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Nonlinear systems which have finite-dimensional $\mathcal{H}_\infty$ suboptimal central controllers

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Abstract

Following up the work of Başar and Bernhard [2], we have recently derived in [12] the nonlinear central controller solving the nonlinear (standard) $\mathcal{H}_\infty$ suboptimal control problem. This nonlinear central controller is an *infinite-dimensional* system, and resembles very much the solution in nonlinear stochastic filtering or nonlinear deterministic filtering. After showing that in the *linear* case the nonlinear central controller reduces to the finite-dimensional central controller as obtained in [4], we consider in the present note the question if there are truly nonlinear systems having finite-dimensional central controllers. Guided by similar considerations in nonlinear stochastic and deterministic filtering, see especially [5], we characterize a specific class of nonlinear systems having finite-dimensional central controllers. This class can be regarded as the deterministic $\mathcal{H}_\infty$ analogue of the class of nonlinear systems admitting finite dimensional filters as identified by Benes [3].

We consider nonlinear control systems of the form

$$\begin{aligned} \dot{x} &= a(x) + b(x)u + g(x)d_1 \\ y &= c(x) + d_2 \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix} \end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates for a smooth state space manifold M. Furthermore, $u \in \mathbb{R}^m$ denote the *control inputs*, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^r$ the *exogenous inputs* (disturbances and/or references), $y \in \mathbb{R}^p$ the *measured outputs*, and $z \in \mathbb{R}^s$ the *to-be-controlled outputs* (tracking errors, cost variables). The maps $a(x), b(x), g(x), c(x), h(x)$ are all assumed to be C^k, with $k \geq 2$. Throughout we assume the existence of a fixed equilibrium x_0, i.e. $a(x_0) = 0$, and without loss of generality we set $x_0 = 0$, and also we let $c(0) = h(0) = 0$. Now let γ be a fixed positive constant. The $\mathcal{H}_\infty$ *suboptimal control problem* (for disturbance attenuation level γ) is to find a compensator

$$\begin{aligned} \dot{\xi} &= k(\xi, y) \\ u &= m(\xi, y) \end{aligned} \tag{2}$$

where $\xi = (\xi_1, \cdots, \xi_\nu)$ are local coordinates for a manifold M_c (the state space of the compensator), with $k(0,0) = 0$ and $m(0,0) = 0$, such that the closed-loop system (1), (2) has L_2-gain *less than or equal* to γ, in the sense that there exists a nonnegative constant K depending on $x(0), \xi(0)$ and zero for $x(0) = 0, \xi(0) = 0$, such that

$$\int_0^T \| z(t) \|^2 dt \leq \gamma^2 \int_0^T \| d(t) \|^2 + K \tag{3}$$

for all $d(\cdot)$ and $T \geq 0$, with $z(\cdot)$ denoting the closed-loop response for initial condition $x(0), \xi(0)$. If some *observability* conditions are satisfied (with regard to the outputs z!), then property (3) will also imply internal stability of the closed-loop system. For further motivation and details we refer e.g. to [1], [7], [9], [10], [12].

The state of the art for this problem is, very roughly, as follows. The state feedback problem (i.e. $y = x$) is reasonably well-understood [9], [10], [7]. For the full (dynamic output feedback) problem appealing necessary conditions have been found generalizing the famous necessary *and* sufficient conditions for linear systems obtained in [4], see [1], [11], [12]. Much effort and ingenuity has been put in obtaining various sufficient conditions, see e.g. [1], [6], [7], but the problem is still largely open.

Following up the work of Başar and Bernhard [2], and using older work on nonlinear deterministic filtering [8], [5], we have recently taken another approach to the full $\mathcal{H}_\infty$ suboptimal control problem. In fact, in [12] we have shown that under suitable technical conditions the $\mathcal{H}_\infty$ suboptimal control problem is solved by the controller

$$\begin{aligned} \dot{\hat{x}} &= \left[a(\hat{x}) - b(\hat{x})b^T(\hat{x})P_x^T(\hat{x}) + \tfrac{1}{\gamma^2} g(\hat{x})g^T(\hat{x})P_x^T(\hat{x})\right] \\ &\quad + \gamma^2 \left[S_{xx}(\hat{x}, t)\right]^{-1} \tfrac{\partial c^T}{\partial x}(\hat{x}) \left[y(t) - c(\hat{x})\right] \\ u &= -b^T(\hat{x})P_x^T(\hat{x}) \end{aligned} \tag{4}$$

with $P \geq 0$ being the minimal solution to the Hamilton-Jacobi equation

$$\begin{aligned} &P_x(x)a(x) + \tfrac{1}{2}P_x(x)\left[\tfrac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x)\right]P_x^T(x) \\ &+ \tfrac{1}{2}h^T(x)h(x) = 0, \quad P(0) = 0, \end{aligned} \tag{5}$$

(implying the solvability of the *state feedback* $\mathcal{H}_\infty$ suboptimal control problem, see [9], [10]), and with R being a solution to the nonstationary Hamilton-Jacobi equation

$$\begin{aligned} &R_t(x,t) + R_x(x,t)a(x) + \tfrac{1}{2}\tfrac{1}{\gamma^2}R_x(x,t)g(x)g^T(x)R_x^T(x,t) \\ &+ \tfrac{1}{2}h^T(x)h(x) - \tfrac{1}{2}\gamma^2 c^T(x)c(x) + \gamma^2 c^T(x)y(t) \\ &+ R_x(x,t)b(x)u(t) - \tfrac{1}{2}\gamma^2 \| y(t) \|^2 + \tfrac{1}{2} \| u(t) \|^2 = 0 \end{aligned} \tag{6}$$

(with $u(t)$ given as the output of (4)), such that $S(x,t) := R(x,t) - P(x)$ has a unique minimum $\hat{x}(t)$ for every t with invertible Hessian matrix $S_{xx}(\hat{x}(t), t)$.

This controller is obtained as the solution of a certain min-max optimization problem with imperfect state measurements, see [2]; and thus is called the nonlinear *central controller*, since in the linear case this optimization problem is known [2] to yield the central controller of [4]. In fact, in the sequel we will *directly* demonstrate how in the linear case the nonlinear central controller reduces to the central controller of [4].

Although the first part of the nonlinear central controller, i.e. (4), has an appealing "worst-case disturbance" observer structure (see [2], [1], [12]), in general the gain matrix $[S_{xx}(\hat{x}, t)]^{-1}$ cannot be computed off-line, since the partial differential equation (6) for $R(x,t)$ is directly driven by the measured outputs $y(t)$, as well as by $u(t)$. Thus the nonlinear central controller is *infinite-dimensional.*

Now suppose (1) is linear, i.e.

$$\begin{aligned} \dot{x} &= Ax + Bu + Gd_1 \\ y &= Cx + d_2 \\ z &= \begin{bmatrix} Hx \\ u \end{bmatrix} \end{aligned} \tag{7}$$

It is readily checked (see [5] for the same argument in the context of deterministic filtering) that the solution $R(x,t)$ of (6) in this case takes the form $R(x,t) = c(t) + \ell^T(t)x + \frac{1}{2}x^TQ(t)x$ with $R_{xx}(x,t) = Q(t) \geq 0$ satisfying

$$\dot{Q}(t) + A^TQ(t) + Q(t)A + \frac{1}{\gamma^2}Q(t)GG^TQ(t) + H^TH - \gamma^2C^TC = 0 \tag{8}$$

It follows that $R_{xx}(x,t) = Q(t)$ can be computed off-line, and in fact can be taken to be the maximal constant matrix $Q > 0$ solving the algebraic Riccati equation

$$A^TQ + QA + \frac{1}{\gamma^2}QGG^TQ + H^TH - \gamma^2C^TC = 0, \tag{9}$$

(Notice that the dual Riccati equation (FARE) of [4] is obtained from (9) by dividing by γ^2 and pre- and post-multiplication by γ^2Q^{-1}!). Furthermore the stationary Hamilton-Jacobi equation (5) reduces to an algebraic Riccati equation, and it is immediately seen that the resulting linear controller (4) is precisely the *central* controller of [4].

A logical question is now: are there any other systems, apart from the linear ones, for which $R_{xx}(x,t)$ and thus $S_{xx}(x,t)$ can be computed off-line, and therefore the nonlinear central controller *reduces* to the finite-dimensional controller (4)? In order to study this problem we take the same approach as used in [5] for the deterministic nonlinear filtering problem, i.e. we consider the *Hamiltonian* function $H(x,p)$ corresponding to the Hamilton-Jacobi equation (6):

$$\begin{aligned} H(x,p) &= p^Ta(x) + \frac{1}{2}\frac{1}{\gamma^2}p^Tg(x)g^T(x)p + \frac{1}{2}h^T(x)h(x) \\ &- \frac{1}{2}\gamma^2c^T(x)c(x) + \gamma^2c^T(x)y(t) \\ &+ p^Tb(x)u(t) - \frac{1}{2}\gamma^2 \parallel y(t) \parallel^2 + \frac{1}{2} \parallel u(t) \parallel^2, \end{aligned} \tag{10}$$

with $p \in \mathbf{R}^n$ denoting the co-state. (Notice that (6) can be rewritten as $R_t + H(x, R_x^T) = 0$.) It is natural (see [5]) to consider the following type of canonical transformations $(x,p) \mapsto (x,\bar{p})$, where

$$p = \bar{p} + V_x^T(x) \tag{11}$$

for some C^k function $V(x)$. Then the Hamiltonian $H(x,p)$ transforms into $\bar{H}(x,\bar{p}) := H(x,\bar{p}+V_x^T(x))$, leading to the *transformed* Hamilton-Jacobi equation $\bar{R}_t + \bar{H}(x,\bar{R}_x^T) = 0$ given as

$$\begin{aligned} &\bar{R}_t(x,t) + \bar{R}_x(x,t)\left[a(x) + \tfrac{1}{\gamma^2}g(x)g^T(x)V_x^T(x)\right] \\ &+\tfrac{1}{2}\tfrac{1}{\gamma^2}\bar{R}_x(x,t)g(x)g^T(x)\bar{R}_x^T(x,t) + \tfrac{1}{2}h^T(x)h(x) \\ &-\tfrac{1}{2}\gamma^2c^T(x)c(x) + \tfrac{1}{2}\tfrac{1}{\gamma^2}V_x(x)g(x)g^T(x)V_x^T(x) + V_x(x)a(x) \quad (12) \\ &+\gamma^2c^T(x)y(t) + \bar{R}_x(x,t)b(x)u(t) + V_x(x)b(x)y(t) \\ &-\tfrac{1}{2}\gamma^2 \parallel y(t) \parallel^2 + \tfrac{1}{2} \parallel u(t) \parallel^2 = 0 \end{aligned}$$

It immediately follows that $\bar{R}$ is a solution of (12) if and only if $\bar{R}+V$ is a solution of (6). Now assume that

$$\begin{aligned} &b(x) \text{ and } g(x) \text{ are constant,} \\ &c(x) \text{ is linear in } x, \end{aligned} \tag{13}$$

and suppose that the nonlinear system (1) has the property that $V(x)$ can be found such that

$$\begin{aligned} &a(x) + \frac{1}{\gamma^2}g(x)g^T(x)V_x^T(x) \text{ is at most linear in } x, \\ &\frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2c^T(x)c(x) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)g(x)g^T(x)V_x^T(x) \\ &+V_x(x)a(x) \text{ is at most quadratic in } x, \\ &V_x(x)b(x) \text{ is at most linear in } x. \end{aligned} \tag{14}$$

Then, *as in the linear case*, it follows that the solution $\bar{R}(x,t)$ of (12) can be written as $\bar{R}(x,t) = \bar{c}(t) + \bar{\ell}(t)x + \frac{1}{2}x^T\bar{Q}(t)x$, where $\bar{Q}(t) = \bar{R}_{xx}(x,t)$ is the solution of a differential Riccati equation of the same type as (8), and thus can be computed *off-line* (without knowing $y(t)$ and $u(t)$). Therefore also the Hessian $R_{xx}(x,t)$ of the solution $R(x,t) = \bar{R}(x,t) + V(x)$ of (6) can be computed off-line, and thus in this case the nonlinear central controller reduces to the finite-dimensional controller (4)! Summarizing:

Theorem *Consider the nonlinear system (1). Suppose the $\mathcal{H}_\infty$ sub-optimal control problem is solvable by the nonlinear central controller (4), (6). Assume that $b(x), g(x)$ and $c(x)$ are as in (13), while $V(x)$ can be found such that (14) holds. Then there exists a solution $R(x,t)$ of (6) for which $R_{xx}(x,t)$ can be computed off-line, and the nonlinear central controller reduces to the finite-dimensional controller (4).*

Example Consider the almost linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u + d_1, & y &= x_2 + d_2 \\ \dot{x}_2 &= -x_2 & z &= \begin{bmatrix} x_2^3 \\ u \end{bmatrix} \end{aligned}$$

Then $V(x_1,x_2) = \frac{1}{12}x_2^6$ satisfies (12), and thus the nonlinear central controller reduces to a finite dimensional controller. □

It would be nice to characterize the class of nonlinear systems covered by the above theorem in a more explicit and coordinate free way. Notice however that (as in nonlinear filtering, see [3], [5]) it is expected to be only a *small* subset of all nonlinear systems. On the other hand, we recall from [12] that even if the central controller is inherently infinite-dimensional, then still there may exist finite dimensional controllers also solving the (sub-)optimal $\mathcal{H}_\infty$ control problem.

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