembly yields the following set of equations

$$\hat{\boldsymbol{K}}_{\boldsymbol{f}}\boldsymbol{U}+\boldsymbol{K^{UP}P}+\boldsymbol{\Phi\lambda}=\boldsymbol{F^f},\quad \left(\boldsymbol{K^{UP}}\right)^{\boldsymbol{T}}\boldsymbol{U}=0,\quad \boldsymbol{\Phi^TU}-\boldsymbol{A}\dot{\boldsymbol{p}}=0. \tag{2.11}$$

The fluid drag forces are now considered as nodal forces, whose virtual work is an additional contribution to Eqn. 2.4. This leads to

$$\delta\boldsymbol{p}^T\left(\boldsymbol{K}\Delta\boldsymbol{p}+\boldsymbol{M}\ddot{\boldsymbol{p}}^{\text{t}+\Delta\text{t}}-\boldsymbol{F}_{\text{ext}}^{\text{t}+\Delta\text{t}}+\boldsymbol{F}_{\text{int}}^{\text{t}}\right)-\boldsymbol{\lambda}^T\boldsymbol{A}\delta\boldsymbol{p}=0, \tag{2.12}$$

where

$$\boldsymbol{\lambda}=\left\{\ \boldsymbol{\lambda}^1,\ \boldsymbol{\lambda}^2\ \right\}^T=\left\{\ \lambda_1^1,\ \lambda_2^1,\ \lambda_1^2,\ \lambda_2^2\ \right\}^T$$

is the Lagrange multiplier vector, λ_i^J is the Lagrange multiplier at the J^{th} node in the i^{th} direction. In general, the node of the solid beam is present inside a fluid element and does not coincide with a fluid node, so that the constraint is satisfied in an approximate sense. As the virtual quantity $\delta\boldsymbol{p}$ is arbitrary, we can write

$$\boldsymbol{K}\Delta\boldsymbol{p}+\boldsymbol{M}\ddot{\boldsymbol{p}}^{\text{t}+\Delta\text{t}}-\boldsymbol{F}_{\text{ext}}^{\text{t}+\Delta\text{t}}+\boldsymbol{F}_{\text{int}}^{\text{t}}-\boldsymbol{A}^T\boldsymbol{\lambda}=0. \tag{2.13}$$

The motion of the film with time is obtained by solving Eqn. 2.13 with appropriate initial and boundary conditions. Newmark's integration scheme is used to integrate Eqn. 2.13 in time, based on which Eqn. 2.13 can be re-written to solve for the nodal velocities as

$$\hat{\boldsymbol{K}}\dot{\boldsymbol{p}}^{\text{t}+\Delta\text{t}}-\boldsymbol{A}^T\boldsymbol{\lambda}=\boldsymbol{F}_s^{\text{t}+\Delta\text{t}}, \tag{2.14}$$

where

$$\hat{\boldsymbol{K}} = \frac{1}{\gamma \Delta t}(\Delta t^2 \beta \boldsymbol{K} + \boldsymbol{M}),$$
$$\boldsymbol{F}_s^{\mathrm{t}+\Delta \mathrm{t}} = \boldsymbol{F}_{\mathrm{ext}}^{\mathrm{t}+\Delta \mathrm{t}} - \boldsymbol{F}_{\mathrm{int}}^{\mathrm{t}} - \Delta t \boldsymbol{K} \left[\dot{\boldsymbol{p}}^{\mathrm{t}} + \frac{1}{2} \Delta t (1 - 2\beta) \ddot{\boldsymbol{p}}^{\mathrm{t}} \right] \tag{2.15}$$
$$+ (\Delta t^2 \beta \boldsymbol{K} + \boldsymbol{M}) \frac{\dot{\boldsymbol{p}}^t + \ddot{\boldsymbol{p}}^t \Delta t (1 - \gamma)}{\gamma \Delta t},$$

and γ and β are integration parameters. Equation 2.14 contains the discretised equations of motion for one beam element. After performing the standard finite element assembly procedure (Cook *et al.*, 2001) we get the discretised equations of motion for the whole film (dropping the superscript $(t + \Delta t)$):

$$\hat{\boldsymbol{K}}_{\boldsymbol{s}} \dot{\boldsymbol{p}} - \boldsymbol{A}^T \boldsymbol{\lambda} = \boldsymbol{F}_{\boldsymbol{s}}. \tag{2.16}$$

Combining the equations of motion for the solid (Eqn. 2.16) and fluid (Eqn. 2.11) results in

$$\begin{bmatrix} \hat{\boldsymbol{K}}_{\boldsymbol{f}} & \boldsymbol{K}^{\boldsymbol{UP}} & \boldsymbol{0} & \boldsymbol{\Phi} \\ (\boldsymbol{K}^{\boldsymbol{UP}})^{\boldsymbol{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{K}}_s & -\boldsymbol{A}^T \\ \boldsymbol{\Phi}^{\boldsymbol{T}} & \boldsymbol{0} & -\boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{P} \\ \dot{\boldsymbol{p}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}^{\boldsymbol{f}} \\ \boldsymbol{0} \\ \boldsymbol{F}_s \\ \boldsymbol{0} \end{Bmatrix}. \tag{2.17}$$

This set of equations is solved to obtain the velocities at the solid and fluid nodal points, the pressure in the fluid and the Lagrange multipliers at the solid nodal points. It is to be noted that in Eqn. 2.17 the velocity of the film and the fluid are simultaneously solved for every time increment. This approach is commonly referred to as the monolithic approach.

2.2.4 Magnetostatics

Maxwell's equations for the magnetostatic problem with no free currents are (Jackson, 1974)

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{2.18}$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = 0, \tag{2.19}$$

with the constitutive relation

$$\boldsymbol{B} = \mu_0(\boldsymbol{M} + \boldsymbol{H}), \tag{2.20}$$

where $\boldsymbol{B}$ is the magnetic flux density (or magnetic induction), $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{M}$ is the magnetization which includes the remnant magnetization, and μ_0 is the permeability of free space. Substituting Eqn. 2.20 into Eqn. 2.18 yields

$$\boldsymbol{\nabla} \cdot \boldsymbol{H} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}. \tag{2.21}$$

As $\boldsymbol{\nabla} \times \boldsymbol{H} = 0$, a scalar potential ϕ exists, such that $\boldsymbol{H} = -\boldsymbol{\nabla}\phi$. Substituting this in Eqn. 2.21 yields a Poisson equation for ϕ, $\nabla^2 \phi = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$. By taking into consideration the effect of discontinuity in the medium, the general solution of the Poisson equation can be found (Jackson, 1974), resulting in

$$\phi(\boldsymbol{x}) = -\frac{1}{4\pi}\oint \frac{\boldsymbol{n}' \cdot \boldsymbol{M}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}dS' + \frac{1}{4\pi}\int \frac{\boldsymbol{\nabla}' \cdot \boldsymbol{M}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}dV'. \tag{2.22}$$

where $\boldsymbol{n}'$ is the outward normal to the surface of V. The magnetic field $\boldsymbol{H}(x)$ can be found from the gradient of $\phi(x)$.

We now discretise the film into a chain of rectangular segments, within which the magnetization is assumed to be uniform. So that $\boldsymbol{\nabla}' \cdot \boldsymbol{M} = 0$ and the volume integral vanishes. The field now is only due to the jump of magnetization across the surface of each segment and is given by the surface integral in Eqn. 2.22. The magnetic field in coordinates local to the segment i (denoted by $\hat{\ }$) due its four surfaces can now be calculated at any position $(\hat{x}, \hat{y})$ by evaluating the surface integral in Eqn. 2.22, resulting in

$$\hat{\boldsymbol{H}}_i \quad = \boldsymbol{G}_i\ \hat{\boldsymbol{M}}_i. \tag{2.23}$$

where $\hat{\boldsymbol{M}} = [\ \hat{M}_x \quad \hat{M}_y\ ]^T$, $\hat{\boldsymbol{H}} = [\ \hat{H}_x \quad \hat{H}_y\ ]^T$ and $\boldsymbol{G}_i$ can be obtained from Eqn. 2.22 (see appendix C). Note that in this section Einstein's summation convention is not applied. The field due to segment i with respect to the global coordinates is

$$\boldsymbol{H}_i = \boldsymbol{R}_i\ \hat{\boldsymbol{H}}_i, \tag{2.24}$$

where $\boldsymbol{R}_i$ is

$$\boldsymbol{R}_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}, \tag{2.25}$$

with θ_i the orientation of the segment i with respect to the global coordinates. The field at any element j because of the magnetization of all the segments throughout the film is

$$\boldsymbol{H}_j \quad = \quad \boldsymbol{H}_0 + \sum_{i=1}^{N} \boldsymbol{R}_i \boldsymbol{G}_{ij} \hat{\boldsymbol{M}}_i = \boldsymbol{H}_0 + \boldsymbol{H}_{\text{self}}, \tag{2.26}$$

where $\boldsymbol{G}_{ij}$ properly accounts for the relative positioning of segments i and j and the geometry of segment i, $\boldsymbol{H}_0$ is the externally applied magnetic field, far away from the film, and N is the total number of segments. Equation 2.26 clearly shows that the magnetic field $\boldsymbol{H}$ in the film is the sum of the external field $\boldsymbol{H}_0$ and the demagnetizing field $\boldsymbol{H}_{\text{self}}$. By rotating $\boldsymbol{H}_j$ back to the local coordinates we get

$$\hat{\boldsymbol{H}}_j \quad = \quad \boldsymbol{R}_j^T \boldsymbol{H}_0 + \sum_{i=1}^{N} \boldsymbol{R}_j^T \boldsymbol{R}_i \boldsymbol{G}_{ij} \hat{\boldsymbol{M}}_i. \tag{2.27}$$

For the situation of a permanently magnetized film with magnetization $\hat{\boldsymbol{M}}_i, i = 1, \ldots, N$, Eqn. 2.27 gives the magnetic field in all the segments.

However, in case of a super-paramagnetic film, the magnetization $\hat{\boldsymbol{M}}_j$ is not known a-priori, but depends on the local magnetic field through

$$\begin{aligned} \hat{\boldsymbol{M}}_j \quad &= \quad \hat{\chi}\ \hat{\boldsymbol{H}}_j \\ &= \quad \hat{\chi} \boldsymbol{R}_j^T \boldsymbol{H}_0 + \sum_{i=1}^{N} \hat{\chi} \boldsymbol{R}_j^T \boldsymbol{R}_i \boldsymbol{G}_{ij} \hat{\boldsymbol{M}}_i, \end{aligned} \tag{2.28}$$

with

$$\boldsymbol{\chi} = \begin{bmatrix} \hat{\chi}_x & \hat{\chi}_{xy} \\ \hat{\chi}_{xy} & \hat{\chi}_y \end{bmatrix}.$$

There are N similar pairs of equations. In total these are $2 \times N$ equations for the $2 \times N$ unknown magnetizations. This set of equations is solved to get the magnetization with respect to the local coordinate frame. From the magnetization, the field can be found from Eqn. 2.27 and the magnetic flux density can be found by using Eqn. 2.20. The advantage of the proposed method is that we need not model the medium around the magnetic film to determine the magnetic field in the film. The verification of the model is performed using a super-paramagnetic film and is given in appendix D. Once the magnetization is calculated by solving Eqn. 2.28, the magnetic body couple and body force per unit volume can be found from $\boldsymbol{N} = \boldsymbol{M} \times \boldsymbol{B}_0$ and $\boldsymbol{f} = \boldsymbol{M} \cdot \boldsymbol{\nabla}\boldsymbol{B}_0$ (with $\boldsymbol{B}_0 = \mu_0 \boldsymbol{H}_0$), and given as input to Eqn. 2.3.

The magneto-mechanical fluid-structure interaction model developed in this section is benchmarked against a reference case in appendix E, and the spatial and temporal convergence of the method in the context of cilia caused flow is shown in appendix F.

2.3 Dimensional analysis

In this section we use the principle of virtual work to identify the dimensionless parameters that govern the deformation behaviour of the artificial cilia. Considering only the transverse deformations and magnetic body torques we have,

$$\int EI \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 \delta v}{\partial x^2} dx + \int \rho A \frac{\partial^2 v}{\partial t^2} \delta v dx - \int N_z \frac{\partial \delta v}{\partial x} A dx - \int \lambda \delta v b dx = 0,$$

where, the first term represents the virtual elastic work done by the internal moments, the second term represents the virtual work done by the inertial forces of the beam, the third term represents the virtual work done by the magnetic couple and the last term represents the work done by the fluid drag forces. In the above equation, λ is the traction due to fluid drag on the film in the transverse direction and has units of force per unit area. This is in contrast to the λ used in section 2.2.3, which has units of force per unit out-of-plane width. We introduce the dimensionless variables V, T and X, such that $v = VL$, $x = XL$ and $t = Tt_{\text{ref}}$, where L is a characteristic length (taken to be the length of the cilia) and t_{ref} is a characteristic time. Substitution yields

$$\int \left(\frac{Ebh^3}{12L^2} \frac{\partial^2 V}{\partial X^2} \frac{\partial^2 \delta V}{\partial X^2} + \frac{\rho b h L^2}{t_{\text{ref}}^2} \delta V \frac{\partial^2 V}{\partial T^2} \right) dX - \int \left(N_z h b \frac{\partial \delta V}{\partial X} - \lambda L \delta V b \right) dX = 0, \quad (2.29)$$

from which the elastic ($Ebh^3/12L^2$), the inertial ($\rho bhL^2/t_{\text{ref}}^2$), the magnetic ($N_z hb$) and the viscous ($\lambda L \delta V b$) terms can be easily identified. By normalising with the elastic term, we get

$$\int \left(\left(\frac{\partial^2 V}{\partial X^2} \frac{\partial^2 \delta V}{\partial X^2} \right) + I_n \left(\frac{\partial^2 V}{\partial T^2} \delta V \right) \right) dX - \int \left(M_n \left(\frac{\partial \delta V}{\partial X} \right) + F_n \delta V \right) dX = 0. \quad (2.30)$$

Here, the three governing dimensionless numbers are defined as the inertia number $I_n = 12(\rho/E)(L/t_{\text{ref}})^2(L/h)^2$ (the ratio of inertial to elastic force), the magnetic number $M_n = 12(N_z/E)(L/h)^2$ (the ratio of magnetic to elastic force) and the fluid number

$F_n = 12(\lambda/E)(L/h)^3$ (the ratio of fluid to elastic force). From dimensional considerations λ should scale with μ/t_{ref}, leading to $F_n = 12(\mu/Et_{\mathrm{ref}})(L/h)^3$. To identify the dimensionless parameters that govern the fluid flow, we start from the Navier-Stokes equations for the fluid

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\frac{d\boldsymbol{u}}{dt} = 0 \quad \text{in } A, \tag{2.31}$$

$$-\|\boldsymbol{\sigma}\|\cdot\boldsymbol{n} + \boldsymbol{\lambda} = 0 \quad \text{on } \Gamma_s. \tag{2.32}$$

The ciliary motion creates a flow that has a dominant velocity in the channel direction, i.e. x-direction. Therefore, we consider only the x-component of Eqn. 2.31. Using the constitutive relation for the fluid gives:

$$\rho^f\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right). \tag{2.33}$$

The relevant length scales can be identified by looking at the mechanism of fluid flow inside the channel (Fig. 1.3). The cilia of length L are placed periodically at a distance a in a channel of height H. The fluid inbetween the cilia is directly driven by them, and the momentum diffuses from this region upwards into the channel. The relevant length scales are the cilia spacing a in the x-direction, L and $H-L$ in the y-direction, and the relevant time scale is t_{ref}. Because the velocity of the fluid after a units will be the same (in the case of uniformly beating cilia), the net velocity and pressure gradients in the x-direction vanish over a distance a. Now, introducing these length and time scales in the above equation leads to:

$$\mathrm{Re_f}\left(\frac{\partial \bar{u}_x}{\partial \bar{t}}\right) \approx \left(\frac{\partial^2 \bar{u}_x}{\partial \bar{y}^2}\right) \quad \text{for } 0 < y < L, \tag{2.34}$$

$$\mathrm{Re_H}\left(\frac{\partial \bar{u}_x}{\partial \bar{t}}\right) \approx \left(\frac{\partial^2 \bar{u}_x}{\partial \bar{y}^2}\right) \quad \text{for } L < y < H, \tag{2.35}$$

where the terms in brackets are nondimensional, $\mathrm{Re_H} = \rho^f(H-L)^2/\mu t_{\mathrm{ref}} = t_{\mathrm{diff}}/t_{\mathrm{ref}}$ is the diffusion Reynolds number and $\mathrm{Re_f} = \rho^f L^2/\mu t_{\mathrm{ref}}$ is the flapping Reynolds number. The diffusion Reynolds number signifies how long it takes for the momentum to diffuse into the fluid (t_{diff}) compared to t_{ref}, whereas the flapping Reynolds number $\mathrm{Re_f}$ quantifies how large the inertia forces are compared to the viscous forces[2]. The question one might ask is, which of these Reynolds numbers is important? We show with the help of a simple analytical model (see appendix G) that $\mathrm{Re_H}$ has two effects; first, it determines the duration of the transient period and secondly, it reduces the fluctuating component of the fluid transported. However, the mean propulsion velocity created by the cilia in the steady state is independent of $\mathrm{Re_H}$, leaving the flapping Reynolds number $\mathrm{Re_f}$ to be the main parameter that governs the fluid transported. In the following, we take Re $\equiv \mathrm{Re_f} = \rho^f L^2/t_{\mathrm{ref}}$. Another interesting observation is that the convection terms in the Navier-Stokes equation do not contribute to the fluid momentum (as also verified by our simulations).

[2] The fluid velocity near the cilia, the velocity gradient and the viscous energy dissipated per unit time scale with $\frac{L}{t_{\mathrm{ref}}}$, $\frac{1}{t_{\mathrm{ref}}}$ and $\mu\frac{1}{t_{\mathrm{ref}}^2}$, respectively. The inertia forces scale with $\rho\frac{L}{t_{\mathrm{ref}}^2}$, hence the kinetic energy input per unit time to the system scales with $\rho\frac{L}{t_{\mathrm{ref}}^2}\frac{L}{t_{\mathrm{ref}}}$. Their ratio gives the flapping Reynolds number $\mathrm{Re_f}$.

Before proceeding, we identify the origin of the magnetic couple N_z for the two magnetic material systems considered in this work: (i) permanently magnetic (PM) and (ii) super-paramagnetic (SPM) materials. For permanently magnetic materials, having a local remanent magnetization ($\hat{M}_x$) pointing along the axial direction of the film: $(\hat{M}_x, \hat{M}_y) = (\hat{M}_x, 0)$, we can write

$$N_z = \hat{M}_x \hat{B}_y - \hat{M}_y \hat{B}_x = \hat{M}_x \hat{B}_y = \hat{M}_x B_0 \hat{f}(\theta), \tag{2.36}$$

where θ is the film orientation and B_0 is the amplitude of the applied magnetic field. To identify the form of the magnetic couple for a SPM film, we have to analyse how the magnetic field inside a SPM film depends on the applied magnetic field, the geometry of the film and the magnetic properties of the film. To this end let us subject the SPM film to an external magnetic field (H_{x0}, H_{y0}). By assuming that the magnetization induced by an applied field is uniform, we can calculate the magnetic field induced by the magnetization $\boldsymbol{H}_{\text{self}}$ (see appendix C):

$$(H_{\text{xself}}, H_{\text{yself}}) = (-\alpha M_x, -\beta M_y)\,,$$

where $\alpha = 2\tan^{-1}(h/L)/\pi$ and $\beta = 2\tan^{-1}(L/h)/\pi$ are positive factors which depend on the geometry of the film. The field $\boldsymbol{H}_{\text{self}}$ is nearly in the direction opposite to the magnetization, hence $\boldsymbol{H}_{\text{self}}$ is called a demagnetizing field. For slender films $h << L$, the factor α approaches zero and the factor β approaches unity. This makes the demagnetizing field to be negligible along the length (or tangential direction) and equal to the magnetization in magnitude along the thickness (or transverse direction). The magnetic field $\boldsymbol{H}$ in the film is the sum of the external field $\boldsymbol{H}_0$ and the demagnetizing field $\boldsymbol{H}_{\text{self}}$, see Eqn. 2.26. As the magnetization $\boldsymbol{M}$ and $\boldsymbol{H}$ are related through $\boldsymbol{\chi}$, we have two equations that can be solved to find the magnetization:

$$(M_x, M_y) = \left(\hat{\chi}_x \frac{H_{x0}}{1+\hat{\chi}_x \alpha}, \hat{\chi}_y \frac{H_{y0}}{1+\hat{\chi}_y \beta} \right).$$

The magnetization along the length is enhanced due to the shape of the film (through α and β) and also due to the magnetic anisotropy (for $\hat{\chi}_y < \hat{\chi}_x$). The magnetic flux density reads $\boldsymbol{B} = \mu_0 (\boldsymbol{M} + \boldsymbol{H}_0 + \boldsymbol{H}_{\text{self}})$. It can be seen that the demagnetizing field $\boldsymbol{H}_{\text{self}}$ has the effect of decreasing the magnetic flux density. For a slender film, the magnetic flux density in the thickness direction is nearly equal to the applied magnetic field (because $H_{\text{yself}} \approx -M_y$), and along the length it is the sum of the magnetization and the external magnetic field (because $H_{\text{xself}} \approx 0$). As we know the magnetization in terms of the applied magnetic field, we can calculate the magnetic couple (N_z) acting on the film:

$$N_z = \frac{\mu_0 H_0^2 \sin 2\theta \left(\hat{\chi}_x - \hat{\chi}_y + \hat{\chi}_x \hat{\chi}_y (\beta - \alpha)\right)}{1 + \alpha\beta\hat{\chi}_x\hat{\chi}_y + \alpha\hat{\chi}_x + \beta\hat{\chi}_y}. \tag{2.37}$$

The magnetic couple is directly proportional to the square of the applied field, depends on twice the angle made by the magnetic field vector with the film, the difference in the susceptibility (magnetic anisotropy) and difference between α and β (geometric anisotropy). The main message from Eqn. 2.37 is that for a magnetic couple to act on the film, we need an anisotropy - either magnetic or geometric. As the magnetic cilia are thin magnetic films, a shape anisotropy is always present.