Asset liability management for pension funds using multistage mixed-integer stochastic programming

Drijver, S.J.

Chapter 6

Numerical experiments

In this chapter we report the first impressions we gained from numerical experiments. In Section 6.1 we consider an *illustrative case*. It is an ALM model for a fictitious pension fund. The values of the deterministic parameters are discussed in relation to the position of the interested parties. The numerical specification of the parameters in the scenario tree are described in the previous chapter and Appendix 5.A. The illustrative case is used to test the heuristic. The results are interpreted in some detail.

The illustrative case is also used as the starting point for *sensitivity analyses*. We consider sensitivity analyses with respect to modeling choices, model justification, and scenario trees. The results of these experiments are presented in Section 6.2.

In order to find the numerical results, we used a personal computer with a Pentium IV 2 GHz processor and 512 MB memory. The multistage stochastic programs with only continuous decision variables (obtained by fixing the values for the binary decision variables) are solved by OSL [71], using the callable library OSL Stochastic Extensions [72]. The heuristic is programmed in Microsoft Visual C++ [66].

6.1 Illustrative case

The illustrative case deals with a fictitious pension fund. Whereas data with respect to nominal liabilities of a Dutch pension fund are used in our scenario generator, as have been described in Chapter 5, all other parameters have been chosen by ourselves, without any relation to this fund. Of course, we have tried to provide realistic specifications. In fact, we aimed to specify the following situation. The sponsor is far from wealthy: he has to borrow money to be able to make a remedial contribution. The retired people are assumed to have relatively much influence, so that a deterioration of indexation is considered as harmful. Finally, the active participants do not have much influence in the decision making process.

As a result of the supposed positions of the supervisor, the sponsor and the retired people, the fixed penalty costs associated with underfunding, a remedial payment and a deterioration of indexation are all high. Are these positions realistic?

In our opinion, the assumed positions of the supervisor and the sponsor adequately represent current practice. Although currently the active participants may have more influence than the retired people, in the (near) future this may change. This more powerful role of the retirees is reflected in the illustrative case.

Before we present the numerical results, we first describe the parameter settings.

6.1.1 Parameter settings

In this illustrative case, the planning horizon is 4 years. The number of branches per year are 6, 6, 5 and 5 respectively. This implies that we consider 900 scenarios and 1,123 states in the scenario tree. The number of continuous and binary decision variables are respectively 28,075 and 6,738, and in the illustrative case the model has 45,147 constraints.

The tolerance, i.e. the upper bound on the required relative gap between the primal and dual solutions, is set equal to 0.0001.

In the description of parameter settings, we make a distinction between parameters which reflect the positions of the parties involved, and other parameters. The various settings of the first group of parameters, corresponding to the illustrative case and subsequent cases, allow for an interpretation in terms of the relative weights that are given to the interests of the parties involved.

Positions interested parties

In Chapter 1 we have seen that there are various interested parties in the ALM decision process. We will consider these parties and indicate what they have agreed on during the negotiations.

- **Active participants**
  Active participants like both a low and stable contribution rate. Moreover, they would like to receive full indexation with respect to increases in the general wage level.

- **Deferred members and retired people**
  Deferred members and retired people have stipulated that a large decrease in the level of the contribution rate is not desirable.

- **Sponsor**
  The sponsor of the fund does not like large increases of the contribution rate between two consecutive years. Moreover, the sponsor is not financially sound: the sponsor has to borrow money to be able to make a remedial payment. As a result, the associated fixed penalty costs will be set high, and large remedial contributions are very expensive.

- **Supervisor**
  Both underfunding and not indexing benefit rights are considered to be undesirable by the supervisor. As a result, the corresponding fixed penalty costs are relatively high. Moreover, the supervisor imposes a restriction on one-year expected shortages.

The specific parameter values for the fixed and variable costs, which are used to penalize undesirable events, together with the maximum decrease and increase in the contribution rate such that no penalty costs are incurred, are presented in Table 6.1. We think that the numerical values of these parameters correspond to the above described characteristics of the interested parties. In particular, the numerical value of ζ_{DZ} is chosen sufficiently high, as we will see in the output of this case in Section 6.1.2. In the remainder of this subsection we will comment on our choices underlying the specification in Table 6.1.

In Table 6.1, the fixed penalty costs are related to the level of the regular contributions c_0W_0 paid to the fund in the previous year (i.e. in year 0). This makes it possible to compare the fixed costs with cash flows. Moreover, one can get a feeling how much the level of the contribution rate should be increased to avoid unfavorable events.

Other parameter settings

Next to the parameter settings which describe the relative position of the interested parties, there are some other parameters in our ALM model. The numerical values of these parameters which are used in the illustrative case are also presented in Table 6.1. We will discuss these values now.

- **Initial positions assets and liabilities**
  The current value of the assets, i.e. the value of the assets just before the portfolio may be changed at time 0, is €18,000 million. This amount is currently divided as follows: 47% is invested in stocks, 25% in bonds, 21% in real estate, and 7% in cash.

  The lower and upper bounds on the value of the liabilities at time 0 are €14,993 million and €15,517 million respectively. This implies that the funding ratio at time 0 is at least 1.16 and at most 1.20.

- **Contribution rate**
  Last year, the contribution rate was set equal to 11% of the pensionable salaries. The minimum level for this rate is 0%, so that a non-contributory pension is not excluded. The maximum level of the contribution rate is set to 21%. Moreover, the interested parties agreed that if the sponsor makes a remedial contribution, the contribution rate should be at least equal to 12.5%.

- **Underfunding and remedial contributions**
  The minimum required level of the funding ratio is 105%. This is similar to the requirements of the Dutch supervisor, described in the FTK [73]. Moreover, if in 2 consecutive years the funding ratio is less than 105%, the sponsor is forced to make a remedial payment. After this payment, the funding ratio should be at least equal to the minimum required level.

  In Chapter 2 we introduced the level θ. If the funding ratio falls below this level, the sponsor is forced to restore the funding ratio with respect to this level immediately. In the illustrative case, we have chosen to set $\theta = 0.90$. As

Initial position			
$A_0/\overline{L}_0$	1.16		
Contribution rate			
$\underline{c}$	0	ρ	0.010
$\overline{c}$	0.210	η	0.015
c^*	0.125	ζ_{ci}	0.750
c_0	0.110	ζ_{cd}	0.500
Underfunding and remedial contributions			
α	1.05	λ_u/c_0W_0	2.20
θ	0.90	λ_z/c_0W_0	3.40
a	2	ζ_Z	1.50
u_{-1}	0	ζ_{ZI}	5.00
u_{-2}	0	ζ_{DZ}	20.00
τ	0.25		
One-year risk constraints			
ψ	0.09		
Indexation			
λ_m/c_0W_0	1.75	ζ_L	0.50
Overfunding and restitutions			
β	2.50	λ_o/c_0W_0	-0.00
b	2	λ_v/c_0W_0	-0.00
o_{-1}	0	ζ_V	-0.02
o_{-2}	0		
Horizon			
Λ	1.25	$\zeta_{\Lambda d}$	0.10
$\zeta_{\Lambda i}$	-0.01		
Portfolio			
$\underline{f}_1$	30	$\%X_{10}/A_0$	47
$\underline{f}_2$	30	$\%X_{20}/A_0$	25
$\underline{f}_3$	10	$\%X_{30}/A_0$	21
$\underline{f}_4$	0	$\%X_{40}/A_0$	7
$\overline{f}_1$	60	$k_1(\times 100)$	0.43
$\overline{f}_2$	60	$k_2(\times 100)$	0.25
$\overline{f}_3$	25	$k_3(\times 100)$	0.43
$\overline{f}_4$	20	$k_4(\times 100)$	0.05

Table 6.1: Data for the illustrative case.

can be seen in Table 6.1, the variable costs associated with a remedial contribution with respect to the level θ, denoted by ζ_{DZ} are very high. This reflects the current financial position of the sponsor of the fund.

On the balance dates of the last two years, the funding ratio was at least equal to 105%, i.e. $u_{-2} = u_{-1} = 0$. However, if underfunding is recorded and the sponsor of the fund makes a remedial contribution, the level of the remedial contribution is also important. This follows from the modeling of remedial contributions in Chapter 2. The marginal costs associated with large remedial payments are much higher than those associated with low ones. In the illustrative case, all remedial payments which are larger than a quarter of the total pensionable salaries are indicated as large.

- **One-year risk constraints**
  In the illustrative case, the maximum allowed expected next year's shortage is set equal to 9 percent of the market value of the liabilities.

- **Overfunding and restitutions**
  In this case, overfunding is present if the funding ratio is greater than 2.5. If overfunding is present in two consecutive years, the pension fund is obliged to restitute the excess wealth to the sponsor. On the last two balance dates, overfunding did not occur.

- **Horizon**
  The board of the pension fund strives for a funding ratio of at least 1.25 in 4 years. If the funding ratio is less than this level, a penalty will be incurred. On the other hand, surplusses with respect to the level 1.25 in 4 years are rewarded. As can be deduced from Table 6.1, the penalty associated with not reaching the target is 10 times higher than the reward associated with reaching it.

- **Portfolios**
  We have already discussed the market value and the composition of the initial portfolio. In our ALM model, more parameters associated with the asset portfolios appear. To be specific, lower and upper bounds on the fraction of assets invested in each asset class are present. Moreover, transaction costs associated with buying and selling assets are taken into account. For the numerical values of these parameters, we refer to Table 6.1.

6.1.2 Output

In this section, we first consider CPU times and computation statistics. Then, we discuss the decisions of the ALM model in detail.

CPU times and computation statistics

The heuristic needed 38 minutes and 29 seconds. The LP-relaxation was solved in 20 seconds. The total number of MSLPs solved is 708. In 654 of these, or in 92%, an improvement, i.e a lower value of the objective function, was found. An overview of the computation statistics is presented in Table 6.2.

In Table 6.2 the values of the objective function associated with the LP-relaxation, the first feasible solution and the heuristic solution are presented under A, B, and C, respectively. In addition, we solved the LP-relaxation with the heuristic decisions at time 0 fixed. This gives a lower bound on the value of the objective function of the heuristic solution, see D. Next to the levels of the objective function A-D, two gaps are presented. The first gap, between the first feasible solution and the heuristic solution, expresses how much the value of the objective function is improved by the execution of the heuristic. We not only present the absolute level, but also the relative improvement, which is calculated as (B-C)/C. The second gap presented in

Table 6.2 is the one between the heuristic solution and the solution obtained by fixing the decisions found in the heuristic and solving the LP-relaxation. The smaller the difference between the values of the objective function of the heuristic solution and this one, the more confidence we have that good recourse actions are found by the heuristic. Also for this gap we not only present an absolute number, but also calculate its relative value (C-D)/C.

Note that from the results presented in Table 6.2 we see the effects of the hot starts. The average time needed to solve a MSLP is 3.26 seconds, while it took 20 seconds to solve the LP-relaxation.

In addition to Table 6.2, in Figure 6.1 the development of the value of the objective function is presented. In this figure it is also shown at which decision moment (time $t = 0, \ldots, 4$) an improvement is searched for. We see that the heuristic performs 4 iterations. In each iteration at time 1 few, but large, improvements are found. On the other hand, at time 4 many small improvements are found. Note that no improvement is found at time 0. This is not surprising, since in the first feasible solution (B) all binary decision variables at $t = 0$ have the value 0.

Comparing the iterations, we see that the total improvement during an iteration is less than that of the previous one. Actually, the very first improvement gave the largest decrease of the objective function. The initial value (B) is not represented in Figure 6.1 because the vertical scaling would then obscure subsequent improvements.

Solution times	
Total solution time (h:mm:ss)	0:38:29
Solution time LP-relaxation (h:mm:ss)	0:00:20
Development value of objective function	
(A) LP-relaxation	1,746
(B) First feasible solution	51,512
(C) Heuristic solution	7,350
(D) Decisions $t = 0$ fixed, LP-relaxation	3,232
Gap (B)-(C) (absolute/relative)	44,164 (86%)
Gap (C)-(D) (absolute/relative)	4,118 (56%)
Number of MSLPs solved and improvements found	
Number of MSLPs solved	708
Number of improvements found (absolute/relative)	653 (92%)

Table 6.2: Computation statistics for the illustrative case.

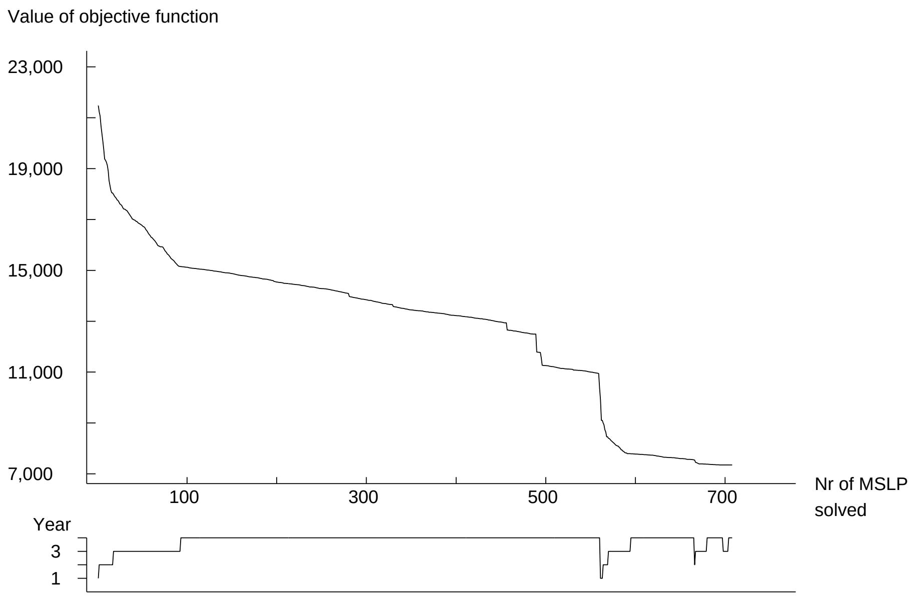

Figure 6.1: Value of the objective function after each MSLP which is solved in the illustrative case. Also the decision moment at which an improvement is searched for is presented.

Decisions

Now, we discuss in detail the decisions generated by the heuristic. A summary of some important characteristics can be found in Table 6.3. Numbers in **bold** face are used to indicate that the corresponding number coincides with a lower or upper bound.

In the last rows of column 't=-1' we have presented the total expected discounted funding costs. Moreover, these costs are split-up in contributions and remedial payments. All these costs are printed in *italics*. These numbers are found by discounting the costs of times $t = 0, \ldots, 4$ by the pension spot curve at time 0.

We will discuss the results in Table 6.3 in more detail now.

- **Development of funding ratios**
  The development of the funding ratios is one of the most important outcomes of the ALM model. Their minimum, mean and maximum values are contained in Table 6.3. More detailed information is given in Figures 6.2 and 6.3.

  At times 0 and 1, the funding ratio is always sufficiently high, i.e. greater than or equal to $\alpha = 105\%$. At later decision moments ($t = 2$, 3, and 4), underfunding does appear. In 17 of the 1,123 states (1.5%) in the scenario tree, the funding ratio is below this critical level. In only 7 states of these 17 (that is, in 41%), the sponsor makes a remedial contribution. Here we see the effect of the flexible modeling of risk. At time 4, the sponsor is forced to make a remedial payment in 2 states, because the funding ratio is less than 105% in

	$t=-1$	$t=0$	$t=1$	$t=2$	$t=3$	$t=T=4$
Funding ratio						
$\mathbb{E}[F]$		1.16	1.16	1.15	1.12	1.14
$\min F$			1.05	0.91	0.90	0.89
$\max F$			1.28	1.48	1.44	1.73
Contributions						
$\mathbb{E}[c]$	0.110	0.120	0.122	0.097	0.059	
$\min c$			0.068	**0**	**0**	
$\max c$			0.164	0.202	**0.210**	
Underfunding and remedial contributions						
$\mathbb{E}[u] \times 100$	0	0	0	5.56	3.33	1.00
$\mathbb{E}[z] \times 100$	0	0	0	2.78	1.67	0.33
$\mathbb{E}[Z/A_0]$	0	0	0	0.00	0.00	0.00
$\min[Z/A_0]$			0	0	0	0
$\max[Z/A_0]$			0	0.15	0.15	0.14
$\mathbb{E}[DZ/A_0]$	0	0	0	0	0	0
$\mathbb{E}[ZI/A_0]$	0	0	0	0	0	0
One-year risk constraints						
P(ICCs binding)		0	0	0	0.01	
Indexation and liabilities						
$\mathbb{E}[m] \times 100$	0	0	0	0	0.56	0
$\mathbb{E}$[degree of indexation]		1	1	1	0.99	1
Horizon						
$\mathbb{E}[Sho\Lambda/A_0 \| A_T < \Lambda L_T]$						0.14
$\mathbb{E}[Sur\Lambda/A_0 \| A_T \geq \Lambda L_T]$						0.02
$P(A_T < \Lambda L_T)$						0.81
Portfolio						
$\mathbb{E}$[% stocks]	47	42	53	44	42	
$\mathbb{E}$[% real estate]	21	**25**	17	15	15	
$\mathbb{E}$[% bonds]	25	**30**	30	32	32	
$\mathbb{E}$[% cash]	7	3	0	9	11	
Expected funding costs in million €	*4,014*	0	1,270	1,372	1,079	652
$\mathbb{E}[cW]$	*3,903*	-	1,270	1,298	1,039	645
$\mathbb{E}[Z]$	*110*	0	0	74	39	7
$\mathbb{E}[DZ]$	*0*	0	0	0	0	0

Table 6.3: Output and decisions for the illustrative case.

two consecutive years.

In Figure 6.2 the probability distributions of the funding ratios are presented for times 1 to 4. The dashed lines represent the minimum required level set by the supervisor. Moreover, the second dashed line at time 4 denotes the target level of the funding ratio at the horizon. These two important levels of the funding ratio are also depicted in Figure 6.3.

From Figure 6.2 we conclude that at times 2, 3 and 4 in many states the funding ratio is only slightly above 105%.

- **Decisions at time 0**

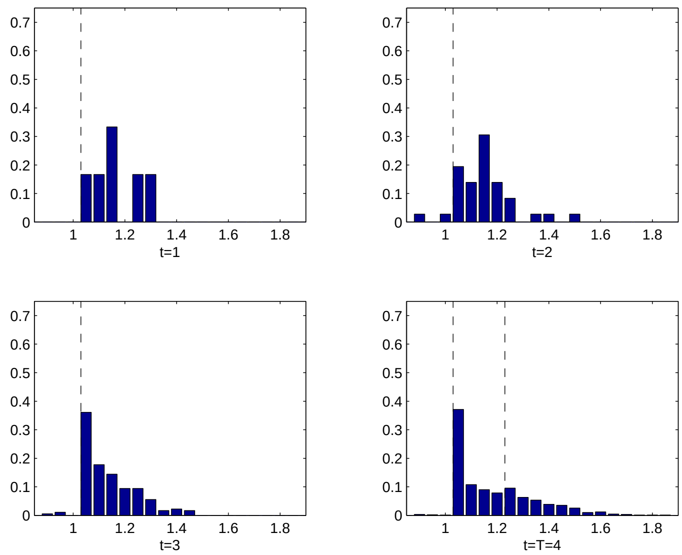

Figure 6.2: Probability distributions of the funding ratios at times 1 to 4 in the illustrative case. Dashed lines indicate levels $\alpha = 1.05$ and $\Lambda = 1.25$.

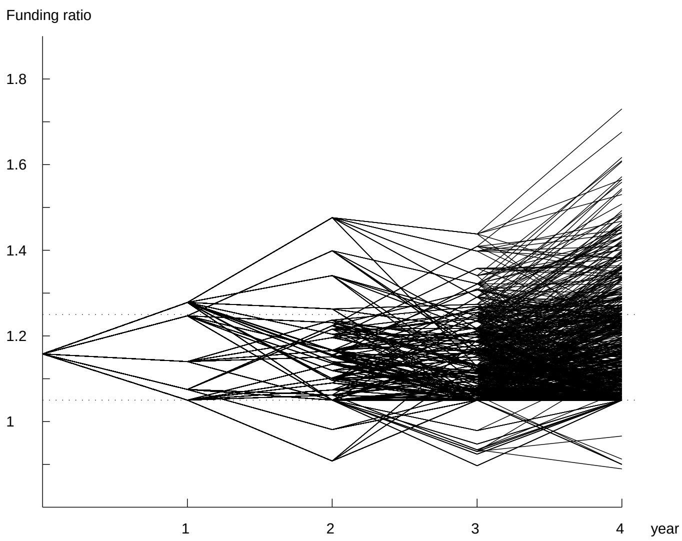

Figure 6.3: Development of the funding ratios in all scenarios in the illustrative case. Dashed lines indicate levels $\alpha = 1.05$ and $\Lambda = 1.25$.

The pension fund under consideration decreases its investments in stocks with 5 percent points. Also the position in cash is reduced (with 4 percent points). These resources are invested in bonds (plus 5 percent points) and real estate (plus 4 percent points).

The contribution rate increases with 1 percent point to 12%. This is the maximum increase, such that no additional penalty costs are incurred.

- **Contribution rate**
At time 1, the expected level of the contribution rate is marginally higher than at time 0. From time 2 on, we see a decrease in the expected level of the contribution rate. Moreover, we see a relatively large range of values of the contribution rate at time 3. At that decision moment, the contribution rate is 0 in some states, and equal to its upper bound (21%) in some other states.

- **Asset portfolios**
We have already discussed the asset portfolios at time 0. At later decision moments, especially the developments of the expected fraction invested in stocks is striking. This expected fraction is 53% at time 1, and slightly above 40% at other times. At the same time, the amount invested in real estate is reduced. This implies that the pension fund takes more risk. At times 3 and 4 we see that the fund invests more conservative: a larger fraction of the assets is invested in cash.

- **Indexation**
In only 1 state (at time 3) the future benefit rights are not indexed fully. In all other states, the rights of the participants are adjusted fully for inflation.

6.2 Sensitivity analysis

In the previous section, we have described an illustrative case in detail. Now, we describe the first impressions we gained by performing sensitivity analyses. We focus on modeling choices, model justification, and scenario trees, respectively.

6.2.1 Sensitivity analysis with respect to modeling choices

From the description of the ALM model in chapters 2 and 3 it follows that this model is rather flexible: a user has to make many choices. In the description of the illustrative case we have presented one such setting. In this section we sketch the outlines of our experiences we obtained by considering some alternative modeling choices.

In the illustrative case, the sponsor of the fund had to make a remedial payment if the funding ratio was too low in two consecutive years. We also examined the effect if this requirement with respect to mid-term risks are strengthened by the supervisor: as soon as the funding ratio is below the minimum required level, the sponsor should restore the financial position of the fund. As a result of these strengthened requirements, the levels of the contribution rate increased considerably in many states of the scenario tree. Moreover, the total level of the expected discounted remedial contributions increased.

We also considered a case in which the supervisor of pension funds imposes more stringent demands with respect to the expected level of underfunding after one year. From numerical experiments we infer that both the number of states in which underfunding is observed, and the number of remedial payments decreased.

On the other hand, the contribution of the active participants increased, and a larger fraction of the asset portfolio was invested less risky. The total cost of funding increased.

To obtain an even better impression of the effects of the integrated chance constraints, we considered a case in which these constraints were omitted. This may be reasonable if the supervisor thinks that requirements with respect to mid-term risks are sufficient. In this case, underfunding and remedial contributions were registered more often. Moreover, the contribution of the active participants increased. As a result, the total funding costs increased.

The last case described briefly here is one in which the interested parties agreed about a higher penalty if the target level of the funding ratio at the horizon is not reached. This higher penalty may be the result of a relative large power of the supervisor and the retired people. Although both the number of states in which underfunding and remedial contributions was observed decreased, the total level of the funding costs increased. Moreover, a larger fraction of the asset portfolio was invested in risky assets.

Summarizing, the first impression of the numerical experiments is that the changes in the outcomes (compared with the illustrative case) are mainly in line with our expectations. However, much more research is needed to make more definite statements about the influences presented in this subsection.

6.2.2 Model justification

In this section we focus on the effects of some characteristics of our model. Recall that the ALM model presented in this thesis is a mixed-integer stochastic program. In this section, we analyze the effects of the introduced fixed penalty costs and the flexible modeling of mid-term risk constraints (by using binary variables) and dynamic rather than static decision strategies. To do so, we first consider an ALM model without binary decision variables. In Section 6.2.4, a static model is considered. In this static model, the fund is only allowed to make decisions with respect to the composition of the asset portfolio and the level of the contribution rate at time 0.

6.2.3 Model without binary decision variables

In order to consider the effect of the binary decision variables, we simplify the model by removing them. That is, in the objective function all fixed-costs terms are deleted, just as all constraints that contain binary decisions. Instead, the inflexible policy rule is implemented, that the sponsor has to restore the funding ratio to the level α =105% as soon as it is too low. Similarly, as soon as the level β =205% is reached, the overflow is restituted. We did not add new constraints on indexation when removing the binaries.

In this case, the active participants get relatively more power in the decision making process. Indeed, by not assigning fixed penalty costs to unfavorable events, these events may occur more often now.

The role of the supervisor is also changed in this case. On the one hand, underfunding is not penalized directly anymore. On the other hand, as soon as the funding ratio is too low, the sponsor has to make a remedial payment. Indeed, the sponsor lost much power, due to these new requirements. Also the position of the current old aged deteriorates: the costs associated with not indexing benefit rights decreased.

In the numerical results, the more powerful role of the active participants can be seen: on average, a lower fraction of the pensionable salaries are transferred to the fund. However, even though this leads to lower total funding costs, one may question whether the supervisor is satisfied. Indeed, underfunding occurs more often in this case. Due to the fact that the sponsor has to restore the funding ratio in those situations immediately, also the weaker role of the sponsor is visible.

Although the fixed costs associated with not indexing benefit rights are removed, we do not see that this instrument is used more often. This may be explained by the relatively high variable costs associated with them.

The board tries to obtain higher returns on its asset portfolio. This is done by investing more in stocks and real estate.

To find numerical results of this model, the computer needed 21 seconds. This is a reduction of 99% compared with the CPU time needed to finish the heuristic for the illustrative case. This is an indicator that larger problems can be solved without binary decision variables, although much flexibility is lost.

6.2.4 Static model

Now we will consider the effect of the dynamic structure in our stochastic program. Due to the multistage character of the model, the fund can react on observed realizations of the stochastic parameters by adjusting the asset portfolio and the contribution rate, for instance. To analyze the effect of static decisions with respect to these decisions, we have set the penalty parameter for a change in contribution rate extremely high. In addition, the transaction costs associated with buying and selling assets also are set at an unrealistic high value. As a consequence, in case of underfunding at a stage after $t = 0$ the only recovery actions that are left for the board are reduction of indexation and a remedial contribution of the sponsor. Obviously, the positions of the active participants and the sponsor deteriorates in the adjusted model.

The total solution time the heuristic needed for this static model, 18 minutes and 36 seconds, is 52% less than in the basic case. This can be explained by the fact that in this case the total number of MSLP problems solved is reduced by (23%).

Numerical results indicate that the active participants have to pay a larger fraction of their pensionable salaries to the fund. However, this is not sufficient to prevent that the funding ratio is too low more often. Indeed, the expected levels of the funding ratios are lower. As a result, it is to be expected that the supervisor will not be satisfied.

Due to the fact that underfunding is observed more often, the sponsor has to make a remedial contribution in more states in the scenario tree. The higher regular and remedial payments resulted in a considerable increase of the funding costs.

Surprisingly, only the retired people would not complain in this case. Their benefit rights are indexed as often as in the illustrative case. This may be the result of the relatively high variable costs associated with not indexing benefit rights

Tree description			
Case	Structure	# Scenarios	# Nodes
Illustrative	6-6-5-5	900	1,124
1	6-6	36	43
2	10-10	100	111
3	15-15	225	241
4	6-6-6	216	259
5	10-10-10	1,000	1,111
6	6-6-6-6	1,296	1,555
7	3-3-3-3-3	243	364
8	4-4-4-4-4	1,024	1,365
9	5-5-5-5-5	3,125	3,906

	CPU time and characteristics					**Discounted funding costs**		
Case	Obj. fun.	LP-rel.	CPU time (h:mm:ss)	# MSLPs	# Impr. (%)	$\mathbb{E}$[tot costs]	$\mathbb{E}$[tot c]	$\mathbb{E}$[tot Z]
Illustrative	7,350	1,746	0:38:29	708	653 (92)	4,014	3,903	110
1	2,184	-542	0:00:01	27	27 (100)	1,255	1,255	0
2	3,361	-413	0:00:05	60	43 (72)	1,233	1,057	176
3	3,284	-129	0:00:39	154	126 (82)	1,639	1,618	21
4	4,018	231	0:01:05	161	149 (93)	2,506	2,438	68
5	5,258	582	0:45:13	810	625 (77)	2,853	2,792	61
6	7,344	1,243	1:31:05	1,239	1,071 (86)	4,141	4,013	128
7	9,768	2,296	0:01:55	210	202 (96)	4,824	4,536	288
8	11,809	1,938	0:44:47	930	889 (96)	4,952	4,040	912
9	9,920	2,541	7:33:09	2,852	2,678 (94)	5,818	5,526	292

	Initial decisions (i.e. at time $t = 0$)					**Performance (total #)**		
Case	$\%X_1$	$\%X_3$	$\%X_2$	$\%X_4$	$\%c$	$\#\{u = 1\}$ (%)	$\#\{z = 1\}$ (%)	$\#\{m = 1\}$ (%)
Illustrative	42	25	30	3	12.00	17 (2)	7 (1)	1 (0)
1	30	25	30	15	9.00	2 (5)	0 (0)	0 (0)
2	40	10	30	20	7.36	11 (10)	2 (2)	0 (0)
3	31	19	30	20	9.50	18 (7)	5 (2)	0 (0)
4	56	13	31	0	10.78	2 (1)	1 (0)	0 (0)
5	45	25	30	0	12.00	38 (3)	13 (1)	0 (0)
6	49	21	30	0	12.00	31 (2)	12 (1)	0 (0)
7	38	25	30	7	12.84	4 (1)	2 (1)	0 (0)
8	52	18	30	0	12.00	13 (1)	10 (1)	0 (0)
9	52	18	30	0	18.00	27 (1)	23 (1)	0 (0)

Table 6.4: Summary statistics for different tree structures: CPU times and characteristics, discounted funding costs, initial decisions, and performance.

completely.

6.2.5 Sensitivity analysis with respect to scenario trees

In this section, we present the results of the experiences we obtained by using different shapes of the scenario tree, other seeds in the scenario generator, and random sampling instead of stratified sampling.

Tree structures

Recall that the tree structure of the model in Section 6.1 is 6-6-5-5; that is, the horizon $T = 4$, the number of branches in nodes of time t is 6 for $t = 0, 1$ and 5 for $t = 2, 3$. Now we formulate nine variants of the illustrative case, that have different tree structures: the horizon varies from 2 to 5, and the number of branches per node varies from 3 to 10. See the top of Table 6.4 for details.

We applied the heuristic to all nine cases, with the same parameter values. Of course, for new scenario trees new scenarios had to be generated. In Table 6.4 the results are presented.

The numerical results indicate that the longer the horizon, the higher the initial contribution rate. For example, in the tree with structure 10-10, this rate is 7.36%, while this is 18% in the tree in which we have 5 time periods and 5 branches per node each time. Also the composition of the asset portfolios differs much. The longer the horizon, the larger the fraction of the assets invested in stocks. The fund invests less in cash if the length of the horizon increases.

These results can be explained by the fact that for longer horizons and more scenarios, the range of possible funding ratios also increases. Because unfavorable circumstances are penalized relatively hard in our ALM model, the fund tries to avoid this. The instruments to do this are the contribution rate, the composition of the asset portfolio and not indexing benefit payments. In case of more time periods, the fund tries to benefit from the higher expected return on investments in stocks. To prevent underfunding in some states, the level of the contribution rate should be increased. Finally, not indexing benefit payments is (very) expensive, so that this instrument is not used. Of course, these results are also influenced by the fact that for different tree structures, new scenarios had to be generated.

The required solution time grows exponentially with the number of scenarios in the ALM model. This can be seen in the CPU times needed to solve models with different tree structures in Table 6.4. We solved one case with 3,125 scenarios and 3,906 nodes. The required CPU time to finish the heuristic was slightly more than seven and a half hours.

Other seeds

In this section, the focus is on the question whether the solutions differ much if the seeds in the scenario generator are adjusted. In these numerical experiments, we still use the stratified sampling procedure. These seeds are for example used in generating returns for stocks, real estate and cash. The closer the solutions of the different cases, the more confidence we have in the way the scenarios are generated. So we repeated the calculations of Section 6.1 with 25 different sets of randomly chosen seeds. A summary of the calculations performed to answer this question can be found in Table 6.5.

Considering the results of Table 6.5, we see that the solutions are very different for different values of the seed. Although the required amount of time needed to finish the heuristic is fairly constant, the range of decisions is very large. For example, the level of the contribution rate at time 0 varies from 12% to 18%. Also the composition of the asset portfolios (and the corresponding risk profiles) differs much from case to case. Only the fraction of assets invested in bonds remains rather constant. Moreover, the fund almost always gives the participants full compensation for increases in the general wage level. This may be the result of the relatively high costs associated with not indexing benefit rights.

The total expected discounted funding costs differ also much from case to case. Also the range of the expected discounted regular contributions and remedial payments are very large. The same conclusion can be drawn if we consider the total number of states in which the funding ratio is less than 105% and the total number of times the sponsor makes a remedial contribution.

The large discrepancies in the outcomes of the different cases are not only the result of the heuristic approach, but also due to the scenarios. This can be concluded from the values of the objective function of the LP-relaxations. The difference between the minimum and maximum value is 51%. The difference between the minimum and maximum value of the objective function of the heuristic solutions is 23%.

Random sampling

Up to now, we used the stratified sampling procedure described in Chapter 5 in generating realizations for the stochastic parameters. We would like to know what the effect of this stratification strategy is. Therefore, we have generated 25 scenario trees, in which the scenarios are generated without stratification. A summary of the numerical results can be found in Table 6.6.

Just as for the cases presented in Table 6.4 (in which 25 cases were considered with other seeds), we conclude that the range of outcomes is very large. This conclusion holds for the level of the contribution rate at time 0, and the composition of the asset portfolios.

Moreover, also the total number of times underfunding is registered (20-181 times) and the number of times the sponsor makes a remedial contribution (6-142 times) are striking. Contrary to the cases discussed in the previous section, there are many cases in which the sponsor makes a remedial contribution as soon as the funding ratio is below its minimum required level.

Also the results of these cases can (at least partially) be assigned to the way the scenarios are generated. The difference between the minimum and maximum value of the objective function associated with the LP-relaxations and the heuristic solutions are 56% and 45% respectively.

	CPU time and characteristics					Discounted funding costs		
Case	Obj. fun.	LP-rel.	CPU time (h:mm:ss)	# MSLPs	# Impr. (%)	$\mathbb{E}$[tot costs]	$\mathbb{E}$[tot c]	$\mathbb{E}$[tot Z]
Illustrative	7,350	1,746	0:38:29	708	653 (92)	4,014	3,903	110
1	7,652	1,734	1:08:02	1,125	964 (86)	4,398	4,267	131
2	7,475	2,105	0:47:08	791	714 (90)	4,593	4,445	148
3	7,176	2,078	0:31:52	756	700 (93)	12,953	4,596	8,357
4	8,457	1,924	0:43:47	809	731 (90)	16,133	3,759	12,374
5	6,624	1,308	0:50:46	809	713 (88)	10,170	4,030	6,140
6	6,512	1,544	0:42:49	835	713 (85)	4,365	4,297	68
7	6,959	1,627	0:50:28	713	655 (92)	6,488	4,422	2,066
8	7,387	1,892	0:41:23	736	634 (86)	11,981	4,205	7,776
9	7,159	1,979	0:39:53	725	646 (89)	6,932	4,354	2,578
10	8,458	1,714	0:34:28	731	623 (85)	12,787	3,412	9,375
11	9,058	1,873	0:29:09	798	661 (83)	11,995	3,640	8,355
12	8,217	1,916	0:38:15	761	663 (87)	8,075	3,880	4,195
13	8,131	2,021	0:21:25	719	635 (88)	3,719	3,455	264
14	9,409	2,274	0:24:39	666	566 (85)	11,224	3,370	7,854
15	9,288	1,974	0:26:44	720	623 (87)	6,682	3,616	3,066
16	7,756	1,843	0:20:33	738	645 (87)	3,967	3,851	116
17	8,883	1,997	0:21:14	689	604 (88)	12,885	3,475	9,410
18	7,299	1,562	0:25:43	688	599 (87)	3,806	3,533	273
19	7,619	1,687	0:26:26	777	667 (86)	3,954	3,841	113
20	7,088	1,224	0:32:26	810	718 (89)	9,046	3,841	5,205
21	7,724	1,610	0:24:13	677	581 (86)	3,832	3,553	279
22	7,185	1,831	0:25:16	662	570 (86)	4,106	3,861	245
23	8,484	1,614	0:28:24	808	691 (86)	8,347	3,702	4,645
24	9,222	1,843	0:33:54	768	679 (88)	20,200	3,555	16,645
25	8,769	2,039	0:26:57	747	636 (85)	6,110	3,678	2,432
$\mathbb{E}$	7,920	1,809	0:50:53	762	666	8,350	3,866	4,484
min	6,512	1,224	0:40:37	662	566	3,719	3,370	68
max	9,409	2,274	1:14:18	1,125	964	20,200	4,596	16,645

	Initial decisions (i.e. at time $t = 0$)					Performance (total #)		
Case	$\%X_1$	$\%X_3$	$\%X_2$	$\%X_4$	$\%c$	$\#\{u = 1\}$ (%)	$\#\{z = 1\}$ (%)	$\#\{m = 1\}$ (%)
Illustrative	42	25	30	3	12.00	17 (2)	7 (1)	1 (0)
1	60	10	30	0	18.00	24 (2)	11 (1)	0 (0)
2	46	22	32	0	16.64	14 (1)	9 (1)	0 (0)
3	51	10	30	9	18.00	5 (0)	2 (0)	0 (0)
4	46	21	33	0	15.50	14 (1)	7 (1)	0 (0)
5	49	21	30	0	15.50	15 (1)	6 (1)	0 (0)
6	45	23	31	1	15.50	19 (2)	11 (1)	4 (0)
7	52	17	31	0	16.55	14 (1)	8 (1)	2 (0)
8	48	22	30	0	16.46	22 (2)	14 (1)	1 (0)
9	56	10	30	4	18.00	17 (2)	10 (1)	1 (0)
10	45	24	31	0	12.00	25 (2)	25 (2)	4 (0)
11	44	25	31	0	12.00	25 (2)	25 (2)	1 (0)
12	49	21	30	0	15.50	22 (2)	10 (1)	1 (0)
13	56	10	30	4	12.00	22 (2)	22 (2)	3 (0)
14	31	23	35	11	12.00	23 (2)	23 (2)	9 (1)
15	50	10	30	10	12.00	27 (2)	27 (2)	1 (0)
16	36	14	32	18	13.73	22 (2)	22 (2)	4 (0)
17	45	25	30	0	12.00	19 (2)	19 (2)	1 (0)
18	59	10	31	0	12.00	20 (2)	20 (2)	3 (0)
19	58	12	30	0	13.94	19 (2)	19 (2)	6 (1)
20	59	11	30	0	12.00	14 (1)	14 (1)	6 (1)
21	60	10	30	0	12.00	26 (2)	26 (2)	0 (0)
22	48	18	34	0	12.00	23 (2)	23 (2)	0 (0)
23	57	10	33	0	12.00	20 (2)	20 (2)	1 (0)
24	49	21	30	0	12.00	21 (2)	21 (2)	2 (0)
25	55	10	34	1	12.00	19 (2)	19 (2)	2 (0)
$\mathbb{E}$	50	16	31	2	13.49	19.64	16.52	2.08
min	31	10	30	0	12.00	5	2	0
max	60	25	35	18	18.00	27	27	9

Table 6.5: Summary statistics for cases in which other seeds are used in the scenario generator: CPU times and characteristics, discounted funding costs, initial decisions, and performance.

	CPU time and characteristics					Discounted funding costs		
Case	Obj. fun.	LP-rel.	CPU time (h:mm:ss)	# MSLPs	# Impr. (%)	$\mathbb{E}$[tot costs]	$\mathbb{E}$[tot c]	$\mathbb{E}$[tot Z]
Illustrative	7,350	1,746	0:38:29	708	653 (92)	4,014	3,903	110
1	6,670	1,415	0:42:43	786	648 (82)	3,518	3,444	74
2	5,913	1,457	0:41:14	770	667 (87)	4,016	4,002	14
3	7,217	1,292	0:44:21	855	730 (85)	3,896	3,650	246
4	6,033	1,485	0:52:17	817	712 (87)	3,902	3,854	48
5	6,219	1,267	0:57:49	789	660 (84)	3,622	3,550	72
6	5,234	1,158	1:02:28	1,137	477 (42)	2,791	2,256	535
7	7,355	1,363	0:45:59	845	703 (83)	3,943	3,726	217
8	5,922	1,244	0:52:17	878	719 (82)	3,840	3,816	24
9	6,568	1,346	0:49:49	829	689 (83)	3,602	3,526	76
10	6,077	1,268	1:14:18	961	817 (85)	4,037	3,992	45
11	6,184	1,362	0:47:43	784	685 (87)	3,427	3,396	31
12	5,995	1,277	0:57:17	925	736 (80)	3,756	3,714	42
13	5,589	1,020	0:45:03	876	714 (82)	3,840	3,784	56
14	6,228	1,291	0:40:37	781	681 (87)	3,846	3,793	53
15	6,288	1,157	0:55:48	851	692 (81)	3,645	3,626	19
16	6,268	1,311	0:42:12	832	689 (83)	3,566	3,533	33
17	6,797	1,148	0:53:34	816	698 (86)	3,556	3,522	34
18	6,090	1,207	1:00:56	947	747 (79)	3,829	3,802	27
19	5,903	1,126	0:42:26	761	672 (88)	3,951	3,917	34
20	6,540	1,439	0:50:48	777	619 (80)	3,674	3,633	41
21	6,168	1,135	0:50:24	819	674 (82)	3,946	3,882	64
22	5,055	1,039	0:56:38	1,140	489 (43)	2,739	2,196	543
23	6,095	1,100	0:49:47	785	648 (83)	3,619	3,523	96
24	6,197	1,187	0:41:48	905	720 (80)	3,655	3,612	43
25	5,558	954	0:53:48	932	763 (82)	4,029	4,029	20
$\mathbb{E}$	6,167	1,241	0:50:53	864	682	3,691	3,591	99
$\min$	5,055	954	0:40:37	761	477	2,739	2,196	14
$\max$	7,355	1,485	1:14:18	1,140	817	4,029	4,029	543

	Initial decisions (i.e. at time $t = 0$)					Performance (total #)		
Case	$\%X_1$	$\%X_3$	$\%X_2$	$\%X_4$	$\%c$	$\#\{u = 1\}$ (%)	$\#\{z = 1\}$ (%)	$\#\{m = 1\}$ (%)
Illustrative	42	25	30	3	12.00	17 (2)	7 (1)	1 (0)
1	51	19	30	0	12.00	68 (6)	19 (2)	1 (0)
2	49	21	30	0	17.00	21 (2)	7 (1)	0 (0)
3	47	21	32	0	12.00	22 (2)	10 (1)	0 (0)
4	48	21	30	1	16.20	27 (2)	6 (1)	0 (0)
5	45	25	30	0	12.00	40 (4)	11 (1)	0 (0)
6	45	25	30	0	9.50	181 (16)	141 (13)	3 (0)
7	43	25	31	1	12.00	22 (2)	10 (1)	0 (0)
8	52	18	30	0	12.00	20 (2)	11 (1)	2 (0)
9	45	25	30	0	12.00	36 (3)	14 (1)	0 (0)
10	49	21	30	0	16.30	25 (2)	8 (1)	7 (1)
11	51	19	30	0	12.00	32 (3)	11 (1)	2 (0)
12	55	15	30	0	12.00	37 (3)	13 (1)	0 (0)
13	49	19	32	0	15.50	26 (2)	9 (1)	3 (0)
14	57	10	33	0	12.00	27 (2)	11 (1)	5 (0)
15	49	21	30	0	12.00	24 (2)	10 (1)	0 (0)
16	51	21	31	0	12.00	37 (3)	12 (1)	6 (1)
17	45	25	30	0	12.00	30 (3)	6 (1)	4 (0)
18	49	21	30	0	12.00	31 (3)	13 (1)	0 (0)
19	42	22	36	0	16.80	23 (2)	11 (1)	4 (0)
20	40	23	30	7	12.00	31 (3)	9 (1)	0 (0)
21	48	22	30	0	17.00	31 (3)	9 (1)	0 (0)
22	41	25	34	0	9.50	179 (16)	142 (13)	5 (0)
23	45	25	30	0	12.00	33 (3)	13 (1)	6 (1)
24	45	25	30	0	12.00	28 (2)	13 (1)	2 (0)
25	48	22	30	0	15.70	30 (3)	12 (1)	0 (0)
$\mathbb{E}$	48	31	21	0	13.02	42 (4)	21 (2)	2 (0)
$\min$	40	30	10	0	9.50	20 (2)	6 (1)	0 (0)
$\max$	57	36	25	7	17.00	181 (16)	142 (13)	7 (1)

Table 6.6: Summary statistics for cases in which random sampling is used in the scenario generator: CPU times and characteristics, discounted funding costs, initial decisions, and performance.