SUPERMEMBRANES AND ELEVEN-DIMENSIONAL SUPERGRAVITY

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We construct an action for a supermembrane propagating in $d=11$ supergravity background. Using the constraints of $d=11$ curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and $d=11$ spacetime spinor. We also discuss the general problem of the coupling of $n$-dimensional extended objects to $d$-dimensional supergravity.

1. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu–Schwarz–Ramond formulation of the spinning string theory.


The generalization of the Hughes et al. model to $n$-extended objects propagating in flat $d$-dimensional superspace is evident. All that is required is the existence of a closer super $(n+2)$-form given by

$$H = E^a E^a E^a \ldots E^a (\gamma_{a_1 \ldots a_n})_{a\bar{a}},$$

where $(E^a, E^a)$ are the basis one-forms in superspace. This form is closed provided that the following $\Gamma$-matrix identity holds:

$$(\gamma^a)_{a\bar{a}} (\gamma_{a_2 \ldots a_{n+1}})_{a\bar{a}} = 0 .$$

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expect that the $n$-extended objects under consideration can consistently propagate only in $d \leq 11$ supergravities whose superspace formulation involves a closed $(n+2)$-form. We further expect that such forms exist in supergravity theories in which a closed bosonic $(n+2)$-form occurs. As far as we know, the following possibilities exist (we include the Yang-Mills couplings whenever possible):

The dual formulation of $d=10$, $N=1$ supergravity involves a closed seven-form. Its dimensional reduction on a $(10-d)$-dimensional torus leads to real closed $(d-3)$-forms in $d$-dimensional supergravities. (These are $N=1$ supergravities in $d=8$, $9$, $10$; $N=2$ in $d=7$ and $N=2$ or $4$ in $d=6$) [9]. Apart from these, there is: (i) A real closed four-form in $d=11$, $N=1$ supergravity, (ii) a real closed three-form in non-chiral $d=10$, $N=2$ supergravity, (iii) a complex closed three-form in chiral $d=10$, $N=2$ supergravity.

Excluding Yang-Mills couplings, as is well known, closed super three-forms exist in $d=3, 4, 6$, and $10$.

Considering the case of the membranes, from the above list it follows that the candidate dimensions are $7$ and $11$. Since the superspace formulation of $d=7$, $N=2$ supergravity is not known at present, we are led to consider the supermembrane propagating in eleven-dimensional spacetime.

Our main result is the construction of an action which describes a consistent coupling of $d=11$ supergravity to a supermembrane. In particular the Kalb-Ramond-like third rank antisymmetric tensor field of $d=11$ supergravity couples to the supermembrane via a Wess-Zumino term.

In the following we focus our attention on the description of the supermembrane action in $d=11$. The extension to the case of $n$-extended objects is given in the appendix.

2. We propose the following action for a supermembrane coupled to $d=11$ supergravity:

$$S = \int \sqrt{-g} \left( \frac{1}{2} \sqrt{-g} g^{ij} E_i^a E_j^b \eta_{ab} + \epsilon^{ijk} E_i^a E_j^b E_k^c B_{cba} - \frac{1}{2} \sqrt{-g} \right).$$

(3)

Here $i=0, 1, 2$ labels the coordinates $\xi^i = (\tau, \sigma, \rho)$ of the world volume with metric $g_{ij}$ and signature $(-, +, +)$. The super three-form $B$ is needed for the superspace description of $d=11$ supergravity [10].

For the Levi-Civita symbol $\epsilon^{ijk}$ we use the same conventions as in ref. [11]. In (3) we have used the notation

$$E_i^a = (\partial_i Z^M) E_M^a,$$

(4)

where $Z^M(\xi)$ are the superspace coordinates, and $E_M^a(Z)$ is the supervielbein.

Note that the action has a cosmological constant with a fixed magnitude. This is so that the field equation of the metric $g_{ij}$ gives the embedding equation

$$g_{ij} = E_i^a E_j^b \eta_{ab} \equiv T_{ij}.$$

(5)

We require that the action $S$ is invariant under a fermionic gauge transformation of the form [5]

$$\delta E_i^a = (1 + \Gamma) \kappa^a,$$

$$\delta g_{ij} = 2 [X_{ij} - g_{ij} X_{kl} (n-1)]$$

$$+ (n=2 \text{ for membrane}),$$

(6)

where the transformation parameter $\kappa^a(\xi)$ is a 32 component Majorana spinor, and a world-volume scalar, and

$$\delta E_i^a = \delta Z^M E_M^a,$$

(7)

$$\Gamma_{\beta}^a = \frac{1}{8} \sqrt{-g} \gamma^{abc} (\gamma_{abc})_{\beta}^a \epsilon.$$ (8)

Here $\gamma^a (a=0, 1, \ldots, 10)$ are the Dirac matrices in eleven dimensions. $X_{ij}$ is a function of $E_i^a$ which will be determined by the invariance of the action. The choice of $\delta g_{ij}$ is due to the fact that, given a variation of the action of the form $\delta S = T_{ij} X^i$, and writing this variation as

$$T_{ij} X^i = g_{ij} X^i + (T_{ij} - g_{ij}) X^i,$$

(9)

the second term on the right-hand side cancels $\delta S / \delta g_{ij} \delta g_{ij}$. Thus we are left with the first term on the right-hand side, which equals the left-hand side upon the use of (5). Effectively, this means that whenever we encounter a variation of the form $T_{ij} X^i$, we can use eq. (5), provided that we add $X^i$ to $\delta g_{ij}$ as in (6).

The matrix $\Gamma_{\beta}^a$ occurring in (8) satisfies the property

$$\Gamma_{\beta}^a \Gamma_{\gamma}^a = (T^i T^j T^k T^l) \delta_{\gamma}^\alpha \equiv \Gamma^2 \delta_{\gamma}^\alpha.$$ (10)

The normalization in (8) is chosen such that upon the use of the equation $T_{ij} = g_{ij}$, the matrix $\Gamma^a_{\beta}$ satisfies the relation

$$\Gamma_{\beta}^a \Gamma_{\gamma}^a = \delta_{\gamma}^\alpha.$$ (10)
Now using (6) the variation of the action (3) is (we consider a closed supermembrane and therefore discard the surface terms)

\[ \delta S = \int d^2 \xi \left[ \sqrt{-g} g^{ij} \left( -\delta g^\beta E_i T^a \right) E_{ja} + \frac{1}{\sqrt{-g}} \left( \delta E^\beta E_i T^a \right) E_{ja} + \epsilon^{ijk} E_i A_j E_k E_{\beta} \right] \]

\[ - \frac{1}{2} \sqrt{-g} \left( \delta g^{ij} T_{ij} - \frac{1}{2} g^{ij} \right) \] (11)

The torsion two-form \( T^a \) and the four-form field strength \( H \) are defined by (our superspace conventions are those of Howe [12])

\[ T^a = dE^a + E^a \sigma^2 B A = \frac{1}{2} E^a E^c T_{CB} A, \]

\[ H = dB = \frac{1}{2} E^a E^c E^d H_{ABCD}. \] (12)

We now organize the terms in (11) according to the number of one-forms \( E^a \) they contain. Those with three \( E^a \) and two \( E^a \) come only from the Wess-Zumino term. They must vanish separately, and this requires the constraints

\[ H_{a \beta \gamma} = H_{a \beta \gamma\delta} = 0. \] (13)

The cancellation of the terms linear in \( E^a \) lead to the constraints

\[ T^a_{\alpha \beta} = (\gamma^a)_{\alpha \beta}, \] (14)

\[ H_{a \beta \alpha b} = - \frac{1}{2} (\gamma_{ab})_{\alpha \beta}, \] (15)

while the cancellation of the terms not containing \( E^a \) require the constraint

\[ \eta_{(a} T_{b) \alpha} = \eta_{ab} A_{\alpha}, \] (16)

\[ H_{a b c d} = - \frac{1}{4} A_{\beta} (\gamma_{abc})_{\beta} \alpha. \] (17)

Here \( A_{\alpha} \) is an arbitrary spinor superfield which is vanishing in \( d=11 \) [10].

It is important to realize that in obtaining (14)–(17) we have used the identity

\[ \delta E^a = \Gamma^a_{\beta \rho} \delta E^\rho + (1 - \Gamma^2) \kappa^a. \] (18)

Using this identity in the variation of the kinetic term, the terms arising from \( \Gamma^a_{\beta \rho} \) in (18) can be shown to cancel similar terms coming from the variation of the Wess-Zumino term, modulo terms which cancel by an appropriate variation of \( g_{ij} \). [Using the argument below (8) once.] In the remaining terms coming from \( (1-\Gamma^2) \), we use the argument given below (8) repeatedly to compute further contributions to \( \delta g_{ij} \). Thus we find the result

\[ X_{ij} = - \frac{1}{4} \sqrt{g} E_k E_i E_j (\gamma_{ab})_{\alpha \beta} \delta E^\beta E_j \alpha \]

\[ + \frac{1}{4} \kappa^\beta E_n (\gamma^d)_{\alpha \beta} E^m g_{ij} (T^k T^l - \frac{1}{2} g^{ij} T^k T^l) \]

\[ + i = j. \] (19)

In summary, the action (3) is invariant under (6) provided that (13)–(17) hold, and \( X_{ij} \) is given by (19). In addition, the following Bianchi identities must hold:

\[ DT^a = - E^b \wedge R^a_{b}, \quad DH = 0. \] (20)

The generalization of the results of this section to the general case of \( n \)-extended objects in \( d \)-dimensional supergravity is straightforward. The result is given in the appendix.

3. We observe that the superspace constraints of \( d=11 \) supergravity given by Cremmer and Ferrara [10] and Brink and Howe [10] do provide a solution to (13)–(17) and the Bianchi identities (20), with \( A_{\alpha} = 0 \).

In conclusion, we have shown that there exists a consistent coupling of a closed supermembrane to eleven-dimensional supergravity. (Note that it is natural to consider a closed supermembrane in eleven dimensions, since there are no matter multiplets in this dimension).

4. There are several directions in which the present work can be extended. We shall name a few.

Firstly, it is of interest to study the quantization of the supermembrane model in eleven dimensions. In particular, the question of whether massless gauge fields can possibly arise is a challenging one. Although usually one encounters difficulties in finding massless excitations of a membrane [13], it is encouraging that, here, we have a spacetime supersymmetric membrane action.

Secondly, it is natural to consider the dimensional reduction of our model from eleven- to ten-dimensional spacetime, and at the same time from three-dimensional world volume to a two-dimensional...
world sheet. It would be interesting to see what kind of $d=10$ string theories could possibly emerge in an infinitely thin membrane limit.

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Appendix. In this appendix we construct the action for an $n$-extended object propagating in $d$-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

$$S = \int \frac{d^2 \xi}{2} \sqrt{-g} \, g^{\alpha \beta} E_\alpha E_\beta \eta_{\alpha \beta}$$

$$+ \epsilon^{i_1 \ldots i_n} E_{i_1} A_1 \ldots E_{i_{a+1}} A_{a+1} B_{A_{a+1} \ldots A_l}$$

$$- \frac{1}{2} (n-1) \sqrt{-g} \, \eta_{\alpha \beta}, \quad (A1)$$

The transformation rules are those in (6), where the matrix $\Gamma^{\alpha \beta}$ is now given by

$$\Gamma^{\alpha \beta} = \eta^{\alpha} \eta^{(n+1)! \sqrt{-g}}$$

$$\times \epsilon^{i_1 \ldots i_n} E_{i_1} A_1 \ldots E_{i_{n+1}} (\gamma_a A_{a+1})^{\alpha \beta}, \quad (A2)$$

where $\eta$ is given by

$$\eta = (-1)^{(n+1)(n-2)/4}. \quad (A3)$$

Invariance of the action (A1) is ensured by imposing the following set of constraints:

$$T_{a \beta} = (\gamma^a)_{\alpha \beta}, \quad \eta_{(a} T^{c \beta b) \alpha} = \eta_{\alpha \beta} A_{\alpha}, \quad (A4)$$

$$H_{\alpha \beta \gamma a_1 \ldots a_n} = 0, \quad (A5)$$

$$H_{a \beta A_{a+1} A_{a+1} \ldots A_l} = 0, \quad (A5\text{cont'd})$$

and by taking $X_{ij}$ occurring in (6) to be

$$X_{ij} = (-\eta/2n!)$$

$$\times \epsilon^{i_1 \ldots i_n} E_{i_1} A_1 \ldots E_{i_{n+1}} (\gamma_{a_1} A_{a+1})^{\alpha \beta}, \quad (A6)$$

References