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On moment matching of transfer functions and their derivatives

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Abstract—**In this paper we study the problem of computing the (family of) reduced order model(s) that satisfy the following properties the systems match the moments of the given to be reduced system and the same property holds for the first order derivatives of the corresponding transfer functions. We prove that such a model exists and compute it. We also study the problem of moment matching for the state-space representations of the first order derivatives of rational transfer functions. Finally, we show that the model that matches the moment of both the given system and its first order derivative is a member of the family of reduced order models whose first order derivatives achieve moment matching.**

I. Introduction

In the problem of model reduction moment matching techniques represent an efficient tool, see e.g. [2], [4], [7], [9], [14], [18], [20] for a complete overview for linear systems. Using a numerical approach based on Krylov projection methods the (reduced order) model is obtained by efficiently constructing a lower degree rational function that approximates a given transfer function (assumed rational). The low degree rational function matches the given transfer function at various points in the complex plane.

If the interpolation points are at zero, then the Padé approximation problem is solved. If the interpolation points are finite, then the general rational interpolation problem is solved. In the case of multiple-input multiple output (MIMO) systems the problem is called tangential interpolation, i.e., finding an approximation that interpolates a transfer matrix at selected points along selected directions, see [3], [11] for further details. If the interpolation points are at infinity, the problem is called partial realization and has been studied in e.g., [19] and references therein. For single-input single-output (SISO) systems, a system theoretic, time-domain approach to moment matching has been taken in [5]. In short, the notion of moment of a linear, minimal system has been related to the unique solution of a Sylvester equation, see also, e.g., [12], [13], for previous results. Furthermore, the moments are in one-to-one relation with the steady-state response (provided it exists) of the given system driven by a signal generator, which ”contains” the interpolation points. The moments have also been connected to the solution of a dual Sylvester equation, and shown to be in one-to-one relation with the well-defined steady-state response of the given system driving a (generalized) signal generator. Used for model reduction, the time-domain approach yields simple and direct characterizations of all parametrized, reduced order models that match a prescribed set of moments of a given system at a set of *finite* interpolation points. The classes of reduced order models that achieve moment matching contain subclasses of models that meet additional constraints, i.e., the free parameters are useful for enforcing properties such as, e.g., passivity, stability or relative degree, irrespective of the choice of interpolation points.

In this paper we study the problem of moment matching for the first derivative of the transfer function. This research paves the way for the study of the solution of the optimal approximation norm in the time-domain moment matching framework, starting with re investigating the first order optimality conditions, see e.g., [1], [10], [15] and the references therein. To this end, in this paper, we propose a notion of moment of a state-space representation of the first order derivative of the transfer function of a linear system. The moments will be defined in terms of the unique solutions of a Sylvester equation and its dual. Furthermore, we compute the class of low order linear systems, whose first order derivatives of the associated transfer functions match the moments of the first order derivative of the transfer function of the given linear system. Finally, we identify from the families of reduced order models that achieve moment matching at a set of interpolation point the unique model that achieves moment matching for the first order derivatives of the transfer functions. We also show that this unique model is a member of the family of reduced order models that achieve moment matching for the first order derivative of the transfer functions at a set of finite interpolation points.

The paper is organized as follows. In Section II we give a brief overview of the recent results of time-domain moment matching and present the families of parametrized reduced order models that achieve moment matching. In Section III-A we present the model that achieves moment matching for both the zero and the first order derivatives of the transfer function simultaneously. Then we formulate and study the problem of moment matching for linear systems characterized by the first order derivative of a transfer functions, giving the definition of moment for such a system and relating it with the (well-defined) steady-state response of an interconnection of linear systems and signal generators. Then we present the notion of a reduced order linear system that satisfies the property that the moments of his first order derivative of the transfer functions matches the moments of the first order derivative of the given system. We compute the parametrized family of models that satisfy this condition. Finally we include the unique model that matches the

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moments of both the zero and the first order derivative of the transfer function of the given linear system at a set of prescribed interpolation points in the family of systems that match the first order derivatives of the transfer function of the given system. The paper ends with some conclusions.

II. Preliminaries

In this section we recall the idea of moment matching for linear, single-input, single-output systems, from a time domain point of view as presented in [6]. Consider the system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

with the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}$, the output $y(t) \in \mathbb{R}$ and the transfer function $K : \mathbb{C} \to \mathbb{C}$,

$$K(s) = C(sI - A)^{-1}B. \tag{2}$$

Assume that Σ is a minimal realization of the transfer function $K(s)$. The moment of (2) is defined as follows.

Definition 1. [2], [5] The 0-moment at $s_1 \in \mathbb{C}$ of system (1) is the complex number $\eta_0(s_1) = C(s_1 I - A)^{-1}B$. The $k-$moment of system (1) at s_1 is the complex number $\eta_k(s_1) = \dfrac{(-1)^k}{k!} \cdot \dfrac{d^k[C(sI-A)^{-1}B]}{ds^k}\Big|_{s=s_1} = \dfrac{(-1)^k}{k!}\dfrac{d^k K(s)}{ds^k}$, $k \geq 1$ and integer. □

Consider the linear system (1) and let the matrices $S \in \mathbb{R}^{\nu\times\nu}$, $L \in \mathbb{R}^{1\times\nu}$ and $Q \in \mathbb{R}^{\nu\times\nu}$, $R \in \mathbb{R}^{\nu}$ be such that the pair (L, S) is observable and the pair (Q, R) is controllable, respectively. Let $\Pi \in \mathbb{C}^{n\times\nu}$ and $\Upsilon \in \mathbb{C}^{\nu\times n}$ be the solutions of the Sylvester equations

$$A\Pi + BL = \Pi S, \tag{3a}$$

$$Q\Upsilon = \Upsilon A + RC, \tag{3b}$$

respectively. Assume that $\sigma(A) \cap \sigma(S) = \emptyset$, where $\sigma(A)$ is the spectrum of the matrix A. Since Σ is minimal, then Π is the unique solution of the equation (3a) and $\operatorname{rank} \Pi = \nu$. Assuming $\sigma(A) \cap \sigma(Q) = \emptyset$, then Υ is the unique solution of the equation (3b) and $\operatorname{rank} \Upsilon = \nu$ (see e.g., [8]).

Definition 2. [5], [6]

1) Let $\phi = [\phi_1\ \phi_2\ \dots\ \phi_\nu] \in \mathbb{C}^{1\times\nu}$ be such that

$$\phi = C\Pi. \tag{4}$$

Then the moments of system (1) at $\sigma(S)$ are the elements ϕ_i, $i = 1, ..., \nu$. The interpolation points are the eigenvalues of S, i.e., $\{s_1, s_2, ..., s_\nu\} = \sigma(S)$.

2) Let $\varphi = [\varphi_1\ \varphi_2\ \dots\ \varphi_\nu]^T \in \mathbb{C}^{\nu}$ be such that

$$\varphi = \Upsilon B. \tag{5}$$

Then the moments of system (1) at $\sigma(Q)$ are the elements φ_i, $i = 1, ..., \nu$. The interpolation points are the eigenvalues of Q, i.e., $\{s_1, s_2, ..., s_\nu\} = \sigma(Q)$. □

Note that the moments as in Definition 1 are equivalent to the notions depicted in Definition 2. Selecting (L, S) and (Q, R) in canonical forms, easy computations yield $[\eta(s_1)\ \dots\ \eta(s_\nu)] = \phi = \varphi$.

Based on Definition 2, we define families of parametrized models of order ν that achieve moment matching at the interpolation points $\sigma(S)$ and $\sigma(Q)$, respectively.

Theorem 1. *[5], [6]*

1) Let the pair (L, S) be observable and assume $\sigma(A) \cap \sigma(S) = \emptyset$. Let $\xi(t) \in \mathbb{R}^{\nu}$ and consider the family of linear models

$$\Sigma_G : \begin{cases} \dot{\xi} = (S - GL)\xi + Gu, \\ \eta = C\Pi\xi, \end{cases} \tag{6}$$

parametrized by $G \in \mathbb{C}^{\nu}$, where Π is the unique solution of (3a). Assume $\sigma(S - GL) \cap \sigma(S) = \emptyset$. Let $\hat{\phi} \in \mathbb{C}^{1\times\nu}$ be the moments of (6) at $\sigma(S)$. Then (6) describes a family of reduced order models of Σ, parametrized in G and achieving moment matching at $\sigma(S)$, i.e., $\phi = \hat{\phi}$.

2) Let the pair (Q, R) be controllable and assume $\sigma(A) \cap \sigma(Q) = \emptyset$. Let $\xi(t) \in \mathbb{R}^{\nu}$ and consider the family of linear models

$$\Sigma_H : \begin{cases} \dot{\xi} = (Q - RH)\xi + \Upsilon Bu, \\ \eta = H\xi, \end{cases} \tag{7}$$

parametrized by $H \in \mathbb{R}^{1\times\nu}$, where Υ is the unique solution of (3b). Assume $\sigma(Q - RH) \cap \sigma(Q) = \emptyset$. Let $\hat{\varphi} \in \mathbb{C}^{1\times\nu}$ be the moments of (7) at $\sigma(Q)$. Then (7) describes a family of reduced order models of Σ, parametrized in H and achieving moment matching at $\sigma(Q)$, i.e., $\varphi = \hat{\varphi}$. □

Remark 1. Any system

$$\begin{aligned} \dot{\xi} &= F\xi + Gu, \\ \eta &= H\xi, \end{aligned} \tag{8}$$

with $F \in \mathbb{R}^{\nu\times\nu}$, $G \in \mathbb{R}^{\nu}$, $H \in \mathbb{R}^{1\times\nu}$ matches the moments of Σ at $\sigma(S)$ if and only if it is equivalent[1] to (6), i.e., $HP = C\Pi$, where the coordinate transformation $P \in \mathbb{C}^{\nu\times\nu}$ is the unique solution of the Sylvester equation $FP + GL = PS$ (see, e.g., [5] for more details). Similar arguments hold for the case of (7) (see, e.g., [6] for more details). □

Note that in practice, equations (3) are never solved and Υ and Π may be constructing using Krylov projections methods, see, e.g., [17] for more details.

III. Moment matching of zero and first order derivatives

A. *Matching further moments - the case $S = Q$*

In this section we study the problem of matching the moments at $\sigma(S)$ and $\sigma(Q)$ simultaneously when $\sigma(S) = \sigma(Q)$,

[1] Two minimal systems described by state-space equations are equivalent if they have the same transfer functions, i.e., the same input-output behaviour.

or without loss of generality $S = Q$. Note that in this case, the results of [16, Proposition 1] are not applicable, i.e., the Sylvester equations are undetermined. Hence a new approach is called for. We show that in this case, a reduced order model that achieves moment matching satisfies the property that its firs derivative achieves moment matching simultaneously. First, we present the model that satisfies both moment matching properties simultaneously and then we show how this model is a member of the class of reduced order models that achieve moment matching in the sense of Theorem 1.

Theorem 2. *Consider the system Σ and assume that the matrix $\Upsilon\Pi$ is invertible. Let*

$$\widehat{K}(s) = C\Pi(sI - (\Upsilon\Pi)^{-1}\Upsilon A\Pi)^{-1}(\Upsilon\Pi)^{-1}\Upsilon B. \quad (9)$$

Then $\widehat{K}(s)$ satisfies the following properties.

1) $\widehat{K}(s_i) = K(s_i)$, for all $s_i \in \sigma(S)$, $i = 1, ..., \nu$, i.e.,$\widehat{K}$ matches the moments of K at $\sigma(S)$;
2) $\dfrac{d\widehat{K}(s_i)}{ds} = \dfrac{dK(s_i)}{ds}$ for all $s_i \in \sigma(S)$, $i = 1, ..., \nu$, i.e., the derivatives of $\widehat{K}$ and K match at s_i. □

Hence we identify the reduced order models Σ_G and Σ_H which achieve moment matching and their derivatives satisfy a similar property.

Corollary 3. *The following statements hold.*

1) The model Σ_G that matches the moments of $K(s)$ and $\frac{dK(s)}{ds}$ at $\sigma(S)$ is given by $G = (\Upsilon\Pi)^{-1}\Upsilon B$.
2) The model Σ_H that matches the moments of $K(s)$ and $\frac{dK(s)}{ds}$ at $\sigma(S)$ is given by $H = C\Pi(\Upsilon\Pi)^{-1}$. □

B. Moment matching of first order derivatives

In this section we start the problem of matching between derivatives of transfer functions by defining the moments of the first derivative of a transfer function at a set of interpolation point. Then we define the family of reduced order linear systems that achieve moment matching and explore its properties with respect to the families of models Σ_G and Σ_H. Let $K(s)$ be as in (2). Then $\frac{dK(s)}{ds} = K'(s) = -C(sI - A)^{-2}B$. Let $s_i \in \sigma(S)$, $i = 1, ..., \nu$. Then the 0-order moment of K' at s_i is given by

$$\begin{aligned}\eta_0(s_i) &= K'(s_i) = -C(s_iI - A)^{-2}B \\ &= -C(s_iI - A)^{-1}(s_iI - A)^{-1}B,\ i = 1, ..., \nu.\end{aligned} \quad (10)$$

Note that $(s_iI - A)^{-1}Bl_i = \Pi_i$, $i = 1, ..., \nu$, where $\Pi = [\Pi_1, ..., \Pi_\nu]$ is the unique solution of the Sylvester equation (3a) and $L = [l_1, ..., l_i, ..., l_\nu]$, such that the pair (L, S) is observable. Similarly, $l_iC(s_iI - A)^{-1} = \Upsilon_i$, $i = 1, ..., \nu$, where $\Upsilon = [\Upsilon_1^T, ..., \Upsilon_\nu^T]^T$ is the unique solution of the Sylvester equation (3b), i.e., $S\Upsilon = \Upsilon A - L^TC$. Hence

$$\eta_0(s_i) = \Upsilon_i\Pi_i = e_i^T\Upsilon\Pi e_i,\ i = 1, ..., \nu, \quad (11)$$

where $e_i = [0, ..., \underbrace{1}_{i}, ..., 0]^T$.

In the sequel we show that the moments defined by equation (11) are in a one-to-one relation[2] with the (well-defined) steady state[3] of the output of an interconnection of subsystems. To this end define

$$\Sigma' : \begin{cases}\dot{x} = Ax + Bu, \\ \dot{z} = Az + x, \\ y = -Cz,\end{cases} \quad (12)$$

where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}$, see also Figure 1.

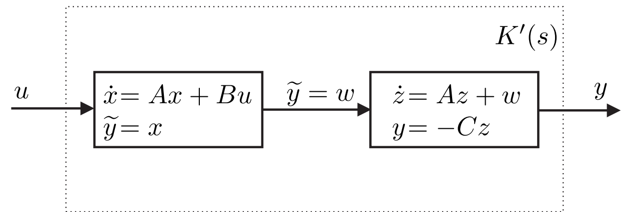

Fig. 1. State-space representation of $K'(s)$

Note that the transfer function of (12) is $\widetilde{K}(s) = K'(s)$, with K as in (2), hence (11) describes the moments of (12) at $\sigma(S)$.

Consider the signal generators

$$\dot{\omega} = S\omega,\ \theta = L\omega,\ \omega(0) \neq 0,\ \omega(t) \in \mathbb{R}^\nu \quad (13)$$

and

$$\dot{\varpi} = S\varpi + L^Tv,\ d = \varpi + \Upsilon z,\ \varphi(0) = 0,\ \varpi(t) \in \mathbb{R}^\nu, \quad (14)$$

where Υ is the unique solution of the equation $S\Upsilon = \Upsilon A - L^TC$. Interconnecting (12) to the signal generators (13), by $u = \theta$ and (14), by $v = y$, we obtain the output signal of the interconnection $d(t)$ as in Figure 2. The next result presents the properties of signal d.

Proposition 4. *Consider the interconnection of systems as in Figure 2, defined by $u = \theta$ and $v = y$. Then the signal $d(t)$ satisfies the equation*

$$\dot{d} = Sd + \Upsilon\Pi\omega + \Upsilon e^{At}(x_0 - \Pi\omega_0), \quad (15)$$

if and only if Π is the unique solution of equation (3a) and Υ is the unique solution of equation (3b). □

Theorem 5. *Consider the system (12). Let Π be the unique solution of the equation (3a). Consider the series interconnection with the signal generators (13), by $u = \theta$ and (14), by $v = y$. Furthermore assume that $\sigma(A) \subset \mathbb{C}^-$ and $\sigma(S) \subset \mathbb{C}^0$. Then the moments of (12) at $\sigma(S)$, defined by*

[2]By one-to-one relation between a set of ν moments $\eta(s_i)$, $i = 1, ..., \nu$ and the well-defined steady state response of the signal $y(t)$ we mean that the moments η uniquely determine the steady-state response of $y(t)$.

[3]Let (A, B, C) be a linear system described by the equations $\dot{x} = Ax + Bu$, $y = Cx$, with $x(t) \in \mathbb{R}^n$ and $\sigma(A) \in \mathbb{C}^-$. Then $x(t) = x_p(t) + x_t(t)$, with $x_p(t) = \lim_{t_0 \to -\infty} \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$. $x_t(t)$ is the transient component of the state $x(t)$, i.e., $\lim_{t\to\infty} x_t(t) = 0$ and $x_p(t)$ is the steady-state. Consequently, $y_p(t) = Cx_p(t)$ is the steady-state response of the linear system (A, B, C).

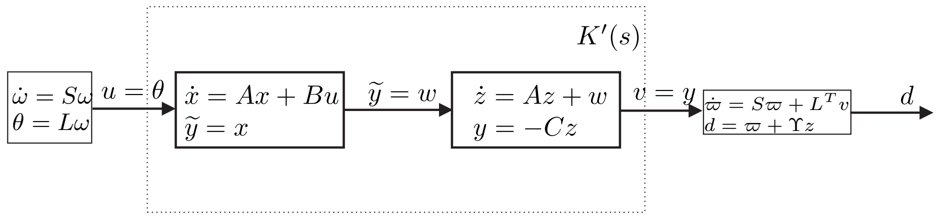

Fig. 2. Diagram illustrating signal d

the diagonal elements of $\Upsilon\Pi$ are in a one-to-one relation with the steady state response of the signal $d(t)$ from (14).$\square$

Now, we are ready to formulate the model reduction problem in this framework. Given the linear minimal system characterized by the transfer function (2) with the state-space realization given by equations (1), of order n and the observable pair (L, S), find the family of linear systems

$$\begin{aligned}\dot\xi &= F\xi + Gu,\\ \eta &= H\xi,\end{aligned} \tag{16}$$

with $\xi(t) \in \mathbb{R}^\nu$, $\nu < n$, such that $\widehat K'(s_i) = K'(s_i)$, for all $s_i \in \sigma(S)$, where $\widehat K(s) = H(sI - F)^{-1}G$. Furthermore, find the unique parameters F, G, H, i.e., the unique system (16) that simultaneously satisfies the following properties, $\widehat K(s_i) = K(s_i)$ and $\widehat K'(s_i) = K'(s_i)$ for all $s_i \in \sigma(S)$.

Note that a state-space representation of $\widehat K(s)$ is given by

$$\widehat\Sigma' : \begin{cases}\dot\xi = F\xi + Gu,\\ \dot\zeta = F\zeta + \xi,\\ y = -H\zeta,\end{cases} \tag{17}$$

with $\zeta(t) \in \mathbb{R}^\nu$.

Consider the interconnection of (17) to the signal generators (13), by $u = \theta$ and (14), by $v = \eta$, we obtain the output signal of the interconnection $\delta(t) = \varpi(t) + \bar P\xi(t)$. Enforcing δ to satisfy the equation $\dot\delta = S\delta + \Upsilon\Pi\xi$ we characterize a family of reduced order models (16) that satisfy the the property $\widehat K'(s_i) = K'(s_i)$, for all $s_i \in \sigma(S)$, where $\widehat K(s) = H(sI - F)^{-1}G$.

Theorem 6. *Consider the observable pair (L, S) and the n-th order linear system Σ, described by equations (1), with the transfer function $K(s)$, as in (2). Let Π be the unique solution of equation (3a) and Υ be the unique solution of equation (3b) and assume that the matrix $\Upsilon\Pi$ is invertible. A family of linear systems of order ν with the transfer function $\widehat K(s)$ which satisfies the property $\widehat K'(s_i) = K'(s_i)$, for all $s_i \in \sigma(S)$, where $\widehat K(s) = H(sI - F)^{-1}G$ is given by a linear system of order ν, described by equations (16) with $F = \bar P^{-1}(S - L^T H)\bar P$, $PL^T H = GLP$, where $\bar P = \Upsilon\Pi$ and $P = (\Upsilon\Pi)^{-1}$, with P an invertible matrix and the unique solution of the equation $FP + GL = PS$.* $\square$

Note that for G and H chosen as in Theorem 2, from Theorem 6 it follows that $L^T C\Pi = \Upsilon BL$, which means that moment matching in the classic sense of the zero order derivatives is achieved as well.

Finally, we mention that the systems (12) and (12) are never computed, since the results of Theorems 2 and 6 give results directly on the parameters F, G, H which are sufficient to construct the reduced order model (16).

IV. Conclusions

In this paper we have studied the problem of computing the (family of) reduced order model(s) that satisfy the following properties the systems match the moments of the given to be reduced system and the same property holds for the first order derivatives of the corresponding transfer functions. We have proven that such a model exists and compute it. We have also studied the problem of moment matching for the state-space representations of the first order derivatives of rational transfer functions. Finally, we have shown that the model that matches the moment of both the given system and its first order derivative is a member of the family of reduced order models whose first order derivatives achieve moment matching. For future work we aim at extending and interpreting these results and their consequences to other cases, such as nonlinear dynamical systems and distributed parameter systems.

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