Essays on dynamic macroeconomics

Romp, Willem Eduard

Chapter 3

Ageing, Schooling, and Growth

3.1 Introduction

It is a well documented fact that the western world is ageing rapidly. Since the post-war period, the ageing process can be attributed both to increased longevity and reduced fertility (Lee, 2003). For example, in the Netherlands, life expectancy at birth rose from 71.5 years in 1950 to 78.6 years in 2003, whilst the annual (crude) birth rate fell from 2.3% to 1.3% of the population. Because infant mortality stayed relatively constant during that period (at 0.8% of the population), the increase in longevity must be attributed to reduced adult mortality. Not surprisingly, the demographic change has led to a dramatic increase in the population share of elderly people over that period—the old-age dependency ratio (measured as the ratio of the population aged 65 years or over to the population aged 15-64) rose from 12.2% in 1950 to 20.1% in 2002. A similar demographic pattern can be observed for most OECD countries.

The objective of this chapter is to investigate the effects on the economic growth performance of a small open economy of substantial demographic shocks of the type and magnitude mentioned above. We use the Blanchard-Yaari model with a realistic mortality process developed in the previous chapter and extend it with a schooling decision. The finitely-lived agents accumulate both physical and human capital. In this model disconnected generations are born at each instant and individual agents face a positive and age-dependent probability of death at each

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moment in time. By making the mortality rate age-dependent, the model can be used to investigate changes in adult mortality.

The other building block of our analysis concerns the engine of growth. Following Lucas (1988), we assume that the purposeful accumulation of human capital forms the core mechanism leading to economic growth. More specifically, like Bils and Klenow (2000), Kalemni-Ozcan et al. (2000), de la Croix and Licandro (1999), and Boucekkine et al. (2002), we assume that individual agents accumulate human capital by engaging in full-time educational activities at the start of life. The start-up education period is chosen optimally and the human capital production function may include an intergenerational external effect of the 'shoulders of giants' variety, as proposed by Azariadis and Drazen (1990).

As we motivate in more detail later on in this chapter, we extend the existing literature in the following directions. First, we generalize Kalemni-Ozcan et al. (2000) by incorporating a realistic (rather than a Blanchard) demographic structure, allowing for non-zero intergenerational spillovers, and by fully characterizing the transitional dynamics. Second, we generalize the analysis by de la Croix and Licandro (1999) and Boucekkine et al. (2002) by incorporating both human and physical capital, by including a concave (rather than linear) felicity function, and by allowing the intergenerational spillover to differ from unity. Third, we generalize the model of Bils and Klenow (2000) by recognising fully-insured-against lifetime uncertainty (rather than a fixed planning horizon), by assuming a more realistic human capital production function, and by characterizing the transitional dynamics. Finally, we generalize all these papers by including an educational subsidy and a labour income tax.

The remainder of this chapter is organized as follows. In Section 3.2 we present the model and demonstrate its main properties. A unique solution for the optimal schooling period is derived which depends on the fiscal parameters and on the mortality process. The mortality process, in combination with the birth rate, also determines a unique path for the population growth rate. For a given initial level of per capita human capital, the model implies a unique time path for all macroeconomic variables. Depending on the strength of the intergenerational external effect, the model either displays exogenous growth (ultimate convergence to constant per capita variables) or endogenous growth (convergence to a constant growth rate).

In Section 3.3 we study the determinants of the optimal schooling decision in detail. An increase in the educational subsidy or the labour income tax leads to an increase in the length of the educational period. Similarly, a reduction in adult mortality also prompts agents to increase the schooling period. In contrast, a reduction in child mortality and a baby bust both leave the optimal schooling period unchanged.

In Section 3.4 we investigate the effects of changes in the birth rate and adult mortality on the population growth rate, both at impact, during transition, and in the long run. A reduction in the birth rate reduces the steady-state population growth rate, whilst an increase in longevity (due to reduced adult mortality) increases this rate because average mortality falls. We use the estimated Gompertz-Makeham mortality process of the previous chapter to illustrate the rather complicated (cyclical) adjustment path resulting from once-off demographic changes. Especially for the embodied mortality shock, convergence toward the new steady state is extremely slow. Indeed, due to the vintage nature of the population, more than 150 years pass until the new demographic steady state is reached.

Section 3.5 deals with the exogenous growth model. In this model there is no or an imperfect intergenerational spillovers and the economy settles at a unique steady state *level* of per capita human capital. We consider this model, on the basis of the empirical evidence, to be the most relevant one. In Section 3.5 we study the effects of fiscal and demographic changes on per capita human capital and the other macroeconomic variables both at impact, in the transition period and in the long run. A positive fiscal impulse leads to an increase in the per capita stock of human capital but leaves the steady-state growth rate of the macro-variables in level terms unchanged (and equal to the steady-state population growth rate). Furthermore, whilst a reduction in the birth rate and an increase in longevity (due to reduced adult mortality) both increase the steady-state per capita human capital stock, the growth effects on level variables are opposite in sign. Again, for both fiscal and demographic shocks, the transitional adjustment is rather slow.

In Section 3.6 we briefly discuss the endogenous growth version of the model. Though this knife-edge case has been studied extensively in the theoretical literature (Azariadis and Drazen, 1990), it is based on an unrealistically strong intergenerational external effect in human capital creation for which very little empirical backing exists. The positive fiscal impulse boosts the steady-state growth rate in per capita human capital due to the scale effect in the growth process. The growth effects of demographic changes are theoretically ambiguous. For a realistic model calibration, however, the asymptotic growth rate is decreasing in the birth rate and in longevity (as measured by life expectancy at birth).

Finally, in Section 3.7 we present some concluding thoughts and give some suggestions for future research. The Appendix contains some key mathematical derivations.

3.2 The model

3.2.1 Households

Individual plans

The core of the model is the same as in Chapter 2. At time t, an individual born at time v ($v \leq t$) has the following (remaining) lifetime utility function:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^{\infty} U[\bar{c}(v,\tau)] e^{-\theta\cdot(\tau-t)-M(\tau-v)} d\tau, \tag{3.1}$$

where $U[\cdot]$ is the felicity function, $\bar{c}(v,\tau)$ is consumption where a bar denotes individual variables and functions as before. θ is the constant pure rate of time preference ($\theta > 0$), and $e^{-M(\tau-v)}$ is the probability that the agent is still alive at time τ. The cumulative mortality rate, $M(\tau - v)$, is defined in Equation (2.3) on page 14 as $M(\tau - v) \equiv \int_0^{\tau-v} m(\alpha) d\alpha$, where $m(\alpha)$ is the instantaneous mortality rate of an agent of age α. As before (see Equation (2.6)), the felicity function is iso-elastic:

$$U[\bar{c}(v,\tau)] = \begin{cases} \dfrac{\bar{c}(v,\tau)^{1-1/\sigma} - 1}{1 - 1/\sigma} & \text{for } \sigma \neq 1 \\ \ln \bar{c}(v,\tau) & \text{for } \sigma = 1 \end{cases}, \tag{3.2}$$

where σ is the constant intertemporal substitution elasticity ($\sigma \geq 0$).

The budget identity is given by:

$$\dot{\bar{a}}(v,\tau) = [r + m(\tau - v)]\bar{a}(v,\tau) + \bar{w}(v,\tau) - \bar{g}(v,\tau) - \bar{c}(v,\tau), \tag{3.3}$$

where $\bar{a}(v,\tau)$ is real financial wealth, r is the constant world interest rate, $\bar{w}(v,\tau)$ is wage income, and $\bar{g}(v,\tau)$ is total tax payments (see below). As usual, a dot above a variable denotes that variable's time rate of change, e.g. $\dot{\bar{a}}(v,\tau) \equiv d\bar{a}(v,\tau)/d\tau$. As in the previous chapter, we follow Yaari (1965) and Blanchard (1985), by assuming the existence of a perfectly competitive life insurance sector which offers actuarially fair annuity contracts to the agents. Since someone's age is directly observable, the annuity rate of interest faced by an individual of age $\tau - v$ is equal to the sum of the world interest rate and the instantaneous mortality rate of that person. In order to avoid having to deal with a taxonomy of different cases, we again restrict attention to the case of a nation populated by patient agents, i.e. $r > \theta$. Financial wealth can be held in the form of claims on domestic capital, $\bar{v}(v,\tau)$, domestic government bonds, $\bar{d}(v,\tau)$, or net foreign assets, $\bar{f}(v,\tau)$.

$$\bar{a}(v,\tau) \equiv \bar{v}(v,\tau) + \bar{d}(v,\tau) + \bar{f}(v,\tau). \tag{3.4}$$

These assets are perfect substitutes in the agents' investment portfolios and thus attract the same rate of return.

To allow for economic growth, we extend this model by postulating that the agent engages in full time schooling during the early stages of life and works full time thereafter. The production function for human capital is given by:[1]

$$\bar{h}(v,\tau) = \begin{cases} 0 & \text{for } v \le \tau \le v + s(v) \\ A_H h(v)^{\phi} s(v) & \text{for } \tau > v + s(v) \end{cases}, \qquad 0 \le \phi \le 1, \tag{3.5}$$

where $\bar{h}(v,\tau)$ is the human capital of the agent upon completion of the schooling period, A_H is an exogenous productivity index, $h(v)$ is *per capita* human capital at time v (see below), ϕ is a parameter regulating the strength of the intergenerational external effect in knowledge creation, and $s(v)$ is the length of the schooling period chosen by an agent born at time v. Special cases of (3.5) are used by de la Croix and Licandro (1999, p. 257) and Boucekkine et al. (2002, p. 347), who set $\phi = 1$, and by Kalemni-Ozcan et al. (2000, pp. 5, 10), who set $\phi = 0$.

Available human capital is rented out to competitive producers so that wage income, $\bar{w}(v,\tau)$, can be written as:

$$\bar{w}(v,\tau) = w(\tau)\bar{h}(v,\tau), \tag{3.6}$$

where $w(\tau)$ is the market-determined rental rate of human capital.

The tax system takes the following form. First, all through life, the agent pays a lumpsum tax. Second, during the educational phase, the agent receives a study grant from the government. Third, during working life, the agent faces a labour

[1] This formulation was first proposed in the context of Diamond-Samuelson style overlapping models by Azariadis and Drazen (1990, p. 510) and Tamura (1991, p. 524). Abstracting from their work experience term and using our notation, Bils and Klenow (2000, p. 1161) model the human capital production function as follows:

$$\bar{h}(v,t) = \bar{h}(v-\bar{u},t)^{\phi} e^{\zeta(s)}, \qquad \text{for } t - v > s, \tag{3.5'}$$

where $\bar{u}$ is interpreted as the age of the teachers (assumed to be fixed), and $\zeta(s)$ captures the productivity effect of schooling ($\zeta'(s) > 0$). Clearly, for $\zeta(s) \equiv \ln s$ the second term on the right-hand side of (3.5′) is equal to s. In our view, Equation (3.5′) does not adequately capture the notion of an intergenerational externality as the link is only operative between generations v and $v - \bar{u}$, which are locked in a tango through time. In (3.5) the *economy-wide* stock of per capita human capital determines the initial condition facing newborns. Hence, every agent alive at time v exerts an external effect on newborns.

income tax on wage earnings. The tax system is thus given by:

$$\bar{g}(v,\tau) = \begin{cases} [z(\tau)-\rho]w(\tau)A_H h(v)^{\phi} & \text{for } v \le \tau \le v+s(v) \\ [z(\tau)+t_L s(v)]w(\tau)A_H h(v)^{\phi} & \text{for } \tau > v+s(v) \end{cases}, \tag{3.7}$$

where ρ is the *educational subsidy* rate ($\rho > 0$), t_L is the labour income tax rate ($0 \le t_L < 1$), and $z(\tau)$ represents the lumpsum part of the tax. All tax instruments are indexed to the value of marginal schooling productivity to the vintage-v individual (i.e. $A_H h(v)^{\phi}$) to ensure that the tax system continues to play a non-trivial role even in the presence of ongoing economic growth.[2]

From the perspective of the planning date t, the agent chooses remaining time in school ($v+s(v)-t$), and sequences for $\bar{c}(v,\tau)$ and $\bar{a}(v,\tau)$ (for $\tau \in [t,\infty)$) in order to maximize $\Lambda(v,t)$ subject to (3.3)–(3.7), a non-negativity constraint $v+s(v) \ge t$,[3] and a transversality condition. By using this transversality condition as well as Equations (3.3)–(3.7), the lifetime budget constraint for an agent with age $u \equiv t-v$ can be written as follows:

$$e^{M(t-v)} \int_t^{\infty} \bar{c}(v,\tau) e^{-r\cdot(\tau-t)-M(\tau-v)} d\tau = \bar{a}(v,t) + \bar{l}i(v,t), \tag{3.8}$$

where we have used the fact that generations are born without financial assets (i.e. $\bar{a}(v,v)=0$) and where $\bar{l}i(v,t)$ is (remaining) lifetime after-tax wage income of the agent:

$$\begin{aligned} \bar{l}i(v,t) \equiv A_H h(v)^{\phi} e^{M(t-v)} \Bigg[\rho \int_t^{\max\{t,v+s(v)\}} w(\tau) e^{-r\cdot(\tau-t)-M(\tau-v)} d\tau \\ + (1-t_L)s(v) \int_{\max\{t,v+s(v)\}}^{\infty} w(\tau) e^{-r\cdot(\tau-t)-M(\tau-v)} d\tau \\ - \int_t^{\infty} z(\tau) w(\tau) e^{-r\cdot(\tau-t)-M(\tau-v)} d\tau \Bigg]. \end{aligned} \tag{3.9}$$

According to (3.8), the present value of consumption expenditure (left-hand side) must equal total lifetime resources (right-hand side). In the presence of actuarially fair annuity contracts, the annuity rate of interest, $r+m(\tau-v)$, is used for discounting purposes in (3.8)–(3.9).

The following two-stage solution approach can now be used. In the first step,

[2] Alternatively, current gross per capita labour income, $w(\tau)h(\tau)$, could have been used for indexing purposes, but this makes the model intractable.

[3] Older agents have already completed the educational phase ($t-v > s(v)$) and only choose paths for consumption and financial assets. Labour market entry is thus assumed to be an absorbing state.

the agent chooses $s(v)$ in order to maximize lifetime wage income, $\bar{li}(v,t)$. This pushes the lifetime budget constraint out as far as possible and fixes the right-hand side of (3.8). In the second step, the agent chooses the optimal sequence for consumption in order to maximize $\Lambda(v,t)$ subject to (3.8).

Schooling period By using (3.9), the first-order condition for the optimal schooling period, $s^*(v)$, is given by $d\bar{li}(v,t)/ds(v) = 0$ which can be written as:

$$\int_{v+s^*(v)}^{\infty} w(\tau)e^{-r(\tau-v)-M(\tau-v)}d\tau = \left[s^*(v) - \frac{\rho}{1-t_L}\right] w(v+s^*(v))e^{-rs^*(v)-M(s^*(v))}. \tag{3.10}$$

For the case studied in this chapter, the wage rate is constant (see below), and Equation (3.10) reduces to:

$$s^* - \frac{\rho}{1-t_L} = \Delta(s^*, r), \tag{3.11}$$

where $\Delta(u,\lambda)$ is defined in Equation (2.12) on page 16 in the previous chapter. Proposition 2.1 describes the main characteristics of this function.

Equation (3.11) determines the age at which the vintage-v individual completes his education. With a constant mortality process, the optimal schooling period is independent of the agent's date of birth. Since the left-hand side of (3.11) is increasing in s^* and (by Proposition 2.1(ii)) the right-hand side is non-increasing in s^*, it follows that the optimal schooling period is positive and unique.[4] In Section 3.3 below we study changes in the tax parameters and the demographic structure which give rise to once-off changes in the optimal schooling period.

Consumption By using (3.1) and (3.8), the first-order conditions for optimal consumption can be written as $\bar{c}(v,\tau) = e^{\sigma\cdot(r-\theta)(\tau-v)}/\lambda_u$, where λ_u (> 0) is the Lagrange multiplier for the lifetime budget constraint (3.8). Since $r > \theta$, it follows that the agent adopts an upward sloping time profile for its consumption provided the intertemporal substitution elasticity is strictly positive ($\sigma > 0$). The growth rate of individual consumption is thus given by the familiar Euler equation:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = \sigma\cdot(r-\theta), \qquad \text{for } \tau \in [t,\infty). \tag{3.12}$$

[4] Indeed, for the Blanchard case with a constant death rate, $\Delta(u,\lambda) = 1/(\lambda+\mu_0)$, and (3.11) simplifies even further to $s(v) = \rho/(1-t_L) + 1/(r+\mu_0)$. Apart from the fiscal parameters, this is the expression found in de la Croix and Licandro (1999, p. 258)

By using (3.12) in (3.8) the expression for the consumption level in the planning period is obtained:

$$\Delta(u,r^*)\bar{c}(v,t) = \bar{a}(v,t) + \bar{l}i(v,t), \tag{3.13}$$

where $r^* \equiv r - \sigma \cdot (r - \theta)$ can be interpreted as the effective discount rate facing the agent.

Demography

We allow for non-zero population growth by employing the analytical framework developed by Buiter (1988) which we extended in the previous chapter to include a non-constant mortality rate. To allow for ageing shocks later on, we must extend this model even further. In the previous chapter we assumed that everybody faces the same mortality profile. Here we drop this assumption and assume instead that different cohorts may face different mortality profiles, but that these cohort specific profiles only depend on the individuals age. The instantaneous mortality rate is $m(\alpha, \psi_m(v))$, where $\psi_m(v)$ is a parameter that only depends on the time of birth. We denote the corresponding cumulative mortality by $M(u, \psi_m(v)) = \int_0^u m(\alpha, \psi_m(v))d\alpha$. Wherever possible, we drop the dependency of ψ_m on v or even the dependency of m and M on ψ_m.

The birth rate varies over time, but is still exogenous by assumption. The size of a newborn generation at time v is proportional to the current population at that time, i.e. $L(v,v) = b(v)L(v)$, where $b(v)$ is the – time varying – crude birth rate ($b(v) > 0$), and $L(v)$ is the population size at time v. The size of cohort v at some later time τ is given by:

$$L(v,\tau) = L(v,v)e^{-M(\tau-v,\psi_m(v))} = bL(v)e^{-M(\tau-v,\psi_m)}. \tag{3.14}$$

By definition, the total population at time t satisfies the following expressions:

$$L(t) \equiv \int_{-\infty}^{t} L(v,t)dv, \tag{3.15}$$

$$L(t) \equiv L(v)e^{N(v,t)}, \qquad N(v,t) \equiv \int_v^t n(\tau)d\tau, \tag{3.16}$$

where $n(\tau)$ is the growth rate of the population at time τ. Finally, by combining

(3.14)–(3.16) we obtain:

$$l(v,t) \equiv \frac{L(v,t)}{L(t)} = b(v)e^{-N(v,t)-M(t-v,\psi_m)}, \qquad t \geq v, \tag{3.17}$$

$$\frac{1}{b(v)} = \int_{-\infty}^{t} e^{-N(v,t)-M(t-v,\psi_m)} dv. \tag{3.18}$$

Equation (3.17) shows the population share of the v-cohort at some later time t. It generalizes the corresponding expression (2.23) on page 21 in Chapter 2 to the case of a non-constant population growth rate, $n(t)$. Equation (3.18) implicitly determines $n(t)$ for given demographic parameters (see also Section 3.4). For an economy which has faced the same demographic environment ($b(v) = b$ and $M(t-v,\psi_m) = M(t-v)$) for a long time, the population growth rate is constant ($n(\tau) = n$) and Equation (3.18) reduces to $1/b = \Delta(0,n)$, which is expression (2.23) on page 21.

Per capita plans

Per capita variables are calculated as the integral of the generation-specific values multiplied by the corresponding generation weights, the same as in the previous chapter, section 2.2.1. For example, per capita human capital is defined as:

$$h(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{h}(v,t)dv, \tag{3.19}$$

where $l(v,t)$ and $\bar{h}(v,t)$ are given in, respectively, (3.17) and (3.5) above.

Turning to the wealth components, per capita financial wealth is defined as $a(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{a}(v,t)dv$. By differentiating this expression with respect to time we obtain the dynamic path of per capita financial assets:[5]

$$\dot{a}(t) = [r - n(t)]a(t) + w(t)h(t) - g(t) - c(t), \tag{3.20}$$

where $g(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{g}(v,t)dv$ is per capita tax payments. We assume that the interest rate net of population growth is positive, i.e. $r > n(t)$. As in the standard Blanchard model, annuity payments drop out of the expression for per capita asset accumulation because they constitute transfers (via the life insurance companies) from the deceased to agents who continue to enjoy life.

[5] In deriving (3.20) we have used Equation (3.3) and noted the fact that agents are born without financial assets ($\bar{a}(t,t) = 0$).

3.2.2 Firms

Perfectly competitive firms use physical and human capital to produce a homogeneous commodity, $Y(t)$, that is traded internationally. The technology is represented by the following Cobb-Douglas production function:

$$Y(t) = K(t)^{\varepsilon}[A_Y H(t)]^{1-\varepsilon}, \qquad 0 < \varepsilon < 1, \tag{3.21}$$

where A_Y is a constant index of labour productivity, $K(t) \equiv L(t)k(t)$ is the aggregate stock of physical capital, and $H(t) \equiv L(t)h(t)$ is the aggregate stock of human capital. The cash flow of the representative firm is given by:

$$\Pi(t) \equiv Y(t) - w(t)H(t) - I(t), \tag{3.22}$$

where $w(t)$ is the rental rate on human capital, and $I(t) \equiv \dot{K}(t) + \delta K(t)$ is gross investment, with δ representing the constant depreciation rate. The (fundamental) stock market value of the firm at time t is equal to the present value of cash flows, using the interest rate for discounting, i.e. $V(t) \equiv \int_t^{\infty} \Pi(\tau) e^{r[t-\tau]} d\tau$. The firm chooses paths for $I(\tau)$, $K(\tau)$, $H(\tau)$, and $Y(\tau)$ (for $\tau \in [t, \infty)$) to maximize $V(t)$ subject to the capital accumulation constraint, the production function (3.21) and the definition of cash flows (3.22). Since there are no adjustment costs on investment, the value of the firm equals the replacement value of the capital stock, i.e. $V(t) = K(t)$. In addition, the usual factor demand equations are obtained:

$$r + \delta = \varepsilon \left[\frac{A_Y h(t)}{k(t)}\right]^{1-\varepsilon} = \frac{\partial Y(t)}{\partial K(t)}, \tag{3.23}$$

$$w(\tau) = (1-\varepsilon) A_Y \left[\frac{A_Y h(t)}{k(t)}\right]^{-\varepsilon} = \frac{\partial Y(\tau)}{\partial H(\tau)}. \tag{3.24}$$

For each factor of production, the marginal product is equated to the rental rate. Since the fixed world interest rate pins down the ratio between human and physical capital, it follows from (3.24) that the wage rate is time-invariant, i.e. $w(\tau) = w$,[6]

[6] With labour-augmenting technological change, $\gamma_A \equiv \dot{A}_Y / A_Y$, the wage rate grows exponentially at rate γ_A and Equation (3.11) changes to:

$$s^* - \frac{\rho}{1 - t_L} = \Delta(s^*, r - \gamma_A).$$

It follows from Proposition 1(i) that $\partial s^* / \partial \gamma_A > 0$, i.e. the schooling period depends positively on anticipated wage growth. See also Bils and Klenow (2000, p. 1161) on this issue.

and that physical capital is proportional to human capital at all time:

$$k(t) = A_Y \left[\frac{\varepsilon}{r+\delta} \right]^{1/(1-\varepsilon)} h(t). \tag{3.25}$$

3.2.3 Government and foreign sector

In the absence of government consumption, the government (flow) budget identity in per capita terms is given by:

$$\dot{d}(t) = [r - n(t)]d(t) - g(t), \tag{3.26}$$

where $d(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{d}(v,t)dv$ is per capita government debt. The government solvency condition is $\lim_{\tau\to\infty} d(\tau)e^{r\cdot(t-\tau)+N(t,\tau)} = 0$, so that the intertemporal budget constraint of the government can be written as:

$$d(t) = \int_{t}^{\infty} g(\tau)e^{r\cdot(t-\tau)+N(t,\tau)}d\tau. \tag{3.27}$$

To the extent that there is outstanding debt (positive left-hand side), it must be exactly matched by the present value of current and future primary surpluses (positive right-hand side), using the net interest rate ($r - n(\tau)$) for discounting purposes.

By using the marginal productivity conditions (3.23)–(3.24) and noting the linear homogeneity of the production function (3.21) and the constancy of factor prices, we find that per capita output, $y(t) \equiv Y(t)/L(t)$, can be written as follows:

$$\begin{aligned} y(t) &= (r+\delta)k(t) + wh(t) \\ &= \left[(r+\delta)^{\varepsilon/(\varepsilon-1)}(\varepsilon A_Y)^{1/(1-\varepsilon)} + w \right] h(t). \end{aligned} \tag{3.28}$$

In going from the first to the second line we have made use of (3.25). It follows from the definition of gross investment that the dynamic evolution of the per capita stock of capital is given by:

$$\dot{k}(t) = i(t) - [\delta + n(t)]k(t), \tag{3.29}$$

where $i(t) \equiv I(t)/L(t)$ is per capita investment. Finally, the current account of the balance of payment, representing the dynamic change in the per capita stock of net foreign assets, $f(t)$, takes the following form:

$$\dot{f}(t) = [r - n(t)]f(t) + y(t) - c(t) - i(t), \tag{3.30}$$

where $f(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{f}(v,t)dv$.[7]

3.2.4 Model solution

The model is recursive and can be solved in three steps. First, for a given demography and with constant tax parameters ρ and t_L, Equation (3.11) determines the optimal schooling period for each agent. Similarly, for a given birth rate, Equation (3.18) can be solved for the population growth rate, $n(t)$. Next, conditional on the optimal value for s^* and the path for $n(t)$, Equation (3.19) can be solved for the equilibrium path of human capital, $h(t)$. Finally, the lumpsum tax z is used to balance the government's intertemporal budget restriction (3.27), after which the values for all remaining variables are fully determined.

In Section 3.3 the effect on the optimal schooling period of both fiscal and demographic shocks are studied. Next, we note that the path for human capital depends critically on the magnitude of the intergenerational externality parameter, ϕ. For values of ϕ in the range $0 \leq \phi < 1$, the model implies a unique steady-state *level* of per capita human capital, i.e. the long-run growth rate in the economy is exogenous (and equal to the population growth rate). This *exogenous growth* case is studied in Section 3.5.

For the knife-edge case with $\phi = 1$, Equation (3.19) gives rise to a unique steady-state *growth rate* in per capita human capital, so that the long-run growth rate is endogenous. This *endogenous growth* model is studied in Section 3.6 below.

3.3 Determinants of schooling

In this section we study the comparative static effect on the optimal schooling period of changes in the fiscal parameters and the demographic process. To keep things simple, only stepwise changes are considered that occur at impact. The time at which the unanticipated and permanent shock occurs is normalised at $t = 0$.

[7] The dynamic expression for per capita assets is given in Equation (3.20), where $a(t) \equiv k(t) + d(t) + f(t)$ (recall that $V(t) = K(t)$). Clearly, total per capita assets $a(t)$ move smoothly over time but its constituting components ($k(t)$, $f(t)$, and $d(t)$) need not. Hence, even in the absence of discrete adjustments in government debt, the capital stock can jump as only $k(t) + f(t)$ moves smoothly over time in that case. A discrete change in $k(t)$ would be engineered by means of an asset swap. Throughout this chapter, however, the world interest rate (r) is held constant so that (via (3.25)) the physical capital stock, $k(t)$, will evolve smoothly because the stock of human capital, $h(t)$, moves smoothly. As a result, the model also gives rise to well-defined current account dynamics—see also Figures 3.4–3.6 below.

3.3.1 Fiscal shocks

The effect of an increase of the educational subsidy on the optimal schooling period have been illustrated in Figure 3.1(a) for the case with a Gompertz-Makeham (G-M) mortality process fitted to actual mortality data for the cohort born in the Netherlands in 1920 (see Chapter 2, Table 2.1 for details).

In terms of Figure 3.1(a), the initial optimum, s_0^*, occurs at the intersection of the line labelled $\Delta + [\rho/(1-t_L)]_0$ and the 45° line. An increase in either ρ or t_L leads to a parallel upward shift in the former line to $\Delta + [\rho/(1-t_L)]_1$ so that the new equilibrium is at s_1^*.

By using (3.11) the comparative static effects of fiscal changes can be computed:

$$\frac{\partial s^*}{\partial \rho} = \frac{1}{(1-t_L)(1-\partial\Delta/\partial s^*)} > 0, \tag{3.31}$$

$$\frac{\partial s^*}{\partial t_L} = \frac{\rho}{(1-t_L)^2(1-\partial\Delta/\partial s^*)} > 0, \tag{3.32}$$

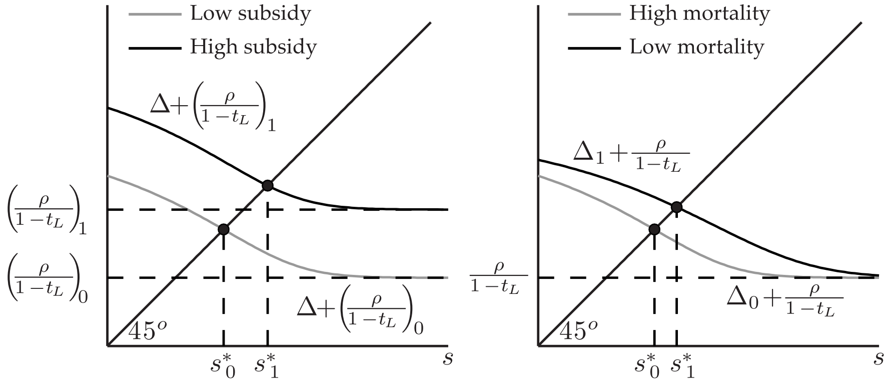
where the signs follow from the fact that $\partial\Delta/\partial s^* \leq 0$ (see Proposition 2.1(ii) on page 17). Not surprisingly, an increase in the educational subsidy leads to a reduction in the opportunity cost of schooling and a longer optimal schooling period. Interestingly, provided the educational subsidy is strictly positive, an increase in the marginal labour income tax also increases the optimal schooling period. Because the educational subsidy is untaxed, the *effective* subsidy affecting the schooling decision is $\rho/(1-t_L)$, which is increasing in t_L.

3.3.2 Demographic shocks

Two types of demographic shocks are considered in our analysis, namely a change in the birth rate and a change in the mortality process. Clearly, in view of (3.11), the birth rate does not affect the optimal schooling period. The mortality process, however, does affect the $\Delta(u,\lambda)$ function and thus the optimal schooling decision. In order to study the effects of changes in the demographic process, we use the notation introduced in section 3.2 and write the instantaneous mortality rate as $m(\alpha,\psi_m)$, where ψ_m is a parameter.[8] In order to investigate the effects of a change in ψ_m we make the following assumptions.

[8] In the Blanchard case, which has only one parameter, μ_0 could be $-\psi_m$ or any decreasing function of ψ_m. For the G-M process, which depends on three parameters (see Table 2.1 on page 31), the parameter *vector* is a function of ψ_m, $(\mu_0,\mu_1,\mu_2) = f(\psi_m)$, and an increase in ψ_m should result in such a change that the G-M mortality function decreases for all ages as ψ_m increases.

Figure 3.1. Effects of educational fiscal shocks (a) and mortality shocks (b) on the optimal schooling period



(a) Fiscal impulse (b) Reduced adult mortality

Assumption 3.1. *The mortality function has the following properties:*

(i) $m(\alpha, \psi_m)$ *is non-negative, continuous, and non-decreasing in age,* $\partial m(\alpha, \psi_m)/\partial\alpha \geq 0$*;*

(ii) $m(\alpha, \psi_m)$ *is convex in age,* $\partial^2 m(\alpha, \psi_m)/\partial\alpha^2 \geq 0$*;*

(iii) $m(\alpha, \psi_m)$ *is non-increasing in* ψ_m *for all ages,* $\partial m(\alpha, \psi_m)/\partial\psi_m \leq 0$*;*

(iv) the effect of ψ_m *on the mortality function is non-decreasing in age,* $\dfrac{\partial^2 m(\alpha, \psi_m)}{\partial\psi_m \partial\alpha} \leq 0.$

An example of a mortality shock satisfying all the requirements of Assumption 3.1 consists of a decrease in μ_1 or μ_2 of the G-M mortality function. In terms of Figure 3.2(a), the shock shifts the mortality function downward, with the reduction in mortality being increasing in age. In panel (b) the function for the surviving fraction of the population shifts to the right. The shock that we consider can thus be interpreted as a reduction in adult mortality. Of course, in view of the terminology of Assumption 1, an increase in ψ_m leads to an increase in the expected remaining lifetime for all ages.

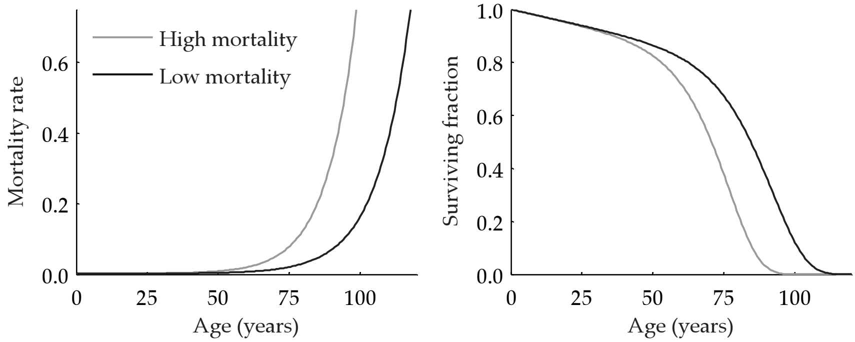
The following results can now be proved.

Proposition 3.1. *Define* $M(u, \psi_m)$ *and* $\Delta(u, \lambda, \psi_m)$ *as:*[9]

$$M(u, \psi_m) \equiv \int_0^u m(\alpha, \psi_m) d\alpha, \tag{3.33}$$

[9] These definitions are generalisations of Equations (2.3) on page 14 and (2.12) on page 16.

Figure 3.2. The effect of reduced adult mortality on the mortality rate (a) and the surviving fraction (b)



(a) Mortality rate, $m(u) = \mu_0 + \mu_1 e^{\mu_2 u}$ (b) Surviving fraction, $S(u) = e^{-M(u)}$

Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values). Low mortality correspond to a 50% decrease of μ_1 and 10% decrease of μ_2.

$$\Delta(u, \lambda, \psi_m) \equiv e^{\lambda u + M(u,\psi_m)} \int_u^\infty e^{-\lambda\alpha - M(\alpha,\psi_m)} d\alpha. \tag{3.34}$$

Under Assumption 3.1, the following results can be established.

(i) $\dfrac{\partial M(u, \psi_m)}{\partial \psi_m} = \displaystyle\int_0^u \frac{\partial m(\alpha, \psi_m)}{\partial \psi_m} d\alpha \leq 0;$

(ii) $\dfrac{\partial^2 M(u, \psi_m)}{\partial u \partial \psi_m} = \dfrac{\partial m(u, \psi_m)}{\partial \psi_m} \leq 0;$

(iii) $\dfrac{\partial \Delta(u, \lambda, \psi_m)}{\partial \psi_m} = e^{\lambda u + M(u,\psi_m)} \displaystyle\int_u^\infty \left[\frac{\partial M(u,\psi_m)}{\partial \psi_m} - \frac{\partial M(\alpha,\psi_m)}{\partial \psi_m}\right] e^{-\lambda\alpha - M(\alpha,\psi_m)} d\alpha > 0.$

Proof. (i) and (ii) follow from simple differentiation and noting assumption 3.1(iii). (iii) follows from differentiation of (3.34) and (i). ■

By using Equation (3.11), and noting the definition (3.34), the comparative static effect on the optimal schooling period of a reduction in adult mortality can be computed:

$$\frac{\partial s^*}{\partial \psi_m} = \frac{\partial \Delta / \partial \psi_m}{1 - \partial \Delta / \partial s^*} > 0, \tag{3.35}$$

where the sign follows from the fact that $\partial \Delta / \partial s^* \leq 0$ (see Proposition 2.1(ii)) and $\partial \Delta / \partial \psi_m > 0$ (see Proposition 3.1(iii)). An increase in longevity prompts agents to increase their human capital investment at the beginning of life. In terms of Figure

3.1(b), the mortality shock shifts the Δ-function to the right, and leads to an increase in the optimal schooling period from s_0^* to s_1^*.

Bils and Klenow argue that a higher life expectancy (as captured in their model by an increase in the exogenous planning horizon) leads to an increase in the optimal schooling period 'since it affords a longer working period over which to reap the wage benefits of schooling' (2000, p. 1164). Similarly, de la Croix and Licandro (1999, p. 258) and Kalemni-Ozcan et al. (2000, p. 11), using the Blanchard demography, show that a decrease in the death probability leads to an increase in the expected planning horizon for all agents and an increase in the optimal schooling period. Our discussion shows that these conclusions are misleading in the presence of lifetime uncertainty *and* age-dependent mortality. In our model, a decrease in child mortality increases expected remaining life time at birth but leaves the optimal schooling period unchanged. In terms of Figure 3.1(b), reduced child mortality flattens the left-hand section of the line $\Delta_0 + \rho/(1 - t_L)$ but the equilibrium solution stays at s_0^*.[10] [11] Of course, with the Blanchard demography one cannot distinguish between child mortality and adult mortality because the death probability is age-independent.

3.4 Demographic shocks and population growth

Demographic changes affect the growth rate of the population, both at impact, during transition, and in the long run. Armed with Proposition 2.1 and 3.1 we can compute the long-run effects of changes in the birth rate and the mortality process. Indeed, since Equation (3.18) reduces in the steady state to $b\Delta(0, \hat{n}, \psi_m) = 1$, it follows that $\hat{n}$ is an implicit function of b and ψ_m, the partial derivatives of which are given by:

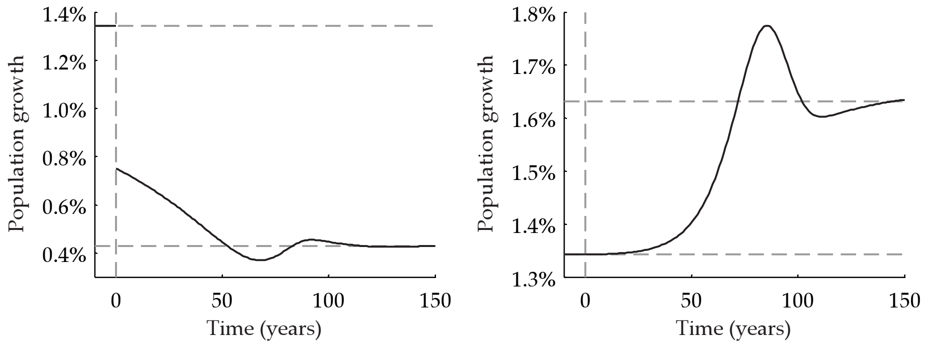
$$\frac{\partial \hat{n}}{\partial b} = -\frac{\Delta(0, \hat{n}, \psi_m)}{b\partial\Delta(0, \hat{n}, \psi_m)/\partial\hat{n}} > 0, \tag{3.36}$$

$$\frac{\partial \hat{n}}{\partial \psi_m} = -\frac{\partial\Delta(0, \hat{n}, \psi_m)/\partial\psi_m}{\partial\Delta(0, \hat{n}, \psi_m)/\partial\hat{n}} > 0, \tag{3.37}$$

[10] Boucekkine et al. also distinguish age-dependent mortality and argue that 'an increase in life expectancy increases the optimal length of schooling' (2002, pp. 352, 370). They thus fail to notice that the mechanism producing this result runs via reduced old-age mortality, not via increased life expectancy in general.

[11] Bils and Klenow (2000, p. 1175) also report that their model implies an unrealistically high sensitivity of the optimal schooling period with respect to life expectancy that is close to unity. In contrast, in the calibrated version of our model, $ds^*/dR(0) = 0.06$ which comes close to the empirical estimate mentioned by Bils and Klenow (2000, p. 1175n27).

Figure 3.3. Population growth rate after a baby bust (a) and an adult mortality shock (b).



(a) Baby bust (b) Reduced adult mortality

Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values), birth rate is 2.36%. Baby bust is a 25% downward jump of the birth rate to 1.78%. Reduced adult mortality is a 50% decrease of μ_1 and 10% decrease of μ_2.

where a hat overstrike designates the steady-state value of a variable, i.e. $\hat{n}$ is the steady-state growth rate of the population. The signs in (3.36)–(3.37) follow from Propositions 2.1(i) and 3.1(iii). Not surprisingly, an increase in the birth rate and an increase in longevity both lead to an increase in the steady-state growth rate of the population.

To compute the transition path for the growth rate of the population we assume that at time $t = 0$ both the mortality process and the birth rate change in a stepwise fashion. The mortality shock is assumed to be *embodied*, i.e. it only affects generations born from time $t = 0$ onwards. Indeed, the mortality process for pre-shock cohorts (with a negative generation index, $v < 0$) is described by $M_0(t-v)$ and $m_0(t-v)$, whereas post-shock cohorts (with $v \geq 0$) face the mortality process described by $M_1(t-v)$ and $m_1(t-v)$. In a similar fashion, the pre-shock and post-shock birth rates are denoted by, respectively, b_0 and b_1. The system is initially in a demographic steady state and the pre-shock population growth rate is denoted by $\hat{n}_0$ (defined implicitly by the condition $1 = b_0\Delta_0(0, \hat{n}_0)$, where $\Delta_0(0, \hat{n}_0)$ is the Δ-function associated with the initial mortality process).

As a consequence of the demographic changes, the path for the population growth rate is implicitly determined by the following expression:

$$1 = b_0 \int_{-\infty}^{0} e^{-M_0(t-v)-N(v,t)} dv + b_1 \int_0^t e^{-M_1(t-v)-N(v,t)} dv, \tag{3.38}$$

where $N(v,t) \equiv \int_v^t n(\tau)d\tau$ (see also (3.16) above). In Box 3.1 we show that Equation (3.38) can be rewritten in the form of a linear Volterra equation of the second kind with a convolution-type kernel for which efficient numerical solution algorithms are available. In Figure 3.3 we plot the transition path for $n(t)$ for both types of demographic shocks. Panel (a) depicts the path for a baby bust. There is an immediate downward jump at impact ($n(0) = \hat{n}_0 - b_0 + b_1$) followed by gradual cyclical adjustment. Adjustment is rather fast because the birth rate change applies to the entire (pre-shock and post-shock) population alike. Panel (b) of Figure 3.3 depicts the adjustment path following a decrease in adult mortality. Nothing happens at impact and the population growth rate only gradually rises to its long-run steady-state value. Transition is much slower than for the baby bust because the ageing shock is embodied, i.e. the shock only applies to post-shock generations and pre-shock generations only die off gradually during the demographic transition.

BOX 3.1

Population growth after demographic shocks

The transition path for $n(t)$ is determined implicitly by Equation (3.38) in the text, but this equation is useless to compute the path of the population growth rate. We can however rewrite this equation into a so-called Volterra Equation of the second kind, for which there are standard (and efficient) solution algorithms. Start by multiplying both sides of this expression by $e^{N(0,t)}$, and noting that $N(v,t) \equiv N(0,t) - N(0,v)$

$$e^{N(t)} = b_0 \int_{-\infty}^{0} e^{-M_0(t-v)+N(v)} dv + b_1 \int_0^t e^{-M_1(t-v)+N(v)} dv,$$

where we define $N(t) \equiv N(0,t)$ for notational convenience. Since $N(v) = \hat{n}_0 v$ for $v < 0$ we find

$$e^{N(t)} = b_0 e^{\hat{n}_0 t} \int_{-\infty}^{0} e^{-\hat{n}_0 \cdot (t-v) - M_0(t-v)} dv + b_1 \int_0^t e^{N(v)-M_1(t-v)} dv$$

$$= b_0 e^{\hat{n}_0 t} \int_t^\infty e^{-\hat{n}_0 u - M_0(u)} du + b_1 \int_0^t e^{N(v)-M_1(t-v)} dv \tag{3.39}$$

$$= b_0 e^{-M_0(t)} \Delta_0(t, \hat{n}_0) + b_1 \int_0^t e^{N(v)-M_1(t-v)} dv, \tag{3.40}$$

where we changed variables from the cohort domain to the age domain in going from the first to the second line, and use the definition of $\Delta_0(t, \hat{n}_0)$ in going from the second to the third line.

Since the long-run population growth rate equals $\hat{n}_1$, it follows that (3.40) can be rewritten in a stationary format by multiplying both sides of the expression by $e^{\hat{n}_1 t}$. We obtain:

$$\xi(t) = \chi(t) + \int_0^t \boldsymbol{K}(t-v)\xi(v)dv, \tag{3.41}$$

where $\xi(t) \equiv e^{N(t)-\hat{n}_1 t}$, $\chi(t) \equiv b_0 e^{-M_0(t)-\hat{n}_1 t}\Delta_0(t, \hat{n}_1)$, and $\boldsymbol{K}(t-v) \equiv b_1 e^{-M_1(t-v)-\hat{n}_1(t-v)}$. Equation (3.41) is a so-called a *renewal equation*, i.e. a linear Volterra equation of the second kind with a convolution type kernel—see *inter alia* Linz (1985, p. 14) and Bellman and Cooke (1963, ch. 7). We use the standard solution algorithm proposed by Linz (1985, p 98) which generates approximations for $\xi(t)$ on a grid with constant step size. If we denote the step size by ϵ, we can approximate $\xi(t)$ at each gridpoint $t = i\epsilon$, $i = 1, 2 \ldots$

$$\xi(i\epsilon) \doteq \chi(i\epsilon) + \epsilon \left[\frac{1}{2}\boldsymbol{K}(i\epsilon)\xi(0) + \sum_{j=1}^{i-1} \boldsymbol{K}([i-j]\epsilon)\xi(j\epsilon) + \frac{1}{2}\boldsymbol{K}(0)\xi(i\epsilon) \right].$$

where we used the simple trapezoidal rule to numerically evaluate the integral. Solving this equation for $\xi(i\epsilon)$ we get

$$\xi(i\epsilon) \doteq \frac{\chi(i\epsilon) + \epsilon \left[\frac{1}{2}\boldsymbol{K}(i\epsilon)\xi(0) + \sum_{j=1}^{i-1} \boldsymbol{K}([i-j]\epsilon)\xi(j\epsilon) \right]}{1 - \epsilon K(0)/2}.$$

The value of $\xi(0)$ is known, $\xi(0) = \chi(0)$. From this we can calculate the value of $\xi(\epsilon)$ and keep on jumping forward in time until we have all the required points. From the path of $\xi(t)$ it is easy to derive path for $n(t)$ by noting that $n(t) \equiv \tilde{\xi}(t) - \hat{n}_1$, where $\tilde{\xi}(t) \equiv d \ln \xi(t)/dt$ can be computed easily with the aid of finite difference methods. The only problem is to determine the initial growth rate, $n(0)$, because the population growth might not be continuous at

$t = 0$ so finite difference methods do not work. By differentiating (3.39)) with respect to time and evaluating the result for $t = 0$ we find that $n(0) = \hat{n}_0 - b_0 + b_1$. In deriving this result we make use of the fact that $M_0(0) = M_1(0) = 0$, $N(0) = 0$, and $b_0\Delta_0(0, \hat{n}_0) = 1$. For our purposes, setting $\epsilon = 1$ is accurately enough, but if more accuracy is required, we can decrease the stepsize ϵ.

3.5 Exogenous growth

In Section 3.3 it was shown that both fiscal and demographic shocks lead to a change in the optimal schooling period, s^*. In this section we study the resulting transitional and long-run effects on human capital formation for the *exogenous growth* case, i.e. we assume that the intergenerational knowledge transfer incorporated in the human capital production function (3.5) is either absent ($\phi = 0$) or subject to diminishing returns ($0 < \phi < 1$). First, in Section 3.5.1 we analytically characterize the steady-state and study its sensitivity with respect to fiscal and demographic shocks. Next, in Section 3.5.2 we visualise the rather complicated transitional dynamics associated with the various shocks for a plausibly parametrized model which incorporates the estimated G-M process introduced above.

3.5.1 Long-run effects

In the long-run equilibrium, Equation (3.19) gives rise to the following expression for the steady-state stock of per capita human capital, $\hat{h}$:

$$\hat{h}^{1-\phi} = A_H s^* b \int_{s^*}^{\infty} e^{-\hat{n}u - M(u,\psi_m)} du. \tag{3.42}$$

Equation (3.42) clearly shows the various mechanisms affecting $\hat{h}$, namely (i) the birth rate, (ii) the optimal schooling decision of agents, s^*, which itself depends on the fiscal and mortality parameters (ρ, t_L, ψ_m), (iii) the population growth rate, $\hat{n}$, which depends on (b, ψ_m), and (iv) the cumulative mortality factor, $M(u, \psi_m)$, which depends on the mortality parameter ψ_m.

Pure schooling shock In order to facilitate the interpretation of our results, we first study the effects of a change in the schooling period in isolation. By differentiating Equation (3.42) with respect to s^* and simplifying we obtain:

$$\begin{aligned}\frac{\partial \hat{h}^{1-\phi}}{\partial s^*} &= A_H b e^{-\hat{n}s^* - M(s^*,\psi_m)} [\Delta(s^*,\hat{n}) - s^*] \\ &= A_H b e^{-\hat{n}s^* - M(s^*,\psi_m)} \left[\Delta(s^*,\hat{n}) - \Delta(s^*,r) - \frac{\rho}{1-t_L}\right], \end{aligned} \tag{3.43}$$

where we have used (3.11) to arrive at the second expression. In the absence of an educational subsidy ($\rho = 0$), a pure schooling shock unambiguously leads to an increase in the per capita stock of human capital. Indeed, since by assumption the interest rate exceeds the steady-state growth rate of the population ($r > \hat{n}$), it follows from Proposition 2.1(i) that $\Delta(s^*,\hat{n}) > \Delta(s^*,r)$ so that $\partial\hat{h}^{1-\phi}/\partial s^* > 0$ in that case. With a non-zero educational subsidy, Equation (3.43) shows that the effect on $\hat{h}$ of a pure schooling shock is no longer unambiguous because a sufficiently high effective educational subsidy will render the term in square brackets negative even for the case with $r > \hat{n}$. Intuitively, in such a case the economy is 'over-educated', i.e. agents study for too long a period and thus have too short a career as productive workers. Because in actual economies r is much greater than $\hat{n}$ and educational subsidies are typically quite low, we make the following assumption which rules out over-education and ensures that $\partial\hat{h}^{1-\phi}/\partial s^*$ is positive.

Assumption 3.2. *The steady-state net interest rate $r - \hat{n}$ is sufficiently positive to ensure that $\Delta(s^*,\hat{n}) > \Delta(s^*,r) + \rho/(1-t_L)$.*

Fiscal shock A fiscal shock, consisting of an increase in either ρ or t_L, affects the steady-state per capita human capital stock according to:

$$\frac{\partial \hat{h}^{1-\phi}}{\partial\left[\rho/(1-t_L)\right]} = \frac{\partial \hat{h}^{1-\phi}}{\partial s^*} \frac{\partial s^*}{\partial\left[\rho/(1-t_L)\right]} > 0, \tag{3.44}$$

where the sign follows from (3.31)–(3.32) above. The fiscal shock leads to an increase in the optimal schooling period which, in view of Assumption 3.2, leads to an increase in $\hat{h}$.

Birth rate shock A change in the birth rate affects steady-state per capita human capital both directly and via its effect on the steady- state population growth rate. By differentiating Equation (3.42) with respect to b and simplifying we obtain:

$$\frac{\partial \hat{h}^{1-\phi}}{\partial b} = A_H s^* \left[\int_{s^*}^{\infty} e^{-\hat{n}u - M(u,\psi_m)} du - b\frac{\partial \hat{n}}{\partial b} \int_{s^*}^{\infty} u e^{-\hat{n}u - M(u,\psi_m)} du\right] < 0, \tag{3.45}$$

where the sign follows from Lemma 3.1 in Appendix 3.B. Intuitively, a higher birth rate leads to an upward shift in the steady-state path of the human capital stock in *level* terms, but also induces an increase in the population growth rate. The latter effect dominates the former so that *per capita* human capital declines in the steady state.

Mortality shock The mortality change is by far the most complicated shock under consideration because it affects the schooling period, s^*, the population growth rate, $\hat{n}$, and the cumulative mortality factor, $M(u, \psi_m)$. By differentiating (3.42) with respect to ψ_m we obtain:

$$\frac{\partial \hat{h}^{1-\phi}}{\partial \psi_m} = \frac{\partial \hat{h}^{1-\phi}}{\partial s^*} \frac{\partial s^*}{\partial \psi_m} + A_H s^* b \frac{\partial}{\partial \psi_m} \int_{s^*}^{\infty} e^{-\hat{n}u - M(u,\psi_m)} du > 0, \tag{3.46}$$

where the sign follows from (3.35), (3.43), and Lemma 3.2 in Appendix 3.B. The first composite term on the right-hand side is straightforward: increased longevity boosts the optimal schooling period which in turn increases per capita human capital in the steady state. The second term on the right-hand side represents the joint effect of increased longevity on the integral appearing on the right-hand side of (3.42). An increase in ψ_m has two effects on the discounting factor of that integral. First, the population growth rate is increased ($\partial \hat{n} / \partial \psi_m > 0$) leading to heavier discounting and a lower value for the integral. Higher population growth constitutes a higher drag on human capital as the cake must be shared over ever more people. This effect leads to a decrease in per capita human capital. Second, the cumulative mortality factor is decreased for higher age levels ($\partial M(u, \psi_m) / \partial \psi_m < 0$) leading to reduced discounting and a higher integral. Educated people live longer as a result of the shock and per capita human capital increases as a result. Lemma 3.2 in Appendix 3.B shows that, under our set of assumptions regarding mortality change, the first effect is dominated by the second and, ceteris paribus the schooling period, *human-capital deepening* occurs as a result of increased longevity, i.e. the second composite term on the right-hand side of (3.46) is positive.

Balanced growth Up to this point attention has been restricted to steady-state per capita human capital. This focus is warranted because all remaining variables are uniquely related to $\hat{h}$. Indeed, it follows directly from, respectively, (3.25) and (3.28), that $\hat{k}$ and $\hat{y}$ are both proportional to $\hat{h}$. The level of per capita human capital determines individual human capita, which fixes the individual lifetime income profile. From this lifetime income profile and the propensity to consume follow consumption and individual assets. The other per capita variables follow from these individual profiles and the generational weights given in (3.17). In the steady state all per capita aggregate variables are constant, so their levels grow at the steady state population growth rate.

3.5.2 Transitional dynamics

In this subsection we compute and visualise the transitional effects of fiscal and demographic shocks using a plausibly calibrated version of the model.[12] The world interest rate is $r = 0.055$, the pure rate of time preference is $\theta = 0.03$, the intertemporal substitution elasticity is $\sigma = 1$, the capital depreciation rate is $\delta = 0.07$, and the efficiency parameter for physical capital is $\varepsilon = 0.3$.

The human capital externality parameter is set at $\phi = 0.3$. We rationalize this choice as follows. In a recent paper, de la Fuente and Doménech (2006, p. 12) formulate an aggregate production function of the form:

$$\ln y_i(t) = \ln TFP_i(t) + \alpha_1 \ln k_i(t) + \alpha_2' \ln s_i(t), \tag{3.47}$$

where i is the country index, TFP_i is total factor productivity, k_i is capital per worker, and s_i measures education attainment, i.e. the average years of education of *employed* workers. Since their data on educational attainment refers to the total (rather than the employed) population, they postulate the relationship $\ln s_i(t) = \beta_1 \ln \bar{s}_i(t) - \beta_2 \ln PR_i(t)$, where $\bar{s}_i$ measures population average education attainment (i.e. average years of schooling in the adult population), and PR_i is the participation rate (i.e. the proportion of employed adults). Substituting this expression into (3.47) they derive the equation to be estimated:

$$\ln y_i(t) = \ln TFP_i(t) + \alpha_1 \ln k_i(t) + \alpha_2 \ln \bar{s}_i(t) + \alpha_3 \ln PR_i(t), \tag{3.48}$$

where $\alpha_2 \equiv \alpha_2'\beta_1$ and $\alpha_3 \equiv -\alpha_2'\beta_2$. They present panel data estimates for the parameters, using different specifications for $\ln TFP_i(t)$, and find large and highly significant values for α_2 ranging from 0.378 to 0.958 (de la Fuente and Doménech, 2006, p. 14). They argue on the basis of meta-estimation that the lower bound for the key parameter of interest, α_2', lies in the range of 0.752 to 0.844 for the fixed-effect regressions. They conclude that '...investment in human capital is an important

[12] Kalemni-Ozcan et al. (2000) restrict attention to the steady state. Boucekkine et al. (2002, pp. 363-365) only show the adjustment path in the endogenous growth rate following a drop in the birth rate.

growth factor whose effect on productivity has been underestimated in previous studies because of poor data quality' (de la Fuente and Doménech, 2006, p. 28).

What does this say about our ϕ parameter? In the steady state our model implies the following relationship:

$$\ln \hat{y} = \alpha_0 + \varepsilon \ln \hat{k} + \frac{1-\varepsilon}{1-\phi} \ln s^*, \tag{3.49}$$

where $\alpha_0 \equiv (1-\varepsilon)\ln A_Y + \frac{1-\varepsilon}{1-\phi}\ln(bA_H \int_{s^*}^{\infty} e^{-\hat{n}u - M(u,\psi_m)}du)$. Ignoring the fact that in Equation (3.49) the constant term itself depends negatively on s^*, we find that $\hat{\alpha}_1$ is an estimate of ε and $\hat{\alpha}_2'$ is an estimate of $(1-\varepsilon)/(1-\phi)$. de la Fuente and Doménech find estimates for $\hat{\alpha}_1$ in the range 0.448 to 0.491, so that the implied estimate for ϕ is given by $\hat{\phi} \equiv 1 + (\hat{\alpha}_1 - 1)/\hat{\alpha}_2'$ which ranges from 0.266 to 0.397.[13] Our chosen value of ϕ falls within this range.

On the demographic side, we use the same specification as in section 2.4 in the previous chapter. We interpret the estimated G-M demography as the truth and choose the birth rate, b, such that $\hat{n} = 0.0134$ (the average population growth rate during the period 1920-1940). This yields a value of $b = 0.0237$. The estimated G-M model yields an expected remaining lifetime at birth of 65.5 years (see Table 2.1 on page 31 for details). We compute the implied wage rate from the factor price frontier and find $w = 1.019$. The initial lumpsum tax follows from the government solvency condition for an initial debt level of $\hat{d}_0 = -2.112$ and fiscal parameters $\rho = 4.915$ and $t_L = 0.15$. The implied value for the lumpsum tax is $z_0 = 0.2645$. Finally, for the scaling variables we use $A_H = A_Y = 1$. The initial age at which agents leave school and enter the labour market is $s_0^* = 21.82$ years. The initial steady state has the following main features: $\hat{a}_0 = 7.8$, $\widehat{li}_0 = 647.2$, $\hat{h}_0 = 36.1$, $\hat{y}_0 = 52.6$, $\hat{c}_0 = 37.2$, $\hat{i}_0 = 10.5$, $\hat{k}_0 = 126.2$, and $\hat{f}_0 = -116.2$. The output shares of consumption, investment, and net exports are, respectively, 0.71, 0.20, and 0.09.

The economy is initially in a steady-state equilibrium, the stepwise shock occurs at time $t = 0$, and we refer to pre-shock ($v < 0$) and post-shock agents ($v \geq 0$). In the interest of brevity, we focus the discussion on the transition path of per capita human capital. As is seen readily from (3.25) and (3.28), the time paths for $k(t)$ and $y(t)$ are proportional to that of $h(t)$. The remaining variables of the model (such as $d(t)$, $i(t)$, $f(t)$, $li(t)$, $a(t)$, and $c(t)$) feature more complicated dynamic adjustment paths but are of less interest for the main purpose of this chapter. Where

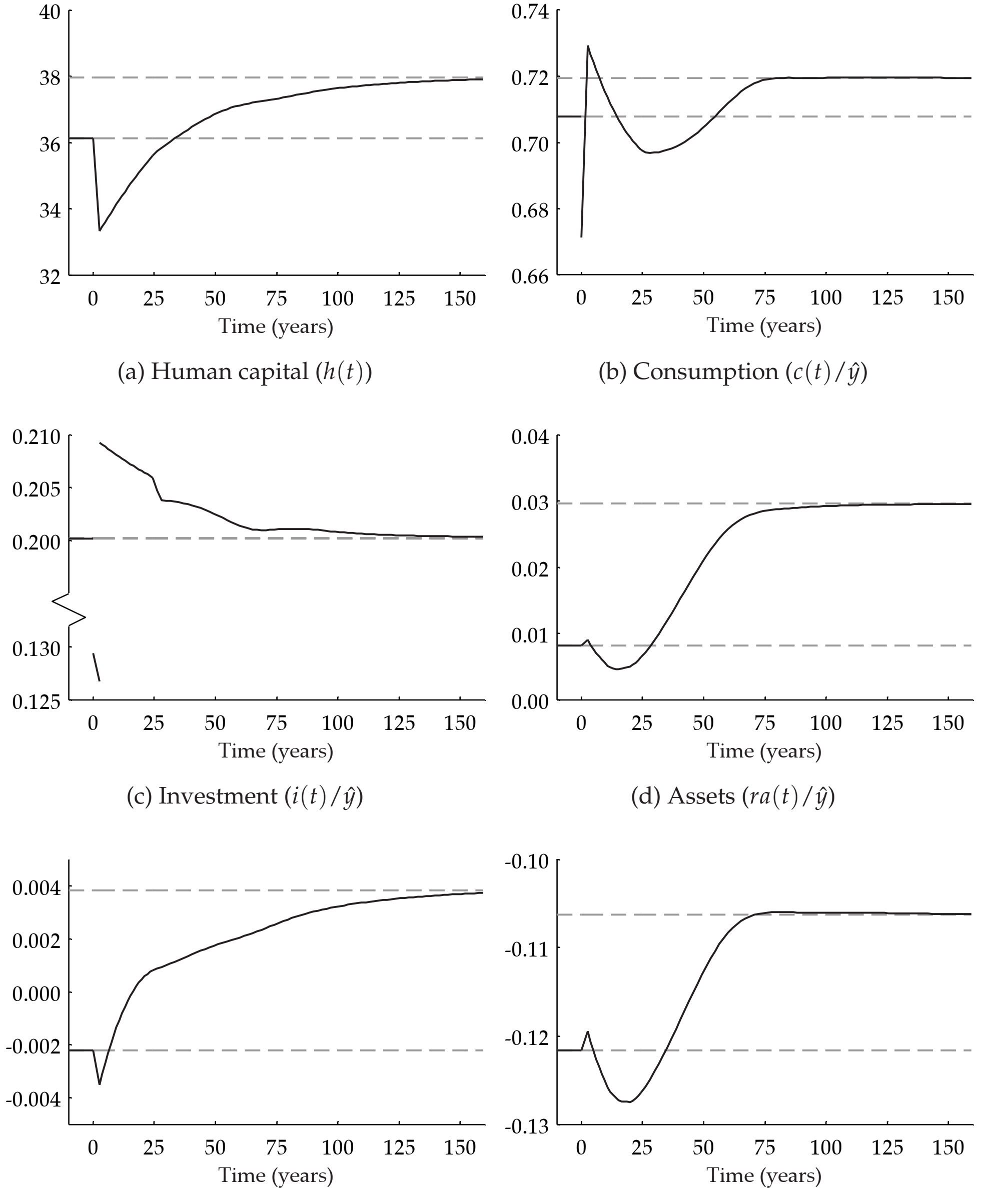
[13] Of course, this is only a very tentative estimate for ϕ for at least two reasons. First, the data may not represent observations for the steady state. Second, the procedure ignores the fact that α_0 itself also depends on s^*. This may lead to an under-estimate for ϕ.

no confusion can arise we drop the 'per capita' adjective in the intuitive discussion of our results.

Fiscal shock In Figure 3.4 we illustrate the transitional dynamics associated with a fiscal education impulse, consisting of a 50% increase in the educational subsidy, from $\rho_0 = 4.915$ to $\rho_1 = 7.372$. There is no effect on the demography so the population growth rate is unchanged ($n(t) = \hat{n}_0$). The human capital of pre-shock workers is unaffected because labour market entry is an *absorbing state*, i.e. workers cannot go back to school by assumption. Pre-shock students, however, react to the improved fiscal incentives by extending their schooling period from $s_0^* = 21.8$ to $s_1^* = 22.9$. As a result, in the time interval $0 \leq t < s_1^* - s_0^*$ there are no new labour market entrants and human capital declines sharply as a result of the mortality process—see Figure 3.4(a). Labour market entry resumes for $t \geq s_1^* - s_0^*$ and the entrants have a higher level of education, so human capital starts to rise as a result. During the interval $s_1^* - s_0^* \leq t < s_1^*$ entry consists entirely of pre-shock students, whereas for $t \geq s_1^*$ only post-shock cohorts enter the labour market. Since these cohorts choose the same schooling period s_1^*, adjustment in human capital is monotonic. For $t \to \infty$, the system reaches a new steady-state which features a higher stock of human capital (see also (3.44) above).

Panels (b)–(f) of Figure 3.4 illustrate the adjustment paths of the other macroeconomic variables. In panel (b) consumption falls at impact due to the once-off increase in the lumpsum tax needed to finance the increase in the educational subsidy. During transition, however, consumption increases non-monotonically as a result of the increase in lifetime income caused by the increase in human capital. In panel (e) the path for government debt is illustrated. Debt fluctuates during transition because the government engages in tax smoothing with respect to the lumpsum tax, z. The current account dynamics is illustrated in panel (f). At impact, the reduction in consumption and investment dominates the reduction in output, so that net exports increase and the stock of net foreign assets rises sharply. During transition, however, net foreign assets gradually fall during the first two decades of adjustment after which they rise to a permanently higher level. In a similar fashion, the path for total assets is non-monotonic due to the population heterogeneity that exists during transition. Indeed, during transition three broad cohort types coexist, namely pre-shock workers (who base their savings decisions on the pre-shock schooling choice s_0^*), pre-shock students (who switched from s_0^* to s_1^* at time $t = 0$ and changed their savings plans accordingly), and post-shock cohorts (who

Figure 3.4. Aggregate effect of a fiscal education impulse.



(a) Human capital ($h(t)$)

(b) Consumption ($c(t)/\hat{y}$)

(c) Investment ($i(t)/\hat{y}$)

(d) Assets ($ra(t)/\hat{y}$)

(e) Government debt ($rd(t)/\hat{y}$)

(f) Foreign assets ($rf(t)/\hat{y}$)

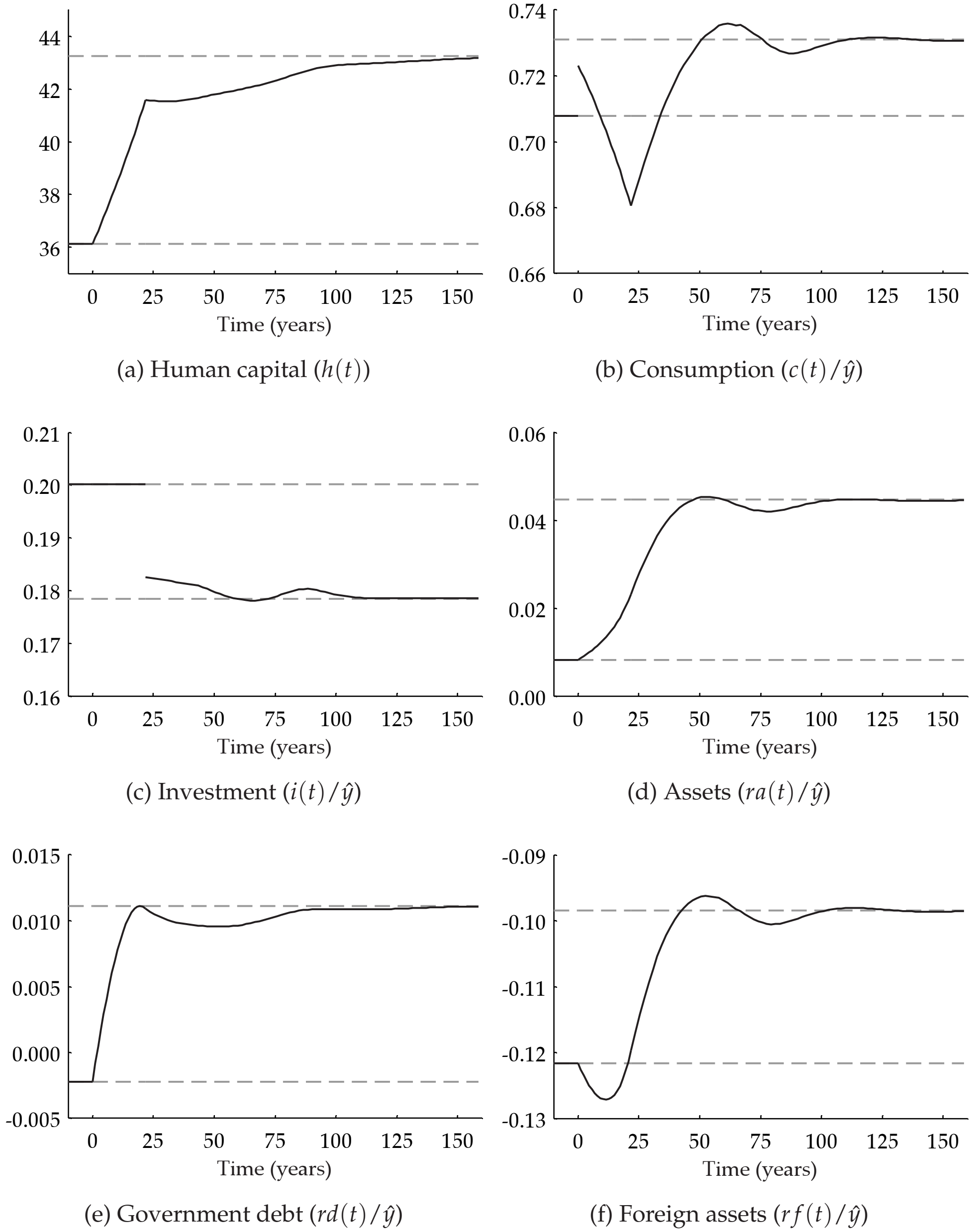
Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values), birth rate is 2.36%. Fiscal education shock is a 50% increase in the educational subsidy, from $\rho_0 = 4.915$ to $\rho_1 = 7.372$. Results are absolute differences relative to the old steady state values.

all choose s_1^* and, provided $\phi > 0$, face changing initial conditions because human capital changes over time).

Birth rate shock In Figure 3.5 we illustrate the transitional dynamics associated with a baby bust, that is the birth rate drops once and for all by 25% from $b_0 = 2.37\%$ to $b_1 = 1.78\%$. Nothing happens to the optimal schooling choice, but the population growth rate falls in a non-monotonic fashion from $\hat{n}_0 = 1.34\%$ to $\hat{n}_1 = 0.43\%$ as is illustrated in Figure 3.3(a). The sharp increase in human capital in Figure 3.5(a) is entirely attributable to the fast reduction in $n(t)$ during the early phase of transition. At time $t = s_0^*$, the population growth rate is close to its new steady state and the slope of the per capita human capital stocks flattens out. This is because the flow of labour market entrants is smaller than before as it consists entirely of post-shock newborns. In the new steady state, per capita human capital increases as a result of the baby bust (see also (3.45) above). For completeness sake, the paths for the remaining macroeconomic variables are also illustrated in panels (b)–(f) of Figure 3.5.

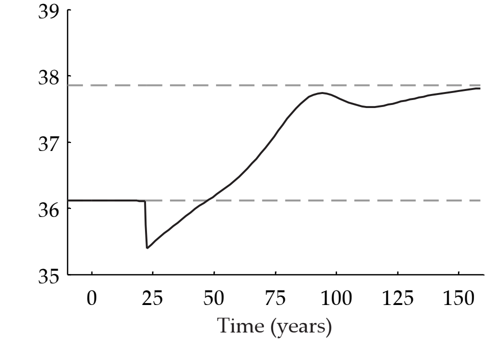
Mortality shocks In Figure 3.6 we illustrate the transitional dynamics associated with an adult mortality shock leading to increased longevity. The μ_1-parameter of the G-M process is reduced by 50% and the μ_2 parameter by 10%, leading to an increase of the expected lifetime at birth from $R_0(0) = 65.45$ to $R_1(0) = 77.57$ years. In the face of increased longevity, post-shock cohorts choose a longer schooling period ($s_1^* = 22.5$ instead of $s_0^* = 21.8$). Furthermore, the shock perturbs the demographic steady-state and causes a rather slow non-monotonic increase in the population growth rate, from $\hat{n}_0 = 1.34\%$ to $\hat{n}_1 = 1.63\%$ as is illustrated in Figure 3.3(b). The transition in human capital passes through the following phases. During the interval $0 \leq t < s_0^*$ nothing happens to human capital because only pre-shock students (facing an unchanged mortality process) enter the labour market and the mortality process for pre-shock workers has not changed. For $s_0^* \leq t < s_1^*$ there are no new labour market entrants at all because post-shock students choose a schooling period s_1^*. Human capital declines sharply because (a) pre-shock cohorts die off at the rate implied by the pre-shock mortality process, and (b) the population growth rate increases. For $t \geq s_1^*$ post-shock cohorts enter the labour market. The closer the birth rate of such cohorts is to s_1^*, the worse are their initial conditions in the human capital formation process. Indeed, the cohort born at time $t = s_1^*$ faces low schooling productivity because $h(s_1^*)$ is quite low. As is clear from Figure 3.6(a), human capital increases in a non-monotonic fashion after $t = s_1^*$, where the

Figure 3.5. Aggregate effect of a baby bust

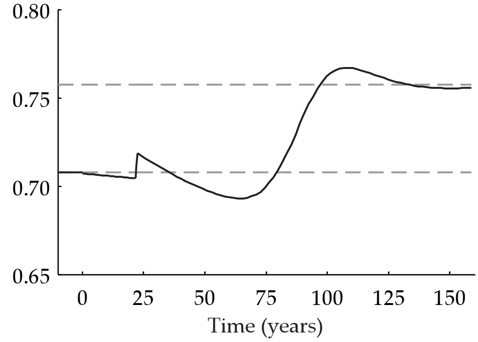


Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values), birth rate is 2.36%. Baby bust is a 25% downward jump of the birth rate to 1.78%. Results are absolute differences relative to the old steady state values.

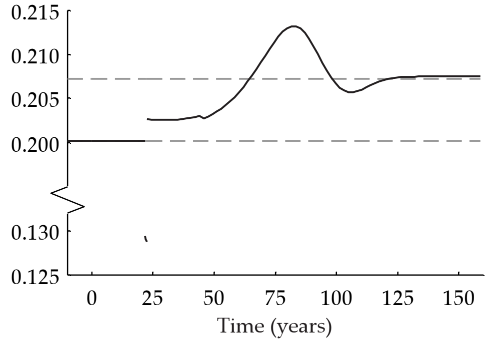
Figure 3.6. Aggregate effect of reduced adult mortality



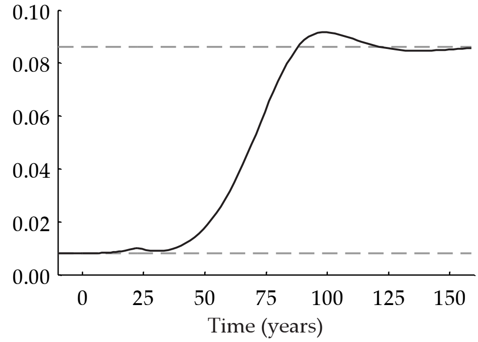
(a) Human capital ($h(t)$)



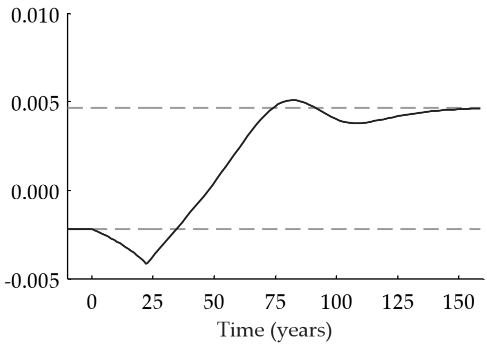
(b) Consumption ($c(t)/\hat{y}$)



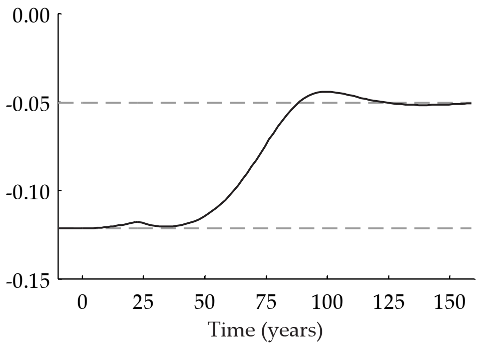
(c) Investment ($i(t)/\hat{y}$)



(d) Assets ($ra(t)/\hat{y}$)



(e) Government debt ($rd(t)/\hat{y}$)



(f) Foreign assets ($rf(t)/\hat{y}$)

Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values, birth rate is 2.36%. Reduced adult mortality is a 50% decrease of μ_1 and 10% decrease of μ_2. Results are absolute differences relative to the old steady state values.

bump after about 95 years is due to the corresponding maximum in the population growth rate at that time—see Figure 3.3(b).

3.5.3 Discussion

The main findings of this section are as follows. Provided the intergenerational externality parameter is below the knife-edge value of unity, the stock of per capita human capital settles at a constant level in the long run. Balanced growth in consumption, investment, output, employment, and human and physical capital is thus entirely due to population growth as in the celebrated Solow-Swan model. Fiscal incentives, though causing permanent level effects, only produce temporary growth effects. In contrast, demographic shocks change both levels and the population growth rate in the long run. In particular, the baby bust reduces long-run growth whilst increased longevity—due to reduced adult mortality—increases it. It is thus an empirical issue whether ageing countries, experiencing the combined demographic shock mentioned in the Introduction, will ultimately converge to a lower or a higher long-run rate of economic growth. Since convergence is extremely slow, time series tests for the exogenous growth model will be hard to conduct given the paucity of data.

3.6 Endogenous growth

Up to this point we have restricted attention to the case for which the intergenerational knowledge externality is relatively weak (i.e. $0 \leq \phi < 1$) and the system reaches a steady state in terms of per capita levels. In this section we study the knife-edge case for which the intergenerational knowledge transfer is very strong and subject to constant returns ($\phi = 1$). This case has been studied extensively in the literature; see among others Azariadis and Drazen (1990) and Boucekkine et al. (2002). However, as we argued in the previous section (p. 75), we do believe that the intergenerational knowledge externality (ϕ) is not unity.

3.6.1 Long-run effects

The steady-state growth path for per capita human capital can be written as follows:

$$\hat{h}(t) = \int_{-\infty}^{t-s^*} l(t-v)\hat{\hat{h}}(t-v)dv$$

$$= A_H s^* b \int_{-\infty}^{t-s^*} e^{-\hat{n}\cdot(t-v)-M(t-v,\psi_m)}\hat{h}(v)dv, \tag{3.50}$$

where we have used (3.5) and (3.17) to arrive at the second expression. In Appendix 3.C we show that there is a unique steady state growth rate of per capita variables. Moreover, after any shock, the growth rate of all per capita variables (most importantly human capital) converge to this unique value.

Denoting the steady-state growth rate by $\hat{\gamma}$, it follows that along the balanced growth path we have $\hat{h}(v) = \hat{h}(t)e^{-\hat{\gamma}\cdot(t-v)}$. By using this result in (3.50) and simplifying we obtain the implicit definition for $\hat{\gamma}$:

$$1 = A_H s^* b \int_{s^*}^{\infty} e^{-(\hat{\gamma}+\hat{n})u-M(u,\psi_m)}du. \tag{3.51}$$

Clearly, the model implies a *scale effect* in the growth process, i.e. a productivity improvement in the human capital production function gives rise to an increase in the steady-state growth rate ($\partial\hat{\gamma}/\partial A_H > 0$). Equation (3.51) can also be used to compute the effect on the asymptotic growth rate of the fiscal and demographic shocks.

Pure schooling shock Just as in Subsection 3.5.1 above, the interpretation of our results is facilitated by first considering a pure schooling shock. By differentiating (3.51) with respect to $\hat{\gamma}$ and s^*, and gathering terms we find:

$$\begin{aligned}\frac{\partial\hat{\gamma}}{\partial s^*} &= \frac{e^{-(\hat{\gamma}+\hat{n})s^*-M(s^*,\psi_m)}\left[\Delta(s^*,\hat{\gamma}+\hat{n})-s^*\right]}{s^*\int_{s^*}^{\infty} ue^{-(\hat{\gamma}+\hat{n})u-M(u,\psi_m)}du} \\ &= \frac{e^{-(\hat{\gamma}+\hat{n})s^*-M(s^*,\psi_m)}}{s^*\int_{s^*}^{\infty} ue^{-(\hat{\gamma}+\hat{n})u-M(u,\psi_m)}du}\left[\Delta(s^*,\hat{\gamma}+\hat{n})-\Delta(s^*,r)-\frac{\rho}{1-t_L}\right] > 0,\end{aligned} \tag{3.52}$$

where we have used quation (3.11) to arrive at the final expression. The sign of $\partial\hat{\gamma}/\partial s^*$ is determined by the term in square brackets on the right-hand side of (3.52). By appealing to the endogenous- growth counterpart of Assumption 3.2 (with $\hat{n}$ replaced by $\hat{n}+\hat{\gamma}$) we find that the steady-state growth rate increases as a result of the pure schooling shock.

Fiscal shock An increase in the educational subsidy or the labour income tax affects the steady-state growth rate via its positive effect on the schooling period.

Indeed, we deduce from (3.31)–(3.32) and (3.52) that:

$$\frac{\partial \hat{\gamma}}{\partial \left[\rho/(1-t_L)\right]} = \frac{\partial \hat{\gamma}}{\partial s^*} \frac{\partial s^*}{\partial \left[\rho/(1-t_L)\right]} > 0. \tag{3.53}$$

Birth rate shock The growth effects of a birth rate change are computed most readily by restating the shock in terms of the steady-state population growth rate, $\hat{n}$, and noting the monotonic relationship between $\hat{n}$ and b stated in (3.36) above. Indeed, by substituting the steady-state version of (3.18) into (3.51) we find an alternative implicit expression for $\hat{\gamma}$:

$$\int_0^\infty e^{-\hat{n}u - M(u,\psi_m)} du = A_H s^* \int_{s^*}^\infty e^{-(\hat{\gamma}+\hat{n})u - M(u,\psi_m)} du. \tag{3.54}$$

Since the birth rate shock leaves the schooling period unchanged, it follows from (3.54) that:

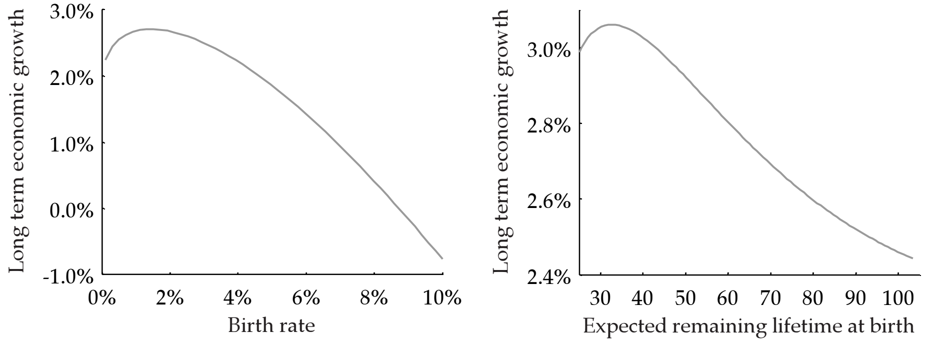
$$\frac{\partial \hat{\gamma}}{\partial b} = \frac{\partial \hat{\gamma}}{\partial \hat{n}} \frac{\partial \hat{n}}{\partial b} = \frac{\partial \hat{n}}{\partial b} \left[\frac{\int_0^\infty u e^{-\hat{n}u - M(u,\psi_m)} du}{A_H s^* \int_{s^*}^\infty u e^{-(\hat{\gamma}+\hat{n})u - M(u,\psi_m)} du} - 1 \right] \gtreqless 0. \tag{3.55}$$

Despite the fact that $\partial \hat{n}/\partial b > 0$, the growth effect of a birth rate change is ambiguously because the term in square brackets on the right-hand side of (3.55) cannot be signed a priori. Indeed, using the calibrated version of the model, we find that the relationship between $\hat{\gamma}$ and b is hump-shaped. As is illustrated in Figure 3.7(a), the growth rate rises with the birth rate for low birth rates, but is decreasing for higher birth rates. For the calibrated model, the maximum growth rate is attained at a birth rate of 1.25% per annum.

Mortality shock Just as in the exogenous growth model, increased longevity constitutes by far the most complicated shock studied here. Indeed, as can be seen from Equation (3.51) above, a mortality shock affects three distinct items featuring in the implicit expression for the steady-state growth rate, $\hat{\gamma}$, namely (a) the optimal schooling period, s^*, (b) the steady-state growth rate of the population, $\hat{n}$, and (c) the cumulative mortality factor, $M(u, \psi_m)$. By differentiating (3.51) with respect to $\hat{\gamma}$ and ψ_m (and recognising the dependence of s^* and $\hat{n}$ on ψ_m) we find after some steps:

$$\frac{\partial \hat{\gamma}}{\partial \psi_m} = \frac{\partial \hat{\gamma}}{\partial s^*} \frac{\partial s^*}{\partial \psi_m} - \frac{\partial \hat{n}}{\partial \psi_m} + \frac{\int_{s^*}^\infty -\frac{\partial M(u,\psi_m)}{\partial \psi_m} e^{-(\hat{\gamma}+\hat{n})u - M(u,\psi_m)} du}{\int_{s^*}^\infty u e^{-(\hat{\gamma}+\hat{n})u - M(u,\psi_m)} du} \gtreqless 0. \tag{3.56}$$

Figure 3.7. The effect of the birth rate (a) and mortality (b) on long-term growth



(a) Growth and the birth rate

(b) Growth and life expectancy at birth

The overall growth effect of increased longevity is ambiguous. The first composite term on the right-hand side of (3.56) represents the schooling effect, which is positive (see (3.35) and (3.52)). The third term on the right-hand side represents the cumulative mortality effect and is also positive (given Proposition 3.1(i)). The ambiguity thus arises because the second term on the right-hand side exerts a negative influence on growth, i.e. increased longevity boosts the steady-state population growth rate (see (3.37) above) which in turn slows down growth.

In Figure 3.7(b) we use the calibrated version of the model to plot the relationship between the steady-state growth rate and a measure of longevity, namely life expectancy at birth, $R(0, \psi_m) \equiv \Delta(0, 0, \psi_m)$. Except for very low values of $R(0, \psi_m)$, there is negative relationship between long-term growth and longevity.

3.6.2 Transitional dynamics

In this subsection we visualise the transitional effects of fiscal and demographic shocks in the endogenous growth model. We restrict attention to the growth rate of per capita human wealth, $\gamma(t) \equiv \dot{h}(t)/h(t)$, since this variable drives all other macroeconomic variables. Except for ϕ and A_H, we use the same calibration values as before (see Subsection 3.5.2). Because the model contains a scale effect, we set $A_H = 0.13$ and obtain a realistic steady-state growth rate, $\hat{\gamma}_0 = 1.096\%$. The discussion here can be quite brief because, following a shock, the transition proceeds along the same phases as in the exogenous growth model.

Fiscal shock Figure 3.8(a) illustrates the path for $\gamma(t)$ following a 20% increase in the educational subsidy. For $0 \leq t < s_1^* - s_0^*$ there are no new labour market entrants and the growth rate collapses. Then, for $s_1^* - s_0^* \leq t < s_1^*$ pre-shock students enter the labour market and the growth rate jumps above its initial steady-state level. Finally, for $t \geq s_1^*$ the growth rate converges in a non-monotonic fashion to its long-run value, i.e. $\lim_{t\to\infty} \gamma(t) = \hat{\gamma}_1 = 1.13\%$, where $\hat{\gamma}_1$ exceeds the initial steady-state growth rate $\hat{\gamma}_0$ (see quation (3.53) above).

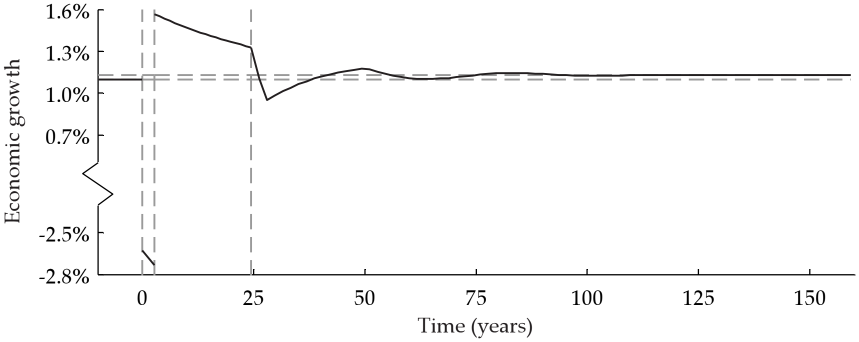
Birth rate shock In Figure 3.8(b) the transitional effects of a baby bust are illustrated. There is no effect on the optimal schooling period but the population growth rate falls from $\hat{n}_0$ to $\hat{n}_1$—see Figure 3.3(a). Growth jumps sharply due to the fast reduction in $n(t)$ that occurs at impact and immediately hereafter. Intuitively, pre-shock students enter the labour market but their human capital is spread out over fewer people than before the shock so that growth in per capita terms increases sharply. About twenty-two years after the shock, $n(t) \approx \hat{n}_1$ and there is a sharp decline in growth. This is because the post-shock students start to enter the labour market. Despite the fact that they have higher human capital than existing workers, as a group they are not large enough to maintain the previous growth in per capita human capital. Thereafter, the growth rate converges in a non-monotonic fashion to its long-run level $\hat{\gamma}_1 = 1.33\%$, which is higher than the initial steady-state growth rate, i.e. $\hat{\gamma}_1 > \hat{\gamma}_0$. Given our calibration, the economy lies to the right of the peak in the curve for $\hat{\gamma}$ in Figure 3.7(a) so that a baby bust increases long-run growth.

Mortality shock In Figure 3.8(c) the effect on the growth rate of increased longevity of generations born after time $t = 0$ is illustrated. Just as for the exogenous growth model, nothing happens to growth for the period $0 \leq t < s_0^*$ because only pre-shock agents enter the labour market and the same type of agents die off. For $s_0^* \leq t < s_1^*$ there are no new labour market entrants and the growth rate collapses. At time $t = s_1^*$ the oldest of the post- shock cohorts enter the labour market and as a result growth is boosted again. For $t > s_1^*$, the growth rate converges non-monotonically towards the new steady-state growth rate $\hat{\gamma}_1 = 1.09\% < \hat{\gamma}_0$. In terms of Figure 3.7(b), the calibration places the economy on the downward sloping segment of the $\hat{\gamma}$ curve so increased longevity reduces the long-run growth rate.

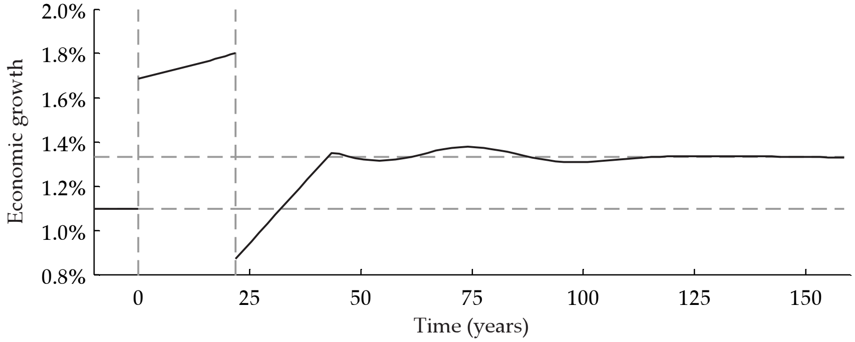
3.6.3 Discussion

The main findings of this section are as follows. For the calibrated model, the long-run growth rate in per capita human capital increases as a result of a positive fiscal

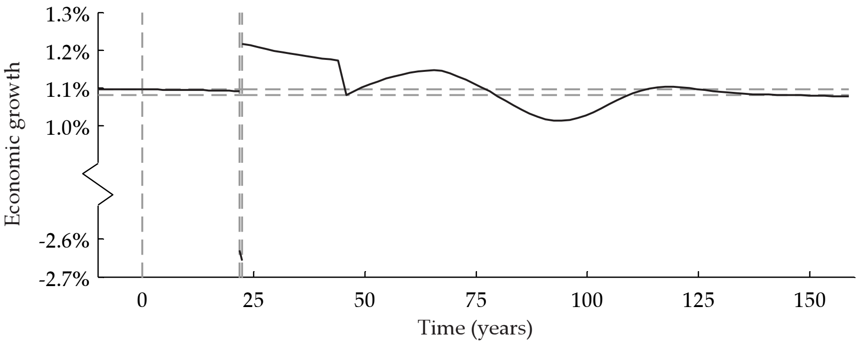
Figure 3.8. Per capita human capital growth



(a) Fiscal impulse



(b) Baby bust



(c) Reduced adult mortality

Notes: Mortality process is a Gompertz-Makeham (see Table 2.1 for parameter values), birth rate is 2.36%. Fiscal education shock is a 50% increase in the educational subsidy, from $\rho_0 = 4.915$ to $\rho_1 = 7.372$. Baby bust is a 25% downward jump of the birth rate to 1.78%. Reduced adult mortality is a 50% decrease of μ_1 and 10% decrease of μ_2. Results are absolute growth rates.

impulse or a fall in the birth rate. Increased longevity, however, reduces this long-run growth rate. The transition path in the growth rate is cyclical and rather complex for all shocks considered, and the new equilibrium is reached only very slowly.

3.7 Conclusion

In this chapter we extended the basic model of Chapter 2 with a schooling decision and we have studied how fiscal incentives and demographic shocks affect the growth performance of a small open economy populated by disconnected generations of finitely-lived agents facing age-dependent mortality and constant factor prices. Our analysis shows that only for a unrealistically strong intergenerational knowledge spillover, policy changes and demographic shocks lead to a permanent higher (or lower) growth rate. Moreover, if the intergenerational spillover is unrealistically large, the link between longevity and economic growth is non-monotonic. For realistic parameter values a higher life expectancy at birth causes a lower long-run growth rate in most developed countries.

This chapter highlights the crucial role played by the strength of the intergenerational external effect in the production of human capital. Also, the vintage nature of the model gives rise to very slow and rather complicated dynamic adjustment. This feature of the model may help explain why robust empirical results linking education and growth have been so hard to come by.

This chapter focused on the decision an individual faces at the beginning his economic life. In the next chapter we focus on the economic decision at the end of an individual's life, the retirement decision. Many OECD countries have experienced an increase in the old-age dependency ratio over the last half century. This has important implications for the feasibility of existing pay-as-you-go pension schemes, a phenomenon that has been abstracted from in this chapter. In Chapter 4 we drop the schooling decision, but we endogenise the agent's labour force participation decision in the presence of a stylized public pension system including realistic institutional features such as the early retirement age and the mandatory retirement age (Gruber and Wise, 1999). With this extended model we hope to contribute to the literature on pension reform in an ageing society.

3.A Optimal schooling period

Consider the household that is still in school. By differentiating (3.9) with respect to $s(v)$ we get:

$$\frac{d\bar{l}i(v,t)}{ds(v)} \equiv A_H h(v)^{\phi} e^{M(t-v)} \Bigg[\rho(v+s(v))w(v+s(v))e^{-r(v+s(v)-t)-M(s(v))} + \int_{v+s(v)}^{\infty} [1-t_L(\tau)]w(\tau)e^{-r\cdot(\tau-t)-M(\tau-v)}d\tau - [1-t_L(v+s(v))]s(v)w(v+s(v))e^{-r\cdot(v+s(v)-t)-M(s(v))} \Bigg].$$

By simplifying and setting $\frac{d\bar{l}i(v,t)}{ds(v)} = 0$ we obtain (for time-invariant ρ and t_L):

$$(1-t_L)\int_{v+s(v)}^{\infty} w(\tau)e^{-r\cdot(\tau-v)-M(\tau-v)}d\tau = [(1-t_L)s^*(v)-\rho]w(v+s^*(v))e^{-rs^*(v)-M(s(v))}.$$

For the time-invariant wage rate we obtain:

$$s^*(v) - \frac{\rho}{1-t_L} = e^{rs^*(v)+M(s^*(v))}\int_{v+s^*(v)}^{\infty} e^{-r(\tau-v)-M(\tau-v)}d\tau \equiv \Delta(s^*(v), r).$$

3.B Useful Lemmas

Birth rate shock To determine effect of a birth rate shock on the level of human capital in Equation (3.45) we need the following lemma

Lemma 3.1. *By using* (3.36) *in* (3.45) *we obtain:*

$$\frac{\partial \hat{h}^{1-\phi}}{\partial b} = \frac{A_H s^*}{b}\psi_m(s^*),$$

where $\psi_m(s)$ is defined as:

$$\psi_m(s) \equiv \frac{\int_s^{\infty} e^{-\hat{n}u-M(u,\psi_m)}du}{\int_0^{\infty} e^{-\hat{n}u-M(u,\psi_m)}du} - \frac{\int_s^{\infty} ue^{-\hat{n}u-M(u,\psi_m)}du}{\int_0^{\infty} ue^{-\hat{n}u-M(u,\psi_m)}du},$$

with $\hat{n} > 0$ and $M(u,\psi_m)$ as defined in equation (2′). The following results can be estab-

lished: (i) $\psi_m(s) \leq 0$ *for all* $s \geq 0$*, (ii)* $\psi_m(0) = 0$*, (iii)* $\lim_{s\to\infty} \psi_m(s) = 0$.

Proof. Results (ii) and (iii) follow directly from the definition of $\psi_m(s)$. Differentiation with respect to s gives

$$\frac{\partial \psi_m}{\partial s} = e^{-\hat{n}s - M(s,\psi_m)} \left[\frac{s}{\int_0^\infty e^{-\hat{n}u - M(u,\psi_m)} du} - \frac{1}{\int_0^\infty u e^{-\hat{n}u - M(u,\psi_m)} du} \right],$$

which is continuous in s and has only one root. The second derivative is positive in this unique stationary point, so it is a global minimum. Together with (ii) and (iii) this implies result (i). ∎

Mortality shock To determine the effect of a mortality shock on the level of human capital in the long run (Equation (3.46), we need the following lemma

Lemma 3.2. *Define* $\Xi(s, \psi_m)$ *for* $s \geq 0$ *as:*

$$\Xi(s, \psi_m) = \int_s^\infty e^{-\hat{n}u - M(u,\psi_m)} du.$$

Then $\dfrac{\partial \Xi(s, \psi_m)}{\partial \psi_m} \geq 0$ *for all* $s > 0$*, where the equality holds if and only if* $\dfrac{\partial^2 m(u, \psi_m)}{\partial u \partial \psi_m} = 0$.

Proof. For the sake of readability define

$$\begin{aligned} \Xi_{\psi_m}(s, \psi_m) &\equiv \frac{\partial \Xi(s, \psi_m)}{\partial \psi_m} \\ &= \int_s^\infty \frac{\partial M(u, \psi_m)}{\partial \psi_m} e^{-\hat{n}u - M(u,\psi_m)} du - \frac{\partial \hat{n}}{\partial \psi_m} \int_s^\infty u e^{-\hat{n}u - M(u,\psi_m)} du, \end{aligned} \tag{3.B.1}$$

Note that $\lim_{s\to\infty} \Xi_{\psi_m}(s, \psi_m) = 0$ and:

$$\frac{\partial \hat{n}}{\partial \psi_m} = \frac{\int_0^\infty \frac{\partial M(u,\psi_m)}{\partial \psi_m} e^{-\hat{n}u - M(u,\psi_m)} du}{\int_0^\infty u e^{-\hat{n}u - M(u,\psi_m)} du}. \tag{3.B.2}$$

By substituting (3.B.2)) into (3.B.1)) we find that $\Xi_{\psi_m}(0, \psi_m) = 0$. The stationary points of $\Xi_{\psi_m}(s, \psi_m)$ with respect to s are determined by the roots of:

$$\frac{\partial \Xi_{\psi_m}(s, \psi_m)}{\partial s} = e^{-\hat{n}u - M(u,\psi_m)} \left[\frac{\partial M(s, \psi_m)}{\partial \psi_m} - s \frac{\partial \hat{n}}{\partial \psi_m} \right]. \tag{3.B.3}$$

From Proposition 3.1 we know that $\frac{\partial M(s,\psi_m)}{\partial \psi_m}$ is non-positive, non-increasing and concave in s. This implies together with $\frac{\partial \Xi_{\psi_m}(0,\psi_m)}{\partial s} = 0$ that (3.B.3) has at most two roots (one at $s = 0$) or is 0 everywhere (if $\frac{\partial \Xi_{\psi_m}(0,\psi_m)}{\partial s} = 0$ on the interval $[0, s^*]$, $0 \leq s^* \ll \infty$, then $\lim_{s\to\infty} \Xi_{\psi_m}(s, \psi_m) = 0$ does not hold). If $\frac{\partial \Xi_{\psi_m}(s,\psi_m)}{\partial s} = 0$ for all $s \geq 0$, then $\Xi_{\psi_m}(s, \psi_m) = 0$ for all $s \geq 0$. This last situation only occurs if $\frac{\partial M(s,\psi_m)}{\partial \psi_m}$ is linear in s, i.e. if $\frac{\partial^2 m(u,\psi_m)}{\partial u \partial \psi_m} = 0$.

If $\frac{\partial^2 m(u,\psi_m)}{\partial u \partial \psi_m} < 0$ for some $s \geq 0$, then $\Xi_{\psi_m}(s, \psi_m)$ has exactly two stationary points for a given ψ_m, one at $s = 0$ and one at $s = s^* > 0$. Concavity of $\frac{\partial M(s,\psi_m)}{\partial \psi_m}$ implies that the stationary point at $s = s^*$ is a maximum. Since $\frac{\partial \Xi(s,\psi_m)}{\partial \psi_m}$ goes to 0 as $s \to \infty$ and is continuous, $\frac{\partial \Xi(s,\psi_m)}{\partial \psi_m}$ must be positive for all $s > 0$, otherwise there would be a minimum somewhere at $s > s^*$. This completes the proof. ■

3.C Convergence of the endogenous growth model

We have already demonstrated that, following a demographic shock, the population growth rate converges to a constant value, $\hat{n}$. In this section we assume for simplicity that $n(t) = \hat{n}$ and consider the stability of the growth rate in per capita human capital for the case with $\phi = 1$, i.e. we prove that $\gamma(t) \equiv \dot{h}(t)/h(t)$ converges to $\hat{\gamma}$ as t gets large. Taking the past as given and focusing on $t > s^*$, we can rewrite the first expression in Equation (3.50) in the form of a normal integral equation. Define:

$$\mathbf{K}(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq s^* \\ A_H s^* l(u) & \text{for } u > s^* \end{cases} \quad \text{and} \quad \chi(t) = \int_{-\infty}^{0} \mathbf{K}(t-v)h(v)dv \quad \text{for } t > s.$$

With these definitions, we can write $h(t)$ for $t > s$ in the form of a renewal equation:

$$h(t) = \chi(t) + \int_0^t \mathbf{K}(t-v)h(v)dv,$$

where the exogenous function $\chi(t)$ is called the forcing equation, and $\mathbf{K}(t-v)$ is the kernel of the integral operator. By definition, human capital in the past ($h(t)$, for $t < 0$) is bounded and continuous. This makes the forcing equation continuous and monotonically decreasing.

We want to show that no matter what the path of human capital was before $t = 0$, human capital growth always converges to a constant. To show this, we closely follow Bellman and Cooke (1963, ch. 7). The integral is the convolution of

$h(t)$ and $\mathbf{K}(t)$, so Laplace techniques are a logical choice to analyse the behaviour of $h(t)$. Taking the Laplace transform of $h(t)$ and using the convolution theorem we obtain

$$\mathcal{L}(h) = \mathcal{L}(\chi) + \mathcal{L}(\mathbf{K})\,\mathcal{L}(h) \quad \Rightarrow \quad \mathcal{L}(h) = \frac{\mathcal{L}(\chi)}{1 - \mathcal{L}(\mathbf{K})}$$

Using the complex inversion formula we find that the solution of $h(t)$ is given by the contour integral

$$h(t) = \int_{(\mathbf{b})} \frac{\mathcal{L}(\chi)(\varsigma)}{1 - \mathcal{L}(\mathbf{K})(\varsigma)} e^{\varsigma t} d\varsigma$$

with $\int_{(\mathbf{b})}$ as in Bellman and Cooke (1963, p. 233):

$$\int_{(\mathbf{b})} F(s)ds = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\mathbf{b}-iT}^{\mathbf{b}+iT} F(s)ds = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} F(\mathbf{b} + it)dt,$$

and $\mathbf{b}$ such that all the singularities of the function under the integral sign lie to the left of the line $\mathrm{Re}(z) < \mathbf{b}$ in the complex plane. That is, $h(t)$ is the contour integral taken over the vertical line in the complex plane defined by $\mathrm{Re}(z) = \mathbf{b}$.

Note that all singularities of the function under the integral sign are determined by the roots of $1 - \mathcal{L}(\mathbf{K})$. As we will see later, it is sufficient to take any $\mathbf{b}$ such that $\mathbf{b} > \hat{\gamma}$, where $\hat{\gamma}$ is implicitly defined as the real solution of:

$$1 = \mathcal{L}(\mathbf{K})(\hat{\gamma}) = A_H s^* \int_{s^*}^{\infty} l(u)e^{-\hat{\gamma}u}du = A_H s^* b \int_{s^*}^{\infty} e^{-[\hat{\gamma}+\hat{n}]u - M(u)}du$$

Following Bellman and Cooke (1963, par. 7.11), it is quite simple to show that $1 - \mathcal{L}(\mathbf{K})$ has only one real root. Denote this real root by $\hat{\gamma}$. Existence and uniqueness of $\hat{\gamma}$ follows directly from continuity and monotonicity of $\mathcal{L}(\mathbf{K})(\varsigma)$ and:

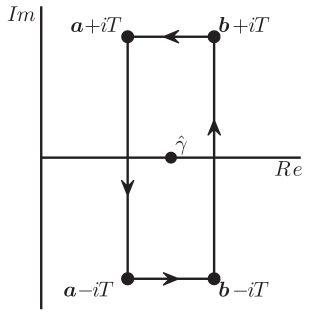
$$\lim_{\varsigma\to-\infty} \mathcal{L}(\mathbf{K})(\varsigma) = -\infty \quad \text{and} \quad \lim_{\varsigma\to\infty} \mathcal{L}(\mathbf{K})(\varsigma) = 0.$$

Note that $\hat{\gamma}$ is not necessarily positive. To prove that $\hat{\gamma}$ is the root with the largest positive real part, suppose there is an other (complex) root $\varsigma = x + i\tau$.

$$1 = \left| \int_{s}^{\infty} e^{-xu} e^{-i\tau u} l(u)du \right| < \int_{s}^{\infty} e^{-xu} l(u)du$$

The only way the term on the right can be larger than one is if $x < \hat{\gamma}$ which means $\mathrm{Re}(\varsigma) < \hat{\gamma}$.

Figure 3.C.1. Contour shifting



To analyse the behaviour of $h(t)$ as $t \to \infty$ we shift the contour $(\mathbf{b} - iT, \mathbf{b} + iT)$ in

$$h(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\mathbf{b}-iT}^{\mathbf{b}+iT} \frac{\mathcal{L}(\chi)(\varsigma)}{1 - \mathcal{L}(\mathbf{K})(\varsigma)} e^{\varsigma t} d\varsigma$$

to the left such that we pick up the pole at $\hat{\gamma}$ and we can (hopefully) write $h(t)$ as

$$h(t) = \beta e^{\hat{\gamma} t} + \int_{(\mathbf{a})} \frac{\mathcal{L}(\chi)(\varsigma)}{1 - \mathcal{L}(\mathbf{K})(\varsigma)} e^{\varsigma t} d\varsigma.$$

That is, we write $h(t)$ as the sum of the residue at $\hat{\gamma}$ and the contour integral taken over the line $\mathrm{Re}(z) = \mathbf{a}$ with $\mathbf{a}$ smaller than $\hat{\gamma}$, but larger than the other poles of $1 - \mathcal{L}(\mathbf{K})$. If for $t \to \infty$ the exponential term dominates the integral, then human capital growth converges to $\hat{\gamma}$.

In Figure 3.C.1, the contour is shifted from the line $(\mathbf{b} - iT, \mathbf{b} + iT)$ to $(\mathbf{a} - iT, \mathbf{a} + iT)$. The residue theorem tells us that the contour integral taken over the square equals the sum of the residues of the singular points within this square. Since $\mathbf{b}$ and $\mathbf{a}$ are chosen in such a way that $\hat{\gamma}$ is the only singular point in this region, the contour integral over the square equals the residue at $\hat{\gamma}$ which is:

$$\lim_{\varsigma\to\hat{\gamma}} \frac{\varsigma - \hat{\gamma}}{1 - \mathcal{L}(\mathbf{K})(\varsigma)} \mathcal{L}(\chi)(\varsigma) e^{\varsigma t} = \frac{\int_0^\infty e^{-\hat{\gamma} t} \chi(t) dt}{\int_0^\infty t e^{-\hat{\gamma} t} \mathbf{K}(t) dt} e^{\hat{\gamma} t}$$

This residue is a function of t and grows at rate $\hat{\gamma}$.

Next we will show that the contribution of the two horizontal contours vanishes as $t \to \infty$. To show that

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\mathbf{b}\pm iT}^{\mathbf{a}\pm iT} \frac{\mathcal{L}(\chi)(\varsigma)}{1-\mathcal{L}(\mathbf{K})(\varsigma)} e^{\varsigma t} d\varsigma = 0, \tag{3.C.1}$$

it is sufficient to show that for $\mathbf{a} \leq x \leq \mathbf{b}$

$$\lim_{T\to\infty} \mathcal{L}(\chi)(x \pm iT) = \lim_{T\to\infty} \mathcal{L}(\mathbf{K})(x \pm iT) = 0.$$

Integration by parts of $\mathcal{L}(\chi)$ on the contour $\varsigma = x \pm iT$ gives

$$\mathcal{L}(\chi) = \frac{1}{T}\left[e^{-xt} f(t) \frac{e^{\pm iT}}{i}\bigg|_{t=0}^{\infty} \pm \frac{x}{i} \int_0^\infty e^{-xt}\chi(t) e^{\pm iT} dt \pm \frac{1}{i}\int_0^\infty e^{-xt}\chi'(t) e^{\pm iT} dt \right].$$

Using the definition of $\chi(t)$ it is easy to see that on $\mathbf{a} \leq x \leq \mathbf{b}$

$$\lim_{t\to\infty} e^{-xt}\chi(t) = 0 \quad \text{and} \quad \int_0^\infty e^{-xt}\chi'(t) dt \ll \infty.$$

This means that the whole term within square brackets is a constant and we can write

$$|\,\mathcal{L}(\chi)| = O(1/T).$$

A similar result holds for $\mathcal{L}(\mathbf{K})$. This means that numerator in the integrand in equation (3.C.1) goes to zero as T goes to infinity and the denominator goes to 1 and the integral vanishes.

The final part we have to show is

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\mathbf{a}-iT}^{\mathbf{a}+iT} \frac{\mathcal{L}(\chi)(\varsigma)}{1-\mathcal{L}(\mathbf{K})(\varsigma)} e^{\varsigma t} d\varsigma = O(e^{\mathbf{a}t}),$$

which follows directly from the the fact that all singularities in $\mathcal{L}(\chi)$ are cancelled by singularities in $\mathcal{L}(\mathbf{K})$ so the fraction $\mathcal{L}(\chi)/\,\mathcal{L}(\mathbf{K})$ remains bounded.

Finally we can write $h(t) = \beta e^{\hat{\gamma}t} + O(e^{\mathbf{a}t})$ as $t \to \infty$, with

$$\beta = \frac{\int_0^\infty e^{-\hat{\gamma}t}\chi(t)dt}{\int_0^\infty te^{-\hat{\gamma}t}\mathbf{K}(t)dt} \quad \text{and } \mathbf{a} < \hat{\gamma}$$

This implies that no matter what the initial path of human capital was in the past, the growth of human capital will always converge towards $\hat{\gamma}$.