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Robust stabilization of nonlinear systems via stable kernel representations with $L_2$-gain bounded uncertainty

A.J. van der Schaft

Abstract

The approach to robust stabilization of linear systems using normalized left coprime factorizations with $\mathcal{H}_\infty$ bounded uncertainty is generalized to nonlinear systems. A nonlinear perturbation model is derived, based on the concept of a stable kernel representation of nonlinear systems. The robust stabilization problem is then translated into a nonlinear disturbance feedforward $\mathcal{H}_\infty$ optimal control problem, whose solution depends on the solvability of a single Hamilton–Jacobi equation.

Keywords: Robust nonlinear control; Perturbation model; Kernel representation; Small-gain theorem; Nonlinear $\mathcal{H}_\infty$ control

1. Introduction

A wealth of literature is available on the problem of robustly stabilizing nonlinear uncertain systems. Here we propose a very particular approach, which directly generalizes the solution of the linear robust stabilization problem via normalized left coprime factorizations, as obtained in Glover and McFarlane [6] (see also [15]), to the nonlinear case. Essential ingredients in our approach are the stable kernel representation of nonlinear state space systems as introduced in [18, 17], the resulting nonlinear perturbation model, and the solution to a particular type of nonlinear $\mathcal{H}_\infty$ control problems. The theory is illustrated with a simple example admitting an explicit solution.

2. A nonlinear perturbation model

A very general perturbation model for linear systems is the numerator–denominator perturbation model, or coprime factor uncertainty model, as it is also known (see e.g. [23, 12, 24]). Let $G(s)$ be the transfer matrix of a linear system (i.e. $G(s)$ is a proper rational matrix). Left factorization of $G(s)$ over the stable proper rational
matrices yields \( G(s) = D^{-1}(s) N(s) \), with \( D(s) \), \( N(s) \) coprime stable proper rational matrices. The stable linear system

\[
e = [N(s) : - D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

(with “inputs” \( u \) and “outputs” \( e \)) will be called a stable kernel representation of \( G(s) \), since by setting \( e = 0 \) in (1) one recovers the input–output map \( y = G(s)u \). In the numerator–denominator perturbation model one considers the following class of perturbations:

\[
\begin{align*}
N(s) & \rightarrow N(s) + A_N(s), \\
D(s) & \rightarrow D(s) + A_D(s),
\end{align*}
\]

with \( A_N(s), A_D(s) \) stable proper rational matrices. (In applications one would normally include some extra weighting filters; however, they can be incorporated in the system transfer matrix \( G(s) \), see [15, 13].) This results in the perturbed stable kernel representation

\[
e_p = [N(s) : - D(s)] \begin{bmatrix} u \\ y \end{bmatrix} + [A_N(s) : - A_D(s)] \begin{bmatrix} u \\ y \end{bmatrix}
\]

and the perturbed transfer matrix \( G_p(s) = [D(s) + A_D(s)]^{-1} [N(s) + A_N(s)] \). Usually it is convenient to normalize the kernel representation (1) by starting with left coprime factors \( D(s), N(s) \) satisfying

\[
N(s)N^T(-s) + D(s)D^T(-s) = I, \quad s \in \mathbb{C}.
\]

A detailed treatment of the robust stabilization problem based on this normalized coprime factor uncertainty model is given in [6, 15], see also [24] for the unnormalized case.

Now let us consider smooth nonlinear systems

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \quad u \in \mathbb{R}^m, \\
y &= h(x), \quad y \in \mathbb{R}^p,
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \) are local coordinates for an \( n \)-dimensional state space manifold \( M \). Throughout we assume the existence of a distinguished equilibrium \( x_0 \), i.e. \( f(x_0) = 0 \). Without loss of generality we assume \( x_0 = 0 \), and furthermore \( h(0) = 0 \).

Before defining a stable kernel representation for \( \Sigma \) and the resulting perturbation model we need some preliminaries. Let \( \gamma > 0 \). \( \Sigma \) is said to have \( L_2 \)-gain \( \leq \gamma \) if there exists a nonnegative solution \( V: M \rightarrow \mathbb{R} \) (a storage function) to the dissipation inequality [25],

\[
V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) \, dt, \quad V(0) = 0,
\]

for all \( t_1 \geq t_0 \) and all \( u \in L_2([t_0, t_1]) \) (with \( x(t_1) \) denoting the solution at time \( t_1 \) for initial condition \( x(t_0) \) at time \( t_0 \)). \( \Sigma \) is said to have \( L_2 \)-gain \( < \gamma \) if there exists some \( \tilde{\gamma} < \gamma \) such that \( \Sigma \) has \( L_2 \)-gain \( \leq \tilde{\gamma} \). Throughout we will assume that if there exists a solution \( V \geq 0 \) to (6) then there also exists a differentiable solution \( V \geq 0 \) to (6), and we will restrict ourselves to these differentiable solutions.

Let \( \Sigma \) have \( L_2 \)-gain \( \leq \gamma \). From [7, 8, 21] we recall that if additionally \( \Sigma \) is zero-state observable (i.e. \( y(t) = 0, u(t) = 0, \forall t \geq 0, \) implying \( x(0) = 0 \)), then necessarily a solution \( V \geq 0 \) to (6) is positive definite \( (V(x) > 0, x \neq 0) \), and 0 is a locally asymptotically stable equilibrium of (5) with Lyapunov function \( V \).

Next we consider the Hamilton–Jacobi–Bellman equation (corresponding to \( \Sigma \) with cost criterion

\[
W_x(x) f(x) + \frac{1}{2} W_x(x) g(x) g^T(x) W_x^T(x) - \frac{1}{2} h^T(x) h(x) = 0, \quad W(0) = 0,
\]

where
Theorem 2.1. Consider the nonlinear system (5), and assume it is zero-state detectable \((y(t) = 0, u(t) = 0, \forall t \geq 0, \text{implying } x(t) \to 0, t \to \infty)\). Suppose there exists a smooth positive definite solution \(W\) to (7), and suppose there exists a smooth solution \(k(x)\) to

\[
W(x)k(x) = h^T(x).
\]

Define the system \(\Sigma\) with inputs \(\begin{bmatrix} u \\ y \end{bmatrix}\) and outputs \(e\):

\[
\dot{x} = \begin{bmatrix} f(x) - k(x)h(x) \\ g(x) : k(x) \end{bmatrix},
\]

\[
\Sigma:\ e = h(x) - y.
\]

Then \(f(x) - k(x)h(x)\) is locally asymptotically stable (w.r.t. the equilibrium \(x = 0\)) with Lyapunov function \(W\), and globally asymptotically stable if \(W\) is proper (i.e. the sets \(\{x \in \mathbb{R}^n | 0 \leq W(x) \leq c\}\) are compact for every \(c \geq 0\)). Furthermore, \(\Sigma\) has \(L_2\)-gain \(= 1\). Setting \(e = 0\) in \(\Sigma\) yields \(\Sigma\), and \(\Sigma\) will be called a (nonlinear) stable kernel representation of \(\Sigma\).

Proof (sketch, see [18, 17] for details). From (7) and (8) we obtain

\[
W_x(x)[f(x) - k(x)h(x)] = -\frac{1}{2}W_x(x)g(x)g^T(x)W^T_x(x) - \frac{1}{2}h^T(x)h(x) \leq 0
\]

and (global) asymptotic stability follows from LaSalle's invariance principle. Similarly,

\[
W_x(x)[f(x) - k(x)h(x)] + g(x)u + k(x)y
\]

\[
= -\frac{1}{2}||u - g^T(x)W^T_x(x)||^2 - \frac{1}{2}||e||^2 + \frac{1}{2}||u||^2 + \frac{1}{2}||y||^2,
\]

proving \(L_2\)-gain \(\leq 1\) by integration (see e.g. [21]). \(\square\)

Remark 2.2. If the linearized system \(\dot{x} = (\partial f/\partial u)(0)x + g(0)u, y = (\partial h/\partial x)(0)x\) is anti-stabilizable, and if the imaginary eigenvalues of \((\partial f/\partial x)(0)\) are \((\partial h/\partial x)(0)\)-observable, then at least locally around 0 there exists a smooth non-negative solution \(W \geq 0\) to (7) (see e.g. [11, 20]), which will be locally positive definite if the linearized system is observable.

Remark 2.3. Consider a star-shaped coordinate neighborhood of \(x = 0\). Since \(W_x(0) = 0\) and \(h(x) = 0\) we can write (see e.g. [16])

\[
W_x(x) = x^T M(x), \quad h(x) = C(x)x
\]

for suitable matrices \(M(x), C(x)\), with entries depending smoothly on \(x\). Assume that \(M(x)\) is invertible for all \(x\) in the coordinate neighborhood of 0; then a solution \(k(x)\) to (8) is given as [18]

\[
k(x) = M^{-1}(x)C^T(x).
\]
(See [9] for similar considerations in a different context.) Note furthermore that \( M(0) = (\partial^2 W/\partial x^2)(0) \) (the Hessian matrix of \( W \) at 0). Thus, under the conditions of Remark 1, \( M(0) \) will be positive definite, implying that \( M(x) \) will be invertible for \( x \) near 0.

**Remark 2.4** ([18]). If \( -V \) is a negative definite solution to (7), then

\[
\dot{\Sigma}_e: \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} -g^T(p)V_p^T(p) \\ h(p) - e \end{bmatrix}, \quad p \in M,
\]

is a right inverse system to \( \Sigma_e \), i.e., if \( x(0) = p(0) \), then the input–output map of \( \Sigma_e \circ \Sigma_e \) is the identity mapping. Furthermore, \( f(p) - g(p)g^T(p)V_p^T(p) \) is locally asymptotically stable (w.r.t. \( p = 0 \)) with Lyapunov function \( V \) (and globally asymptotically stable if \( V \) is proper). Hence \( \Sigma_e \) has a stable right inverse, generalizing the linear notion of coprimeness.

**Remark 2.5.** For a linear system \( \Sigma \), the stable kernel representation \( \Sigma_e \) reduces to the left normalized coprime factorization (1), (4).

Analogously to the linear case (cf. (3)), we will now consider perturbed nonlinear stable kernel representations

\[
\dot{x} = \begin{bmatrix} f(x) - k(x)h(x) \\ g(x) \end{bmatrix} + \begin{bmatrix} u \\ y \end{bmatrix},
\]

\( e_p = e + w, \quad (15) \)

where \( w \) is the output of an arbitrary nonlinear state space system with input \( \begin{bmatrix} u \\ y \end{bmatrix} \),

\[
A: \begin{cases} \phi = \alpha(p, u, y), & \alpha(0, 0, 0) = 0, \\ w = \beta(p, u, y), & \beta(0, 0, 0) = 0, \end{cases} \quad (16)
\]

having finite \( L_2 \)-gain. (More generally we could consider families of nonlinear input–output maps from \( \begin{bmatrix} \gamma \end{bmatrix} \) to \( w \), parametrized by the set of initial conditions.) Setting \( e_p = 0 \) in (15) yields the perturbed system

\[
\dot{x} = f(x) + g(x)u + k(x)w, \\
y = h(x) + w, \quad (17)
\]

with \( w \) the output of (16).

### 3. The robust stabilization problem

Consider the nonlinear system \( \Sigma \) given by (5), and its perturbed model \( \Sigma_p \) given by (17), (16). The **robust stabilization problem** is to find a controller

\[
C: \begin{cases} \dot{\xi} = l(\xi, y), & l(0, 0) = 0, \\ u = m(\xi, y), & m(0, 0) = 0, \end{cases} \quad (18)
\]

with \( \xi \in \mathbb{R}^r \) the controller state, such that the \( L_2 \)-gain of the closed-loop system (17), (18), from \( w \) to \( z \), is minimized, say equal to \( \gamma^* \geq 0 \).
By the small-gain theorem (see e.g. [3]) this will mean that the overall closed-loop system (16)–(18) will be \(L_2\)-stable for all perturbations \(\Delta\) with \(L_2\)-gain strictly less than \(1/\gamma^*\).

In state space terms, if there exist proper positive definite solutions \(V_{\Sigma C}, V_{\Delta}\) to the dissipation inequalities

\[
\begin{align*}
V_{\Sigma C}(x(t_1), \xi(t_1)) - V_{\Sigma C}(x(t_0), \xi(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \||w(t)||^2 - \||z(t)||^2\) dt, \quad \gamma \geq \gamma^*, \\
V_{\Delta}(\varphi(t_1)) - V_{\Delta}(\varphi(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \||z(t)||^2 - \||w(t)||^2\) dt, \quad \gamma_{\Delta} < 1/\gamma,
\end{align*}
\]

(19)

then, assuming zero-state detectability [7], the overall closed-loop system (16)–(18) will be globally asymptotically stable with Lyapunov function \(\gamma_{\Delta} V_{\Sigma C} + \gamma V_{\Delta}\), as can be readily checked from (19).

The problem of minimizing the \(L_2\)-gain from \(w\) to \(z = [\gamma]\) for (17) is a standard \(\mathcal{H}_\infty\) optimal nonlinear control problem [19, 21, 10, 1, 22, 9]. Usually one first considers the suboptimal \(\mathcal{H}_\infty\) problem of finding for given \(\gamma > 0\) a controller \(C\) (if existing!) which makes the \(L_2\)-gain from \(w\) to \(z = [\gamma]\) less than or equal to \(\gamma\). For the solution to the suboptimal \(\mathcal{H}_\infty\) control problem we follow the approach of [2, 22]. For the state feedback suboptimal \(\mathcal{H}_\infty\) control problem we consider the pseudo-Hamiltonian

\[
K(x, p, u, w) = p^T[f(x) + g(x)u + k(x)w] \\
- \frac{1}{2}\gamma^2||w||^2 + \frac{1}{2}||u||^2 + \frac{1}{2}||h(x) + w||^2.
\]

(20)

Solving \(\partial K/\partial u = 0, \partial K/\partial d = 0\) leads to the saddle point \(u^* = -g^T(x)p, w^* = (\gamma^2 - 1)^{-1}[h(x) + k^T(x)p]\). Substitution of \(u^*, w^*\) into \(K\) yields the Hamiltonian \(H(x, p) = K(x, p, u^*, w^*)\) and the Hamilton-Jacobi-Isaacs equation \(H(x, P^T_\Sigma(x)) = 0\) given as

\[
\begin{align*}
P_x(x)[f(x) + (\gamma^2 - 1)^{-1}k(x)h(x)] + \gamma^2(\gamma^2 - 1)^{-1}h^T(x)h(x) \\
+ \frac{1}{2}P_x(x)[(\gamma^2 - 1)^{-1}k(x)k^T(x) - g(x)g^T(x)]P^T_\Sigma(x) = 0, \quad P(0) = 0.
\end{align*}
\]

(21)

If there exists a solution \(P \geq 0\) to (21) then the suboptimal state feedback \(\mathcal{H}_\infty\) control problem (for \(\gamma\)) is solvable by the state feedback

\[
u = -g^T(x)P^T_\Sigma(x).
\]

(22)

Following the certainty equivalence principle of [2] the solution to the output feedback suboptimal \(\mathcal{H}_\infty\) control problem is, under appropriate conditions, given as

\[
u = -g^T(\hat{x})P^T_\Sigma(\hat{x}),
\]

(23)

with \(\hat{x}(t)\) denoting the worst-case estimate of \(x(t)\) given the measurements \(y(\tau), -\infty < \tau \leq t\), see [2, 22]. (Currently there is intense research activity about the precise conditions for the validity of the worst-case certainty equivalence principle, but we will not elaborate on this.) In general (see e.g. [22]), this will yield an infinite dimensional controller. In the present case, however, the situation is much simpler. Indeed, the suboptimal \(\mathcal{H}_\infty\) control problem for (17) with \(z = [\gamma]\) is an example of the so-called disturbance feedforward problem, discussed for the linear case in [5], and for the nonlinear case in [14]. In fact, by asymptotic stability
of $\dot{x} = f(x) - k(x)h(x)$ it follows that for a given control function $u(\tau)$, $-\infty < \tau \leq t$, the measurement record $y(\tau)$, $-\infty < \tau \leq t$, uniquely specifies the disturbance $\psi(\tau) = y(\tau) - h(\dot{x}(\tau))$ and the state trajectory $x(\tau)$. Indeed the state trajectory $\dot{x}(\cdot)$ is generated by the differential equations

$$\dot{x}(\tau) = f(\dot{x}(\tau)) + g(\dot{x}(\tau))u(\tau) + k(\dot{x}(\tau))[y(\tau) - h(\dot{x}(\tau))], \quad x(-\infty) = 0. \tag{24}$$

Hence a controller solving the suboptimal $\mathcal{H}_\infty$ control problem for (17) is given as (substitute (23) into (24))

$$\begin{align*}
\dot{x} &= f(\dot{x}) - g(\dot{x})g^T(\dot{x})P^T_x(\dot{x}) + k(\dot{x})[y - h(\dot{x})], \\
u &= -g^T(\dot{x})P_x(\dot{x}). \tag{25}
\end{align*}$$

Based on the linear case, the same controller for the general nonlinear disturbance feedforward problem has been recently proposed in [14]. In this paper also a direct proof is provided showing that (25) solves the suboptimal $\mathcal{H}_\infty$ control problem at least locally, i.e. for initial states in a neighborhood of the origin and for disturbances $w(\cdot)$ which keep the state trajectories within this neighborhood. Summarizing, we have the following theorem.

**Theorem 3.1.** Suppose (cf. Theorem 2.1) that $\Sigma_\gamma$ given by (9) is a stable kernel representation of $\Sigma$ such that $f(x) - k(x)h(x)$ is globally asymptotically stable. Suppose there exists a solution $P \succ 0$ to (21) for given $\gamma > 0$, and assume the certainty equivalence principle for the suboptimal $\mathcal{H}_\infty$ control problem for (17) holds. Then the controller (25) stabilizes the closed-loop system (16), (17), (25) for every perturbation system $\Delta$ as in (16), having $L_2$-gain $< 1/\gamma$.

**Remark 3.2.** From the linear theory (cf. [6, 15]) and the local existence of solutions to (21) based on existence of solutions to the corresponding Riccati equation (cf. [20, 21]), it follows that the minimal $\gamma^*$, such that locally around 0 there exist solutions $P \succ 0$ to (21) for $\gamma > \gamma^*$, is given by

$$\gamma^* = [1 + \sigma_{\max}(X^T Z)]^{1/2}, \tag{26}$$

with $X$ the Hessian matrix $(\partial^2 V/\partial x^2)(0)$ and $Z$ the inverse Hessian matrix $[(\partial^2 W/\partial x^2)(0)]^{-1}$ of the solutions $W \succ 0$ and $-V \preceq 0$ to (7).

**Remark 3.3.** A related approach to nonlinear robust stabilization will be found in [4].

**Example 3.4.** Let $\Sigma$ be a lossless system, i.e. there exists $H : M \to \mathbb{R}$, $H(0) = 0$, $H(x) > 0$, $x \neq 0$, called the internal energy, such that $(d/dt)H = u^T y$ or, equivalently,

$$H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x). \tag{27}$$

Clearly, positive and negative definite solutions to (7) are given as $H$, and $-H$, respectively. Furthermore, $k(x)$ solving (8) is given as $g(x)$, and thus the perturbed system $\Sigma_\gamma$ is given as

$$\begin{align*}
\dot{x} &= f(x) + g(x)[u + w], \quad y = g^T(x)H^T_x(x) + w. \tag{28}
\end{align*}$$

The Hamilton–Jacobi–Isaacs equation (21) takes the form

$$P_x(x)[f(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H^T_x(x)] + \frac{1}{2}[(\gamma^2 - 1)^{-1} - 1] \cdot P_x(x)g(x)g^T(x)P^T_x(x)$$

$$+ \frac{1}{2}[\gamma^2(\gamma^2 - 1)^{-1}H_x(x)g(x)g^T(x)H^T_x(x)] = 0, \quad P(0) = 0, \tag{29}$$

having the positive definite solution $P(x) = (\gamma^2/(\gamma^2 - 2))H(x)$ for $\gamma > \sqrt{2}$. It follows that the controller

$$\begin{align*}
\dot{x} &= f(\dot{x}) - \frac{\gamma^2}{\gamma^2 - 2}g(\dot{x})g^T(\dot{x})H^T_x(\dot{x}) + g(\dot{x})[y - g^T(\dot{x})H^T_x(\dot{x})], \\
u &= -\frac{\gamma^2}{\gamma^2 - 2}g^T(\dot{x})H^T_x(\dot{x}) \tag{30}
\end{align*}$$
robustly stabilizes $\Sigma$ for every perturbation $A$ with $L_2$-gain $< \frac{1}{\gamma}$. By Remark 3.2, $\gamma^*$ is given by (26). Since $W = H$ and $V = -H$ we conclude that $\gamma^* = \sqrt{2}$, in accordance with the lower bound $\gamma > \sqrt{2}$ as derived above. From a physical point of view, if in (28) $u$’s denote external forces and $y$’s are the corresponding (disturbed) generalized velocities, then (30) corresponds to adding damping with regard to the estimated generalized velocities with a damping factor $\gamma^2(\gamma^2 - 2)^{-1}$, tending to $\infty$ for $\gamma \downarrow \gamma^* = \sqrt{2}$.

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