Essays on dynamic macroeconomics

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Part I

Realistic Demographics in Overlapping Generations Models

Chapter 2

The Basic Model

> It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration or an increased inability to withstand destruction.
> *(Gompertz, 1825)*

2.1 Introduction

The opening quotation is a verbal introduction to a phenomenon that is now often called Gompertz' law of mortality. In his path-breaking paper, Benjamin Gompertz[1] (1825) identified two main causes of death, namely one due to pure chance and another depending on the person's age. He pointed out that if only the first cause were relevant, then 'the intensity of mortality' would be constant and the surviving fraction of a given cohort would decline in geometric progression. In contrast, if only the second cause would be relevant, and 'if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death' then the force of mortality would increase with age. Gompertz' law was subsequently generalized by Makeham (1860) who argued that the instantaneous

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This chapter is based on joint work with Ben Heijdra, 'A Life-Cycle Overlapping-Generations Model of the Small Open Economy', forthcoming in the Oxford Economic Papers.

[1] As Hooker (1965) points out, Benjamin Gompertz can be seen as one of the founding fathers of modern demographic and actuarial theory. See also Preston et al. (2001, p. 192). Blanchard (1985, p. 225) and Faruqee (2003, p. 301) incorrectly refer to the non-existing 'Gomperty's law'.

mortality rate depends both on a constant term (first cause) and on a term that is exponential in the person's age (second cause).

The microeconomic implications for consumption behaviour of lifetime uncertainty—resulting from a positive death probability—were first studied in the seminal paper by Yaari (1965). He showed that, faced with a positive mortality rate, individual agents will discount future felicity more heavily due to the uncertainty of survival. Furthermore, with lifetime uncertainty the consumer faces not only the usual solvency condition but also a constraint prohibiting negative net wealth at any time—the agent is simply not allowed by capital markets to expire indebted. Yaari assumes that the household can purchase (annuity) or sell (life insurance) actuarial notes at an actuarially fair interest rate. In the absence of a bequest motive, the household will use such notes to fully insure against the adverse effect of lifetime uncertainty.

The Yaari insights were embedded in a general equilibrium growth model by Blanchard (1985). In order to allow for exact aggregation of individual decision rules, Blanchard simplified the Yaari model by assuming a constant death probability, i.e. only the first cause of death is introduced into the model and households enjoy a perpetual youth. Because of its flexibility, the Blanchard-Yaari model has achieved workhorse status in the last two decades.[2] As Blanchard himself points out, his modelling approach has the disadvantage that it cannot capture the life-cycle aspects of consumption and saving behaviour—the age-independent mortality rate ensures that the propensity to consume out of total wealth is the same for all households.[3]

Blanchard's modelling dilemma is clear: exact aggregation is 'bought' at the expense of a rather unrealistic description of the demographic process.[4] Of course, in a closed-economy context, the aggregation step is indispensable because equilibrium factor prices are determined in the aggregate factor markets. However, in the context of a small open economy, factor prices are typically determined in world markets so that the aggregation step is not necessary and life-cycle effects can be modelled. The main objective in this chapter is to elaborate on exactly this point.

[2] For our purposes, the most important extension is due to Buiter (1988) who allows for non-zero population growth by using the insights of P. Weil (1989). For a textbook treatment of the Blanchard-Yaari model, see Blanchard and Fischer (1989, ch. 3) or Heijdra and van der Ploeg (2002, ch. 16).

[3] Blanchard shows that a 'saving-for-retirement' effect can be mimicked by assuming that labour income declines with age. Faruqee and Laxton (2000) use this approach in a calibrated simulation model.

[4] Blanchard suggests that a constant mortality rate may be more reasonable if the model is applied to 'dynastic families' rather than to individual agents (1985, p. 225, fn.1). Under this interpretation the mortality rate refers to the probability that the dynasty literally becomes extinct or that the chain of bequests between the generations is broken.

As we demonstrate below, provided we restrict attention to the case of a small open economy, it is quite feasible to construct and analytically analyse a Blanchard-Yaari type overlapping-generations model incorporating a realistic description of demography. In addition we show that such a model gives rise to drastically different impulse-response functions associated with various macroeconomic shocks—the demographic realism matters.

The remainder of this chapter is organized as follows. Section 2.2 sets out the model. Following Calvo and Obstfeld (1988) and Faruqee (2003), we assume that the mortality rate is age-dependent and solve for the optimal decision rules of the individual households.[5] We establish that the propensity to consume out of total wealth is an increasing function of the individual's age, provided the mortality rate is non-decreasing in age. Next, we postulate a constant birth rate and characterize both the population composition and the implied aggregate population growth rate associated with the demographic process. Still using the general demographic process we characterize the steady-state age-profiles for consumption, human wealth, and asset holdings.

In Section 2.3 we employ actual demographic data for the Netherlands to estimate the parameters of various demographic processes, among which the Blanchard and Gompertz-Makeham demographic models. Not surprisingly, the latter model provides by far the superior fit with the data. Interestingly, the estimated Gompertz-Makeham (G-M hereafter) model distinguishes two 'phases' of life, namely youth and old-age. During youth, Gompertz' first cause of death dominates and the mortality rate is virtually constant, but during old-age it rises exponentially with age. In our view, the G-M model is interesting for at least two reasons. First, it presents a continuous-time generalization of the Diamond (1965) model, allowing individuals to differ even within each 'phase' of life. Second, it gives rise to relatively simple analytical expressions for the propensity to consume and the steady-state age profiles for consumption, human wealth, and financial assets. In the remainder of the section we show that the G-M model also gives rise to a bell-shaped age profile for financial assets (Modigliani's life-cycle pattern).

In Section 2.4 we compute and visualize the effects on the key variables of three typical macroeconomic shocks affecting the small open economy, namely a

[5] The relationship between these papers and this chapter is as follows. Calvo and Obstfeld (1988) recognize age-dependent mortality but do not solve the decentralized model. Instead, they characterize the dynamically consistent social optimum in the presence of time- and age-dependent lumpsum taxes. Faruqee (2003) models age-specific mortality in a decentralized setting but is ultimately unsuccessful. Indeed, he confuses the cumulative density function with the mortality rate (by requiring the death rate to go to unity in the limit; see Faruqee (2003, p. 302)). Furthermore, he is unable to solve the transitional dynamics.

balanced-budget spending shock, a temporary tax cut (Ricardian equivalence experiment), and an interest rate shock. We compare and contrast the results obtained for the Blanchard and G-M models. In the second part of Section 2.4 we also present the welfare effects associated with the shocks and demonstrate that the G-M model may give rise to non-monotonic welfare effects on existing generations, something which is impossible in the Blanchard case. We conclude Section 2.4 by showing that the two models also give rise to significantly different impulse-response functions for the aggregate variables (especially for asset holdings)—the heterogeneity does not 'wash out' in the aggregate. Finally, in Section 2.5 we mention a number of possible model applications and extensions and we draw some conclusions.

2.2 The model

2.2.1 Households

Individual consumption

From the perspective of birth, the expected lifetime utility of an agent is given by:

$$\Lambda(v,v) \equiv \int_v^\infty [1-\Phi(\tau-v)]U[\bar{c}(v,\tau)]e^{-\theta\cdot(\tau-v)}d\tau, \tag{2.1}$$

where the first argument of $\Lambda(\cdot,\cdot)$ denotes the the birth date, the second denotes the moment of evaluation, $U[\cdot]$ is 'felicity' (or instantaneous utility), $\bar{c}(v,\tau)$ is consumption of a vintage-v agent at time τ ($\geq v$), and θ is the constant pure rate of time preference ($\theta > 0$). Intuitively, $1-\Phi(\tau-v)$ is the probability that an agent born at time v is still alive at time τ (at which time the agent's age is $\tau - v$). The instantaneous mortality rate (or death probability) of an agent of age s is given by the hazard rate of the stochastic distribution of the age of death $m(s) \equiv \phi(s)/[1-\Phi(s)]$, where $\phi(s)$ and $\Phi(s)$ denote, respectively, the density and distribution (or cumulative density) functions. These functions exhibit the usual properties, i.e. $\phi(s) \geq 0$ and $0 \leq \Phi(s) \leq 1$ for $s \geq 0$. Since, by definition, $\Phi'(s) = \phi(s)$ and $\Phi(0) = 0$, it follows that the first term on the right-hand side of (2.1) can be simplified to:

$$1-\Phi(\tau-v) = e^{-M(\tau-v)}, \tag{2.2}$$

where

$$M(\tau-v) \equiv \int_0^{\tau-v} m(s)ds \tag{2.3}$$

is the cumulative mortality factor. By using (2.2) in (2.1) we find that the utility function of a newborn agent can be written as:

$$\Lambda(v,v) \equiv \int_v^\infty U[\bar{c}(v,\tau)]e^{-\theta\cdot(\tau-v)-M(\tau-v)}d\tau. \tag{2.4}$$

As was pointed out by Yaari (1965), future felicity is discounted both because of pure time preference (as $\theta > 0$) and because of life-time uncertainty (as $M(\tau - v) > 0$).[6]

From the perspective of some later time period t ($> v$), the utility function of the agent born at time v takes the following form:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^\infty U[\bar{c}(v,\tau)]e^{-\theta\cdot(\tau-t)-M(\tau-v)}d\tau, \tag{2.5}$$

where the discounting factor due to life-time uncertainty ($M(\tau - v)$) depends on the age of the household at time τ.[7] The felicity function is iso-elastic:

$$U[\bar{c}(v,\tau)] = \begin{cases} \dfrac{\bar{c}(v,\tau)^{1-1/\sigma} - 1}{1-1/\sigma} & \text{for } \sigma \neq 1 \\ \ln \bar{c}(v,\tau) & \text{for } \sigma = 1 \end{cases}, \tag{2.6}$$

where σ is the constant intertemporal substitution elasticity ($\sigma \geq 0$). As explained in more detail in Box 2.1, for the Blanchard model ($m(\cdot)$ constant) the choice of σ is far from innocuous in an open economy model with an exogenous interest rate. If the intertemporal substitution elasticity is too high, the model has no solution.

The household budget identity is given by:

$$\dot{\bar{a}}(v,\tau) = [r + m(\tau - v)]\bar{a}(v,\tau) + \bar{w}(\tau) - \bar{z}(\tau) - \bar{c}(v,\tau), \tag{2.7}$$

where $\bar{a}(v,\tau)$ is real financial wealth, r is the exogenously given (constant) world rate of interest, $\bar{w}(\tau)$ is the real wage rate, and $\bar{z}(\tau)$ is the lumpsum tax (the latter two variables are assumed to be independent of age and we assume that $\bar{w}(\tau) > \bar{z}(\tau)$). Labour supply is exogenous and each household supplies a single unit of labour. As usual, a dot above a variable denotes that variable's time rate of change,

[6] Yaari (1965, p. 143) attributes the latter insight to Fisher (1930, pp. 216–7).

[7] The appearance of the term $e^{M(t-v)}$ in front of the integral is a consequence of the fact that the distribution of expected remaining lifetime is not memoryless in general. Blanchard (1985) uses the memoryless exponential distribution for which $M(s) = \mu_0 s$ (where μ_0 is a constant) and thus $M(t-v) - M(\tau - v) = -M(\tau - t)$. Equation (2.5) can then be written in a more familiar format as $\Lambda(v,t) \equiv \int_t^\infty U[\bar{c}(v,\tau)]e^{-(\theta+\mu_0)(\tau-t)}d\tau$.

e.g. $\dot{\bar{a}}(v,\tau) \equiv d\bar{a}(v,\tau)/d\tau$. Following Yaari (1965) and Blanchard (1985), we postulate the existence of a perfectly competitive life insurance sector which offers actuarially fair annuity contracts to the households. Since household age is directly observable, the annuity rate of interest faced by a household of age $\tau - v$ is equal to the sum of the world interest rate and the instantaneous mortality rate of that household.[8]

Abstracting from physical capital, financial wealth can be held in the form of domestic government bonds ($\bar{d}(v,\tau)$) or foreign bonds ($\bar{f}(v,\tau)$).

$$\bar{a}(v,\tau) \equiv \bar{d}(v,\tau) + \bar{f}(v,\tau). \tag{2.8}$$

The two assets are perfect substitutes in the households' portfolios and thus attract the same rate of return.

In the planning period t, the household chooses paths for consumption and financial assets in order to maximize lifetime utility (2.5) subject to the flow budget identity (2.7) and a solvency condition, taking as given its initial level of financial assets $\bar{a}(v,t)$. The household optimum is fully characterized by:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = \sigma\cdot(r-\theta), \tag{2.9}$$

$$\Delta(u,r^*)\bar{c}(v,t) = \bar{a}(v,t) + \bar{h}(v,t), \tag{2.10}$$

$$\bar{h}(v,t) \equiv e^{ru+M(u)} \int_u^\infty [\bar{w}(s+v) - \bar{z}(s+v)] e^{-rs-M(s)} ds, \tag{2.11}$$

where $u \equiv t - v$ is the age of the household in the planning period, $r^* \equiv r - \sigma\cdot(r-\theta)$, and $\Delta(u,\lambda)$ is defined in general terms as:[9]

$$\Delta(u,\lambda) \equiv e^{\lambda u+M(u)} \int_u^\infty e^{-\lambda s-M(s)} ds, \qquad (\text{for } u \geq 0). \tag{2.12}$$

Equation (2.9) is the 'consumption Euler equation', relating the optimal time profile of consumption to the difference between the interest rate and the pure rate of time preference. The instantaneous mortality rate does not feature in this expression because households fully insure against the adverse effects of lifetime uncertainty (Yaari, 1965). In order to avoid having to deal with a taxonomy of different cases, we restrict attention in the remainder of this chapter (and throughout Part I of this

[8] See Mitchell et al. (1999) for a discussion on the availability of 'actuarial fair' annuities.

[9] As we demonstrate below, $\Delta(u,\lambda)$ plays a very important role in the model. Proposition 2.1 covers its main properties.

dissertation) to the case of a nation populated by patient agents, i.e. $r > \theta$.[10] Equation (2.10) shows that consumption in the planning period is proportional to total wealth, consisting of financial wealth ($\bar{a}(v,t)$) and human wealth ($\bar{h}(v,t)$). The marginal (and average) propensity to consume out of total wealth equals $1/\Delta(u, r^*)$, where r^* can be seen as the 'effective' discount rate facing the consumer. Clearly, $\Delta(u, r^*)$ depends only on the household's age in the planning period and not on time itself, because r and $M(\cdot)$ are not time dependent. For future reference, Proposition 2.1 establishes the important properties of the $\Delta(u, \lambda)$ function. Finally, human wealth is defined in (2.11) and represents the market value of the unit time endowment, i.e. the present value of after-tax wage income, using the annuity rate of interest for discounting purposes. Unless after-tax wage income is time-invariant, human wealth depends on both time and on the household's age in the planning period.

Proposition 2.1. *Let $\Delta(u, \lambda)$ be defined as in* (2.12) *and assume that the mortality rate is non-decreasing, i.e. $m'(s) \geq 0$ for all $s \geq 0$. Then the following properties can be established for $\Delta(u, \lambda)$:*

(i) *decreasing in λ, $\partial\Delta(u, \lambda)/\partial\lambda < 0$;*

(ii) *non-increasing in household age, $\partial\Delta(u, \lambda)/\partial u \leq 0$;*

(iii) *upper bound, $\Delta(u, \lambda) \leq 1/[\lambda + m(u)]$ (if $\lambda + m(u) > 0$);*

(iv) $\Delta(u, \lambda) > 0$;

(v) $\lim_{\lambda \to \infty} \Delta(u, \lambda) = 0$;

(vi) *for $m'(s) > 0$ and $m''(s) \geq 0$, the inequalities in (ii)-(iii) are strict and* $\lim_{u \to \infty} \Delta(u, \lambda) = 0$.

Proof. By definition, $M(u) \equiv \int_0^u m(s)ds$ so that $M(0) = 0$, $M'(u) = m(u) > 0$, and $M''(u) = m'(u) \geq 0$. First consider $\lambda + m(u) > 0$. Since $M(s)$ is a convex function of s we have $M(s) \geq M(u) + m(u) \cdot (s - u)$ and thus:

$$\Delta(u, \lambda) \leq e^{\lambda u + M(u)} \int_u^\infty e^{-\lambda s - m(u) \cdot (s-u) - M(u)} ds = \frac{1}{\lambda + m(u)}. \tag{2.13}$$

This establishes part (iii). Part (i) follows by straightforward differentiation:

$$\frac{\partial\Delta(u, \lambda)}{\partial\lambda} = -e^{\lambda u + M(u)} \int_u^\infty (s - u) e^{-\lambda s - M(s)} ds < 0. \tag{2.14}$$

[10] The results for the other cases (with $r < \theta$ or $r = \theta$) are easily deduced from our mathematical expressions.

Similarly, part (ii) is obtained by differentiating $\Delta(u, \lambda)$ with respect to u:

$$\frac{\partial \Delta(u, \lambda)}{\partial u} = [\lambda + m(u)]\Delta(u, \lambda) - 1 \leq 0, \tag{2.15}$$

where the sign follows from (2.13). For the alternative case, with $\lambda + m(u) < 0$, (2.13) no longer holds but (2.14)–(2.15) do. For $m'(u) > 0$ the inequalities in (2.14)–(2.15) are strict. Parts (iv)–(vi) are obvious. ∎

BOX 2.1

Iso-elastic felicity and intertemporal optimisation

The apparent simplicity of the iso-elastic felicity function as defined in Equation (2.6) can give quite unexpected results. Specifically, if the intertemporal elasticity of substitution is too high, optimal consumption is not defined. As an example, consider a simplified version of the model developed so far. If we look at the optimisation problem for a newborn, but impose Blanchard's perpetual youth model ($m(u) = m$) we have

$$\begin{aligned} \max_{\bar{c}(u)} & \int_0^\infty \frac{\bar{c}(s)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-(\theta+m)s} ds \\ \text{s.t.} \quad & \dot{\bar{a}}(u) = (r + m)\bar{a} - \bar{c}(u), \\ & \bar{a}(0) = \bar{a}_0, \ \bar{a}(u) \geq 0, \ \bar{c}(u) \geq 0 \text{ for all } u. \end{aligned}$$

Where we assumed that an individual has no income, but that he receives the present value of his future after tax earnings at birth. This problem is mathematically equivalent to the infinite horizon optimal growth problem with a constant capital–output ratio, which is analysed in depth by, among others, Tinbergen (1956) (See Takayama, 1985, ch. 5, sec. D for a discussion).

Following Takayama (1985) we obtain the Euler equation (2.9), which must hold for any *feasible* Euler path of consumption. Solving the linear differential equation (2.9) gives the equivalent condition

$$\bar{c}(u) = \bar{c}_0 e^{\sigma \cdot (r - \theta)} \tag{2.16}$$

where $\bar{c}_0$ is initial consumption. Substitution into the budget identity gives for the path of assets

$$\bar{a}(u) = e^{(r+m)u}\left[\bar{a}_0 - \bar{c}_0 \int_0^u e^{-[r-\sigma\cdot(r-\theta)+m]s} ds\right] \tag{2.17}$$

The integrand is positive, so non-negativity of $\bar{a}(u)$ implies $\bar{c}_0 = \bar{a}_0/\Delta(0, r^*)$, with $\Delta(\cdot)$ defined in (2.12) and $r^* = r - \sigma\cdot(r-\theta)$, provided that the integral converges as $u \to \infty$.

This is exactly the problem. The integral diverges for $\sigma\cdot(r-\theta) \geq r+m$, so $\bar{c}_0 > 0$ is not eligible since assets will eventually become negative. This leaves just one option, $\bar{c}_0 = 0$. However, this solution implies that $\bar{c}(u) = 0$ for all u, which is the worst possible solution. The only conclusion is that the solution path for the infinite horizon problem does not exist for $\sigma\cdot(r-\theta) \geq r+m$. There are two solutions to this problem: (1) impose an (usually) arbitrary upper limit to consumption, or (2) use an upward sloping mortality function.

Upper bound on consumption Suppose that for some reason, individual consumption has an upper bound $\bar{c}_M$ (exogenous or from another part of the model). To prevent trivial solutions, assume that $\bar{c}_M$ is too high to be able to afford consuming $\bar{c}_M$ the entire life, but low enough that if the consumption profile is declining, that for young ages the upper bound is binding.

If $r > \theta$, consumption increases at rate $\sigma\cdot(r-\theta)$ as long as the upper bound is not binding. If consumption hits the upper bound, it stays at this level until the individual dies. Note that these results hold for all $r > \theta \geq 0$, there is no restriction on the parameters as in the model without an upper bound. In the linear utility model, $\sigma \to \infty$, we have a bang-bang solution. Consumption is zero until total wealth is just sufficient to finance maximum consumption indefinitely.

If $r < \theta$, consumption will start at its maximum until a certain age. After this age consumption will decrease at rate $|\sigma\cdot(r-\theta)|$. Again we have a bang-bang solution if felicity is linear in consumption. Consumption is first maximal until the consumer cannot afford to consume any more. Consumption jumps to zero and the consumer spends the rest of his life paying off his debt.

Increasing mortality Another possibility to overcome the rather strange implication that consumption is postponed forever is to assume that the mortality rate $m(u)$ increases with age and has no upper bound (this excludes the perpetual youth model). Note that these assumptions imply that the mortality rate becomes infinite, possibly for a finite age as in Boucekkine et al. (2002). The optimisation problem for a newborn becomes

$$\max_{\bar{c}(u)} \int_0^\infty \frac{\bar{c}(s)^{1-1/\sigma}-1}{1-1/\sigma} e^{-\theta s - M(s)} ds$$
$$\text{s.t.} \quad \dot{\bar{a}}(u) = [r+m(u)]\bar{a} - \bar{c}(u),$$
$$\bar{a}(0) = \bar{a}_0,\ \bar{a}(u) \geq 0,\ \bar{c}(u) \geq 0 \text{ for all } u.$$

which gives the same Euler equation (2.16) and (2.17) becomes

$$\bar{a}(u) = e^{ru+M(u)} \left[\bar{a}_0 - \bar{c}_0 \int_0^u e^{-[r-\sigma\cdot(r-\theta)]s - M(s)} ds \right] \tag{2.18}$$

The integral in Equation (2.18) always converges as $u \to \infty$ (as long as $\sigma \ll \infty$), so we do not have the restriction on the parameters as in the Blanchard model. To show this, note that by assumption there always exists an age s^* above which the cumulative mortality rate $M(s)$ is higher than $-r^*u$, so we can split the integral in two parts, $0 \leq s \leq s^*$ and $s^* \leq s$. Boundedness of the integrand assures that the integral over $[0, s^*]$ exists and the second part exists because for $s > s^*$ the $e^{-M(s)}$ part suppresses the exponential term e^{-r^*s}, so the integral over $[s^*, \infty)$ converges.

Demography

In order to allow for non-zero population growth, we employ the analytical framework developed by Buiter (1988) which distinguishes the instantaneous mortality rate $m(s)$ and the birth rate b (> 0) and thus allows for net population growth or decline.[11] The population size at time t is denoted by $L(t)$ and the size of a newborn generation is assumed to be proportional to the current population:

$$L(v,v) = bL(v). \tag{2.19}$$

[11] The birth rate b is the crude birth rate, i.e. the number of newborns per capita. A more realistic assumption would be that only women between (approximately) 20 and 40 can give birth, but this makes the model intractable.

The size of cohort v at some later time τ is:

$$L(v,\tau) = L(v,v)[1-\Phi(\tau-v)] = bL(v)e^{-M(\tau-v)}, \tag{2.20}$$

where we have used (2.2) and (2.19). The aggregate mortality rate, $\bar{m}$, is defined by

$$\bar{m}L(t) = \int_{-\infty}^{t} m(t-v)L(v,t)dv, \tag{2.21}$$

and it is assumed that the system is in a 'demographic steady state' so that $\bar{m}$ is constant. Despite the fact that the expected remaining lifetime of each individual is stochastic, there is no aggregate uncertainty in the economy. In the absence of international migration, the growth rate of the aggregate population, n, is equal to the difference between the birth rate and the aggregate mortality rate, i.e. $n \equiv b - \bar{m}$. It follows that $L(v) = A_0 e^{nv}$, $L(t) = A_0 e^{nt}$ and thus $L(v) = L(t)e^{-n\cdot(t-v)}$. Using this result in (2.20) we obtain the generational population weights:

$$l(v,t) \equiv \frac{L(v,t)}{L(t)} = be^{-n\cdot(t-v)-M(t-v)}, \qquad \text{for } t \geq v. \tag{2.22}$$

The key thing to note about (2.22) is that the population proportion of generation v at time t only depends on the age of that generation and not on time itself.

The growth rate of the population in the demographic steady state is computed by combining (2.21) and (2.22) and simplifying:

$$\frac{1}{b} = \Delta(0,n). \tag{2.23}$$

For a given birth rate b, eq. (2.23) implicitly defines the coherent solution for n and thus for the aggregate mortality rate, $\bar{m} \equiv b - n$.[12]

Per capita household sector

Per capita variables are calculated as the integral of the generation-specific values weighted by the corresponding generation weights. For example, per capita consumption, $c(t)$, is defined as:

$$c(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{c}(v,t)dv, \tag{2.24}$$

[12] For a constant mortality rate m, we have $1/\Delta(0,n) = n + m$ so that (2.23) implies $n = b - m$. Blanchard (1985) sets $b = m$ so that $n = 0$ (constant population).

where $l(v,t)$ and $\bar{c}(v,t)$ are defined in, respectively, (2.22) and (2.10) above. Exact aggregation of (2.10) is impossible because both $\Delta(u,r^*)$ and the wealth components, $\bar{a}(v,t)$ and $\bar{h}(v,t)$, depend on the generations index v. The 'Euler equation' for per capita consumption can nevertheless be obtained by differentiating (2.24) with respect to time and noting (2.9) and (2.22):

$$\dot{c}(t) = b\bar{c}(t,t) + \sigma\cdot(r-\theta)c(t) - \int_{-\infty}^{t}[n+m(t-v)]l(v,t)\bar{c}(v,t)dv. \tag{2.25}$$

Per capita consumption growth is boosted by the arrival of new generations who start to consume out of human wealth (first term on the right-hand side) and by individual consumption growth (second term). The third term on the right-hand side of (2.25) corrects for population growth and (age-dependent) mortality.[13]

Per capita financial wealth is defined as $a(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{a}(v,t)dv$. By differentiating this expression with respect to t we obtain:

$$\dot{a}(t) = (r-n)a(t) + w(t) - z(t) - c(t), \tag{2.26}$$

where the wage rate $w(t) = \bar{w}(t)$, taxes $z(t) = \bar{z}(t)$, and we have used eq. (2.7) and noted the fact that newborns are born without financial assets ($\bar{a}(v,v) = 0$). The interest rate net of population growth is assumed to be positive, i.e. $r > n$. As in the standard Blanchard model, annuity payments drop out of the expression for per capita asset accumulation because they constitute transfers (via the life insurance companies) from those who die to agents who stay alive.

Finally, per capita human wealth is defined as $h(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{h}(v,t)dv$ so that $\dot{h}(t)$ can be written as:

$$\dot{h}(t) = (r-n)h(t) + b\bar{h}(t,t) - w(t) + z(t). \tag{2.27}$$

In the standard Buiter model per capita human wealth is the same for all generations and accumulates at the constant annuity rate of interest ($r+m$). In contrast, in the present model the effects of the net interest rate ($r-n$) and the birth rate (b) are separate, with the former applying to per capita human wealth and the latter applying to the human wealth of newborn generations.

[13] If the mortality rate were constant, as in Blanchard (1985) and Buiter (1988), then $n \equiv b - m$ and Equation (2.25) would simplify to $\dot{c}(t) = \sigma\cdot(r-\theta)c(t) - b[c(t) - c(t,t)]$.

2.2.2 Firms, government, and foreign sector

Following Buiter (1988) we keep the production side of the model in this chapter as simple as possible by abstracting from physical capital altogether.[14] Competitive firms face the technology $Y(t) = k_0 L(t)$ where k_0 is an exogenous productivity index and $L(t)$ is the aggregate supply of labour. The real wage rate is then given by $w(t) = k_0$.

The government budget identity is given by:

$$\dot{d}(t) = (r-n)d(t) + g(t) - z(t), \tag{2.28}$$

where $d(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{d}(v,t)dv$ is the per capita stock of domestic bonds, and $g(t)$ is per capita government goods consumption. The government solvency condition is $\lim_{\tau\to\infty} d(\tau)e^{[r-n][t-\tau]} = 0$, so that the intertemporal budget constraint of the government can be written as:

$$d(t) = \int_{t}^{\infty} [z(\tau) - g(\tau)]e^{-(r-n)(\tau-t)}d\tau. \tag{2.29}$$

To the extent that there is outstanding debt (positive left-hand side), it must be exactly matched by the present value of current and future primary surpluses (positive right-hand side), using the net interest rate $(r-n)$ for discounting.

Finally, the evolution of the per capita stock of net foreign assets is explained by the current account:

$$\dot{f}(t) = (r-n)f(t) + w(t) - c(t) - g(t), \tag{2.30}$$

where we have used that $y(t) \equiv Y(t)/L(t) = w(t)$ and where $f(t)$ is defined as usual as $f(t) \equiv \int_{-\infty}^{t} l(v,t)f(v,t)dv$ denotes the per capita stock of net foreign bonds in the hands of domestic households.

2.2.3 Steady-state equilibrium

It is relatively straightforward to characterize the steady state of the model. The steady-state values for all variables are designated by means of a hat overstrike, e.g. $\hat{c}$ is steady-state per capita consumption. Where no confusion can arise, the

[14] In the context of a small open economy with firms facing convex investment adjustment costs, our approach does not entail much loss of generality because the investment and savings systems decouple in that case. See Matsuyama (1987); Bovenberg (1993, 1994); Heijdra and Meijdam (2002) and Heijdra and van der Ploeg (2002, pp. 571-581).

time index is also suppressed. Since technology is held constant, the steady-state wage rate is time-invariant, i.e. $w(t) = \hat{w} = k_0$. If the government variables are also held constant, so that $z(t) = \hat{z}$, $g(t) = \hat{g}$, and $d(t) = \hat{d} \equiv (\hat{z} - \hat{g})/(r - n)$, then the economy settles into a unique saddle-point stable steady-state equilibrium in which $c(t) = \hat{c}$, $h(t) = \hat{h}$, $a(t) = \hat{a}$, and $f(t) = \hat{f}$.[15]

In the steady-state equilibrium, all variables applying to individuals can be re-written solely in terms of their age, $u \equiv t - v$ (as is also the case outside the steady state for $\Delta(u, r^*)$—see eq. (2.12) above). After some straightforward substitutions we find:

$$\hat{\hat{h}}(u) \equiv \hat{\hat{h}}(v,t) = (\hat{w} - \hat{z})\Delta(u,r), \tag{2.31}$$

$$\hat{\hat{c}}(u) \equiv \hat{\hat{c}}(v,t) = \frac{\hat{\hat{h}}(0)}{\Delta(0,r^*)} e^{\sigma\cdot(r-\theta)u}, \tag{2.32}$$

$$\hat{\hat{a}}(u) \equiv \hat{\hat{a}}(v,t) = \Psi(u,r,r^*)\hat{\hat{h}}(0), \tag{2.33}$$

where $r^* \equiv r - \sigma\cdot(r - \theta)$, $\Delta(u, \lambda)$ is defined in eq. (2.12), and $\Psi(u, r, r^*)$ is given by:

$$\Psi(u,r,r^*) \equiv e^{ru+M(u)} \left[\frac{\int_u^\infty e^{-r^*s-M(s)}ds}{\Delta(0,r^*)} - \frac{\int_u^\infty e^{-rs-M(s)}ds}{\Delta(0,r)} \right]. \tag{2.34}$$

Deferring the economic intuition behind (2.31)–(2.33) to Section 2.3.2, we simply note that human wealth is positive (since $\hat{w} > \hat{z}$) and proportional to $\Delta(u,r)$, the properties of which are covered in Proposition 2.1. Human wealth at birth is an important determinant for the age profiles for both consumption and financial assets. In the absence of initial financial wealth (e.g. received bequests), $\hat{\hat{h}}(0)$ is the key 'initial condition' facing agents. Consumption rises monotonically with age but the age profile of financial assets depends critically on the demographic process, i.e. on the $\Psi(u,r,r^*)$ function. The main properties of this function are stated in Lemma 2.1. If rate of time preference θ equals the interest rate r, than individuals do not save or borrow (Lemma 2.1(i)). With a constant mortality rate, financial wealth rises monotonically with age (Lemma 2.1(iv)). When the mortality rate increases with age, however, the assets are positive and increasing early on in life, but return to zero at higher ages provided the condition in Lemma 2.1(iii) is satisfied. For the general case, the asset profile may display multiple peaks though there is only a

[15] Saddle-point stability follows trivially from the fact that all agents in the economy satisfy their respective solvency conditions. Consumption and human wealth are forward-looking variables (able to feature discrete jumps) whilst total financial assets and net foreign assets are predetermined (non-jumping) variables.

single peak for the empirically most relevant G-M model studied in Section 2.3.2 below.

Lemma 2.1. *Let* $\Psi(u,r,r^*)$ *be defined as in* (2.34) *and note that* $r^* = r \Leftrightarrow \sigma\cdot(r-\theta) = 0$. *The following properties can be established for* $\Psi(u,r,r^*)$*:*

(i) $\Psi(u,r,r) = 0$ *for all* $u \geq 0$*;*

(ii) for $r > r^*$, $\Psi(u,r,r^*) \geq 0$ *with the equality sign only holding for* $u = 0$*;*

(iii) if $r > r^*$ *then* $\lim_{u\to\infty} \Psi(u,r,r^*) = 0$ *if and only if* $\lim_{u\to\infty} e^{(r-r^*)u}/[r+m(u)] = 0$*;*

(iv) if $m(u) = m_0$ *(Blanchard) then* $\Psi(u,r,r^*) \equiv e^{\sigma\cdot(r-r^*)u} - 1$ *is a strictly increasing function in* u*.*

Proof. We denote the term in square brackets on the right-hand side of (2.34) by $\Omega(u,r,r^*)$ and note that, $\Omega(0,r,r^*) = \lim_{u\to\infty} \Omega(u,r,r^*) = 0$. Taking the derivative with respect to u we find:

$$\Omega'(u,r,r^*) = e^{-M(u)}\left[\frac{e^{-ru}}{\Delta(0,r)} - \frac{e^{-r^*u}}{\Delta(0,r^*)}\right], \tag{2.35}$$

which clearly has a single root (at $\bar{u} \equiv 1/(r-r^*)\ln\left(\Delta(0,r)/\Delta(0,r^*)\right) > 0$) and satisfies $\Omega'(0,r,r^*) > 0$ (for $r > r^*$). This in combination with continuity of $\Omega(u,r,r^*)$ shows that $\Omega(u,r,r^*) > 0$ for $u > 0$ (and $r > r^*$). Since $\Psi(u,r,r^*) \equiv e^{ru+M(u)}\Omega(u,r,r^*)$, this proves part (ii).

To show part (iii), rewrite $\lim_{u\to\infty} \Psi(u,r,r^*)$ and use l'Hopital's rule

$$\begin{aligned}
\lim_{u\to\infty} \Psi(u,r,r^*) &= \lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{\int_u^\infty e^{-r^*s-M(s)}ds}{e^{-ru-M(u)}} - \frac{1}{\Delta(0,r)}\frac{\int_u^\infty e^{-rs-M(s)}ds}{e^{-ru-M(u)}}\right\} \\
\ldots &= \lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{e^{-r^*u-M(u)}}{[r+m(u)]e^{-ru-M(u)}} - \frac{1}{\Delta(0,r)}\frac{e^{-ru-M(u)}}{[r+m(u)]e^{-ru-M(u)}}\right\} \\
\ldots &= \lim_{u\to\infty}\left\{\frac{1}{\Delta(0,r^*)}\frac{e^{(r-r^*)u}}{r+m(u)} - \frac{1}{\Delta(0,r)}\frac{1}{r+m(u)}\right\},
\end{aligned}$$

from which (iii) follows immediately. Part (iv) is obvious. ■

Simple expressions for the steady-state per capita variables can also be found:

$$\hat{c} = \hat{\bar{c}}(0)\frac{\Delta(0,n^*)}{\Delta(0,n)}, \tag{2.36}$$

$$\hat{h} = \frac{\hat{w} - \hat{z}}{r - n}\left[1 - \frac{\Delta(0,r)}{\Delta(0,n)}\right] \tag{2.37}$$

$$\hat{a} \equiv \hat{d} + \hat{f} = \frac{\hat{w} - \hat{z}}{r - n}\left[\frac{\Delta(0,r)}{\Delta(0,r^*)}\frac{\Delta(0,n^*)}{\Delta(0,n)} - 1\right], \tag{2.38}$$

where $n^* \equiv n - \sigma\cdot(r - \theta)$ and the term in square brackets on the right-hand side of (2.38) is positive. Not surprisingly, per capita consumption exceeds consumption by newborns (because $n > n^*$ so that $\Delta(0,n^*) > \Delta(0,n)$), and both per capita human and financial wealth are positive.

Armed with these expressions it is straightforward to derive the long-run effects of various shocks impacting the economy.[16] A *balanced-budget increase in government consumption* ($d\hat{z} = d\hat{g} > 0$) leads to a decrease in steady-state human wealth and consumption for all cohorts:

$$\frac{d\hat{\bar{h}}(u)}{d\hat{z}} = -\Delta(u,r) < 0, \tag{2.39}$$

$$\frac{d\hat{\bar{c}}(u)}{d\hat{z}} = -\frac{\Delta(0,r)}{\Delta(0,r^*)}e^{\sigma\cdot(r-\theta)u} < 0. \tag{2.40}$$

Obviously, per capita steady-state consumption and human wealth also fall (see eqs (2.36) and (2.37)). It follows from (2.38) that per capita steady-state financial assets decline:

$$\frac{d\hat{a}}{d\hat{z}} = \frac{1}{r - n}\left[1 + \frac{d\hat{c}}{d\hat{z}}\right] < 0. \tag{2.41}$$

This implies that consumption is crowded out more than one for one. Finally, since government debt is unchanged (by design) it follows from the first equality in (2.38) that $d\hat{f}/d\hat{z} = d\hat{a}/d\hat{z}$. The balanced-budget increase in government consumption thus leads to a long-run reduction in financial assets and a reduction in net imports, just as in the standard open-economy Blanchard model with $r > \theta$ (1985, p. 230-1).

A *long-run tax-financed increase in public debt* ($[r - n]d\hat{d} = d\hat{z} > 0$) leads to a decrease in generation-specific and per capita steady-state consumption and human wealth (see (2.39)–(2.40)). It follows from (2.38) that:

$$(r - n)\,\frac{d\hat{f}}{d\hat{z}} \equiv -[r - n]\frac{d\hat{d}}{d\hat{z}} + \frac{d\hat{c}}{d\hat{z}} + 1 = \frac{d\hat{c}}{d\hat{z}} < -1. \tag{2.42}$$

As in the standard Blanchard model (with $r > \theta$), government debt more than displaces foreign assets in the households' portfolios (1985, p. 242).

[16] The impact and transitional effects of these shocks are studied in Section 2.4.

An *increase in the world interest rate* leads to higher discounting of after-tax wages and a reduction in both individual and aggregate human wealth:

$$\frac{d\hat{\hat{h}}(u)}{dr} = [\hat{w} - \hat{z}]\frac{\partial\Delta(u,r)}{\partial r} < 0, \tag{2.43}$$

$$\frac{d\hat{h}}{dr} = \int_0^\infty l(u)\frac{d\hat{\hat{h}}(u)}{dr}du < 0, \tag{2.44}$$

where we have used Proposition 2.1(i) to establish the sign in (2.43). The interest elasticity of individual consumption is given by:

$$\frac{r}{\hat{\hat{c}}(u)}\frac{d\hat{\hat{c}}(u)}{dr} = \frac{r}{\hat{\hat{h}}(0)}\frac{d\hat{\hat{h}}(0)}{dr} + r\sigma u - (1-\sigma)\frac{r}{\Delta(0,r^*)}\frac{\partial\Delta(0,r^*)}{\partial r^*}. \tag{2.45}$$

The effect on individual consumption is ambiguous in general because it results from the interplay of three effects, namely the (initial) human-wealth effect (HWE), the consumption-growth effect (CGE), and the (initial) consumption-propensity effect (CPE). The HWE is represented by the first term on the right-hand side of (2.45) and is negative as after-tax income is discounted more heavily. The CGE effect (the second term on the right-hand side) is positive and increasing in the household's age. An increase in the interest rate causes agents to adopt a steeper age-profile for consumption. Finally, the third term on the right-hand side represents the CPE, i.e. the effect of the interest rate change on a newborn's propensity to consume, $1/\Delta(0,r^*)$. In the empirically plausible case, with $\sigma < 1$, the CPE is positive, thus partially offsetting the negative HWE. For the case with a logarithmic felicity function, which we focus on from Section 2.3.2 onward, $\sigma = 1$ and the CPE is zero ($\Delta(0,r^*) = \Delta(0,\theta)$ in that case).

The effect on per capita consumption can be written as:

$$\frac{r}{\hat{c}}\frac{d\hat{c}}{dr} = \frac{r}{\hat{\hat{c}}(0)}\frac{d\hat{\hat{c}}(0)}{dr} - \sigma\frac{r}{\Delta(0,n^*)}\frac{d\Delta(0,n^*)}{dn^*}. \tag{2.46}$$

and is thus also ambiguous in general. The sign of first term on the right-hand side is ambiguous for $\sigma < 1$, because the HWE is negative and the CPE is positive (see (2.45)). For the logarithmic case ($\sigma = 1$), however, the first term must be negative. Since the second term on the right-hand side is positive, it is nevertheless possible for per capita consumption to rise (as is the case in the simulations performed in Section 2.4). Finally, the effect on individual and per capita assets is ambiguous for the general specification of the model.

2.3 Demography

As was stressed by Blanchard (1985, p. 223), exact aggregation of the consumption function is generally impossible because both the propensity to consume (our $1/\Delta(u, r^*)$) and the wealth components (our $\bar{a}(v, t)$ and $\bar{h}(v, t)$) are age dependent. Blanchard cuts this Gordian knot by assuming the mortality rate to be constant, i.e. $m(s) = \mu_0 > 0$ so $M(u) = \mu_0 u$. The advantages of his approach are its simplicity and flexibility—the expected remaining planning horizon is $1/\mu_0$ so, by letting $\mu_0 \to 0$, the infinite-horizon Ramsey model is obtained as a special case. The main disadvantage of the Blanchard approach is that it cannot capture the life-cycle aspect of consumption behaviour. In addition, the perpetual youth assumption is easily refuted empirically as it runs foul of the Gompertz-Makeham law of mortality (Preston et al. (2001) and Section 2.3.1 below).

In the context of a small open economy, however, it is quite feasible to incorporate a realistic demographic structure because the aggregation step is not necessary. Since both the interest rate and the wage rate are exogenous, the macroeconomic equilibrium can be studied directly at the level of individual households (see Sections 2.3.2 and 2.4).

2.3.1 Estimates

In this section we estimate the survival function $(1 - \Phi(\tau - v))$ by using actual demographic data for the Netherlands taken from the Human Mortality Database (2006). We will use these estimates throughout this and the following two chapters. The data are annual and apply to the population cohort born in 1920. Actual mortality figures are available up to 2003, implying that demographic projections have only been used to compute the survival probabilities for the age range 84–105.[17] Denoting the actual surviving fraction up until age u_i of the people born in 1920 by $S(u_i)$, we estimate the parameters of a given parametric distribution function by means of non-linear least squares. Denoting the parameter vector by $\boldsymbol{\mu}$, the model to be estimated is thus:

$$S(u_i) = 1 - \Phi(u_i, \boldsymbol{\mu}) + \varepsilon_i = e^{-M(u_i, \boldsymbol{\mu})} + \varepsilon_i, \tag{2.47}$$

[17] Child mortality was still a real issue in the 1920s—almost 11 percent of the 1920 cohort died during their first year. Since it is not the phenomenon that we wish to focus on, we adjust the mortality figures by assuming the death probability for ages 0–14 to be equal to that of a 15 year old. This takes out the downward sloping segment of the mortality function at the start of life.

where $M(u_i, \boldsymbol{\mu}) = \int_0^{u_i} m(s, \boldsymbol{\mu}) ds$ and ε_i is the stochastic error term. The estimates are reported in Table 2.1 for various specifications of the mortality process. In that table, $\hat{\sigma}$ is the estimated standard error of the regression, $\bar{m}$ is the average mortality rate, the t-statistics are given in round brackets below the estimates, and $1 - \widehat{\Phi(100)}$ represents the estimated proportion of centenarians.

We consider five different functional forms for the instantaneous mortality rate and the associated $M(u_i)$ functions. The Blanchard model based on a *constant* mortality rate (model 1) yields an estimated mortality rate of 0.7% per annum and displays the worst fit of all cases considered–the estimated standard error is 0.22 which far exceeds the standard errors for the other models.

The second and third models are the linear and piecewise linear mortality rates models. The linear model is based on the notion that the mortality rate increases with age. This *linear-in-age* model with the mortality rate defined by $m(u) = \mu_0 + 2\mu_1^2 u$ fits a little better than the constant mortality rate model but it predicts a negative mortality rate for newborns. Constraining the constant to zero gives a standard error of 0.13, better than the Blanchard model, but still quite high. A combination of the Blanchard model and the linear model, the piecewise linear model fits the data much better with $\sigma = 0.024$. According to the piecewise linear model, a human life can be divided in two parts; for young people mortality is constant and low, for old people mortality increases linearly with age. The mortality function can be written as

$$m(u) \equiv \begin{cases} \mu_0 & \text{for } 0 < u < \bar{u} \\ \mu_0 + 2\mu_1^2(u - \bar{u}) & \text{for } u \geq \bar{u} \end{cases} \tag{2.48}$$

For the 1920-generation, the kink in the mortality profile lies around the age of 55. Below this age, the first cause of death dominates, beyond this age, biological wear and tear starts to increase the probability of death.

The fourth model we estimate is the mortality process used by Boucekkine et al. (2002). Their proposed survival law follows

$$S(u) = 1 - \Phi(u) = \frac{e^{-\beta u} - \alpha}{1 - \alpha}, \quad 0 \leq u \leq A \tag{2.49}$$

with $A = -\frac{1}{\beta} \ln \alpha$. Beyond age A no one is supposed to be alive, it is the maximum attainable age. For the estimated parameter values the maximum age is 86 years. This immediately shows the weakness of this model. Although the fit is quite good,

$\hat{\sigma}$ is 0.016, the model predicts that nobody from the 1920 generation will survive this year (2006), however, about 8% is still alive.[18]

Finally, the last model postulates the instantaneous mortality rate to follow the Gompertz Makeham process:

$$m(u) \equiv \mu_0 + \mu_1 e^{\mu_2 u}, \tag{2.50}$$

with $\mu_i > 0$. As Table 2.1 shows, the parameter estimates are all highly significant. The standard deviation is very small and the model features a realistic prediction for the fraction of centenarians (0.1% rather than the unrealistic prediction of almost 32% for the Blanchard model). This model does not suffer from the doubtfull maximum attainable age of the previous model. The G-M model has no maximum age, but mortality rates increase exponentianally, so at very old ages, it is highly unlikely to survive another year. This closely resembles the idea of Gavrilov and Gavrilova (1991) who argued that people die before the age of infinity, not because they cannot pass bounding age, but because the probability of a person avoiding the ever-present risk of death for that long is infinitesimal.

In the top panel of Figure 2.1 we illustrate data points at five-year intervals (stars) as well as the estimated survival functions for the five models. The poor fit of the Blanchard model is confirmed–the surviving fraction is underestimated up to about age 73 and overestimated thereafter. In contrast, the G-M model tracks the data quite well. Another way to visualize the difference between the two models makes use of their predicted mortality rates (middle panel) and expected remaining lifetimes (bottom panel of Figure 2.1). After about age 60, the mortality rate of the G-M model rises exponentially with age. The estimated G-M model thus distinguishes two phases of life, namely 'youth', lasting until about age 60, and 'old age' thereafter. Of course, for the Blanchard model expected remaining lifetime is constant (and equal to 87 years) so the agent enjoys a 'perpetual youth'.

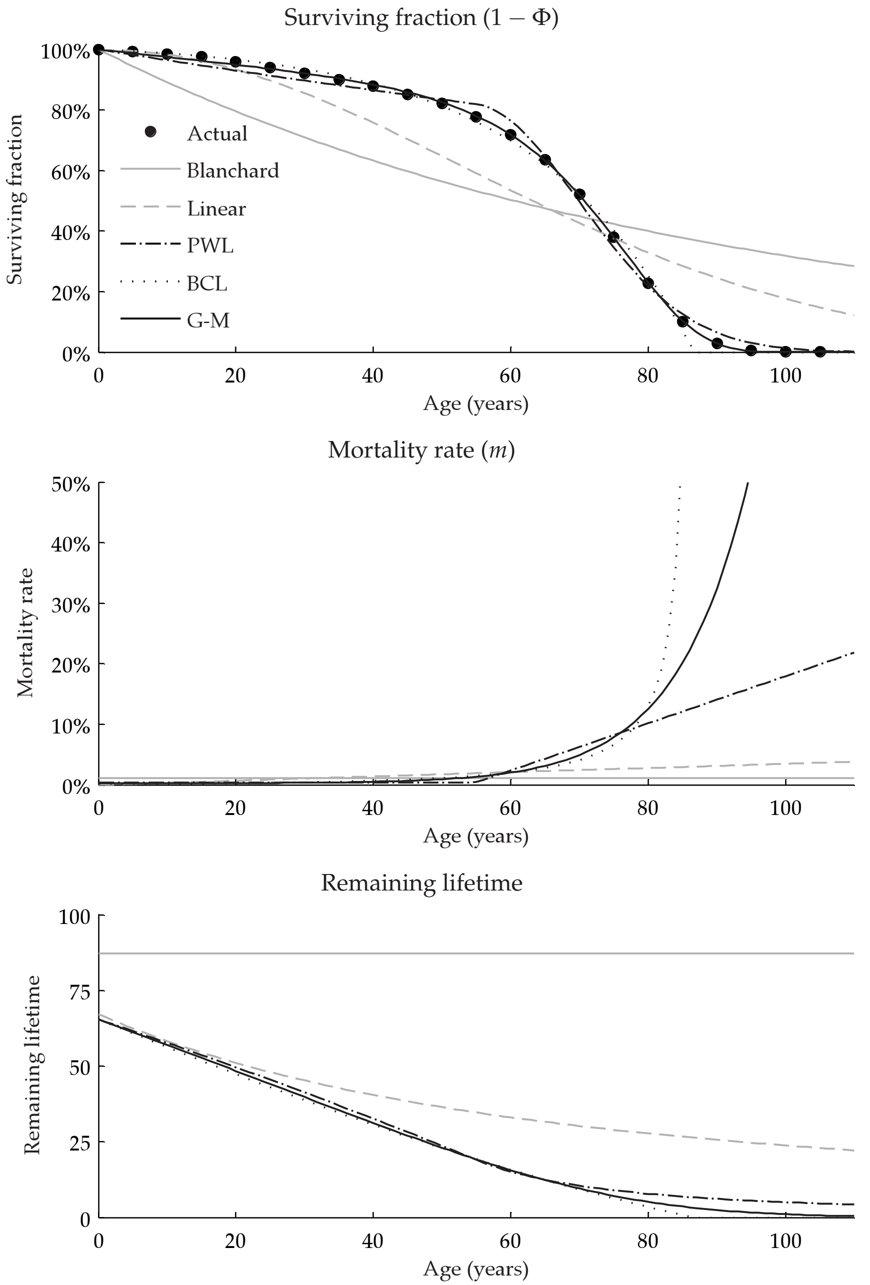
[18] There is a large literature on the maximum length of life. An interesting overview of this literature is given by Kirkwood (2001) who states: 'The truth is that the idea of a fixed limit to human longevity was always a little questionable but it is only now, as understanding of the ageing process improves, that the reason has become apparent. There is no mechanism that measures man's span of time and then activates a destructive process. In fact, quite the reverse is true and nearly every system in the body does its best to preserve life. Even apoptosis is directed mostly at protecting the body by deleting cells that might cause harm. These systems are not perfect, however, and ageing occurs because myriad tiny faults accumulate. Eventually the viability of various organs is compromised, the weakest link is revealed, and so goodbye.'

Table 2.1. Estimated survival functions

1. *Blanchard:*	$M(u) \equiv \mu_0 u$		
$\hat{\sigma} = 0.2213$	$\hat{\mu}_0$		
$\bar{m} = 1.15\%$	0.01147		
$1 - \widehat{\Phi(100)} = 31.8\%$	(14.3)		
2. *Linear:*	$M(u) \equiv \mu_0 u + \mu_1^2 u^2$		
$\hat{\sigma} = 0.1312$	$\hat{\mu}_0$	$\hat{\mu}_1$	
$\bar{m} = 1.11\%$	–	0.0132	
$1 - \widehat{\Phi(100)} = 17.6\%$	–	(42.24)	
3. *Piecewise linear (PWL):*	$M(u) \equiv \begin{cases} \mu_0 u & \text{for } 0 < u < \bar{u} \\ \mu_0 u + \mu_1^2 (u - \bar{u})^2 & \text{for } u \geq \bar{u} \end{cases}$		
$\hat{\sigma} = 0.0243$	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\bar{u}}$
$\bar{m} = 1.04\%$	3.63×10^{-3}	0.0441	54.8
$1 - \widehat{\Phi(100)} = 1.3\%$	(32.14)	(37.74)	(97.61)
4. *Boucekkine et al. (2002):*	$M(u) \equiv \ln\left(\frac{1-\mu_0}{e^{-\mu_1 u} - \mu_0}\right), \quad 0 < u < \ln\left(-\frac{\mu_0}{\mu_1}\right)$		
$\hat{\sigma} = 0.0162$	$\hat{\mu}_0$	$\hat{\mu}_1$	
$\bar{m} = 1.01\%$	41.06	-0.0429	
$1 - \widehat{\Phi(100)} = 0.0\%$	(23.711)	(-78.84)	
5. *Gompertz-Makeham (G-M):*	$M(u) \equiv \mu_0 u + (\mu_1 / \mu_2)[e^{\mu_2 u} - 1]$		
$\hat{\sigma} = 4.852 \times 10^{-3}$	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$
$\bar{m} = 1.02\%$	2.437×10^{-3}	5.52×10^{-5}	0.0964
$1 - \widehat{\Phi(100)} = 0.1\%$	(65.8)	(20.5)	(138.2)

Notes: All function fitted to data for the cohort born in the Netherlands, 1920 (male and female). Observed survival rates for ages 0–85, projected survival rates otherwise. To correct for child mortality, the death probability for ages 0–14 is assumed to be equal to that of a 15 year old. t-statistics between brackets, $\hat{\sigma}$ is the standard deviation, $1 - \widehat{\Phi(100)}$ is the predicted fraction of centenarians. *Source:* Human Mortality Database (2006) and own calculations.

Figure 2.1. Actual and estimated survival rates



Note: All survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values).

2.3.2 Steady-state profiles

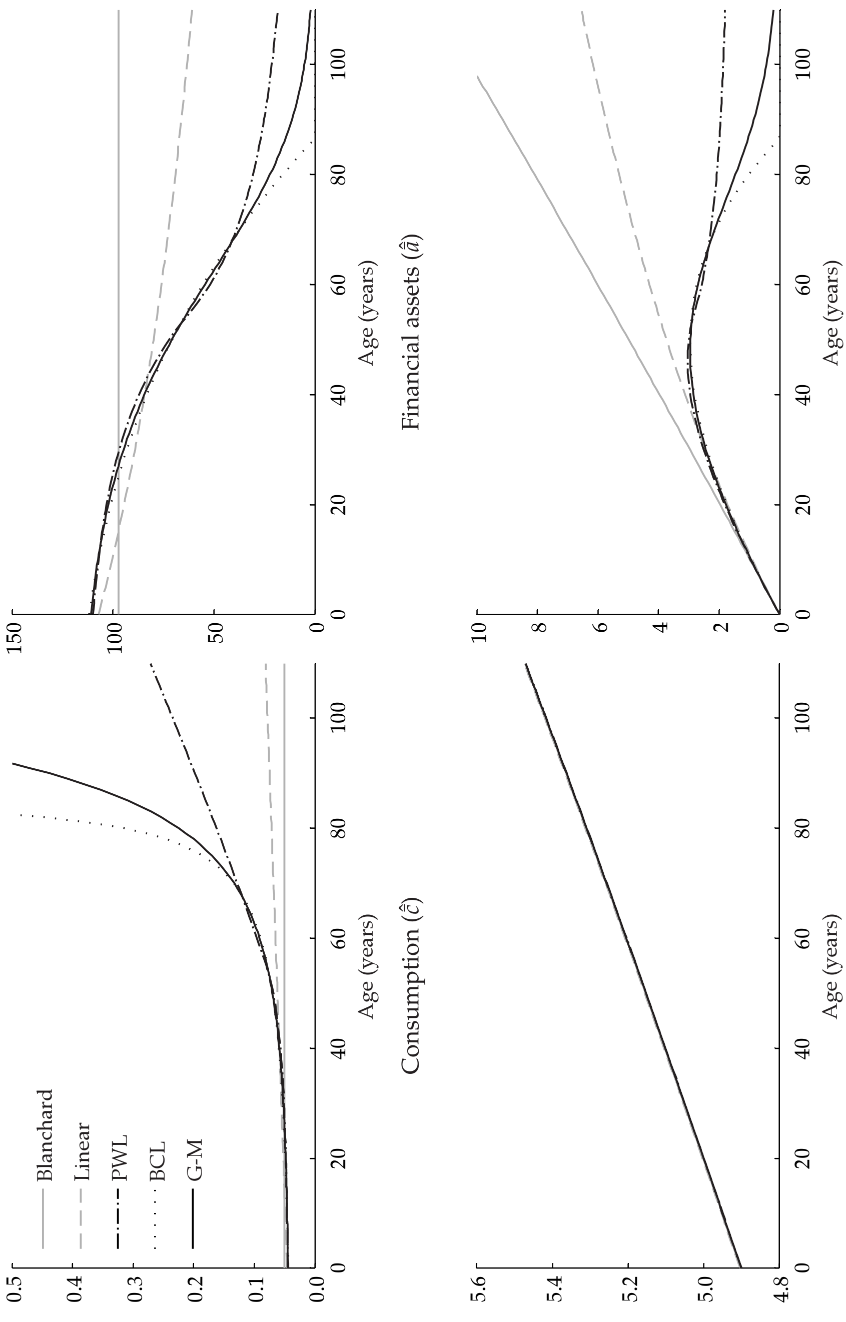
In Figure 2.2 we visualize (for all estimated models) the steady-state age profiles for the propensity to consume, human wealth, consumption, and financial assets. The analytical expressions for these variables are given in, respectively, eqs (2.12), (2.31), (2.32), and (2.33). In order to avoid a taxonomy of cases, we restrict attention to the 'unit-elastic case' in the remainder of this chapter, i.e. we set the intertemporal substitution elasticity equal to one ($\sigma = 1$). This implies that $r^* = \theta$ so that the marginal propensity to consume out of total wealth equals $1/\Delta(u,\theta)$ and is thus independent of the interest rate.

Clearly the $\Delta(u,\lambda)$ function (defined in (2.12)) plays a key role in the model. Fortunately, for all demographic specifications, easily computed closed-form solutions for $\Delta(u,\lambda)$ can be derived. Indeed, for the Blanchard model it reduces to $\Delta(u,\lambda) = 1/(\lambda + \mu_0)$ and is thus independent of the age of the household. We show in Appendix 2.A that the solution for the linear model and the piecewise linear model can be written in terms of the error function and the G-M model can be written in terms of the upper-tailed incomplete gamma function (Kreyszig, 1999, p. A55). Not surprisingly, since all models satisfy the assumptions stated in Proposition 2.1, it follows that the marginal propensity to consume, $1/\Delta(u,\theta)$, increases with age, except for the Blanchard model, for which the marginal propensity to consume is constant (the borderline case in Proposition 2.1(ii) and (iii)). This is confirmed in the top left-hand panel of Figure 2.2.

In the top right-hand panel of Figure 2.2 the age profile for steady-state human wealth is plotted.[19] For the standard Blanchard model the annuity rate of interest is age-independent because the mortality rate is constant. As a result, human wealth is age-independent also. In stark contrast, for the G-M model the annuity rate of interest rises with age so that discounting of after-tax wage income is heavier the older the household is. Human wealth gradually falls with age as a result. Indeed, it follows from (2.31) that $\hat{\hat{h}}(u)$ is proportional to $\Delta(u,r)$ which is downward sloping in u for any demography with a non-decreasing mortality rate (see Proposition 2.1). Exploiting the proportionality between $\hat{\hat{h}}(u)$ and $\Delta(u,r)$, we find that the slope of

[19] Following Gardia (1991, p. 423) we set $r = 0.04$ and $\theta = r^* = 0.039$. We interpret the G-M demography as the truth and choose b such that $n = 0.0134$ (the average population growth rate during the period 1920–1940). This yields a value of $b = 0.0236$ (which falls in between the observed birth rates for 1920 ($= 0.028$) and 1940 ($= 0.02$)). The estimated G-M model yields an expected remaining lifetime at birth of 65.5 years, which is very close to the value used by Cardia ($= 67$). Finally, for the unimportant scaling variables we use $w = 5$ and $z = 0$. The simulation results are quite robust for different parameter values.

Figure 2.2. Steady-state profiles for individuals



Note: All survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values).

the human wealth profile is given by:

$$\frac{d\hat{\bar{h}}(u)}{du} = (\hat{w} - \hat{z})\big([r + m(u)]\Delta(u,r) - 1\big) < 0, \tag{2.51}$$

where the term in square brackets on the right-hand side of Equation (2.51) is equal to $\partial\Delta(u,r)/\partial u$. During the early phase of life, for the linear, the piecewise linear and the G-M model, the annuity rate $r + m(u)$ is relatively low, $\Delta(u,r)$ is relatively high, and human wealth falls only slightly as young agents are still on the flat part of the mortality curve. At high ages, $r + m(u)$ is high, $\Delta(u,r)$ is low, and $d\hat{\bar{h}}(u)/du$ is again relatively low. These models thus give to an inverse-S-shaped profile for human wealth with a point of inflexion located at the approximate age of 55.

In the bottom left-hand panel of Figure 2.2 the age profile of steady-state consumption is visualized. As follows readily from eq. (2.32), the growth rate of individual consumption is the same for both demographic models. Interestingly, the estimated mortality models all predict very similar steady-state consumption paths (in level terms).

Finally, in the bottom right-hand panel of Figure 2.2 the age profile of steady-state financial assets is visualized. For the Blanchard model financial assets rise with age—see Lemma 2.1(iv). Matters are vastly different for the G-M model. Indeed, for that model financial asset holdings follow the classic life-cycle pattern stressed by Modigliani and co-workers, i.e. households start life with zero assets, then save up until middle age, after which dissaving takes place. Despite the fact that very old agents have hardly any financial assets or human wealth left, the annuity rate of interest is so high for them that a high consumption level can nevertheless be maintained.[20] The results for the piecewise linear model is in between the Blanchard model and the G-M model. For all normal ages, the asset profile shows a humped shape, but after 140 years, assets start to increase again. The asset profile for BCL mortality process hits zero at the maximum age 86 because no one wants to die with assets (we abstract from bequest motives).

The upshot of the discussion so far is as follows. The Blanchard specification tracks the demographic data very poorly and predicts unrealistic age patterns for the consumption propensity, human wealth, and financial wealth. In contrast, the G-M model tracks the data rather well and predicts the relevant life-cycle patterns in these variables. A further theoretical advantage of the G-M model is that it en-

[20] Only the estimated G-M model satisfies the condition stated in Lemma 2.1(iii) so that assets go to zero as the agent gets very old. In addition, the model gives rise to a single peak in the asset profile, a result we have been unable to prove analytically in general.

ables a conceptual distinction between youth and old age (just as is possible in the two-period Diamond (1965) model). The other models have some characteristics of both the Blanchard model and the G-M model, but none of them predicts an actual life-cycle pattern in assets. For this reason, and because the G-M model tracks observed survival rates best, we will focus on these two models from now onward.

2.4 Visualizing shocks with realistic demography

In this section we compute and visualize the effects on the different variables of a number of prototypical shocks affecting a small open economy at time $t = 0$.[21] For reasons mentioned above and to prevent an taxonomy of different models, we will focus on the Blanchard model and the G-M model from now on.

2.4.1 Shocks

Balanced-budget fiscal policy

The first shock consists of an unanticipated and (believed to be) permanent increase in government consumption which is financed by means of lumpsum taxes (i.e. $d\hat{g} = d\hat{z} > 0$). The effects of this shock on individual human wealth and financial assets are illustrated in Figure 2.3. In that figure, the left-hand panels depict the Blanchard case whilst the right-hand panels illustrate the results for the G-M model.

In the Blanchard case, the increase in the lumpsum tax causes a once-off decrease in human wealth which is the same for all existing and future generations. In stark contrast, in the G-M model the fall in human wealth depends both on time and on the generations index. The top right-hand panel of Figure 2.3 shows the effects for two existing households (aged, respectively, 40 and 20 at the time of the shock) and two future households (born respectively one second and 40 years after the shock). As a result of the shock there is a once-off change in the age profile of human wealth. This profile itself does not depend on time because there is no transitional dynamics in after-tax wages (see eq. (2.31) above).

In the bottom two panels of Figure 2.3 the paths for financial assets are illustrated. In the Blanchard case these assets rise monotonically over time for each household. The shock induces a slight kink (at time $t = 0$) in the profile for each generation. For the G-M model in the right-hand panel, the crowding-out effect due to the tax increase is much more visible. The peak in financial asset holdings

[21] These shocks do not have to be infinitesimal as no linearisation techniques have been used.

is higher, the older the existing household is (compare, for example, the 40 and 20 years old households). The profiles for the future households born, respectively, in 0 and 40 years time are identical in shape. Again, this is because of the lack of transitional dynamics in after-tax wages, i.e. in terms of eq. (2.33) the effect operates entirely via steady-state human wealth at birth for post-shock generations.

Temporary tax cut

The second shock consists of a typical Ricardian equivalence experiment. At impact ($t = 0$), the lumpsum tax is reduced, which is financed by issuing bonds. As a result, the stock of government debt gradually increases over time. In order to ensure that government solvency is maintained, the tax is gradually increased over time and ultimately rises to a level higher than in the initial situation. The shock that is administered thus takes the following form:

$$dz(t) = -dz_0 e^{-\chi t} + d\hat{z}\left(1 - e^{-\chi t}\right), \qquad (\text{for } t \geq 0), \tag{2.52}$$

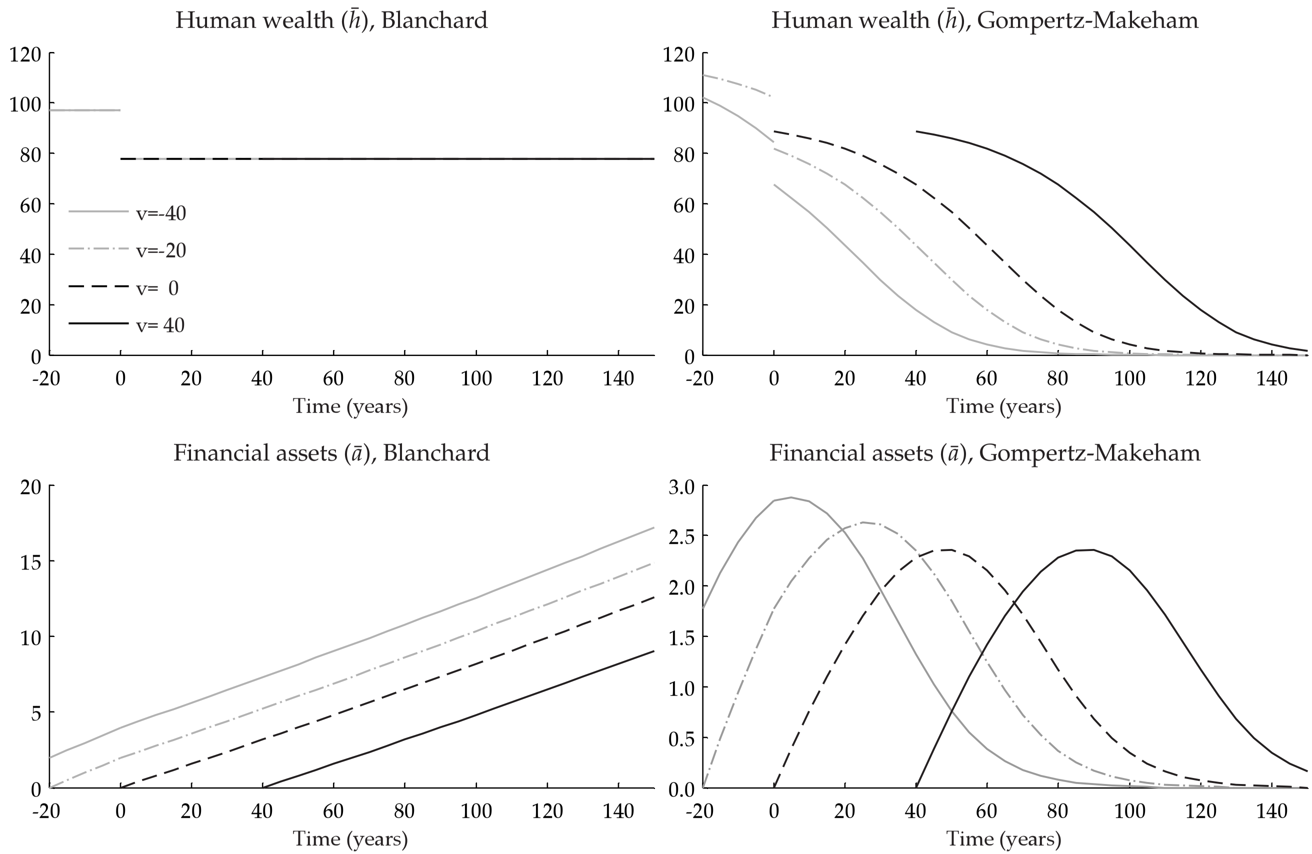
where $0 < \chi \ll \infty$, $dz_0 > 0$, and $d\hat{z} = [(r-n)/\chi]dz_0 > 0$. At impact, the lumpsum tax falls by dz_0 but in the long run it rises by $d\hat{z}$. (The long-run effect on public debt equals $d\hat{d} = dz_0/\chi > 0$.) In the simulations, the persistence parameter is set at $\chi = 0.1$ implying that the tax reaches its pre-shock level after about 15 to 16 years.

The effects on human and financial wealth are illustrated for the two cases in Figure 2.4. In the Blanchard case, human wealth is age-independent. It nevertheless features transitional dynamics because the path of lumpsum taxes is time dependent. Human wealth increases at impact (because of the tax cut), but during transition it gradually falls again (because of the gradual tax increase). In the long run, the permanently higher taxes (needed to finance interest payments on accumulated debt) ensure that human wealth is less than before the shock.

In the G-M model, the effect on human wealth is both time- and age-dependent. At impact, all existing households experience an increase in their human wealth because of the tax cut. For each household, human wealth declines during transition both because of ageing (gradual increase in the annuity rate of interest) and because the tax rises over time. For the future household born 40 years after the shock, the human wealth profile is virtually in the new steady state as most of the shock has worn out by then.

In the bottom panels of Figure 2.4 the profiles for financial assets are illustrated. In the Blanchard case the tax cut causes an acceleration in asset accumulation at

Figure 2.3. Balanced-budget fiscal policy



Note: Both survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values). Birth rate is 2.36%. At time $t = 0$ lumpsum taxes and government consumption jump from 0 to 1 (20% of gross wage income).

impact. This kink also occurs for the G-M model in the bottom right-hand right panel. The G-M case illustrates quite clearly that the Ricardian equivalence experiment redistributes resources from distant future generations toward near future and existing generations. Especially members of the generation born at the time of the shock react strongly to the tax cut as far as their savings behaviour is concerned. Indeed, their maximum asset holding peaks at a much higher level than that of 40 year old existing generations and generations born 40 years after the shock.

Interest rate shock

The final shock analysed in this chapter consists of an unanticipated and permanent increase in the world interest rate (i.e. $dr > 0$ for $t \geq 0$). The effects of this shock on human and financial wealth are illustrated in Figure 2.5. In the Blanchard case the shock causes a once-off decrease in age-independent human wealth. The higher annuity rate of interest leads to stronger discounting of future after-tax wages. For the G-M model there is a once-off downward shift in the age profile of human wealth. Like the shock itself, this age profile displays no further transitional dynamics over time.

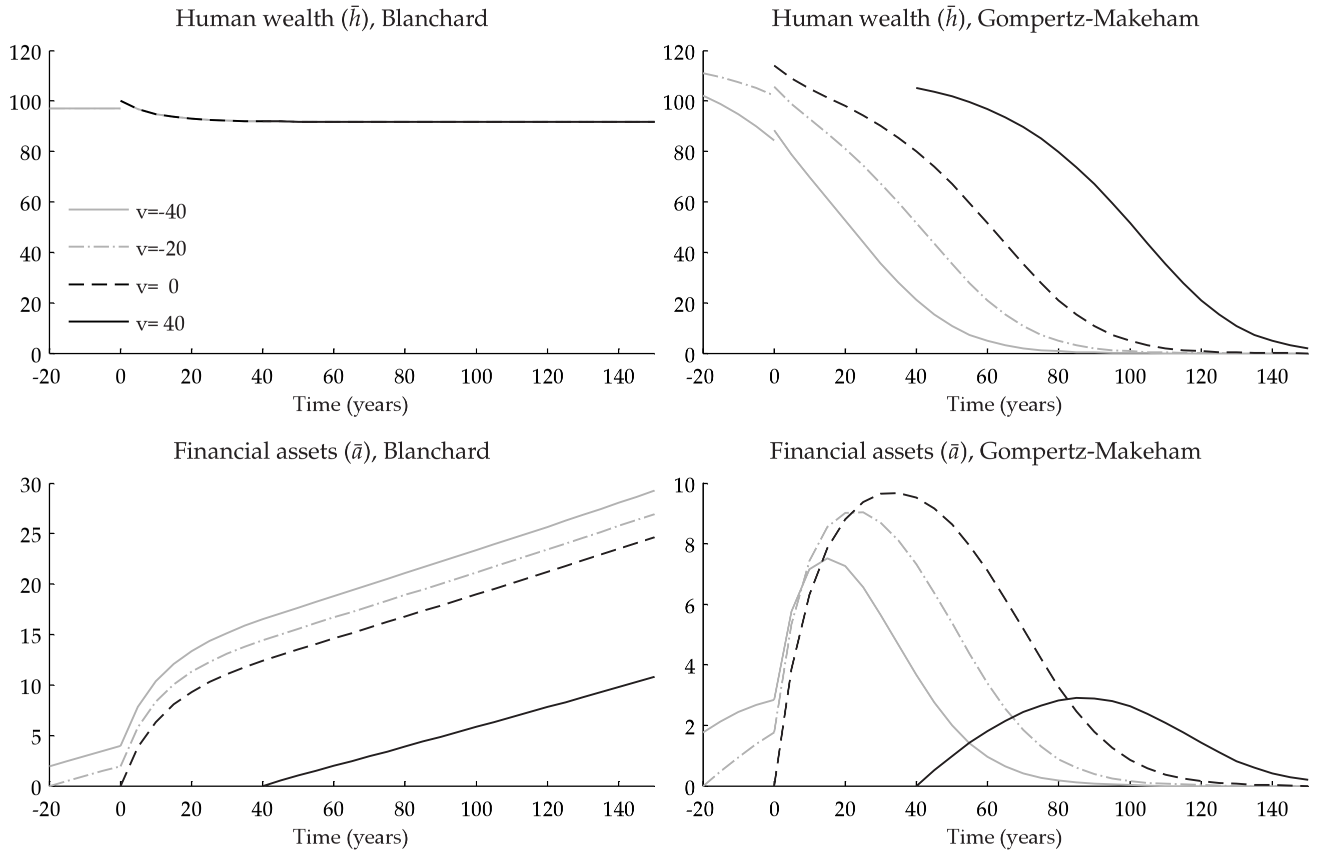
The bottom panels of Figure 2.5 illustrate the effects on financial assets. Whilst the effects for the Blanchard case speak for themselves, those for the G-M model warrant some further comment. For future generations, the age profile of financial assets features a once-off upward shift at impact and displays no further transitional dynamics thereafter. In contrast, for existing generations the time path of assets depends both on their age and on time. This transitional dynamics is caused by the fact that the consumption path for such generations depends on both t and v separately. Existing generations are affected by the interest rate hike both via their human wealth and via their accumulated financial assets which attract a higher rate of return after the shock.

2.4.2 Welfare effects

The Blanchard model is often used to investigate the intergenerational welfare effects of various policy measures.[22] In this section we visualize the intergenerational welfare effects associated with the three shocks studied above. For existing house-

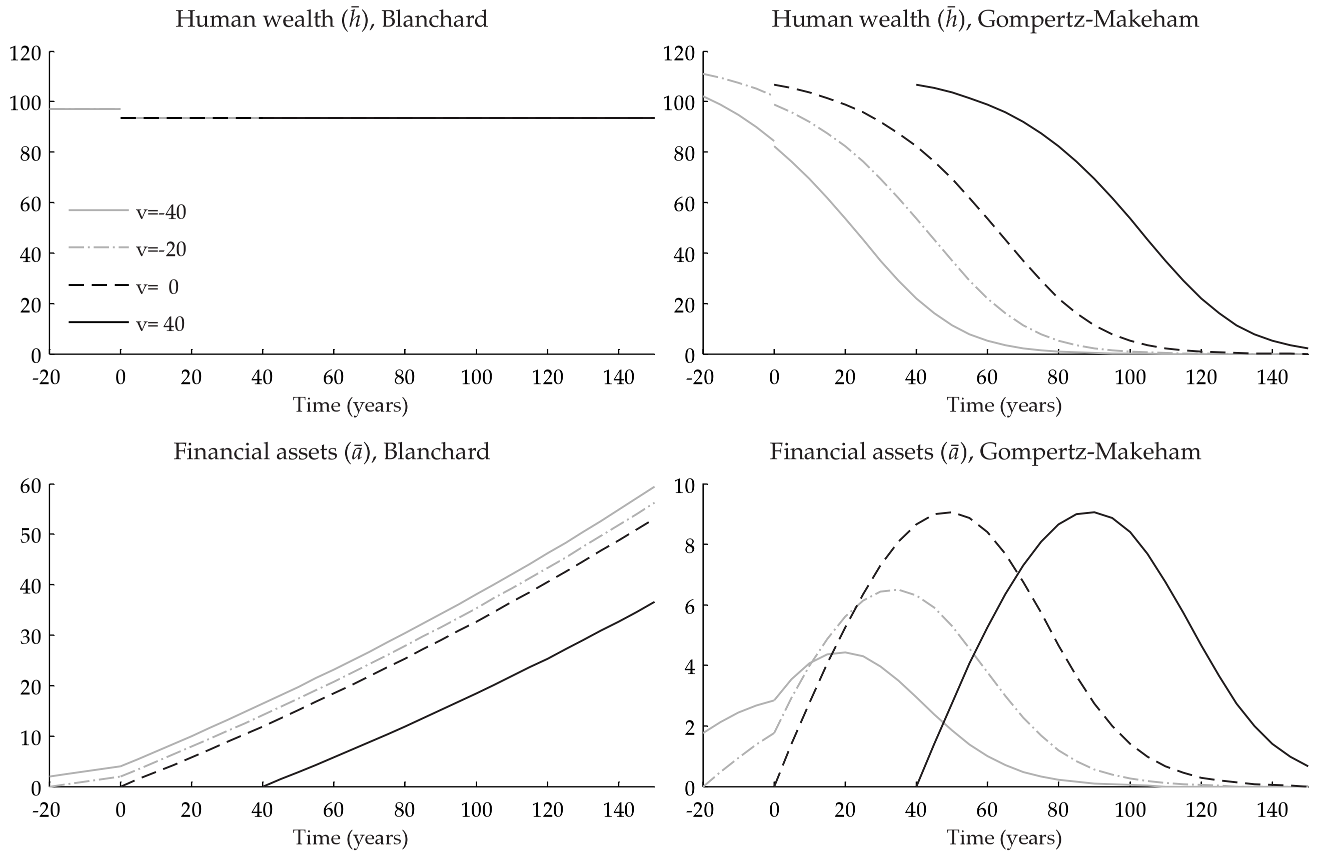
[22] See, for example, Bovenberg (1993, 1994) on capital taxation and investment subsidies, Bettendorf and Heijdra (2001b) and Bettendorf and Heijdra (2001a) on product subsidies and tariffs under monopolistic competition, and Heijdra and Meijdam (2002) on government infrastructure. All these studies are set in the context of a small open economy.

Figure 2.4. Ricardian equivalence experiment: temporary tax cut



Note: Both survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values). Birth rate is 2.36%. At time $t = 0$ lumpsum taxes drop from 0 to -1 (a subsidy of 20% of gross wage income) and increase exponentially to its new steady state value. v is the date of birth relative to the time of the shock.

Figure 2.5. Increase in the world interest rate



Note: Both survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values). Birth rate is 2.36%. At time $t = 0$ the exogenous world interest rate jumps from 4% to 4.2%.

holds, the change in welfare from the perspective of the shock period $t = 0$ is evaluated ($d\Lambda(v,0)$ for $v \leq 0$) whereas for future agents the welfare change from the perspective of their birth date is computed ($d\Lambda(v,v)$ for $v > 0$). The welfare effect for existing agents ($v \leq 0$) can be written as:

$$d\Lambda(v,0) = dr \int_0^\infty \tau e^{-\theta\tau - M(\tau-v)+M(-v)} d\tau + \Delta(-v,\theta) \ln \Gamma_E(v), \quad (\text{for } v \leq 0), \tag{2.53}$$

where $\Delta(-v,\theta)$ is defined in eq. (2.12) above and where $\Gamma_E(v)$ is defined as:

$$\Gamma_E(v) \equiv \frac{\hat{a}(-v) + \bar{h}(v,0)}{\hat{a}(-v) + \hat{\bar{h}}(-v)}, \quad (\text{for } v \leq 0). \tag{2.54}$$

Intuitively, $\Gamma_E(v)$ captures the effect of the impact change in human wealth for existing generations. The welfare effect consists of two separate components. The first term on the right-hand side of (2.53) represents the 'consumption growth effect' and is only relevant for the world interest rate shock (i.e., if $dr > 0$). Individual consumption growth is equal to $r - \theta$ and an increase in r leads to a steeper consumption time profile. The mortality process exerts a non-trivial influence on the consumption growth effect via the utility function. The second term on the right-hand side of (2.53) summarizes the welfare effect of the change in the level of consumption caused by the impact change in human wealth. This 'human wealth effect' is relevant for all shocks and is equal to the product of $\ln \Gamma_E(v)$ (defined in (2.54)) and the inverse propensity to consume of a v-year old agent ($\Delta(-v,\theta)$).

The welfare effect for future generations can be written as:

$$d\Lambda(v,v) = dr \int_0^\infty s e^{-\theta s - M(s)} ds + \Delta(0,\theta) \ln \Gamma_F(v), \quad (\text{for } v > 0), \tag{2.55}$$

where $\Delta(0,\theta)$ is the inverse propensity to consume of a newborn and $\Gamma_F(v)$ is defined as:

$$\Gamma_F(v) \equiv \frac{\bar{h}(v,v)}{\hat{\bar{h}}(0)}, \quad (\text{for } v > 0). \tag{2.56}$$

Here, $\Gamma_F(v)$ represents the effect on the human wealth of a future newborn. Just as for existing generations, the welfare effect for future generations consists of a consumption growth effect (first term on the right-hand side of (2.55)) and a human wealth effect (second term).

The welfare effects of the different shocks are illustrated in Figure 2.6. The left-hand panels present the results for the Blanchard case whilst the right-hand panels visualize those for the G-M model. The welfare effects of balanced-budget fiscal policy are illustrated in the top panels. All present and future generations experience a reduction in human wealth and as a result the welfare effect is negative for all generations. The effect is the same for all future generations because there is no transitional dynamics in human wealth (see above). For existing generations the welfare loss declines with the age of the generation. The human wealth effect decreases with age because both the inverse propensity to consume ($\Delta(-v, \theta)$) and the relative importance of human wealth ($\ln \Gamma_E(v)$ in (2.53) above) decline with age. The Blanchard and G-M models thus give qualitatively similar welfare results for the spending shock. A key difference between the two models concerns the slope of the welfare profile for existing generations. In the G-M model (right-hand panel) the welfare effect is practically zero for all generations older than 100 years. In contrast, for the Blanchard case (left-hand panel) there is still a noticeable welfare effect for 200 year old generations. This low 'generational adjustment speed' of the Blanchard model is also observed for the other shocks. Intuitively, in the Blanchard case, the population share of the old generations is too large because the expected lifetime if too high (see also the top panel of Figure 2.1).

The middle two panels of Figure 2.6 illustrate the welfare effects for the Ricardian tax cut experiment. All existing generations as well as future generations born close to the time of shock benefit at the expense of more distant future generations. For future generations the welfare loss is larger the later they are born. For existing generations the welfare profile is monotonically decreasing in age for the Blanchard case but non-monotonic for the G-M model. In the Blanchard case, $\Delta(-v, \theta) = \Delta(0, \theta) = 1/(\theta + \mu_0)$ is constant and $\ln \Gamma_E(v)$ declines monotonically with age. In contrast, for the G-M model, $\Delta(-v, \theta)$ decreases with age but $\ln \Gamma_E(v)$ is non-monotonic. Indeed, $\ln \Gamma_E(v)$ is increasing in age for all generations up to about 90 years and only decreases in age thereafter. [23] As a result, the welfare profile for existing generations displays a bump around the age of 55 in the middle right-hand panel of Figure 2.6. At that point, the drop in $\Delta(-v, \theta)$ just matches the increase in $\ln \Gamma_E(v)$.

In the bottom two panels of Figure 2.6 the welfare effects for the interest rate shock are illustrated. Since the shock induces no transitional dynamics in the age

[23] Of course, there are virtually no centenarians predicted by the G-M model so the downward sloping part of the $\ln \Gamma_E(v)$ function is practically irrelevant. In contrast, the estimated Blanchard demography predicts that about 32 percent of newborns will still be alive at age 100. See Table 2.1 and the top panel of Figure 2.1.

profile of human wealth for future generations, the welfare effect is the same for all future generations in both models. For existing generations the welfare effect increases with age in the Blanchard model, but is non-monotonic for the G-M model. For an interest rate shock both the consumption growth effect and the human wealth effect are relevant. The shock induces a decrease in $\ln \Gamma_E(v)$ which falls with age in both models. In the Blanchard case, the consumption growth effect is constant (and positive) for all generations. In contrast, for the G-M model, the consumption growth effect is positive and constant for future generations, but falling in age for existing generations. As a result, the total effect on welfare displays a bump around the age of 15 for the G-M model (see the bottom right-hand panel of Figure 2.6).

2.4.3 Aggregate effects

As was pointed out above, Blanchard (1985) assumes a constant mortality rate in order to allow for exact aggregation of the consumption function. With the more general mortality process considered in our model, only numerical aggregation is possible. This section visualizes the aggregate effects on the key variables of the three shocks considered above. To what extent do the aggregate results predicted by the Blanchard and G-M models differ?

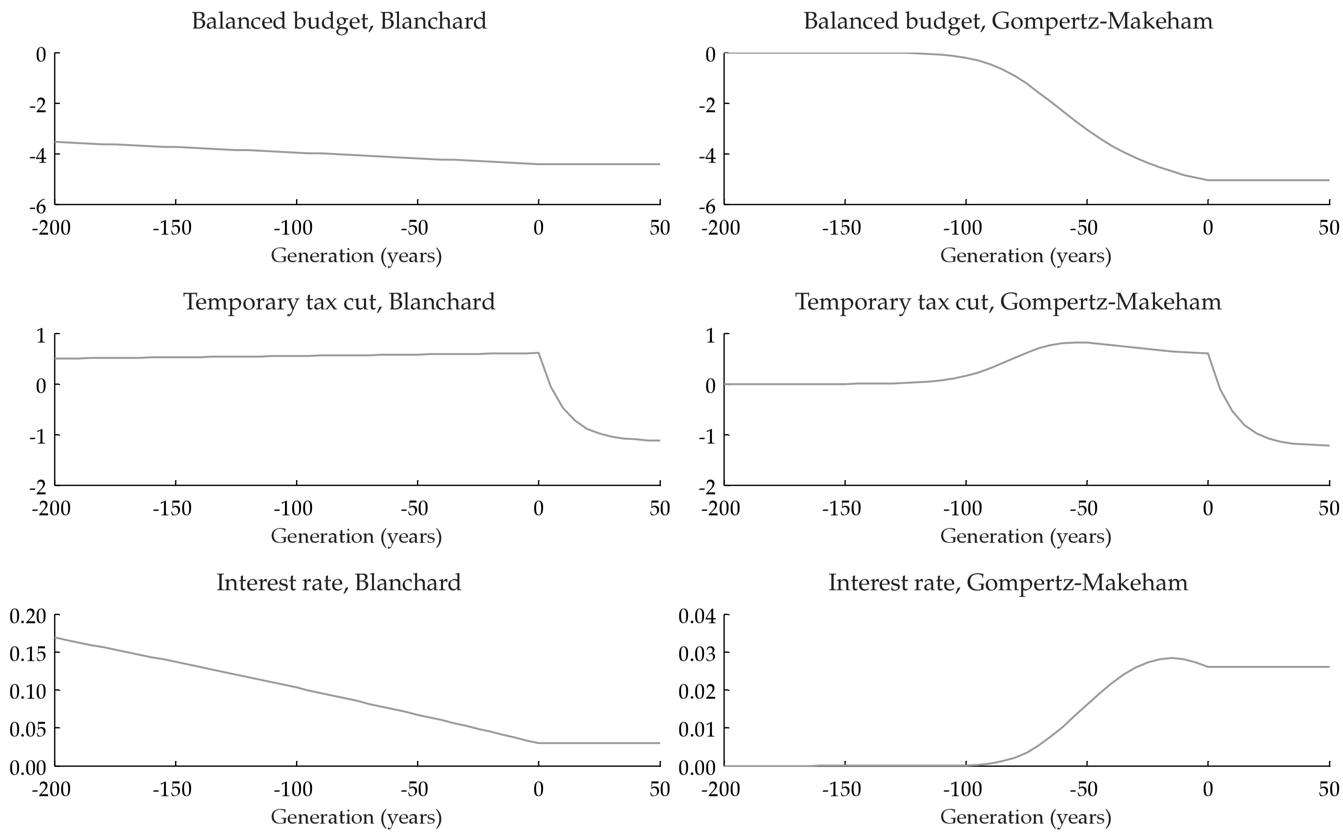
In Figure 2.7 we illustrate the effects on human wealth (first row), consumption (second row), and financial assets (third row) for the spending shock (first column), the Ricardian tax cut (second column), and the interest rate shock (third column). To make it easier to compare the two models, we show the percentage deviations from the steady state for all variables in Figure 2.7, i.e. we show $(h(t) - \hat{h})/\hat{h}$, $(c(t) - \hat{c})/\hat{c}$, and $(a(t) - \hat{a})/\hat{a}$.

For the spending shock, the results for human wealth are identical and those for consumption and financial assets are qualitatively very similar but differ in terms of the speed of adjustment towards the new steady state. The slow speed of convergence is also a feature of the Blanchard results for the other two shocks.

For the Ricardian tax cut, the effects on human wealth are again similar but those on consumption and financial wealth are not. For the G-M model, the impact effect on consumption is much larger, and the slope of the aggregate Euler equation is much steeper during transition, than for the Blanchard model. Similarly, the savings response is much more pronounced for the G-M model.

Finally, for the interest rate shock the effect on human wealth is qualitatively the same for the two models, though the effect is stronger for the Blanchard model.

Figure 2.6. Welfare effects (absolute change



Note: Both survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values). Birth rate is 2.36%. Balanced budget shock is a jump of lumpsum taxes and government consumption from 0 to 1 (20% of gross wage income). Ricardian Temporary tax cut is a drop of lumpsum taxes from 0 to -1 (a subsidy of 20% of gross wage income) and they increase exponentially to its new steady state value. Interest rate shock is a jump of the exogenous world interest rate jumps from 4% to 4.2%.

The impact reduction in consumption is virtually the same for the two models but transition is much faster for the G-M model. Again, the savings response at impact is stronger for the G-M model.

2.5 Concluding remarks

In this chapter we showed that it is quite feasible to incorporate a realistic demographic structure in an overlapping generation model of a small open economy facing an exogenous world interest rate. At the level of individual households, a realistic description of the mortality process instead of Blanchard's perpetual youth reinstates the classic life-cycle saving insights of Modigliani and co-workers.

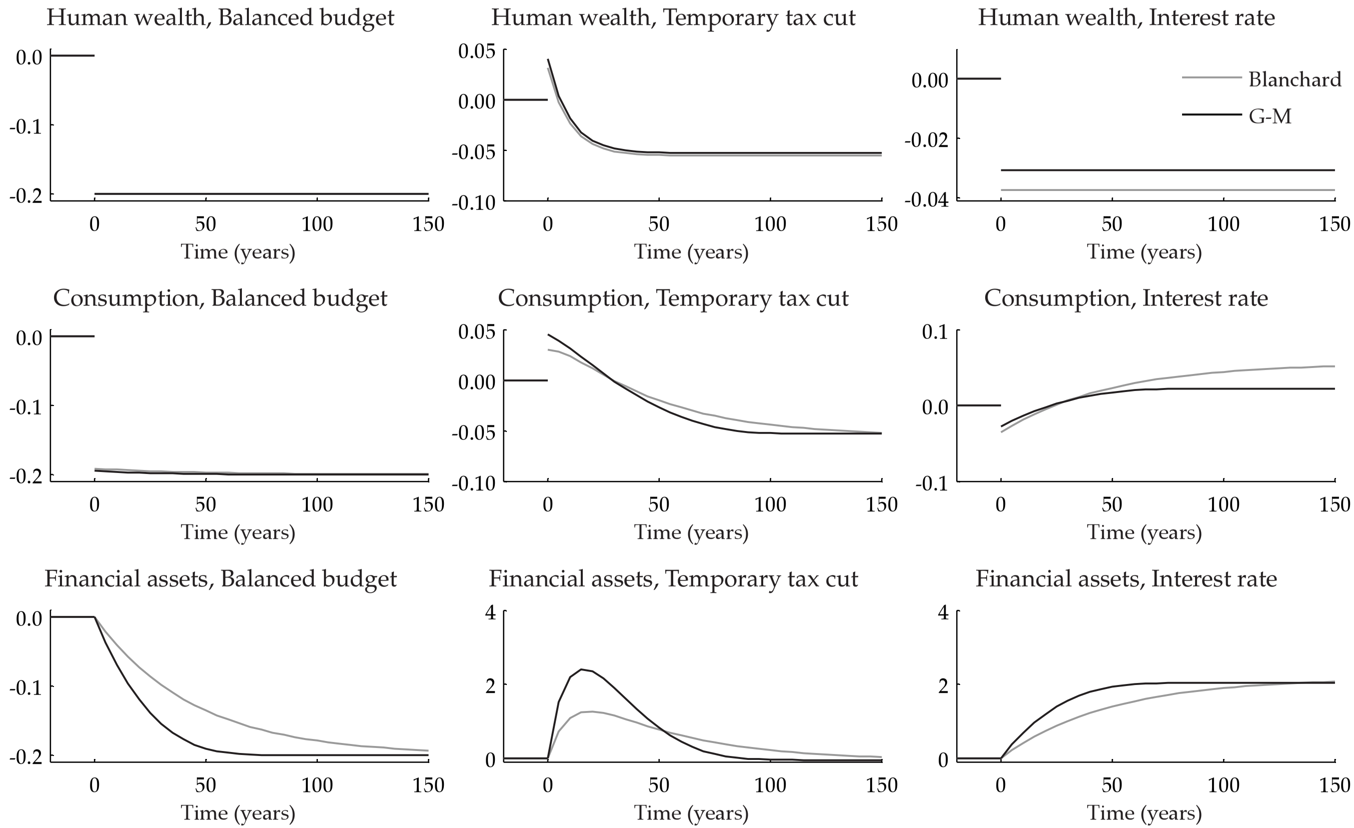
The welfare effects associated with the different shocks are also potentially affected in a non-trivial manner by a more realistic demography. Two key differences stand out between the Blanchard and G-M models. First, the G-M model predicts a much faster (and in our view more realistic) 'generational convergence speed' of the welfare effects than the Blanchard model. Second, the G-M model incorporates more extensive age-dependency and as a result may give rise to a non-monotonic welfare effect on existing generations—something which is impossible in the Blanchard case (for the shocks studied).

Finally, we have demonstrated that the demographic details do not 'wash out' at the aggregate level. The impulse-response functions for the different shocks are quite different for the Blanchard and G-M models, especially the ones for per capita consumption and financial assets.

In some applications of our model, individual behaviour may depend in part on aggregate variables so that knowledge of the latter is crucial. For example, if the revenue of a consumption tax (t_C) is recycled in a lumpsum fashion to households (i.e. $\bar{z}(t) = z(t) = -t_C c(t)$) then individual consumption, human wealth, and financial assets will all depend on the aggregate tax revenue. This complication can be easily dealt with by using an iterative procedure in the simulations. In the first step the initial tax revenue and implied lumpsum transfer are guessed and individual and aggregate consumption levels are computed. In subsequent steps, the aggregate information is used to update the guess for transfers until convergence is achieved. We will use this procedure to solve the extended models in Chapters 3 and 4

The framework developed in this chapter can be extended in a number of directions. First, in order to investigate the effects of demographic change, it is neces-

Figure 2.7. Aggregate effects of the shocks



Note: Both survival functions are fitted to the cohort born in the Netherlands in 1920 (see Table 2.1 for parameter values). Birth rate is 2.36%. Balanced budget shock is a jump of lumpsum taxes and government consumption from 0 to 1 (20% of gross wage income). Ricardian Temporary tax cut is a drop of lumpsum taxes from 0 to -1 (a subsidy of 20% of gross wage income) and they increase exponentially to its new steady state value. Interest rate shock is a jump of the exogenous world interest rate jumps from 4% to 4.2%.

sary to generalize the stochastic distribution for expected remaining lifetimes. Two possibilities can be distinguished. 'Embodied' mortality change can be studied by assuming the instantaneous mortality rate to be generation-specific, i.e. by writing it as $m(v,s)$. An example of embodied mortality change could be the ability to extract and store embryonic stem cells to be used for future organ repairs. In contrast, 'disembodied' demographic change can be modelled by writing the mortality rate $m(t,s)$, i.e. by postulating a time-dependent mortality process. An example of disembodied mortality change would be a comprehensive cure for cancer or heart and vascular diseases.

Second, the age profile for individual consumption could be generalized by introducing shift factors in the utility function. In the current model (with $r > \theta$) consumption is increasing in the age of the household. There are reasons to believe that in reality consumption is hump-shaped, i.e. $\bar{c}(v,t)$ features a rising time profile early on in life followed by a falling profile later on. A simple way to capture this effect is to assume that a household's 'needs' get smaller the older they get. In the diminishing-needs model, lifetime utility is given by:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^\infty \left[\frac{\bar{e}\,(v,\tau)^{1-1/\sigma} - 1}{1 - 1/\sigma} \right] e^{-\theta\cdot(\tau-t)-M(\tau-v)} d\tau, \tag{2.57}$$

where $\sigma > 0$ is the intertemporal substitution elasticity and $\bar{e}(v,\tau)$ is *effective* consumption:

$$\bar{e}(v,\tau) \equiv \bar{c}(v,\tau) \exp\left\{ \frac{\zeta_0(\tau-v)^{1+\zeta_1}}{1+\zeta_1} \right\}, \tag{2.58}$$

with $\zeta_0 > 0$ and $\zeta_1 > 0$. According to (2.58), a given amount of *actual* consumption, $\bar{c}(v,\tau)$, yields more effective consumption (featuring in the felicity function), the older the household is. Using this specification of preferences, it is straightforward to show that the individual consumption Euler equation is generalized to:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = \sigma\cdot(r-\theta) - (1-\sigma)\zeta_0(\tau-v)^{\zeta_1}. \tag{2.59}$$

For the empirically relevant case (with $0 < \sigma < 1$), consumption rises during the early phase of life ($\tau - v$ low) and falls during the later stages of life ($\tau - v$ high).

In the next chapter, we present a third extension. We extend the basic framework developed here with an optimal schooling decision. Individuals spend the first years of life at school, accumulating knowledge, which increases their productivity and hence their wage rate later in life. Furthermore, we allow for embodied mortality change and a time varying birth rate and we use the extended model to analyse the effects of embodied mortality changes and baby busts on the aggregate productivity growth rate.

In the final chapter of part I, we endogenise the household's labour supply and retirement decisions. By including leisure hours into the felicity function, the agent has an additional choice variable which determines optimal labour income and lifetime utility. Two approaches can be considered. In the 'divisible labour' case, the agent can freely choose the number of working hours at each instant. In the typical formulation, consumption and leisure are both normal goods so that, as the agent gets older and richer, labour supply gradually declines to its lower bound (of zero). Hence, the agent gradually retires from the labour market. In the 'indivisible labour' case, employment is assumed to be a participation decision, i.e. the agent either works a fixed number of hours (full time) or not at all. In such a setting the retirement decision constitutes a withdrawal from the labour market altogether. In both types of labour supply models, the most interesting shocks that can be studied are ageing shocks and pension reform. In Chapter 4 we focus on the second case and we will analyse how ageing will effect the retirement decision and a nation's retirement system.

2.A Computation of the Δ-function

The model used in this chapter makes extensive use of the Δ-function as defined in Equation (2.12). To be able to solve the model in a reasonable amount of time, we need efficient methods to evaluate this function for the specified mortality processes.

Blanchard: The Δ-function for Blanchard's mortality process can be written as

$$\Delta(u,\lambda) = e^{(\lambda+\mu_0)u}\int_u^\infty e^{-(\lambda+\mu_0)s}ds = \frac{1}{\lambda+\mu_0}$$

Linear and piecewise linear: The demographic discount function for the linear mortality process can be written as:

$$\Delta(u,\lambda) = e^{(\lambda+\mu_0)u+\mu_1^2u^2}\int_u^\infty e^{-(\lambda+\mu_0)s-\mu_1^2s^2}ds$$

We define $\beta(u) = \mu_1 u + \frac{\lambda+\mu_0}{2\mu_1}$ and $t = \beta(s)$. Changing the integrand we obtain:

$$\Delta(u,\lambda) = \frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\mu_1 u + \frac{\lambda+\mu_0}{2\mu_1}\right)$$

where $\operatorname{erfcx}(x)$ is the scaled complementary error function (defined in general terms as $\operatorname{erfcx}(x) = e^{x^2}\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$; see Kreyszig, 1999). The scaled complementary error function is well documented and, more importantly, most software packages have very fast routines to calculate it accurately enough for our purposes.

Some tedious, but otherwise straightforward math shows that the expression for $\Delta(u,\theta)$ for the piecewise linear model features two branches, depending on whether the household is still 'young' ($0 < u < \bar{u}$) or has entered 'old age' ($u > \bar{u}$):

$$\Delta(u,\theta) = \begin{cases} \dfrac{1-e^{-(\lambda+\mu_0)(\bar{u}-u)}}{\lambda+\mu_0} \\ \qquad + e^{-(\lambda+\mu_0)(\bar{u}-u)}\dfrac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\dfrac{\lambda+\mu_0}{2\mu_1}\right) & \text{for } 0 \le u < \bar{u} \\ \dfrac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\mu_1\cdot(u-\bar{u}) + \dfrac{\lambda+\mu_0}{2\mu_1}\right) & \text{for } u \ge \bar{u} \end{cases}$$

Boucekkine et al. (2002): Boucekkine et al. (2002) introduce a survival function

$$S(u) = 1-\Phi(u) = \frac{e^{-\beta u}-\alpha}{1-\alpha}, \quad 0 \le u \le A$$

with $A = -\frac{1}{\beta}\ln\alpha$. This gives for the mortality rate (only for the relevant domain)

$$m(u) = \frac{\Phi'(u)}{1-\Phi(u)} = \frac{\beta e^{-\beta u}}{e^{-\beta u}-\alpha} \qquad \text{for } 0 \le u < A$$

and integrated mortality rate

$$\begin{aligned} M(u) &= \beta \int_0^u \frac{e^{-\beta s}}{e^{-\beta s}-\alpha} ds = -\int_1^{e^{-\beta u}} \frac{x}{x-\alpha}\frac{1}{x} dx \\ &= \ln|1-\alpha| - \ln|e^{-\beta u}-\alpha| = -\ln S(u) \qquad \text{for } 0 \le u < A \end{aligned}$$

The demographic discount function can be written as (also only for $0 \le u \le A$)

$$\begin{aligned} \Delta(u,\lambda) &= e^{\lambda u + \ln|1-\alpha| - \ln|e^{-\beta u}-\alpha|} \int_u^A e^{-\lambda s - \ln|1-\alpha| + \ln|e^{-\beta s}-\alpha|} ds \\ &= e^{\lambda u} \frac{|1-\alpha|}{|e^{-\beta u}-\alpha|} \int_u^A e^{-\lambda s} \frac{|e^{-\beta s}-\alpha|}{|1-\alpha|} ds \\ &= \frac{1}{e^{-\beta u}-\alpha} \left\{ \frac{1}{\lambda+\beta} \left[e^{-\beta u} - e^{-\lambda[A-u]-\beta A} \right] + \frac{\alpha}{\lambda} \left[e^{-\lambda[A-u]} - 1 \right] \right\} \end{aligned}$$

This expression is easy to evaluate efficiently using any standard mathematical software package. For $\lambda = 0$, we need l'Hopital's rule

$$\Delta(u,0) = \frac{1}{e^{-\beta u}-\alpha} \left\{ \frac{1}{\beta} \left[e^{-\beta u} - e^{-\beta A} \right] - \alpha[A-u] \right\}.$$

Gompertz-Makeham: The demographic discount function for the G-M process can be written as:

$$\Delta(u,\lambda) = e^{(\lambda+\mu_0)u + \frac{\mu_1}{\mu_2}e^{\mu_2 u}} \int_u^\infty e^{-(\lambda+\mu_0)s - \frac{\mu_1}{\mu_2}e^{\mu_2 s}} ds.$$

We define $\beta(u) \equiv \frac{\mu_1}{\mu_2}e^{\mu_2 u}$ and $t = \beta(s)$. Changing the integrand we obtain:

$$\Delta(u,\lambda) = \frac{\mu_2^{\alpha-1}}{\mu_1^{\alpha}} e^{(\lambda+\mu_0)u+\beta(u)} \Gamma\big(\alpha, \beta(u)\big),$$

where $\alpha \equiv -(\lambda+\mu_0)/\mu_2$ and $\Gamma\big(\alpha,\beta(u)\big)$ is the upper tailed incomplete gamma function (defined in general terms as $\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}dt$; see Kreyszig (1999, p. A55)). The incomplete gamma function is just like the scaled incomplete error function well documented (see e.g. Kreyszig (1999, p. A78)) and software packages have very fast routines to calculate it.

There is one slight complication; the incomplete gamma function is usually only defined for $\alpha \geq 0$, whereas we also need to evaluate it for $\alpha < 0$. We can solve this problem by using the 'functional relation of the incomplete gamma function'. Indeed, by integrating the incomplete gamma function by parts we obtain the following recursion formula:

$$\Gamma(\alpha, x) = \frac{1}{\alpha} e^{-t} t^{\alpha} \Big|_{t=x}^{\infty} + \frac{1}{\alpha} \int_x^{\infty} t^{\alpha} e^{-t} dt = -\frac{1}{\alpha} e^{-x} x^{\alpha} + \frac{1}{\alpha} \Gamma(\alpha + 1, x).$$

Repeated application gives for $k = 0, 1, 2, \ldots$:

$$\Gamma(\alpha, x) = -e^{-x} x^{\alpha} \left[\frac{1}{\alpha} + \frac{1}{\alpha} \frac{1}{\alpha + 1} x + \frac{1}{\alpha} \frac{1}{\alpha + 1} \frac{1}{\alpha + 2} x^2 + \cdots \right.$$
$$\left. + \frac{1}{\alpha} \frac{1}{\alpha + 1} \cdots \frac{1}{\alpha + k - 1} x^{k-1} \right] + \frac{1}{\alpha} \frac{1}{\alpha + 1} \cdots \frac{1}{\alpha + k - 1} \Gamma(\alpha + k, x).$$

Hence, by choosing the smallest integer k such that $\alpha + k$ is non-negative, the value of $\Gamma(\alpha, x)$ can be computed in a standard fashion.

Others: For instances of the mortality function that can not be solved explicitly or rewritten in terms of well-documented (and easily calculated) functions, we can evaluate the Δ-function using standard numerical integration techniques. To evaluate the Δ-function for more than one age (u), we can make use of the following algorithm:

1. Initialise; sort the u's, such that $u_1 < u_2 < \ldots < u_n$, calculate $\Delta(u_n, \lambda)$ using any (adaptive) quadrature and set $i = n - 1$.

2. Calculate $\Delta(u_i, \lambda)$ using

$$\Delta(u_i, \lambda) = e^{-\lambda \cdot (u_{i+1} - u_i) - M(u_{i+1}) + M(u_i)} \Delta(u_{i+1}) + \int_{u_i}^{u_{i+1}} e^{-\lambda \cdot (s - u_i) - M(s) + M(u_i)} ds.$$

   The integral can be evaluated using any (adaptive) quadrature.

3. If $i = 1$, then exit, else set $i = i - 1$ and go to step 2.

It is important first to construct the exponents and then to evaluate the e-power, to prevent numerical problems. Furthermore, sorting in descending order ensures that u_{i+1} is always larger than u_i, so the exponential terms are always smaller than 1, which prevents inaccuracies.