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Published in:
Nuclear Physics B

DOI:
10.1016/0550-3213(85)90041-0

Nuclear Physics B249 (1985) 93–130

SUPERSYMMETRIC SKYRMIONS IN FOUR DIMENSIONS

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Received 31 July 1984

Possible topological solitons (skyrmions) of four-dimensional supersymmetric nonlinear σ-models are investigated. The requirements of supersymmetry limit our study to the CP^1 model. A stable soliton seems possible, but in the absence of a demonstrated lower-bound for the mass, the stability of the soliton is unproved. The semi-classical spectrum of the CP^1 skyrmion, as well as its supersymmetric extension, is studied.

1. Introduction

A good approximation to the low-energy physics ($\lesssim \Lambda_{QCD}$) of QCD is provided by the nonlinear σ-model. The small chiral perturbations about the vacuum describe soft pions, and the solitons of the model (skyrmions) represent baryons [1–4]. Further, chiral fluctuations about the skyrmions can be constructed so that all soft-pion skyrmion threshold theorems are automatically satisfied [5]. For an odd number of colors it can be shown that the skyrmions are fermions, as a result of the non-trivial topological properties of the Wess-Zumino term [6]. Significantly, quarks are never explicitly mentioned; nevertheless the effective soft-pion skyrmion lagrangian gives an excellent description of the static properties of baryons and their interactions.

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† Supported in part by the US Department of Energy under contract no. DE-AC03-75-ER03232-A011.

†† John S. Guggenheim Memorial Fellow 1983–84.

††† Supported in part by the National Science Foundation under grant no. PHY-82-15249.

The Skyrme model therefore gives a new way of describing bound states (baryons) in QCD, and the question arises whether similar insights can be obtained about the bound state structures of other field theories. A particularly interesting class of theories to consider are those with supersymmetry, as very little is known about the bound states of such models*. Moreover, bound states of supersymmetric theories may have phenomenological applications. In particular, attention has recently been given to possible supersymmetric preon theories [8]. In the supersymmetric limit, the low-energy ($\lesssim \Lambda_{\text{preon}}$) effective dynamics might be described by a supersymmetric nonlinear σ-model. If the preons form baryon-like bound states, these could appear as soliton supermultiplets (supersymmetric skyrmions) of the model. The issue of supersymmetric skyrmions is the focus of this paper.

In sect. 2 of this paper we state the criteria to be satisfied by four-dimensional supersymmetric nonlinear σ-models admitting topological solitons. As a result of these considerations, and those of appendix A, we find but one: the supersymmetric CP^1 model. A careful description of this model is presented in sect. 3. According to Derrick's theorem [9], if the lagrangian of the model is only quadratic in derivatives, then the solitons of the model are unstable and shrink to zero size. This problem is typically resolved by adding terms to the action which are quartic in derivatives [1, 10]. In the usual Skyrme model, there exists a unique term which is fourth order in derivatives, but only quadratic in time derivatives. For the case at hand the higher-derivative terms must appear in special combinations so as to maintain supersymmetry. We find that there is *no* supersymmetric higher-derivative contribution to the lagrangian which is at most quadratic in time derivatives. That is, there is no supersymmetric Skyrme term, a feature which is a direct result of the chiral properties of supersymmetry in four dimensions. However, there are other quartic terms which offer the possibility of stability, which we study.

A convenient soliton ansatz is discussed in sect. 4, and its stability is investigated in sect. 5. If the most general supersymmetric lagrangian fourth order in derivatives is considered, or if supersymmetry is broken, then a stable soliton seems possible. However, in the absence of a lower bound for the mass, the stability of the soliton is unproved.

In the remainder of the paper we assume soliton stability, and we turn our attention to semi-classical static soliton spectra. As a prelude to the supersymmetric case, we investigate in sect. 6 solitons of the bosonic CP^1 model. Collective coordinates for rotational and internal symmetries are introduced in order to describe the corresponding soliton excitations. We find that the excitations are those of a type of quantum mechanical rotor. The supersymmetric extension of the system (the "supersymmetric rotor") is presented in sect. 7. We discuss the possibility of deriving this model by also introducing collective coordinates for supersymmetry rotations.

* A discussion of Regge behavior in supersymmetric gauge theories was given by Grisaru and Schnitzer [7].

In the final section we discuss our results, and indicate new problems that were raised. In appendix A we compute $\Pi_3(\mathrm{M})$ for M a compact complex Grassmann manifold, and in appendix B we present the Maurer-Cartan equations of the CP^1 model. Appendix C is devoted to technical details of the canonical quantization of the collective coordinates for the CP^1 model. Attention is paid to the constraints satisfied by the collective coordinates. Finite supersymmetry transformations for the soliton ansatz are given in appendix D.

2. Models admitting supersymmetric solitons

What is the class of four-dimensional models that admits supersymmetric topological solitons? As remarked in the introduction, we have in mind an effective theory for some strongly interacting (renormalizable) preon dynamics. We therefore restrict our attention to nonlinear sigma models with fields that take values on some manifold M. (In particular, we do not consider supersymmetric Yang-Mills-Higgs systems, which are known to admit supersymmetric monopoles [11].)

As shown in [12], in order that such four-dimensional models have $N=1$ supersymmetry, M must be a Kähler manifold. Moreover, static topologically stable field configurations are possible only if $\Pi_3(\mathrm{M})$ is nontrivial. Hence, we are faced with the purely mathematical task of classifying all Kähler manifolds with nontrivial Π_3. Although we are unable to provide a complete classification, we observe that a large class of Kähler spaces is the set of compact complex Grassmann manifolds $\mathrm{G}(k,n) \equiv \mathrm{U}(n)/\mathrm{U}(k)\times \mathrm{U}(n-k)$, which includes the complex projective spaces $\mathrm{CP}^{n-1} \equiv \mathrm{U}(n)/\mathrm{U}(1)\times\mathrm{U}(n-1)$ as special cases. A straightforward exercise in exact sequences of homotopy groups, presented in appendix A, shows that the only compact complex Grassmann manifold with nontrivial Π_3 is $\mathrm{CP}^1 = \mathrm{S}^2$; indeed, $\Pi_3(\mathrm{CP}^1) = \mathrm{Z}$. We consider only compact manifolds; it is known that the homotopy group of a noncompact group is the same as that of its maximal compact subgroup. Thus, the search for a model which admits supersymmetric solitons leads rather directly to the four-dimensional supersymmetric CP^1 model*.

3. The CP^1 model and its supersymmetric extension

Before discussing the CP^1 model, it is useful to first briefly review the nonlinear $\mathrm{SU}(2)\times\mathrm{SU}(2)$ chiral model, to which it is closely related. The quadratic part of the effective lagrangian is given by**

$$\mathfrak{L}_2(\text{chiral}) = -\tfrac{1}{16} f_\pi^2 \mathrm{Tr}\left[\partial^\mu U^\dagger(x)\, \partial_\mu U(x)\right]. \tag{3.1}$$

* A similar conclusion has been reached independently by E. Rabinovici, A. Schwimmer and S. Yankielowicz (private communication).

** Our metric convention is $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. Indices follow the convention $i,j,\ldots = 1,2$; $a,b,\ldots = 1,2,3$; $\mu,\nu,\ldots = 0,1,2,3$.

Here $U(x)$ is a field with values in SU(2), and f_π is the pion decay constant. The chiral lagrangian (3.1) is invariant under independent global SU(2) transformations from the left and from the right $(\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R})$:

$$U(x) \to AU(x)B^{-1}, \qquad A \in \mathrm{SU}(2)_\mathrm{L}, \qquad B \in \mathrm{SU}(2)_\mathrm{R}. \tag{3.2}$$

In the CP[1] model one gauges the U(1) subgroup of $\mathrm{SU}(2)_\mathrm{R}$ generated by τ_3, which we shall henceforth call $\mathrm{U}(1)_\mathrm{R}$. This is achieved by replacing ordinary derivatives of $U(x)$ by right covariant derivatives:

$$D_\mu U \equiv \partial_\mu U - iV_\mu U\tau_3, \tag{3.3}$$

where $V_\mu(x)$ is the corresponding gauge field. Indeed the quadratic part of the lagrangian for the CP[1] model is given by (cf. (3.1))

$$\mathcal{L}_2(\mathrm{CP}^1) = -\tfrac{1}{16} f_\pi^2 \mathrm{Tr}\left[D^\mu U^\dagger(x)\, D_\mu U(x)\right], \tag{3.4}$$

and is invariant under local $\mathrm{U}(1)_\mathrm{R}$ and global $\mathrm{SU}(2)_\mathrm{L}$ transformations:

$$\begin{aligned} &U(x) \to AU(x)\mathrm{e}^{i\lambda(x)\tau_3}, \qquad A \in \mathrm{SU}(2)_\mathrm{L}, \\ &V_\mu(x) \to V_\mu(x) + \partial_\mu \lambda(x). \end{aligned} \tag{3.5}$$

Note that there are no global SU(2) rotations from the right. Since there is no kinetic term for V_μ, it is a dependent field. Its field equation yields an algebraic expression for V_μ in terms of U:

$$V_\mu = -\tfrac{1}{2} i\, \mathrm{Tr}\left(U^\dagger \partial_\mu U \tau_3\right). \tag{3.6}$$

For later convenience we give an alternative formulation of the CP[1] model. First we parametrize the SU(2) matrix $U(x)$ in terms of the complex scalars A_i $(i = 1,2)$:

$$U = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix} \quad \text{with } \bar{A}^i A_i \equiv A_1^* A_1 + A_2^* A_2 = 1. \tag{3.7}$$

Substituting this expression into (3.4) we obtain

$$\mathcal{L}_2(\mathrm{CP}^1) = -\tfrac{1}{8} f_\pi^2 D^\mu \bar{A}^i D_\mu A_i, \tag{3.8}$$

where the covariant derivative is given by

$$D_\mu A_i = \left(\partial_\mu - iV_\mu\right) A_i, \tag{3.9}$$

and the field equation for V_μ now becomes (cf. (3.6))

$$V_\mu = -\tfrac{1}{2}i\bar{A}^i\overleftrightarrow{\partial}_\mu A_i \equiv -\tfrac{1}{2}i\left[\bar{A}^i\partial_\mu A_i - \left(\partial_\mu \bar{A}^i\right)A_i\right]. \tag{3.10}$$

The Kähler structure of the CP^1 model can be made manifest by solving the constraint $\bar{A}^iA_i = 1$ in terms of complex projective coordinates Z (thereby fixing the U(1) gauge) in the following way:

$$(A_1, A_2) = \frac{1}{(1+ZZ^*)^{1/2}}(1, Z) \quad \text{or} \quad Z = A_2/A_1. \tag{3.11}$$

For $A_1 = 0$ these coordinates are singular. There is a similar set of coordinates which singles out $A_2 = 0$. The two sets of coordinates, which are related by a gauge transformation, together provide a nonsingular global coordinate system for S^2. In terms of projective coordinates the lagrangian (3.8) reads

$$\mathfrak{L}_2(\mathrm{CP}^1) = -\tfrac{1}{8}f_\pi^2 g(Z, Z^*)\, \partial^\mu Z^*\, \partial_\mu Z, \tag{3.12}$$

where $g(Z, Z^*)$ is the Kähler metric

$$g(Z, Z^*) = \frac{1}{(1+ZZ^*)^2} = \frac{\partial}{\partial Z}\frac{\partial}{\partial Z^*}\ln(1+ZZ^*). \tag{3.13}$$

Finally, we mention that the CP^1 model can be reformulated as the O(3)/O(2) nonlinear σ-model in terms of the gauge-invariant variables $q_a(x) \equiv \bar{A}^i(x)(\tau_a)_i{}^jA_j(x)$.

Having discussed the CP^1 model, we now proceed to the supersymmetric case [12,13]. To this end we replace the complex scalars A_i by chiral scalar multiplets $(A_i, \psi_{\alpha i}, F_i)$ $(i, \alpha = 1,2)$ and the vector V_μ by a real vector multiplet $(V_\mu, \lambda_\alpha, D)$. Here the fields F_i are complex scalars, D is a real scalar and $\psi_{\alpha i}$, λ_α are Majorana spinors. For these spinors we will use the two-component notation*. In particular $\psi_{\alpha i}$ corresponds to a left-handed chiral spinor, whereas the complex conjugate field $\bar{\psi}^{\dot{\alpha} i} \equiv (\psi^\alpha{}_i)^*$ corresponds to a right-handed one. The lagrangian of the supersymmetric CP^1 model is given by

$$\begin{aligned}\mathfrak{L}_2 = \tfrac{1}{8}f_\pi^2\Big\{ &-D^\mu\bar{A}^iD_\mu A_i - \tfrac{1}{2}i\bar{\psi}^{\dot{\alpha} i}(\sigma_\mu)_{\alpha\dot{\alpha}}\overleftrightarrow{D}^\mu\psi^\alpha{}_i + \bar{F}^iF_i \\ &- i\bar{A}^i\lambda^\alpha\psi_{\alpha i} + iA_i\bar{\lambda}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}{}^i + D\left(\bar{A}^iA_i - 1\right)\Big\},\end{aligned} \tag{3.14}$$

* We follow the notation and conventions of [14].

with

$$D_\mu A_i = (\partial_\mu - iV_\mu) A_i ,$$

$$D_\mu \psi_i^\alpha = (\partial_\mu - iV_\mu) \psi_i^\alpha . \tag{3.15}$$

The model is invariant under the following set of supersymmetry transformations:

$$\delta A_i = -\varepsilon^\alpha \psi_{\alpha i} ,$$

$$\delta \psi_{\alpha i} = -i\bar{\varepsilon}^{\dot\alpha} (\sigma^\mu)_{\alpha\dot\alpha} D_\mu A_i + \varepsilon_\alpha F_i ,$$

$$\delta F_i = -i\bar{\varepsilon}^{\dot\alpha} (\sigma^\mu)^\alpha{}_{\dot\alpha} D_\mu \psi_{\alpha i} - i\bar{\varepsilon}^{\dot\alpha} A_i \bar{\lambda}_{\dot\alpha} , \tag{3.16a}$$

$$\delta V_\mu = -\tfrac{1}{2} i (\sigma_\mu)^{\alpha\dot\alpha} (\bar{\varepsilon}_{\dot\alpha} \lambda_\alpha + \varepsilon_\alpha \bar{\lambda}_{\dot\alpha}) ,$$

$$\delta \lambda_\alpha = \varepsilon^\beta (\sigma^{\mu\nu} C)_{\beta\alpha} F_{\mu\nu} + i\varepsilon_\alpha D ,$$

$$\delta D = \tfrac{1}{2} (\sigma^\mu)_{\alpha\dot\alpha} \partial_\mu (\bar{\varepsilon}^{\dot\alpha} \lambda^\alpha - \varepsilon^\alpha \bar{\lambda}^{\dot\alpha}) . \tag{3.16b}$$

The field equations of (3.14) and their supersymmetry transformations lead to the following constraints on $(A_i, \psi_{\alpha i}, F_i)$:

$$\bar{A}^i A_i = 1 ,$$

$$\bar{A}^i \psi_{\alpha i} = 0 ,$$

$$\bar{A}^i F_i = 0 . \tag{3.17}$$

Moreover they lead to the following algebraic expressions for $(V_\mu, \lambda_\alpha, D)$ in terms of $(A_i, \psi_{\alpha i}, F_i)$:

$$V_\mu = -\tfrac{1}{2} \left[i\bar{A}^i \overleftrightarrow{\partial}_\mu A_i + (\sigma_\mu)^{\alpha\dot\alpha} \bar{\psi}_{\dot\alpha}{}^i \psi_{\alpha i} \right] ,$$

$$\lambda_\alpha = -i \left[\bar{F}^i \psi_{\alpha i} + i (\sigma^\mu)_{\alpha\dot\alpha} (D_\mu A_i) \bar{\psi}^{\dot\alpha i} \right] ,$$

$$D = D^\mu \bar{A}^i D_\mu A_i + \tfrac{1}{2} i \bar{\psi}^{\dot\alpha i} (\sigma^\mu)_{\alpha\dot\alpha} \overleftrightarrow{D}_\mu \psi^\alpha{}_i - \bar{F}^i F_i . \tag{3.18}$$

In analogy to the CP^1 model we can solve the constraints (3.17) by introducing chiral projective coordinates (Z, ψ_α, F) in the following way:

$$A_i = \frac{1}{(1+ZZ^*)^{1/2}}(1, Z),$$

$$\psi_{\alpha i} = \frac{1}{(1+ZZ^*)^{3/2}}(-Z^*, 1)\psi_\alpha,$$

$$F_i = \frac{1}{(1+ZZ^*)^{3/2}}(-Z^*, 1)\left[F - \frac{Z}{1+ZZ^*}\psi^\alpha\psi_\alpha\right]. \tag{3.19}$$

In terms of these projective coordinates the lagrangian (3.14) reads [12]

$$\mathcal{L}_2 = \tfrac{1}{8}f_\pi^2\left\{\tfrac{1}{2}g(Z, Z^*)\left[-\partial^\mu Z^*\,\partial_\mu Z - \tfrac{1}{2}i\bar{\psi}^{\dot\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\overleftrightarrow{\partial}_\mu\psi^\alpha + \bar{F}F\right]\right.$$
$$\left. + \frac{\partial g}{\partial Z}\left[\psi^2\bar{F} - \tfrac{1}{2}i\bar{\psi}_{\dot\alpha}(\sigma^\mu)^{\alpha\dot\alpha}(\partial_\mu Z)\psi_\alpha\right] + \frac{1}{2}\frac{\partial^2 g}{\partial Z\,\partial Z^*}\psi^2\bar{\psi}^2\right\} + \text{h.c.}, \tag{3.20}$$

with the Kähler metric $g(Z, Z^*)$ given by (3.13).

Although we have introduced the supersymmetric CP^1 model in order to study its topological solitons, the quadratic model as such does not admit stable soliton solutions. Indeed, let us first recall the SU(2) × SU(2) chiral model: according to Derrick's theorem [9] the solitons of this model are unstable, and shrink to zero size. One can stabilize the solitons by adding a term with four derivatives to the lagrangian (3.1). Such a quartic term, first introduced by Skyrme, is

$$\mathcal{L}_4(\text{chiral}) = \frac{1}{32e^2}\mathrm{Tr}\left[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\right]^2, \tag{3.21}$$

where e is a dimensionless parameter. This is the unique term with four derivatives that is only second order in time derivatives. Also, it leads to a positive definite hamiltonian.

Similarly, the corresponding term to stabilize the solitons of the nonsupersymmetric CP^1 model is

$$\mathcal{L}_4(CP^1) = \frac{1}{32e^2}\mathrm{Tr}\left[U^\dagger D_\mu U, U^\dagger D_\nu U\right]^2, \tag{3.22}$$

with the covariant derivative defined in (3.3) and the gauge field V_μ given by (3.6). Using the Maurer-Cartan equations (see appendix B) this term can be rewritten in the following form:

$$\mathcal{L}_4(CP^1) = -\frac{1}{16e^2}F_{\mu\nu}^2(V), \tag{3.23}$$

where $F_{\mu\nu}(V) = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the field strength of the dependent field V_μ.

The form (3.23) clearly suggests that in the supersymmetric case we use instead the supersymmetric kinetic term for the entire vector multiplet (V_μ, λ_α, D):

$$\mathcal{L}_4 = \frac{1}{8e^2}\left\{-\tfrac{1}{2}F_{\mu\nu}^2(V) + i\bar{\lambda}^{\dot\alpha}(\sigma^\mu)^\alpha{}_{\dot\alpha}\partial_\mu\lambda_\alpha + D^2\right\}. \tag{3.24}$$

Here it is understood that V_μ, λ_α and D are the dependent fields given by their expressions (3.18) in terms of A_i, $\psi_{\alpha i}$ and F_i. With this identification we verify that $\mathcal{L}_4$ is supersymmetric under the transformations (3.16). We immediately see that (3.24) is not a "minimal" supersymmetric extension of the bosonic term (3.23). The additional bosonic term D^2 is unavoidably introduced by the requirement of supersymmetry. This will become significant later in the paper. For now, simply note that the bosonic sector of (3.24), obtained by taking $\psi_{\alpha i} = F_i = 0$, is

$$\mathcal{L}_4 \to \frac{1}{8e^2}\left\{-\tfrac{1}{2}F_{\mu\nu}^2(V) + \left(D^\mu\bar{A}^i D_\mu A_i\right)^2\right\}. \tag{3.25}$$

Further, observe that the second term in (3.25) is fourth order in time derivatives.

Once one allows fourth-order time derivatives, the bosonic term (3.23) is no longer unique. Indeed, it can be shown that in the bosonic CP^1 model there exist three independent terms with four derivatives:

$$\mathcal{L}_4(CP^1) = aF_{\mu\nu}^2(V) + b\left(D^\mu\bar{A}^i D_\mu A_i\right)^2 + c\Box\bar{A}^i\Box A_i, \tag{3.26}$$

where $\Box \equiv D^\mu D_\mu$ is the gauge covariant d'alembertian.

One must now investigate whether there are CP^1 invariants other than the one given in (3.25), which allow for a supersymmetric extension. To answer this question it is convenient to use a superfield formulation*. In such a formulation the supersymmetric CP^1 action is given by

$$S_2 = \tfrac{1}{8}f_\pi^2\int d^4x\, d^4\theta\left(\bar{\Phi}^i\Phi_i - V\right). \tag{3.27}$$

Here Φ_i ($i = 1,2$) are *covariantly* chiral superfields (i.e. $\bar{\nabla}_{\dot\alpha}\Phi_i = 0$) with components ($A_i$, $\psi_{\alpha i}$, F_i), and V is a real vector superfield, whose components in the Wess-Zumino gauge are given by V_μ, λ_α and D. The equation of motion with respect to V gives

$$\bar{\Phi}^i\Phi_i = 1. \tag{3.28}$$

In components, this corresponds to eqs. (3.17), (3.18).

* For more details on notation, see [14].

Given the constraint (3.28), we now try to construct in terms of the Φ_i the most general supersymmetric CP^1 invariant expression with four derivatives. It appears that there are only two independent structures:

$$S_4 = -\frac{1}{32e^2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\Big\{\alpha\big[\overline{\nabla}^{\dot\alpha}\overline{\Phi}^i\overline{\nabla}_{\dot\alpha}\overline{\Phi}^j\,\nabla^\alpha\Phi_i\,\nabla_\alpha\Phi_j\big] + \beta\big[\overline{\nabla}^{\dot\alpha}\overline{\nabla}_{\dot\alpha}\overline{\Phi}^i\,\nabla^\alpha\nabla_\alpha\Phi_i\big]\Big\}. \tag{3.29}$$

One can verify that the bosonic sector of (3.29) (taking $\psi_{\alpha i} = F_i = 0$) is

$$S_4 \to -\frac{1}{8e^2}\int \mathrm{d}^4x\Big\{\alpha\Big[-\tfrac{1}{2}F^2_{\mu\nu}(V) + \big(D^\mu\overline{A}^iD_\mu A_i\big)^2\Big] + \beta\Big[\Box\overline{A}^i\Box A_i - \big(D^\mu\overline{A}^iD_\mu A_i\big)^2\Big]\Big\}. \tag{3.30}$$

Clearly the first term is the one we considered earlier (see (3.25)). The second represents a new possible stabilizing term. The stability of solitons with these interactions will be studied in sect. 5.

It should be noted that the supersymmetric action (3.27) together with the higher-derivative term (3.29), does admit solutions of the F-field equation with $F_i \neq 0$. In fact, F_i becomes a propagating field. We have investigated whether these nonzero solutions are able to cancel terms in (3.30), leaving only the desired $F^2_{\mu\nu}(V)$ term; it seems that this is not the case. Therefore we have not pursued this direction.

We observe that it is the chirality of the Φ_i which restricts us to the two independent structures given in (3.29). Indeed, in two and three dimensions where no chiral superfields exist, one can construct a supersymmetric higher-derivative term that is second order in time derivatives.

4. Soliton ansatz and topological charge

In sect. 3 we have described the supersymmetric CP^1 model. We now look for classical soliton (skyrmion) solutions. We set the Fermi fields (and also the auxiliary fields F_i) to zero, and consider static, topologically stable boson field configurations. Such configurations describe mappings from S^3 to $\mathrm{CP}^1 = \mathrm{S}^2$ with nonzero winding number (recall that $\Pi_3(\mathrm{CP}^1) = \mathrm{Z}$), the prototype of which is the Hopf map. For clarity, we exhibit these field configurations in two steps: first we identify S^3 in terms of the three spatial coordinates x^a, and then we describe the Hopf map $\mathrm{S}^3 \to \mathrm{CP}^1$.

We can identify an S^3 in physical space by way of

$$U_0 \colon \boldsymbol{x} \to U_0(\boldsymbol{x}) = e^{if(r)\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}} = \cos f(r) + i\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}\sin f(r), \tag{4.1}$$

where $f(0)=\pi$ and $f(\infty)=0$, and $\hat{\boldsymbol{x}} \equiv \boldsymbol{x}/r$. Clearly, this maps once from $[R^3 + \{\infty\}]$ to $SU(2) = S^3$. This is not the most general such map, but is adequate for our purposes (see below).

Next, let us consider the Hopf map, which can be conveniently described by introducing a pair of complex coordinates (A_1, A_2), with $A_1 A_1^* + A_2 A_2^* = 1$, to parametrize S^3; the desired map is then

$$(A_1, A_2) \to [A_1, A_2], \tag{4.2}$$

where $[A_1, A_2]$ denotes the equivalence class with respect to multiplication by a complex phase (i.e., $[A_1', A_2'] \sim [A_1, A_2]$ if and only if $(A_1', A_2') = e^{i\theta}(A_1, A_2)$.) Equivalently, representing S^3 by the SU(2) matrix $U = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}$ as in sect. 3, the Hopf map becomes

$$U \to [U], \tag{4.3}$$

where $[U]$ denotes the equivalence class with respect to right multiplication by $e^{i\theta\tau_3}$ (i.e. a gauge transformation of the CP^1 model). This is precisely the natural projection $SU(2) \to SU(2)/U(1)$.

Combining these results, we see that a suitable ansatz for a CP^1 model soliton is the coset $[U_0]$, with U_0 given by (4.1); i.e., it is simply the Skyrme ansatz [1], modulo a U(1) transformation on the right.

Let us recall that the winding number ("Hopf invariant") for a map $U(\boldsymbol{x})$: $S^3 \to S^3$ is given by

$$Q = -\frac{1}{24\pi^2}\int d^3x\, \varepsilon_{abc} \mathrm{Tr}\left[\left(U^\dagger \partial_a U\right)\left(U^\dagger \partial_b U\right)\left(U^\dagger \partial_c U\right)\right]. \tag{4.4a}$$

One can check that this is invariant under the local $U(1)_R$ transformations $U(\boldsymbol{x}) \to U(\boldsymbol{x})e^{i\theta(x)\tau_3}$ for nonsingular $\theta(x)$ falling off sufficiently rapidly at spatial infinity, i.e. for "small" gauge transformations. Indeed, in terms of the dependent gauge fields (3.6) introduced earlier, one can show with the help of the Maurer-Cartan equations (see appendix B) that the expression for Q takes the simple form [15]

$$Q = -\frac{1}{8\pi^2}\int d^3x\, \varepsilon_{abc} V_a F_{bc}(V), \tag{4.4b}$$

from which the local U(1) invariance is more evident. Hence, we have the important observation (see e.g. [16]) that the formulas (4.4) also measure the winding number of the coset map $[U(\boldsymbol{x})]$: $S^3 \to S^2$*. In particular, one can readily check that our CP^1 soliton ansatz, with the stated boundary conditions, has unit topological charge, as does the Skyrme ansatz.

A current for which Q, given by (4.4), is the charge is

$$J^{\mu} = -\frac{1}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}V_{\nu}F_{\rho\sigma}(V). \tag{4.5}$$

Even though this current is conserved ($\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \equiv 0$), it is not invariant under $U(1)_R$ transformations. There is no current with charge (4.4) which is both conserved and gauge invariant. Note also that the current vanishes identically when written in projective coordinates, (3.11).

Finally, let us address the generality of our ansatz. As already noted, U_0 (4.1) is not the most general map from $[R^3 + \{\infty\}]$ to S^3. In general, the vacuum has more symmetry than a soliton ansatz. One attempts to find a nontrivial soliton ansatz with the maximal symmetry consistent with the model. For the Skyrme model, the ansatz is invariant under $\boldsymbol{J}+\boldsymbol{I}$; correspondingly, for the CP^1 model, only a $J_3 + I_3$ invariance of the ansatz is expected. A map which is only $J_3 + I_3$ invariant is [17]

$$U_0(\boldsymbol{x}) = e^{if(\rho, x^3)\hat{\omega}\cdot\tau}, \tag{4.6}$$

where

$$\hat{\omega} \equiv \left(\frac{x'}{\rho}\cos g, \frac{x^2}{\rho}\cos g, \sin g\right), \qquad g \equiv g(\rho, x^3), \qquad \rho^2 \equiv (x')^2 + (x^2)^2.$$

Since the map (4.1) has more symmetry than this, it is much more convenient for performing calculations.

5. Stability

Our soliton ansatz $[U_0]$, with U_0 given by (4.1), depends on a single unknown function $f(r)$. The function $f(r)$ is determined by requiring that the energy of the field configuration is minimized. For static configurations of the supersymmetric CP^1 model (with auxiliary and Fermi fields set to zero), the energy is [see (3.8), (3.30)]

$$E_{\text{stat}} = \int d^3x \left\{ \tfrac{1}{8}f_\pi^2 D_a\bar{A}^i D_a A_i + \frac{1}{8e^2}\left[\alpha\left(-\tfrac{1}{2}F_{ab}^2 + \left(D_a\bar{A}^i D_a A_i\right)^2\right) \right.\right.$$
$$\left.\left. + \beta\left(D_a D_a \bar{A}^i D_b D_b A_i - \left(D_a\bar{A}^i D_a A_i\right)^2\right)\right]\right\}. \tag{5.1}$$

* In fact, this provides an explicit demonstration that $\Pi_3(S^3) = \Pi_3(S^2)$.

Substituting our ansatz yeilds, after some algebra, an expression for the soliton mass

$$M = 4\pi \frac{f_\pi}{e} \int_0^\infty \mathrm{d}r\, r^2 \left\{ \tfrac{1}{12}\left[(f')^2 + 2\frac{\sin^2 f}{r^2} \right] + (\alpha+\beta)\tfrac{1}{15}\left[(f')^2 - \frac{\sin^2 f}{r^2} \right]^2 \right.$$
$$\left. + \beta \tfrac{1}{12}\left[f'' + 2\frac{f'}{r} - 2\frac{\sin f \cos f}{r^2} \right]^2 \right\}, \tag{5.2}$$

where we have changed to dimensionless variables $r \to r/ef_\pi$. We take $\alpha \neq 0$, and consider in turn, the two cases (subsect. 5.1) $\beta = 0$, and (subsect. 5.2) $\beta \neq 0$.

5.1. $\beta = 0$

We single out the case $\beta = 0$, since it corresponds to the supersymmetric generalization of the CP^1 model defined by (3.8) and (3.22), which hereafter will be called the "minimal" CP^1 model. Let us consider an $f(r)$ for which the term in (5.2) proportional to α vanishes. That is,

$$f'(r) = -\frac{\sin f(r)}{r}. \tag{5.3}$$

A solution that satisfies the correct boundary conditions $f(0) = \pi$, $f(\infty) = 0$ is

$$\cos f(r) = -\frac{1 - r^2/R^2}{1 + r^2/R^2}, \tag{5.4}$$

where R is an arbitrary dimensionless constant. With this choice for $f(r)$, M becomes

$$M = 4\pi \frac{f_\pi}{e} \int_0^\infty \mathrm{d}r\, \tfrac{1}{4}\sin^2 f(r) = \pi^2 \frac{f_\pi}{e} R. \tag{5.5}$$

Hence, by scaling $R \to 0$, one scales the soliton mass to zero. That is, the soliton is in fact unstable, and shrinks to zero size and mass. (See fig. 1.)

This result is not difficult to understand. The reason for introducing fourth-derivative terms in the action was to have a $1/r^4$ contribution to the mass density, which (if of the appropriate sign) disfavors field configurations with small radius. This is the essence of Derrick's theorem. We see from (5.2) that in the supersymmetric model the "repulsive" effect of these higher-derivative terms is cancelled, and the stability problem persists. It should be stressed that this instability occurs because there exist nontrivial field configurations for which the entire contribution of the fourth-derivative terms to the soliton mass vanishes.

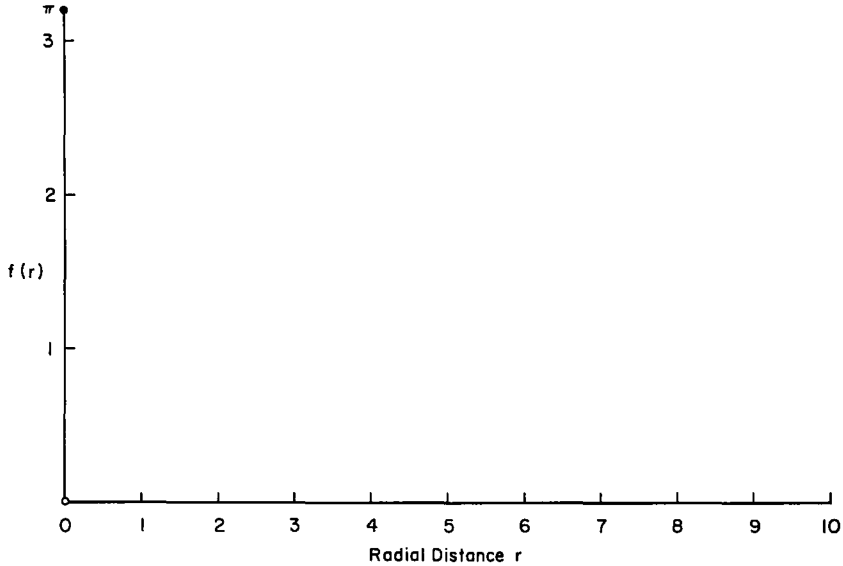

Fig. 1. Plot of $f(r)$ for the singular soliton field configuration.

As a check on this reasoning, we performed numerical studies on a model with an explicit supersymmetry breaking term, and studied the approach to the supersymmetric limit. That is, we added to (5.1) the term $(\varepsilon\alpha/8e^2)(D_a\bar{A}^i D_a A_i)^2$, giving an additional contribution ΔM to the soliton mass M (5.2):

$$\Delta M = 4\pi\frac{f_\pi}{e}\frac{\varepsilon\alpha}{10}\int_0^\infty \mathrm{d}r\, r^2\left[\tfrac{2}{3}(f')^4 + \frac{2}{r^2}(\sin^2 f)(f')^2 + \frac{7}{3}\frac{\sin^4 f}{r^2}\right].$$

The differential equation for $f(r)$ following from the variational principle $\delta(M+\Delta M)/\delta f$, with $\alpha = 1$ and $\beta = 0$, is

$$\begin{aligned} &f''\left\{\tfrac{1}{2}r^2 - \tfrac{2}{5}(2-3\varepsilon)\sin^2 f + \tfrac{12}{5}(1+\varepsilon)(rf')^2\right\} + rf' \\ &-\tfrac{1}{5}(2-3\varepsilon)(\sin 2f)(f')^2 - \tfrac{1}{2}\sin 2f \\ &+\tfrac{1}{5}(-2-7\varepsilon)(\sin 2f)\frac{\sin^2 f}{r^2} + \tfrac{8}{5}(1+\varepsilon)r(f')^3 = 0. \qquad (5.6) \end{aligned}$$

As already noted, $\varepsilon = 0$ corresponds to the supersymmetric limit. We solved this differential equation numerically for various values of ε. In fig. 2 we plot $f(r)$ versus r for the three cases $\varepsilon = 1.0, 0.1, 0.01$. Clearly, as ε approaches the critical value 0, $f(r)$ approaches the singular configuration of fig. 1. Moreover, the soliton masses

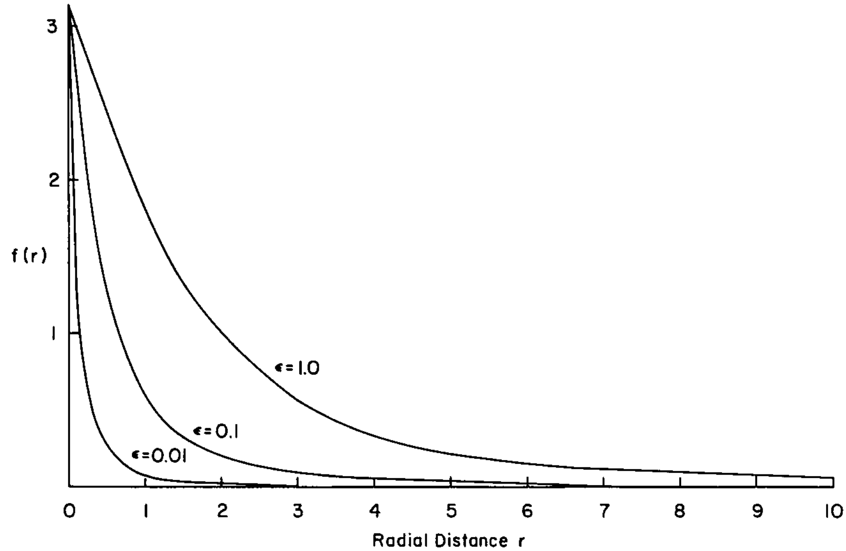

Fig. 2. Plot of $f(r)$ for soliton field configurations for models with three values of ε (see eq. (5.6)). The radial distance is given in the dimensionless variable r.

corresponding to these three values of ε are proportional to 1.79, 0.58, 0.20, respectively, with proportionality factor $\pi^2(f_\pi/e)$; these approach zero, as expected.

One point in the above discussion deserves further clarification. We use a soliton ansatz that is too restrictive (i.e., it has a $\boldsymbol{J}+\boldsymbol{I}$ invariance not present in the action). Therefore, there is no guarantee that the stationary point of M, as given in (5.2), is also a stationary point of the action, which for time-independent configurations is E_{stat} given by (5.1). (See [18].) The minimum of M provides only an upper bound for the minimum of E_{stat}. However, since we find field configurations for which $M=0$, these configurations in fact also minimize E_{stat}.

In short, the supersymmetric CP^1 model with $\alpha \neq 0$, $\beta = 0$ does not have stable solitons. Of course, if supersymmetry is explicitly broken (e.g., taking $\varepsilon \neq 0$ in (5.6)), then a stable soliton is not ruled out by these considerations.

5.2. $\beta \neq 0$

We have seen that if there is any possibility of finding stable solitons for the supersymmetric CP^1 model, one must take $\beta \neq 0$ in (5.1). In this case, we cannot find an upper bound for the soliton mass which vanishes. For instance, taking again the special case (5.3), (5.4), we find from (5.2) that

$$M = \pi^2 \frac{f_\pi}{e}\left\{R + \frac{\beta}{6}\frac{1}{R}\right\}. \tag{5.7}$$

This is stationary for $R=\sqrt{\frac{1}{6}\beta}$; correspondingly, we have the estimate $M=(\pi^2 f_\pi/e)\sqrt{\frac{2}{3}\beta}$. A lower upper bound for M could be found by numerically solving the variational differential equation $\delta M/\delta f=0$, which now involves fourth-order derivatives of $f(r)$. We have not performed this calculation.

Thus far, we have discussed upper bounds for the soliton mass. Now we consider the possibility of establishing lower bounds, relying on the fact that the soliton has nonvanishing topological charge. This technique was pioneered by Skyrme [1], and was later exploited by others in monopole [19] and instanton [20] physics. For the case at hand, observe that

$$\int \mathrm{d}^3x \left(f_\pi V_a \pm \frac{1}{2e}\varepsilon_{abc}F_{bc}\right)^2 \geqslant 0, \tag{5.8a}$$

$$\int \mathrm{d}^3x \left|f_\pi B_a \pm \frac{1}{2e}\varepsilon_{abc}\partial_{[b}B_{c]}\right|^2 \geqslant 0, \tag{5.8b}$$

from which follows, using (4.4) and (B.3), that

$$\int \mathrm{d}^3x \left(f_\pi^2 V_a^2 + \frac{1}{2e^2}F_{ab}^2\right) \geqslant 8\pi^2\frac{f_\pi}{e}|Q|, \tag{5.9a}$$

$$\int \mathrm{d}^3x \left(f_\pi^2 B_a^* B_a + \frac{1}{2e^2}\partial_{[a}B_{b]}^*\partial_{[a}B_{b]}\right) \geqslant 16\pi^2\frac{f_\pi}{e}|Q|. \tag{5.9b}$$

These are the fundamental inequalities. To demonstrate their use, consider the Skyrme model (3.1), (3.21), whose static energy can be expressed as

$$E_{\mathrm{stat}}(\mathrm{Skyrme}) = \int \mathrm{d}^3x \left\{\tfrac{1}{8}f_\pi^2\left(V_a^2 + B_a^* B_a\right) + \frac{1}{16e^2}\left[F_{ab}^2 + \left(\partial_{[a}B_{b]}^*\right)\left(\partial_{[a}B_{b]}\right)\right]\right\}. \tag{5.10}$$

From the two inequalities, it is clear that

$$E_{\mathrm{stat}}(\mathrm{Skyrme}) \geqslant 3\pi^2\frac{f_\pi}{e}|Q|,$$

which is Skyrme's result.

Now consider our model. In terms of the variables V_a and B_a, the static energy (5.1) reads

$$E_{\mathrm{stat}} = \int \mathrm{d}^3x \left\{\tfrac{1}{8}f_\pi^2 B_a^* B_a + \frac{1}{8e^2}\left[\alpha\left(-\tfrac{1}{2}F_{ab}^2 + \left(B_a^* B_a\right)^2\right) + \beta\left(D_a B_a^*\right)\left(D_b B_b\right)\right]\right\}. \tag{5.11}$$

Since this model is gauge invariant, no explicit V_a terms (in particular, V_a^2) appear in this expression. Hence, the inequality (5.9a) cannot be readily used to give a nontrivial lower bound for the soliton mass. Similarly, the inequality (5.9b) evidently does not make such an estimate possible. Thus, the problem of providing a lower bound for the mass of a supersymmetric skyrmion remains open. Of course, if one were to modify the model by adding the gauge symmetry breaking term V_a^2 to the action, then a lower bound could be easily obtained.

The conclusions of this section are as follows:

(i) The supersymmetric CP^1 model in which the quartic term is the supersymmetric extension of $F_{\mu\nu}^2$ [i.e., the case $\beta = 0$ in (5.1)] has no stable solitons. However, if in this model supersymmetry is explicitly broken, then a stable soliton is not ruled out.

(ii) For the supersymmetric CP^1 model with the most general quartic term [i.e., $\alpha \neq 0$, $\beta \neq 0$ in (5.1)], a stable soliton solution is not excluded. However, in the absence of a nontrivial lower bound for the soliton mass, the stability of such a solution is unproven.

(iii) For models in which gauge invariance (and hence, also supersymmetry) is explicitly broken, we find nontrivial upper and lower bounds for the soliton mass. Thus, these models evidently do admit stable soliton solutions.

For those models which may admit stable solitons, clearly it would be interesting to find solutions based on the more general ansatz (4.6); however, we have not attempted this. Also, as already noted, these models involve higher time derivatives. Moreover, the actions for these models are in general not bounded from below. To ensure that the soliton mass be positive definite requires the values $\alpha, \beta > 0$ (see e.g. (5.2)); however, for these values the action (3.8), (3.30) is not bounded from below. Since these models are considered only as effective theories, the higher-derivative terms can presumably be treated as small perturbations. However, it remains to be verified that these apparent dynamical stability difficulties are in fact not realized in the low-energy regime for which these models are valid.

Finally, for later convenience, we briefly consider here the stability of solitons of a related model: namely the minimal model (3.8), (3.22). As in the supersymmetric case, we cannot provide a nontrivial lower bound for the soliton mass. Nevertheless, an upper bound for the mass can be found: using again the ansatz (4.1), we obtain the following expression for the mass:

$$M = 4\pi \frac{f_\pi}{e} \int_0^\infty \mathrm{d}r\, r^2 \left\{ \tfrac{1}{12} \left[(f')^2 + 2\frac{\sin^2 f}{r^2} \right] + \frac{1}{6} \frac{\sin^2 f}{r^2} \left[\frac{\sin^2 f}{r^2} + 2(f')^2 \right] \right\}. \tag{5.12}$$

The solution $f(r)$ of the corresponding differential equation has been determined numerically and is plotted in fig. 3; this, in turn, gives $M = (\pi^2 f_\pi / e)1.74$. As already

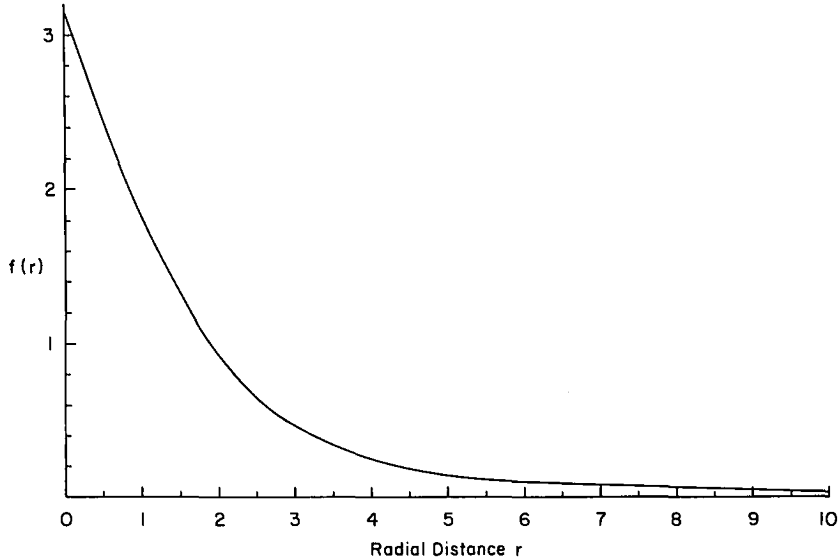

Fig. 3. Plot of $f(r)$ for the soliton field configuration for the CP^1 model.

emphasized, this configuration is not expected to be a solution of the field equations of the CP^1 model. In passing, we note that a cruder mass estimate, requiring no numerical computations, can be found by substituting the expression for $f(r)$ [(5.3), (5.4)] into (5.12), yielding

$$M = \pi^2 \frac{f_\pi}{e}\left\{R + \frac{1}{R}\right\}. \tag{5.13}$$

This is stationary for $R = 1$, for which $M = 2\pi^2 f_\pi/e$, in reasonable agreement with the previous estimate.

6. Semiclassical spectrum of CP^1 soliton

In the previous section, we discussed the purely classical problem of soliton stability. There is a possibility that stable soliton solutions exist, both for the supersymmetric and nonsupersymmetric CP^1 models. We shall now assume that such solutions do exist, and use semi-classical methods to investigate the corresponding quantum states. In the present section we treat the "minimal" nonsupersymmetric CP^1 model (3.4), (3.22)

$$\mathcal{L} = -\tfrac{1}{16} f_\pi^2 \mathrm{Tr}\left(D^\mu U^\dagger D_\mu U\right) - \frac{1}{16e^2} F_{\mu\nu}^2, \tag{6.1}$$

leaving the supersymmetric case for the following section.

One method of quantizing a classical soliton solution involves use of collective coordinates (see e.g. [21] and references therein). In general, one such coordinate is introduced for each global symmetry of the vacuum which does not leave the solution invariant. As a result, once these collective coordinates are quantized, the symmetries "broken" by the solution are restored: the soliton states are towers of states labeled by the maximal set of commuting generators of the symmetries of the vacuum.

A collective coordinate analysis for static $SU(N)\times SU(N)$ skyrmions, with $N=2,3$ has been given in [3,4] respectively. As we shall see, the corresponding analysis for CP^1 solitons is similar, but involves some new features as well. Indeed, already at the outset there is a difficulty: we do not have at our disposal a classical soliton solution; rather, we have only a field configuration (see fig. 3) based on the ansatz (4.1), which is adequate for finding an upper bound for the soliton mass, but which in fact is not a solution of the field equations. Nonetheless, it is instructive to proceed using the above field configuration as if it were a true, stable solution.

With this in mind, observe that the vacuum of the CP^1 model has the global symmetries $SU(2)_J\times U(1)_{I_3}$, as well as a local $U(1)_R$ symmetry. (Here, J denotes the generator of space rotations, and I_3 is the generator of the L + R transformation $U\to e^{i\alpha\tau_3}Ue^{-i\alpha\tau_3}$.) Of course, since we are concerned only with static solitons, we do not consider translations. Our soliton ansatz (4.1) is invariant under the diagonal J_3+I_3 subgroup of this invariance group. Hence, it would seem that we need to introduce three collective coordinates: one each for J_1, J_2, and J_3-I_3. This is conveniently done by writing the "rotated" soliton U as

$$U=e^{iO_{ab}\hat{x}_a\tau_b f(r)}, \tag{6.2}$$

where O_{ab} is the SO(3) matrix of collective coordinates. Equivalently, one can represent the collective coordinates by an SU(2) matrix A, related to O_{ab} by

$$O_{ab}=\tfrac{1}{2}\mathrm{Tr}\left(A^\dagger\tau_a A\tau_b\right), \tag{6.3}$$

so that (6.2) reads

$$U=A^\dagger U_0 A. \tag{6.4}$$

From (6.4), it is seen that spatial and internal rotations of U can be realized on the collective coordinates A. Thus, under spatial rotations, A is multiplied on the left by an SU(2) matrix B:

$$A\to BA, \tag{6.5a}$$

whereas under arbitrary isospin rotations, A is multiplied by an SU(2) matrix C on the right:

$$A\to AC. \tag{6.5b}$$

Significantly, these collective coordinates are invariant under gauge transformations. One could represent these gauge transformations by introducing yet another collective coordinate c, thereby replacing (6.2) by

$$U = \mathrm{e}^{iO_{ab}\hat{x}_a\tau_b f(r)}\mathrm{e}^{ic\tau_3}.$$

From this it is clear that the collective coordinates O_{ab} are gauge invariant, and only c transforms under gauge transformations. Of course, once U is substituted into the gauge invaraint action (6.1), the new coordinate c drops out; hence, we do not consider it further. That no more than 3 collective coordinates are required can also be seen simply by working in a formulation of the model in which gauge invariance is absent (e.g., the O(3)/O(2) version).

The collective coordinates are now promoted to dynamical variables, and as such, become functions of the time, t. The lagrangian for the collective coordinates is found by substituting the rotated ansatz (6.4) into the lagrangian (6.1), yielding

$$\begin{aligned} L &= \int \mathrm{d}^3x\, \mathcal{L}(x) \\ &= -M + \tfrac{1}{4}\lambda_1 \mathrm{Tr}(\dot{A}^\dagger \dot{A}) - \tfrac{1}{8}\lambda_2\left[\mathrm{Tr}(\tau_3 A^\dagger \dot{A})\right]^2. \end{aligned} \tag{6.6}$$

Here, M is the expression (5.12), and the coefficients λ_1 and λ_2 are

$$\lambda_1 = \frac{4\pi}{e^3 f_\pi}\int_0^\infty \mathrm{d}r\, \tfrac{1}{15} r^2 \left\{\sin^2\! f\left[5 + 4(f')^2\right] + 4\sin^4\! f\left[1 + 4(f')^2 + \frac{4}{r^2}\right]\right\},$$

$$\lambda_2 = \frac{4\pi}{e^3 f_\pi}\int_0^\infty \mathrm{d}r\, \tfrac{1}{15} r^2 \left\{\sin^2\! f\left[5 + 28(f')^2\right] - 4\sin^4\! f\left[3 + 12(f')^2 + \frac{2}{r^2}\right]\right\}. \tag{6.7}$$

As previously noted, for the configuration of fig. 3, $M = (\pi^2 f_\pi/e)1.74$; furthermore, $\lambda_1 = (4\pi/e^3 f_\pi)7.76$ and $\lambda_2 = -(4\pi/e^3 f_\pi)1.51$. Also, we remark that if the gauge symmetry breaking term

$$\gamma\left\{-\tfrac{1}{8}f_\pi^2 V_\mu^2 - \frac{1}{16e^2}\left(\partial^{[\mu}B^{\nu]}\right)\left(\partial_{[\mu}B^*_{\nu]}\right)\right\} \tag{6.8}$$

had been added to the lagrangian (6.1), then the corresponding lagrangian for the collective coordinates would have the same structure as (6.6), differing only in the expressions for λ_1 and λ_2. In particular, adding such a term with $\gamma = 1$ to (6.1) gives the Skyrme model (3.1), (3.21), and yields $\lambda_2 = 0$, with an appropriate change in λ_1.

We observe that for the special case $\lambda_1 + \lambda_2 = 0$, the lagrangian (6.6) can be rewritten in the form

$$L = -M + \tfrac{1}{4}\lambda_1 \mathrm{Tr}\left(D_0 A^\dagger D_0 A\right), \tag{6.9}$$

with the covariant derivative $D_0 A$ given by

$$D_0 A = \dot{A} - \tfrac{1}{2} A \tau_3 \mathrm{Tr}(\tau_3 A^\dagger \dot{A}) .$$

The lagrangian (6.9) is invariant under gauge transformations with a real parameter $\omega(t)$

$$A(t) \to A(t) e^{i\omega(t)\tau_3} . \tag{6.10}$$

This gauge invariance does not appear to be related to the $U(1)_R$ gauge invariance of the original field theoretic model, as the collective coordinates A are already invariant with respect to that transformation. The model (6.9) can be reformulated in terms of the gauge invariant variables

$$q_a = \tfrac{1}{2} \mathrm{Tr}(A^\dagger \tau_a A \tau_3) , \qquad q_a^2 = 1 , \tag{6.11}$$

leading to

$$L = -M + \tfrac{1}{8} \lambda_1 \dot{q}_a \dot{q}_a . \tag{6.12}$$

It is useful to know the generators J_a of spatial rotations and I_b of isospin transformations in terms of the collective coordinates. From (6.6) and the transformation properties (6.5), we easily find via the Noether prescription that the generators are

$$J_a = \tfrac{1}{4} i \left\{ \lambda_1 \mathrm{Tr}(\tau_a \dot{A} A^\dagger) + \lambda_2 O_{a3} \mathrm{Tr}(\tau_3 A^\dagger \dot{A}) \right\} ,$$

$$I_b = \tfrac{1}{4} i \left\{ \lambda_1 \mathrm{Tr}(\tau_b A^\dagger \dot{A}) + \lambda_2 \delta_{b3} \mathrm{Tr}(\tau_3 A^\dagger \dot{A}) \right\} . \tag{6.13}$$

(A tedious alternative is to first find the symmetry generators for the model (6.1) in terms of fields, and then make the substitution (6.4). We have explicitly verified that both procedures lead to the same result.) As in the Skyrme model, the generators J_a and I_b are related by a rotation:

$$J_a = O_{ab} I_b , \tag{6.14a}$$

so that

$$J_a^2 = I_b^2 . \tag{6.14b}$$

Note that the lagrangian (6.6) is invariant only under spatial and I_3 rotations, and hence, only these generators are (on-shell) time independent.

To pass from the lagrangian (6.6) to the hamiltonian, we first define the canonical momenta conjugate to $A^\dagger$:

$$\Pi \equiv 2\frac{\delta L}{\delta \dot{A}^\dagger} = \lambda_1 \dot{A} + \tfrac{1}{2}\lambda_2 (A\tau_3)\mathrm{Tr}(\tau_3 A^\dagger \dot{A}), \tag{6.15}$$

implying the identity

$$\dot{A} = \frac{1}{\lambda_1}\left[\Pi - \tfrac{1}{2}\frac{\lambda_2}{\lambda_1+\lambda_2} A\tau_3 \mathrm{Tr}(\tau_3 A^\dagger \Pi)\right]. \tag{6.16}$$

The hamiltonian is therefore

$$\begin{aligned} H &\equiv -L + \tfrac{1}{2}\mathrm{Tr}(\dot{A}^\dagger \Pi) \\ &= M + \frac{1}{4\lambda_1}\mathrm{Tr}(\Pi^\dagger \Pi) + \frac{\lambda_2}{8\lambda_1(\lambda_1+\lambda_2)}\left[\mathrm{Tr}(\tau_3 A^\dagger \Pi)\right]^2, \end{aligned} \tag{6.17}$$

where $\Pi^\dagger = -A^\dagger \Pi A^\dagger$. Since the dynamical variables $A^\dagger$ obey constraints, naive commutation relations cannot be used. A discussion of the canonical quantization of this system, following the method of Dirac [23], is provided in appendix C.

The generators (6.13) can be re-expressed in terms of the canonical momenta by means of (6.16):

$$\begin{aligned} J_a &= \tfrac{1}{4} i \mathrm{Tr}(\tau_a \Pi A^\dagger) \\ I_b &= \tfrac{1}{4} i \mathrm{Tr}(\tau_b A^\dagger \Pi). \end{aligned} \tag{6.18}$$

Using the fundamental commutation relations presented in appendix C, one can verify that these generators form an $\mathrm{SU}(2)_J \times \mathrm{SU}(2)_I$ algebra.

The hamiltonian (6.17) can be expressed simply in terms of these generators as

$$\begin{aligned} H &= M + \frac{2}{\lambda_1}\left\{ J^2 - \frac{\lambda_2}{\lambda_1+\lambda_2} I_3^2 \right\} \\ &= M + 2\left\{ \frac{1}{\lambda_1}(I_1^2 + I_2^2) + \frac{1}{\lambda_1+\lambda_2} I_3^2 \right\}. \end{aligned} \tag{6.19}$$

For the special case $\lambda_1 + \lambda_2 = 0$, eq. (6.13) implies

$$I_3 = 0, \tag{6.20a}$$

and correspondingly,

$$H = M + \frac{2}{\lambda_1} J^2 . \tag{6.20b}$$

The hamiltonian (6.19) describes a system familiar from elementary quantum mechanics: that of a "symmetrical" top having one symmetry axis (i.e., two equal moments of inertia). This is seen [2] by identifying J_a as components of the angular momentum in the space-fixed frame, and I_b (which are related to J_a by (6.14)), as the corresponding components in the body-fixed frame. Energy eigenstates are labeled by quantum numbers $j\ (=i)$, j_3, i_3; energy levels have either a $(2j+1)$-fold degeneracy ($i_3 = 0$) or a $(4j+2)$-fold degeneracy ($i_3 \neq 0$). Depending on whether the top is quantized as a boson or fermion, j assumes integer or half-integer values, respectively.

In particular, the case $\lambda_2 = 0$ in (6.19) corresponds to a "spherical" (isotropic) top; or equivalently, a point particle constrained to move on a sphere S^3. The spectrum of this model is given by the spherical harmonics on S^3, as discussed in [1–3].

Similarly, the quantum mechanical system related to (6.20) is an infinitely thin rigid rod, having zero moment of inertia about the symmetry axis; or, equivalently, a point particle constrained to move on a sphere S^2. (See also (6.12).) As is well known, its spectrum is the set of spherical harmonics on S^2.

Our collective coordinate analysis for solitons of the CP^1 model gives $\lambda_1 + \lambda_2 = (4\pi/e^3 f_\pi)6.25 \neq 0$, corresponding to the system (6.19). However, it is clear that the values of λ_1, λ_2 are sensitive to the specific classical soliton field configuration used in the analysis. As already stressed, in the present study, our choice of field configuration was not optimal. It is possible that a more careful analysis of CP^1 solitons could lead to $\lambda_1 + \lambda_2 = 0$, and hence, the system (6.20).

To make this conjecture plausible, consider again the family of models consisting of (6.1) and an added gauge-breaking term (6.8) parametrized by γ, such that $\gamma = 1$ corresponds to the Skyrme model. One expects, for $\gamma \neq 0$, that this set of models describes very similar physics. Indeed, it is clear that these models all lead to the collective coordinate hamiltonian (6.19), with values of λ_1, λ_2 depending on the choice of γ. Hence, as γ is varied continuously, the semi-classical soliton spectrum also changes in a continuous fashion. Only for $\gamma = 0$ could one expect a discontinuity in the spectrum, which in a more accurate calculation could correspond to the case (6.20). In other words, one suspects that the value $\gamma = 0$, which is singular for the above class of field theoretic models, corresponds to the value $\lambda_1 + \lambda_2 = 0$, which is singular for the class of collective coordinate hamiltonians (6.19). A further argument based on supersymmetry is provided in sect. 7.

Since $\Pi_4(SU(2)) = \Pi_4(CP^1) = Z_2$, the solitons can be quantized as either bosons or fermions for arbitrary values of γ. The choice cannot be dictated by adding a

Wess-Zumino term [6] to the action, since such a term vanishes identically for SU(2), and hence, also for CP^1. (See however [24].)

7. Supersymmetric solitons

In the previous section, a collective coordinate analysis for CP^1 solitons was performed, under the assumption of classical soliton stability. It was shown that the collective coordinate lagrangian has the form (6.6)

$$L = -M + \tfrac{1}{2}\lambda_1 \dot{\bar{A}}^i \dot{A}_i - \tfrac{1}{2}\lambda_2 \left(\bar{A}^i \dot{A}_i\right)^2, \qquad \bar{A}^i A_i = 1, \tag{7.1}$$

where the variables $A_i(t)$ are related to the SU(2) matrix of collective coordinates $A(t)$ in analogy with (3.7). (See also (7.15).) Explicit calculation with a trial classical soliton configuration gives $\lambda_1 + \lambda_2 \neq 0$. However, it was suggested that a more precise treatment could yield $\lambda_1 + \lambda_2 = 0$, corresponding to a semi-classical soliton spectrum given by the spherical harmonics on S^2.

We now proceed to the supersymmetric case. Naturally, one expects the soliton states to form linear massive irreducible representations of $N = 1$ supersymmetry in four dimensions ($d = 4$). Such representations are in fact representations of the little subalgebra which leaves static states invariant. (See e.g. [25].) Included in this subalgebra are the generators Q_i and $\bar{Q}^j$, satisfying

$$\left\{Q_i, \bar{Q}^j\right\} = P_0 \delta_i^j, \quad i, j = 1, 2. \tag{7.2}$$

However, this is exactly the algebra of $N = 4$ supersymmetric quantum mechanics [26]*. Hence, one might expect that the collective coordinate lagrangian describing supersymmetric skyrmions includes the $N = 4$ supersymmetric extension of (7.1) (with possibly different values for M, λ_1, λ_2 and also higher-derivative terms). We shall first construct this model, deferring until later the discussion of how such a quantum mechanical model might be derived directly from a collective coordinate analysis of the original supersymmetric field theory.

Supersymmetric quantum mechanical systems can be conveniently described using superfields. The superfield formalism for supersymmetric quantum mechanics (i.e. $d = 1$ field theory) is quite similar to the more familiar formalism for $d = 4$ field theory [14]. A superspace is introduced with one bosonic time coordinate t and, for $N = 4$, two complex fermionic coordinates θ^i ($i = 1, 2$)**. Supersymmetry transfor-

* Some authors refer to the algebra (7.2) as $N = 2$ supersymmetric quantum mechanics.

** We use the convention $(\theta^i)^* \equiv \bar{\theta}_i$.

mations are realized on these coordinates as follows:

$$\theta'^i = \theta^i - i\bar{\alpha}^i,$$

$$t' = t + \tfrac{1}{2}\left(\bar{\alpha}^i\bar{\theta}_i - \alpha_i\theta^i\right),$$

where α_i are two complex spinor parameters. The supersymmetry generators Q_i and spinor derivatives D_i that anticommute with the Q_i may be represented as differential operators in this superspace:

$$Q_i = i\frac{\partial}{\partial\theta^i} - \tfrac{1}{2}\bar{\theta}_i\frac{\mathrm{d}}{\mathrm{d}t},$$

$$D_i = \frac{\partial}{\partial\theta^i} - \tfrac{1}{2}i\bar{\theta}_i\frac{\mathrm{d}}{\mathrm{d}t}. \tag{7.3}$$

They satisfy the anticommutation relations

$$\left\{Q_i, \bar{Q}^j\right\} = -\left\{D_i, \bar{D}^j\right\} = i\delta_i^j\frac{\mathrm{d}}{\mathrm{d}t}, \tag{7.4}$$

and the remaining anticommutators are zero.

One can distinguish two types of superfields, subject to different constraints. Chiral superfields satisfy the condition

$$\bar{D}^i\Phi(t,\theta,\bar{\theta}) = 0. \tag{7.5}$$

In an appropriate frame this condition implies that Φ depends only on the spinor coordinates θ:

$$\Phi = A + \theta^i\psi_i + \tfrac{1}{2}i\varepsilon_{ij}\theta^i\theta^j F. \tag{7.6}$$

Real superfields satisfy the reality condition $\Phi^* = \Phi$. The θ-expansion of such a superfield runs up to $\theta^2\bar{\theta}^2$.

Consider the collective coordinates A_i. These coordinates should be the first components of some superfield Φ_i. A minimum number of additional components is introduced by taking Φ_i to be chiral. Since the A_i satisfy the constraint $\bar{A}^iA_i = 1$, Φ_i must correspondingly obey $\bar{\Phi}^i\Phi_i = 1$. As in the four-dimensional field theory, this can be done consistently only by introducing a U(1) gauge invariance and taking Φ_i to be *covariantly* chiral:

$$\Phi_i = \left(A_i, \psi_{ji}, F_i\right)$$

with

$$\bar{\nabla}^i\Phi_j = 0, \qquad \bar{\Phi}^i\Phi_i = 1. \tag{7.7}$$

Here ∇_i is a gauge covariant spinor derivative which is analogous to the $d=4$ operator ∇_α. In particular, the U(1) gauge field V is the first component of a real $N=4$ vector multiplet (V, q^i_j, λ_i, D), where q^i_j is hermitian, traceless, λ_i is complex and D is real.

The above chiral and vector multiplets are precisely those obtained by "trivial" dimensional reduction of corresponding $d=4$ scalar and vector multiplets to $d=1$. That is, one takes the fields of the $d=4$ model to depend only on time, and makes the identifications

$$\{A_i, \psi_{\alpha i}, F_i\} \to \{A_i, \psi_{ji}, F_i\},$$

$$\{V_\mu, \lambda_\alpha, D\} \to \left\{\left(-2V, (\tau_a)^i{}_j q^j_i\right), \varepsilon_{ij}\bar{\lambda}^j, D\right\}. \tag{7.8}$$

The transformation rules for the $d=1$ scalar and vector multiplets therefore follow straightforwardly from the $d=4$ results (3.16):

$$\delta A_i = i\bar{\alpha}^j\psi_{ji},$$

$$\delta\psi_{ji} = -\alpha_j D_0 A_i + \varepsilon_{jk}\bar{\alpha}^k F_i - i\alpha_k q^k_j A_i,$$

$$\delta F_i = i\varepsilon^{kl}\alpha_k D_0\psi_{li} - \varepsilon^{kl}\alpha_m q^m_k\psi_{li} - \varepsilon^{kl}A_i\alpha_k\lambda_l, \tag{7.9}$$

$$\delta V = -\tfrac{1}{2}i\left(\bar{\alpha}^i\lambda_i + \alpha_i\bar{\lambda}^i\right),$$

$$\delta q^i_j = -i\left[\left(\bar{\alpha}^i\lambda_j + \alpha_j\bar{\lambda}^i\right) - \text{trace}\right],$$

$$\delta\lambda_i = \alpha_k\dot{q}^k_i - i\alpha_i D,$$

$$\delta D = \tfrac{1}{2}\left(\bar{\alpha}^i\dot{\lambda}_i - \alpha_i\dot{\bar{\lambda}}^i\right), \tag{7.10}$$

respectively. In the above, covariant time derivatives are defined as follows:

$$D_0 A_i = \dot{A}_i - iVA_i,$$

$$D_0\psi_{ji} = \dot{\psi}_{ji} - iV\psi_{ji}.$$

Moreover, the lagrangian reads (cf. (3.14))

$$L = D_0\bar{A}^i D_0 A_i + i\bar{\psi}^{ij}D_0\psi_{ij} + \bar{F}^i F_i$$

$$-\tfrac{1}{2}q^i_j q^j_i \bar{A}^k A_k + q^i_j\bar{\psi}^{jk}\psi_{ik}$$

$$-\bar{A}^j\bar{\lambda}^i\psi_{ij} + A_j\lambda_i\bar{\psi}^{ij} + D\left(\bar{A}^i A_i - 1\right). \tag{7.11}$$

One can verify that this model has an SU(2) $\times$ SU(2) invariance; spatial rotations in $d=4$ are realized on the collective coordinates as the diagonal subgroup.

As in $d=4$, the constraints on the components (A_i, ψ_{ji}, F_i) follow from the field equations of the vector multiplet, which serves as a set of Lagrange multipliers. These are

$$\begin{aligned}
&\bar{A}^i A_i = 1\,, \qquad \bar{A}^i \psi_{ji} = 0\,, \qquad \bar{A}^i F_i = 0\,, \\
&\quad V = -i\bar{A}^i \dot{A}_i - \tfrac{1}{2}\bar{\psi}^{ij}\psi_{ij}\,, \qquad q^i_j = \bar{\psi}^{ik}\psi_{jk} - \text{trace}\,, \\
&\quad \lambda_i = -i\varepsilon_{ij} F_k \bar{\psi}^{jk} - i\left(D_0 \bar{A}^j\right)\psi_{ij} + q_i^j \bar{A}^k \psi_{jk}\,, \\
&\quad D = -D_0 \bar{A}^i D_0 A_i - \tfrac{1}{2} i \bar{\psi}^{ij} \overleftrightarrow{D}_0 \psi_{ij} - \bar{F}^i F_i - \tfrac{1}{2}\left(\bar{\psi}^{ik}\psi_{jk}\right)\left(\bar{\psi}^{jl}\psi_{il}\right) + \tfrac{1}{4}\left(\bar{\psi}^{ij}\psi_{ij}\right)^2.
\end{aligned} \tag{7.12}$$

We note that with these constraints, the spinors ψ_{ij} can be replaced by four unconstrained spinor parameters ε_i in the following way:

$$\psi_{ij} = \varepsilon_{ik}\bar{A}^k \varepsilon_j \quad \text{or} \quad \varepsilon_i = -\varepsilon^{jk} A_j \psi_{ik}\,. \tag{7.13}$$

We have seen that an $N=4$ supersymmetric extension of (7.1) requires U(1) gauge invariance, and hence $\lambda_1 + \lambda_2 = 0$, [cf. (6.9]. We have verified that a supersymmetric extension of (7.1) with arbitrary values of λ_1, λ_2 does exist only for $N=1$. However, in that model $[Q, J] = 0$, which leads to an unacceptable spectrum.

Our approach to supersymmetric skyrmions has been to seek the supersymmetric extension of the CP^1 collective coordinate lagrangian (7.1), and this has led to the model (7.11). (One is free to add the constant $-M$ to (7.11); correspondingly, the ground state has mass M.) It would be satisfying to obtain this result directly from the $d=4$ supersymmetric field theory (3.14), (3.30) and to identify the spinor coordinates as collective coordinates for supersymmetry rotations. Indeed, as the classical soliton ansatz (4.1) is not invariant under supersymmetry, one expects that the supersymmetry of the semi-classical states can be "restored" by introducing corresponding collective coordinates, as was done for rotational and internal excitations. We now discuss this possibility.

Generalizing the procedure of sect. 6, we introduce collective coordinates by performing combined spatial/internal and supersymmetry rotations on the soliton ansatz (4.1), with corresponding time-dependent parameters $A_i(t)$ and $\varepsilon_\alpha(t)$, respectively. That is, we introduce a rotated superfield Φ given by*

$$\Phi = \exp\left[i\left\{\eta_a(t) G_a + \varepsilon^\alpha(t) Q_\alpha + \bar{\varepsilon}^{\dot{\alpha}}(t)\bar{Q}_{\dot{\alpha}}\right\}\right]\Phi_0(x)\,, \tag{7.14}$$

* In (7.14), it is understood that the generators do not act on the parameters. Also, as before, we do not introduce collective coordinates for translations, since we consider only static solitons. However, it is not clear whether for supersymmetry this procedure is entirely consistent. (See below.)

where $\Phi_0(x)$ is the soliton ansatz $(A_i(x), \psi_{\alpha i}=0, F_i=0)$, G_a are the operators (J_1, J_2, J_3-I_3), and the parameters $\eta_a(t)$ are related to the collective coordinates $A_i(t)$ according to

$$A(t)=\mathrm{e}^{i\eta_a(t)\tau_a/2}, \qquad A=\begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}. \tag{7.15}$$

Finite supersymmetry transformations are presented in appendix D. In particular, finite supersymmetry transformations of the ansatz lead to equivalent ansätze, with nonzero $\psi_{\alpha i}$ and F_i.

To obtain the effective lagrangian for the collective motions, one must substitute the rotated ansatz (7.14) into the supersymmetric field theory (3.14), (3.29) and perform the spatial integrations. However, the resulting expression does not resemble the quantum mechanical model (7.11). It is not difficult to see why in this approach the collective coordinate lagrangian is a priori not invariant under supersymmetry transformations of the form (7.9), (7.10). The basic point is that the supersymmetry transformations of the original field theory cannot be realized as conventional supersymmetry transformations on the collective coordinates introduced above.

To illustrate this point, we recall from sect. 6 how spatial rotations of the bosonic CP^1 field theory are realized on the collective coordinates. (For simplicity, we ignore isospin rotations.) In this case, the collective coordinates can be introduced by performing a rotation on the ansatz U_0:

$$U=\mathrm{e}^{i\boldsymbol{\eta}(t)\cdot \boldsymbol{J}}U_0(x), \tag{7.16}$$

where $\boldsymbol{\eta}(t)$ are related to the SU(2) matrix of collective coordinates $A(t)$ by (7.15). Under a global space rotation with parameter $\boldsymbol{\omega}$, U becomes

$$U'=\mathrm{e}^{i\boldsymbol{\omega}\cdot \boldsymbol{J}}\mathrm{e}^{i\boldsymbol{\eta}(t)\cdot \boldsymbol{J}}U_0(x)\equiv \mathrm{e}^{i\boldsymbol{\eta}'(t)\cdot \boldsymbol{J}}U_0(x). \tag{7.17}$$

By the Baker-Campbell-Hausdorff identity,

$$\boldsymbol{\eta}'(t)=\boldsymbol{\eta}(t)+\boldsymbol{\omega}-\tfrac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\eta}(t)+\cdots. \tag{7.18}$$

Equivalently, defining $B\equiv \mathrm{e}^{i\boldsymbol{\omega}\cdot\boldsymbol{\tau}/2}$, we again see that a spatial rotation on U can be represented on the collective coordinates by the transformation

$$A'=BA. \tag{7.19}$$

Since the field theory is invariant under rotations of U, the collective coordinate lagrangian is guaranteed to be invariant under the corresponding transformation (7.19).

Now let us return to the supersymmetric case. Under a supersymmetry transformation with parameter δ, the field Φ (7.14) becomes

$$\Phi' = \exp\left[i\left(\delta^{\alpha} Q_{\alpha} + \bar{\delta}^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}}\right)\right] \exp\left[i\left\{\eta_a(t) G_a + \varepsilon^{\alpha}(t) Q_{\alpha} + \bar{\varepsilon}^{\dot{\alpha}}(t) \overline{Q}_{\dot{\alpha}}\right\}\right] \Phi_0(x). \tag{7.20}$$

We are immediately faced with two problems. First, since $\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\} = (\sigma^{\mu})_{\alpha\dot{\alpha}} P_{\mu}$, it would seem that a new collective coordinate for translations is required in order to represent the supersymmetry transformation on the collective coordinates. One would expect that such translational collective coordinates could be avoided in a study of static solitons. Secondly, if the operators $Q_{\alpha}, \overline{Q}_{\dot{\alpha}}$ are assumed not to act on the collective coordinates themselves, then the transformations induced on the collective coordinates by (7.20) do not contain time derivatives*. As such, these are not conventional supersymmetry transformations. Similar transformations have been considered in recent studies [27] of quantum fluctuations about supersymmetric instantons. It is under such unconventional transformations that the collective coordinate lagrangian is guaranteed to be invariant. In particular, the collective coordinate lagrangian is not of the form (7.11), which is invariant under the conventional supersymmetry transformation rules (7.9), (7.10). Hence, it is not apparent that the semi-classical soliton spectrum corresponds to linear massive representations of $N=1$ supersymmetry in $d=4$.

The second difficulty would be resolved if the collective coordinate lagrangian could be recast in the form (7.11). Indeed, it has been shown [28] that a related quantum-mechanical model invariant under such unconventional transformations can be reformulated so as to be invariant under conventional supersymmetry transformations. This is achieved by performing suitable redefinitions (involving time derivatives) of the variables. It remains to be investigated whether such a reformulation is possible for our model.

We have seen that a straightforward generalization of the collective coordinate method may, in principle, be applied to supersymmetric solitons. However, this procedure is unsatisfactory, as the collective coordinate lagrangian is not manifestly invariant under conventional supersymmetry transformations. It would be interesting to see if this procedure could be improved.

8. Discussion

We have investigated solitons of the four-dimensional CP^1 model and its supersymmetric extension, focusing on two main issues: the classical stability of soliton

* Allowing the supersymmetry generators to act on the collective coordinates is also not satisfactory, since then the algebra does not close.

solutions, and the corresponding semi-classical quantization. In this section we briefly discuss our results, emphasizing new problems that were raised.

We sought supersymmetric higher-derivative terms that could be added to the nonlinear sigma model action to stabilize its soliton solutions, and found two such candidates. Surprisingly, one of these terms cannot alone provide stability. Moreover, both terms contain quartic time derivatives and lead to actions that are not bounded from below. Presumably, these higher-derivative terms can be treated as perturbations, as is typically done for effective nonlinear models. Still, one should verify that these terms lead to a consistent low-energy dynamics.

Since it is technically cumbersome to work with the soliton ansatz that is appropriate for the CP^1 model, all of our calculations were performed with a simplified ansatz. As a result, our stability analysis is incomplete. In view of the absence of a lower bound for the soliton mass, it is now important that soliton solutions based on the more general ansatz be studied.

Quite generally, the static semi-classical spectrum of a soliton consists of towers of rotational and internal excitations, corresponding to symmetries broken by the classical soliton solution. Our results for the CP^1 soliton spectrum are best understood by first considering a wider class of models: namely, the one-parameter (γ) family of nonlinear models, with $\gamma = 0$ corresponding to the CP^1 model, and $\gamma = 1$ corresponding to the usual Skyrme model. The soliton spectrum of the Skyrme model ($\gamma = 1$) coincides with that of a quantized spherical top. We find that solitons of models with $\gamma \neq 1$ in general have the spectra of symmetrical tops. In particular, one expects that the CP^1 case ($\gamma = 0$) should be special, corresponding to an infinitely thin rigid rod, having zero moment of inertia about the symmetry axis. This expectation is not borne out by our explicit collective coordinate calculation. Clearly this analysis should be repeated using a soliton solution based on the more general ansatz, mentioned above.

We see that, in our application of the collective coordinate method, qualitative aspects of the spectrum can depend sensitively on features of the soliton solution other than its symmetry properties. However, we cannot rule out a different approach in which all qualitative features of the spectrum emerge more directly.

Using a straightforward generalization of the collective coordinate method to the supersymmetric CP^1 model leads to a collective coordinate lagrangian which, a priori, is not invariant under conventional supersymmetry. We find this situation rather unsatisfactory. Collective coordinates should instead be introduced in such a way that the supersymmetry of the field theory is realized as a conventional supersymmetry on these coordinates. This possibility deserves further attention.

Our search for four-dimensional supersymmetric nonlinear sigma models admitting topological solitons was not exhaustive; thus, CP^1 may not be the only such model. It would be interesting to see whether these models indeed correspond to the low-energy limit of renormalizable supersymmetric field theories, such as supersymmetric Yang-Mills.

Note added in proof

It has recently been shown by Moore and Nelson [33] that certain nonlinear sigma models with fermions have gauge anomalies which arise from (nonpropagating) composite gauge fields. Models with this difficulty include the four-dimensional supersymmetric Grassmann models. In particular, the supersymmetric CP^n models have abelian anomalies proportional to $F_{\mu\nu}\tilde{F}_{\mu\nu}$. However, since $F_{\mu\nu}\tilde{F}_{\mu\nu} \equiv 0$ in CP^1 (see text after eq. (4.5)), supersymmetric CP^1 theory is well-defined.

In related work Nemeschansky and Rohm [34] have argued that a four-dimensional supersymmetric Wess-Zumino term is possible only on a noncompact Kähler manifold. This result is consistent with our work since we also do not find a Wess-Zumino term in the supersymmetric CP^1 model.

One of us (H.J.S.) wishes to thank the Department of Physics of Harvard University for its hospitality and the Guggenheim Foundation for their support during 1983–84. He is indebted to Professor L. Alvarez-Gaumé for several conversations during the course of this work. For one of us (E.A.B.) this work is part of the research program of the Netherlands Organization for the Advancement of Pure Research (ZWO). Two of us (E.A.B. and R.I.N.) acknowledge useful discussions with Professors M. Grisaru and D. Zanon. We are grateful to Professor C. Nappi for making available the computer program used in ref. [3].

Appendix A

HOMOTOPY GROUPS OF GRASSMANN MANIFOLDS

Here we compute Π_3 for the compact complex Grassmann manifold $G(k,n) \equiv U(n)/U(k) \times U(n-k)$, $n > k$. We distinguish two cases:

Case (i): $n > k+1$.

For this case, we observe that

$$G(k,n) = V(k,n)/U(k), \tag{A.1}$$

where $V(k,n) \equiv U(n)/U(n-k)$ is the so-called complex Stiefel manifold (see e.g. [29]). Correspondingly, we have the homotopy exact sequence*

$$\to \Pi_3(U(k)) \to \Pi_3(V(k,n)) \to \Pi_3(G(k,n)) \to \Pi_2(U(k)) \to \Pi_2(V(k,n)) \to . \tag{A.2}$$

* We remind the reader that a sequence

$$\to A \xrightarrow{f} B \xrightarrow{g} C \to$$

is exact if $\operatorname{Im} f = \operatorname{Ker} g$. For more details, see e.g. [30].

Now we need the following.

Lemma:

$$\Pi_m(\mathrm{V}(k,n)) = 0, \qquad m < 2(n-k)+1. \tag{A.3}$$

Proof: (following e.g. [29]). Observe that there is a natural imbedding i: $\mathrm{V}(k,n) \to \mathrm{V}(k+1,n+1)$ and also a projection p: $\mathrm{V}(k+1,n+1) \to \mathrm{V}(1,n+1) = \mathrm{S}^{2n+1}$ These maps induce the exact sequence

$$\Pi_{m+1}(\mathrm{S}^{2n+1}) \to \Pi_m(\mathrm{V}(k,n)) \to \Pi_m(\mathrm{V}(k+1,n+1) \to \Pi_m(\mathrm{S}^{2n+1}). \tag{A.4}$$

Since $\Pi_{m+1}(\mathrm{S}^{2n+1}) = 0$ for $m < 2n$, we obtain the recurrence relation

$$\Pi_m(\mathrm{V}(k,n)) = \Pi_m(\mathrm{V}(k+1,n+1)), \qquad m < 2n. \tag{A.5}$$

It is a fact that

$$\Pi_m(\mathrm{V}(1,n)) = \Pi_m(\mathrm{S}^{2n-1}) = 0, \qquad m < 2n-1.$$

Hence, by repeated application of (A.5), we finally obtain

$$0 = \Pi_m(\mathrm{V}(1,n-k+1)) = \Pi_m(\mathrm{V}(k,n)), \qquad m < 2(n-k+1)-1.$$

This proves the lemma.

Now, since $n > k+1$, by the lemma we just proved,

$$\Pi_3(\mathrm{V}(k,n)) = 0 = \Pi_2(\mathrm{V}(k,n)), \qquad n > k+1. \tag{A.6}$$

Hence, from the exact sequence (A.2), we immediately conclude

$$\Pi_3(\mathrm{G}(k,n)) = \Pi_2(\mathrm{U}(k)) = 0, \qquad n > k+1, \qquad k \geqslant 1. \tag{A.7}$$

That is, all compact complex Grassmann manifolds, with the possible exception of $\mathrm{G}(k,k+1)$ (which we consider below) have trivial Π_3.

Case (ii): $n = k+1$.

In this case, we write

$$\mathrm{G}(k,n) = \mathrm{G}(k,k+1) = \mathrm{CP}^k = \mathrm{S}^{2k+1}/\mathrm{U}(1). \tag{A.8}$$

Consider the corresponding exact sequence

$$\to \Pi_3(\mathrm{U}(1)) \to \Pi_3(\mathrm{S}^{2k+1}) \to \Pi_3(\mathrm{CP}^k) \to \Pi_2(\mathrm{U}(1)) \to . \tag{A.9}$$

Recalling that $\Pi_m(\mathrm{U}(1))=0$ for $m \geqslant 2$, one sees

$$\Pi_3(\mathrm{CP}^k)=\Pi_3(\mathrm{S}^{2k+1})=\begin{cases} Z, & k=1 \\ 0, & k>1 \end{cases}. \tag{A.10}$$

Hence, CP^1 is the only compact complex Grassmann manifold with nontrivial Π_3.

Appendix B

MAURER-CARTAN EQUATIONS

Consider the following SU(2) matrix:

$$U=\begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}$$

with

$$\bar{A}^i A_i \equiv A_1^* A_1 + A_2^* A_2 = 1 . \tag{B.1}$$

The matrix $iU^\dagger \partial_\mu U$ is an element of the Lie algebra of SU(2) and

$$iU^\dagger \partial_\mu U=\begin{pmatrix} -V_\mu & B_\mu^* \\ B_\mu & V_\mu \end{pmatrix}, \tag{B.2}$$

with V_μ and B_μ given by

$$V_\mu = -\tfrac{1}{2} i \bar{A}^i \overleftrightarrow{\partial}_\mu A_i ,$$

$$B_\mu = i\varepsilon^{ij} A_i \partial_\mu A_j . \tag{B.3}$$

The Maurer-Cartan equations follow from the observation that $iU^\dagger\partial_\mu U$ has the form of a trivial SU(2) gauge field, whose curvature vanishes:

$$\partial_{[\mu}\left(U^\dagger i \partial_{\nu]} U\right) - i\left(U^\dagger i \partial_{[\mu} U\right)\left(U^\dagger i \partial_{\nu]} U\right) = 0 . \tag{B.4}$$

Substituting (B.2) into (B.4) then leads to the equations

$$F_{\mu\nu}(V) \equiv \partial_{[\mu} V_{\nu]} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu = -iB^*_{[\mu} B_{\nu]} ,$$

$$F_{\mu\nu}(B) \equiv \left(\partial_{[\mu} - 2iV_{[\mu}\right) B_{\nu]} = 0 . \tag{B.5}$$

For a more geometrical interpretation of the above formulae we refer to [31]. Furthermore, note that the $U(1)_R$-covariant matrix $iU^\dagger D_\mu U$, with V_μ given by (B.3), takes the following form:

$$iU^\dagger D_\mu U = \begin{pmatrix} 0 & B_\mu^* \\ B_\mu & 0 \end{pmatrix}. \tag{B.6}$$

Finally, we record some useful identities:

$$D_\mu A_j = i\varepsilon_{jk}\bar{A}^k B_\mu ,$$

$$(\tau_a)_i{}^j(\tau_a)_k{}^l = 2\delta_i{}^l\delta_k{}^j - \delta_i{}^j\delta_k{}^l. \tag{B.7}$$

Appendix C

CANONICAL QUANTIZATION

In this appendix we discuss the canonical quantization of the model, defined in (6.6) as

$$L = -M + \tfrac{1}{4}\lambda_1 \mathrm{Tr}(\dot{A}^\dagger \dot{A}) - \tfrac{1}{8}\lambda_2\left[\mathrm{Tr}(\tau_3 A^\dagger \dot{A})\right]^2. \tag{C.1}$$

It suffices to take the SU(2) matrix $A^\dagger$ as the set of dynamical variables; the hermitian conjugate matrix A is not another independent set, owing to the relation

$$A^\dagger_{\beta\alpha} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}A_{\gamma\delta}, \qquad \alpha, \beta = 1,2. \tag{C.2}$$

Clearly, the dynamical variables satisfy the constraint

$$\varphi_1 \equiv \det A^\dagger - 1 = 0. \tag{C.3}$$

A quantization procedure for systems obeying such constraints has been given by Dirac [23], which we follow. (Its application to nonlinear sigma models has been reviewed in [32].) As outlined in sect. 6, the canonical momenta $\Pi_{\alpha\beta}$ conjugate to $A^\dagger_{\beta\alpha}$ are

$$\Pi_{\alpha\beta} \equiv 2\frac{\delta L}{\delta \dot{A}^\dagger_{\beta\alpha}} = \lambda_1 \dot{A}_{\alpha\beta} + \tfrac{1}{2}\lambda_2 (A\tau_3)_{\alpha\beta}\mathrm{Tr}(\tau_3 A^\dagger \dot{A}), \tag{C.4}$$

and the hamiltonian is (6.17)

$$H = M + \frac{1}{4\lambda_1}\mathrm{Tr}(\Pi^\dagger \Pi) + \frac{\lambda_2}{8\lambda_1(\lambda_1+\lambda_2)}\left[\mathrm{Tr}(\tau_3 A^\dagger \Pi)\right]^2. \tag{C.5}$$

Next, define the naive (equal-time) Poisson brackets

$$\{\Pi_{\alpha\beta}, A^\dagger_{\gamma\delta}\} = 2\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{C.6}$$

and require that the constraint (C.3) be maintained in time; i.e., that it commutes with the hamiltonian. Using the identity

$$\{\Pi, \det A^\dagger\} = 2A, \tag{C.7}$$

one sees that this leads to the additional constraint

$$\varphi_2 \equiv \mathrm{Tr}(A^\dagger \Pi) = 0, \tag{C.8}$$

which is itself preserved in time.

It is clear that our model does not lead to further restrictions on the canonical variables, except for the special case $\lambda_1 + \lambda_2 = 0$. In that case, the lagrangian (C.1) is invariant under the gauge transformations

$$A(t) \to A(t)e^{i\omega(t)\tau_3}, \tag{C.9}$$

as discussed in the text (see (6.8) ff.). In Dirac's method this gauge invariance would lead to an additional constraint

$$\mathrm{Tr}(\tau_3 A^\dagger \Pi) = 0. \tag{C.10}$$

For general polynomial functions of the dynamical variables and conjugate momenta $F(A^\dagger, \Pi)$ and $G(A^\dagger, \Pi)$, Dirac brackets are defined by

$$\{F, G\}^* \equiv \{F, G\} - \{F, \varphi_i\} C^{-1}_{ij} \{\varphi_j, G\}, \tag{C.11}$$

where

$$C_{ij} \equiv \{\varphi_i, \varphi_j\}. \tag{C.12}$$

In our case C^{-1}_{ij} is the 2×2 matrix $\frac{1}{4}i(\tau_2)_{ij}$, which leads to the following Dirac brackets:

$$\begin{aligned} \{\Pi_{\alpha\beta}, A^\dagger_{\gamma\delta}\}^* &= 2\delta_{\alpha\delta}\delta_{\beta\gamma} - A_{\alpha\beta}A^\dagger_{\gamma\delta}, \\ \{\Pi_{\alpha\beta}, \Pi_{\gamma\delta}\}^* &= A_{\alpha\beta}\Pi_{\gamma\delta} - \Pi_{\alpha\beta}A_{\gamma\delta}, \\ \{A^\dagger_{\alpha\beta}, A^\dagger_{\gamma\delta}\}^* &= 0. \end{aligned} \tag{C.13}$$

One can verify that these brackets effectively remove the constraints

$$\{F(A^\dagger, \Pi), \varphi_i\}^* = 0. \tag{C.14}$$

The transition from classical to quantum theory is now made by replacing Dirac brackets by commutators.

Appendix D

FINITE SUPERSYMMETRY TRANSFORMATIONS

In this appendix we derive the finite supersymmetry transformations of a soliton ansatz $(A_i, \psi_{\alpha i} = 0, F_i = 0)$. The fields A_i, $\psi_{\alpha i}$, F_i are the components of a covariantly chiral superfield Φ_i. Under a finite supersymmetry transformation with (time-independent) parameter ε_α such a superfield transforms according to [14]:

$$\Phi'_i = \exp\left[i\left(\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}^{\dot\alpha}\overline{Q}_{\dot\alpha}\right)\right]\Phi_i. \tag{D.1}$$

The components of Φ_i can be defined in terms of covariant projections

$$\begin{aligned} A_i &= \Phi_i| ,\\ \psi_{\alpha i} &= \nabla_\alpha \Phi_i| ,\\ F_i &= \nabla^2 \Phi_i| , \end{aligned} \tag{D.2}$$

where $\Phi_i|$ means the superfield Φ_i evaluated at $\theta = 0$, and ∇_α is the covariant spinor derivative. For $\theta = 0$ we have the identity $Q_\alpha = i\nabla_\alpha$; thus,

$$\begin{aligned} A'_i &= \exp\left[-\left(\varepsilon^\beta \nabla_\beta + \bar{\varepsilon}^{\dot\beta}\overline{\nabla}_{\dot\beta}\right)\Phi_i\right]\Big| ,\\ \psi'_{\alpha i} &= \exp\left[-\left(\varepsilon^\beta \nabla_\beta + \bar{\varepsilon}^{\dot\beta}\overline{\nabla}_{\dot\beta}\right)\nabla_\alpha\Phi_i\right]\Big| ,\\ F_i' &= \exp\left[-\left(\varepsilon^\beta \nabla_\beta + \bar{\varepsilon}^{\dot\beta}\overline{\nabla}_{\dot\beta}\right)\nabla^2\Phi_i\right]\Big| . \end{aligned} \tag{D.3}$$

Using the basic commutation relations

$$\begin{aligned} \left\{\nabla_\alpha, \overline{\nabla}_{\dot\beta}\right\} &= i\nabla_{\alpha\dot\beta} ,\\ \left[\nabla_\alpha, \nabla_{\beta\dot\beta}\right] &= C_{\alpha\beta}\overline{W}_{\dot\beta} , \end{aligned} \tag{D.4}$$

and expanding the exponents in (D.3) one can now derive that the soliton ansatz

$A_i(x)$ is transformed into a field configuration $(A_i', \psi_{\alpha i}', F_i')$ given by

$$A_i' = A_i + \tfrac{1}{2}i\varepsilon^{\alpha}\bar{\varepsilon}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}A_i + \tfrac{1}{4}\varepsilon^2\bar{\varepsilon}^2(\Box A_i + 2DA_i),$$

$$\psi_{\alpha i}' = -i\bar{\varepsilon}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}A_i - \tfrac{1}{2}i\varepsilon^{\beta}\bar{\varepsilon}^2 f_{\alpha\beta}A_i - \tfrac{1}{2}\bar{\varepsilon}^2\varepsilon_{\alpha}(\Box A_i + 2DA_i),$$

$$F_i' = -\bar{\varepsilon}^2(\Box A_i + DA_i). \tag{D.5}$$

As usual, vector and spinor indices are related by

$$V_{\mu} = \tfrac{1}{2}(\sigma_{\mu})^{\alpha\dot{\alpha}}V_{\alpha\dot{\alpha}}, \quad V_{\alpha\dot{\alpha}} = (\sigma^{\mu})_{\alpha\dot{\alpha}}V_{\mu},$$

$$f_{\alpha\beta} = (\sigma^{\mu\nu}C)_{\alpha\beta}F_{\mu\nu}. \tag{D.6}$$

Also, it is understood that the dependent fields V_{μ} and D are given by

$$V_{\mu} = -\tfrac{1}{2}i\bar{A}^i\overleftrightarrow{\partial}_{\mu}A_i,$$

$$D = D^{\mu}\bar{A}^iD_{\mu}A_i, \tag{D.7}$$

and $\Box$ denotes the gauge covariant d'alembertian. In deriving (D.5) we have also used the fact that the untransformed field configuration is given by the set $(A_i, 0, 0)$ and hence $\nabla_{\alpha}\Phi_i| = \nabla^2\Phi_i| = 0$.

Finally we calculate the finite supersymmetry transformations of the vector multiplet with dependent components $(V_{\mu}, \lambda_{\alpha}, D)$ given in (3.18) with $\psi_{\alpha i} = F_i = 0$. Using the finite transformation rules of the scalar multiplet we find that

$$A_{\alpha\dot{\alpha}}' = A_{\alpha\dot{\alpha}} - \varepsilon_{\alpha}\bar{\varepsilon}_{\dot{\alpha}}D + \tfrac{1}{2}i\varepsilon_{\alpha}\bar{\varepsilon}^{\dot{\beta}}\bar{f}_{\dot{\alpha}\dot{\beta}}$$

$$- \tfrac{1}{2}i\varepsilon^{\beta}\bar{\varepsilon}_{\dot{\alpha}}f_{\alpha\beta} - \tfrac{1}{4}\varepsilon^2\bar{\varepsilon}^2\nabla_{\dot{\alpha}}{}^{\beta}f_{\alpha\beta},$$

$$\lambda_{\alpha}' = -\varepsilon^{\beta}f_{\beta\alpha} + i\varepsilon_{\alpha}D + \tfrac{1}{2}\varepsilon^2\bar{\varepsilon}^{\dot{\alpha}}\left(\nabla_{\alpha\dot{\alpha}}D + i\nabla_{\dot{\alpha}}{}^{\beta}f_{\alpha\beta}\right),$$

$$D' = D - \tfrac{1}{2}\varepsilon^{\beta}\bar{\varepsilon}^{\dot{\beta}}\nabla_{\dot{\beta}}{}^{\alpha}f_{\alpha\beta} - \tfrac{1}{4}\varepsilon^2\bar{\varepsilon}^2D. \tag{D.8}$$

References

[1] T.H.R. Skyrme, Proc. Roy. Soc. A260 (1961) 127; A262 (1961) 237; Nucl. Phys. 31 (1962) 556;
D. Finkelstein and J. Rubenstein, J. Math. Phys. 9 (1968) 1762;
N.K. Pak and H.C. Tze, Ann. of Phys. 117 (1979) 164

[2] E. Witten, Nucl. Phys. B160 (1979) 57; B223 (1983) 422, 433;
A.P. Balachandran et al., Phys. Rev. Lett. 49 (1982) 1124; Phys. Rev. D27 (1983) 1153
[3] G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552
[4] E. Guadagnini, Nucl. Phys. B236 (1984) 35
[5] H.J. Schnitzer, Phys. Lett. 139B (1984) 217
[6] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95
[7] M.T. Grisaru and H.J. Schnitzer, Nucl. Phys. B204 (1982) 267
[8] W. Buchmüller, R.D. Peccei and T. Yanagida, Phys. Lett. 124B (1983) 67;
O.W. Greenberg, R.N. Mohapatra and M. Yasue, Phys. Rev. Lett. 51 (1983) 1737;
Y.J. Ng and B.A. Ovrut, Phys. Lett. 125B (1983) 147;
J.C. Pati and A. Salam, Nucl. Phys. B234 (1984) 223
[9] R. Hobart, Proc. Phys. Soc. A82 (1963) 201;
G.H. Derrick, J. Math. Phys. 5 (1964) 1252
[10] S. Deser, M.J. Duff and C.J. Isham, Nucl. Phys. B114 (1976) 29
[11] A. d'Adda, R. Horsley and P. di Vecchia, Phys. Lett. 76B (1978) 298;
J. Hruby, Nucl. Phys. B162 (1980) 449;
H. Osborn, Phys. Lett. 83B (1979) 321;
F.A. Bais and W. Troost, Nucl. Phys. B178 (1981) 125;
M. Claudson and M.B. Wise, Nucl. Phys. B221 (1983) 461
[12] B. Zumino, Phys. Lett. 87B (1979) 203;
L. Alvarez-Gaumé and D.Z. Freedman, Comm. Math. Phys. 91 (1983) 87
[13] E. Cremmer and J. Scherk, Phys. Lett. 74B (1978) 341;
A. d'Adda, M. Lüscher and P. di Vecchia, Nucl. Phys. B146 (1978) 63; B152 (1979) 125;
E. Witten, Nucl. Phys. B149 (1979) 285;
I. Bars and M. Günaydin, Phys. Rev. D22 (1980) 1403
[14] S. Gates, M. Grisaru, M. Roček and W. Siegel, Superspace (Benjamin/Cummings, London, 1983)
[15] F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250;
S. Deser, R. Jackiw and S. Templeton, Ann. of Phys. 140 (1982) 372
[16] G. Woo, J. Math Phys. 18 (1977) 1756
[17] J.M. Gipson, Nucl. Phys. B231 (1984) 365
[18] L.D. Faddeev, in Nonlocal, nonlinear and nonrenormalizable field theories, Proc. Int. Symp., Alushta (Joint Inst. for Nuclear Research, Dubna, 1976);
S. Coleman, in New phenomena in subnuclear physics, Proc. 1975 Int. School of Physics 'Ettore Majorana', ed. A. Zichichi (Plenum, New York, 1975); Phys. Rev. D11 (1975) 2088
[19] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449
[20] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85
[21] R. Rajaraman, Solitons and instantons (North-Holland, Amsterdam, 1982)
[22] M. Bander and F. Hayot, Saclay preprint
[23] P.A.M. Dirac, Lectures on quantum mechanics (Yeshiva University, New York, 1964)
[24] E. D'Hoker and E. Farhi, Phys. Lett. 134B (1984) 86
[25] D.Z. Freedman, in Cargèse 1978, ed. M. Lévy and S. Deser (Plenum, New York, 1979).
[26] P. Di Vecchia and F. Ravndal, Phys. Lett. 73A (1979) 371;
E. Witten, Nucl. Phys. B188 (1981) 513;
M. de Crombrugghe and V. Rittenberg, Ann. of Phys. 151 (1983) 99;
D. Lancaster, Nuov. Cim. 79A (1984) 28;
E. D'Hoker and L. Vinet, Phys. Lett. 137B (1984) 72;
M. Claudson and M.B. Halpern, Berkeley preprint (1984)
[27] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B229 (1983) 394;
H. Bohr, E. Katznelson and K.S. Narain, Nucl. Phys. B238 (1984) 407;
I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B241 (1984) 493
[28] L. Brink and J.H. Schwarz, Phys. Lett. 100B (1981) 310;
J.H. Schwarz, Nucl. Phys. B185 (1981) 221;
W. Siegel, Phys. Lett. 128B (1983) 397
[29] S. Kobayashi and K. Nomizu, Foundations of differential geometry (Wiley Interscience, New York, 1963);

D. Husemoller, Fibre bundles (Springer, New York, 1975)
[30] P.J. Hilton, An introduction to homotopy theory (Cambridge Univ. Press, Cambridge, 1953); L. Castellani, L.J. Romans and N.P. Warner, Nucl. Phys. B241 (1984) 429
[31] B. de Wit, in Lectures at fourth adriatic meeting on Particle physics (1983), NIKHEF preprint
[32] A.C. Davis, A.J. MacFarlane and J.W. van Holten, Nucl. Phys. B232 (1984) 473
[33] G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519
[34] D. Nemeschansky and R. Rohm, Anomaly constraints on supersymmetric effective lagrangians, Princeton University preprint