Asset liability management for pension funds using multistage mixed-integer stochastic programming

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Chapter 5

Scenario generation

In this chapter we describe how the scenarios for the multistage stochastic program are generated. Before we do that, we first recall which stochastic parameters appear in our ALM model. We compare these parameters with those which appear in some other ALM models for pension funds in the literature. Then, we will indicate some properties that scenarios for multistage stochastic programs should satisfy. It will be argued that both consistency with historical data, and consistency with financial theory is important. After a discussion on the interdependencies between the stochastic parameters, we devote the remainder of this chapter to describe how we actually find realizations for all stochastic parameters.

5.1 Probabilities and stochastic parameters

Before we describe the probabilities and the stochastic parameters, we first recall some notation which is used regularly in this chapter. In Chapter 2, we introduced the definitions of the time sets $\mathcal{T}_0 := \{0, \ldots, T-1\}$ and $\mathcal{T}_1 := \{1, \ldots, T\}$.

As we have seen in Chapter 2, a scenario tree is needed to define multistage stochastic programs. Moreover, *probabilities* have to be assigned to each node of the tree. We assume that the branching structure is specified in advance and that each node at time $t \in \mathcal{T}_0$ has the same number of child nodes. This number is denoted by $branch_t$. In addition, we assume that given a certain state, the conditional probabilities associated with every child node are the same, i.e. they are given by $(branch_t)^{-1}$. As a result of this choice, all scenarios have equal probability S^{-1}, and the probability of a state (t, s) is given by

$$p_t^s = \left(\Pi_{q=1}^{t-1} branch_q\right)^{-1} \quad t \in \mathcal{T}_1, s \in \mathcal{S}_t.$$

As we will see in Section 5.3, the choice of assigning equal probabilities to all child nodes is convenient in our method to match returns on stocks and real estate with returns on bonds.

Given the specification of the scenario tree and their corresponding probabilities, we have to find numerical values for the vector of stochastic parameters in each state (t, s) in the scenario tree. In Chapter 2, we have denoted the vector of stochastic parameters by ω_t^s, defined as:

$$\omega_t^s = (r_{1t}^s, r_{2t}^s, \dots, r_{Nt}^s, w_t^s, \underline{L}_t^s, \overline{L}_t^s, \underline{B}_t^s, \overline{B}_t^s, \gamma_t^s, W_t^s).$$

In this chapter, we focus on the first five elements of ω_t^s, and call the new vector

$$\varpi_t^s := (r_{1t}^s, r_{2t}^s, r_{3t}^s, r_{4t}^s, w_t^s).$$

The stochastic parameters which appeared in ω_t^s, but not in ϖ_t^s, are (deterministic) functions of the elements of ϖ_t^s and other data.

In our ALM model, more stochastic parameters (which are present in ω_t^s) appear than in Consigli and Dempster [17], Dert [24], and Kouwenberg [55]. The main differences are due to the fact that we use indexation as a decision in the stochastic program, whereas the other authors use fixed values for the level of the liabilities. In addition, Consigli and Dempster consider exchange rates and borrowing rates, while we do not use these parameters; we assume that the exchange rates are considered in the asset returns. Borrowing money, which is also considered by Kouwenberg, is not a possibility in our model. We assume that the fund only invests money.

Pflug [77] describes a method to generate realizations of a stochastic process for multiperiod financial problems, using optimal discretization. However, he considers a one-dimensional stochastic process, while we deal with a 5-dimensional vector ϖ_t^s.

Dupačová et al. [28] describe properties which scenarios for multistage stochastic programs should have. In building representative scenario trees, one should keep in mind underlying probability assumptions, the existing data, and the purpose of the application. On the one hand, trees must represent the underlying distribution, on the other hand they should be such that the model produces good decisions. Explicitly formulated additional requirements concerning properties of the probability distribution can help. The statistical properties can be made specific through a suitable manipulation of the data to obtain the prescribed moments, given a fixed tree structure. According to Dupačová et al. [28], building a scenario tree should be done such that some statistical properties of the data process are retained. For instance, one should take into account specified expectations, other moments, and correlations between the stochastic variables. Moreover, Dupačová et al. [28] argue that one should also consider the purpose of the model under consideration. For example in financial optimization problems, one should build trees which are arbitrage free (see Section 5.3.4).

The difficulty in generating scenarios for ϖ_t^s, $t \in \mathcal{T}_1$, $s \in \mathcal{S}$, are the dependencies between the components. These dependencies may be necessary from a theoretical point of view, or may be the result of wishes of the model user. These issues will be discussed now briefly.

Consistency with historical data

We would like that the sample we use to represent returns on stocks, real estate, and the bank account in the scenario tree are consistent with empirical data. Therefore, we specify the stochastic processes for these parameters, and estimate the parameters of such models using empirical data. To obtain numerical values for ϖ_t^s in the scenario tree, we sample (in a deterministic or stochastic way) from the specified stochastic processes.

We use econometric models to describe these stochastic processes. In these econometric models, special attention is paid to empirically based autocorrelation and lower and upper bounds. Because the return on the bank account and the change in the general wage level both depend on the level of the inflation, and are cointegrated of the first order as we shall see, we implement these processes in an error-correction model. In addition, the variances of the historical returns on stocks and real estate vary over time and are modeled as a GARCH(1,1) process. These returns are assumed to be lognormal. Finally, we consider excess returns of stocks over bonds and excess returns of real estate over bonds to obtain dependencies between these stochastic parameters which are in accordance with historical data.

Consistency with financial theory

Because some stochastic parameters which appear in our ALM model are interrelated, we have to be aware that consistency is obtained. Given the parameters of ϖ_t^s, the following wishes arise with respect to the specification of realistic values of other stochastic parameters in the scenarios.

- The numerical values of the lower and upper bounds on the value of the liabilities and benefit payments are based on discounting future streams of cash flows. In order to be able to compare the asset value with the value of the liabilities, one should use appropriate discount factors to find market values of the liabilities.

- Discount rates, which are used to compare cash flows over time, should be consistent with the ones found in valuing the liabilities. Moreover, they should be consistent with expected returns on the bank account.

- Bond returns should be consistent with the yield curve, and should be generated taking into account coupon and principal payments.

- Returns on stocks, bonds, real estate, and the bank account should be assigned to the nodes of the scenario tree, such that no arbitrage opportunities are present.

Vector Autoregressive Models

In the financial literature, Vector Autoregressive (VAR) models are often used to find realizations for stochastic parameters, see for example Boender et al. [7], Dert [24], and Kouwenberg [55]. VAR models were popularized by Sims [89]. However, VAR models have some drawbacks, see Maddala and Kim [63]. In practice it has been found that the unrestricted VAR model gives very erratic estimates, because of high multicollinearity among the explanatory variables (which are parameters in the ALM model). In addition, if the variables in the VAR model are cointegrated, this imposes restrictions on the parameters of this model. In this case, standard estimation techniques do not lead to a good description of the stochastic process. Moreover, using a VAR model may even lead to inconsistencies, for example in generating bond returns. This will be explained in Section 5.3.1.

Because of these disadvantages of VAR models, we propose an alternative method to find numerical values for the stochastic parameters of the ALM model. This method is based on dependencies between some of the parameters. Now, we will describe these dependencies in more detail.

Dependencies between the stochastic parameters in ω_t^s

As we have argued, there are good reasons to pay attention to dependencies between various stochastic parameters, when one is generating possible realizations of them. In Table 5.1 we give an overview of the relationships we have modeled in our scenario generation algorithm. Each row in this table gives a relationship between two or more stochastic parameters, by placing a $\bullet$ on the corresponding position. Moreover, a brief comment is given, indicating why these parameters should not be specified independently. We describe these relationships in the next subsections. They are indicated here to show which numerical values have to be generated simultaneously.

r_4	w	r_2	r_1	r_3	$\underline{L}$	$\overline{L}$	$\underline{B}$	$\overline{B}$	γ	W	Relationship
$\bullet$	$\bullet$										Cointegration
$\bullet$		$\bullet$									Yield curve
$\bullet$			$\bullet$								Bounds on stock returns
		$\bullet$	$\bullet$								Probability of outperformance (stocks/bonds)
$\bullet$				$\bullet$							Bound real estate returns
		$\bullet$		$\bullet$							Probability of outperformance (real estate/bonds)
$\bullet$		$\bullet$	$\bullet$	$\bullet$							No arbitrage
$\bullet$									$\bullet$		Consistent discount rates
$\bullet$					$\bullet$					$\bullet$	Definition $\underline{L}$
$\bullet$	$\bullet$					$\bullet$				$\bullet$	Definition $\overline{L}$
					$\bullet$		$\bullet$				Definition $\underline{B}$
	$\bullet$							$\bullet$			Definition $\overline{B}$

Table 5.1: Relationships between the stochastic parameters which are present in ω_t^s. The first five elements are also part of ϖ_t^s.

The relationships also determine the order in which numerical values have to be found, since for some parameters the values of other parameters are needed. Given the relationships between the parameters, as presented in Table 5.1, the following order is used to find numerical values for the stochastic parameters for a fixed state (t, s), assuming that all parameters of all predecessors already have numerical values:

1. Because of lack of data, we use $W_t^s = W_t$.

2. Find return on the bank account r_{4t}^s and the development in the general wage level W_t^s.

3. Generate bond return r_{2t}^s.

4. Generate stock return r_{1t}^s and returns on the real estate portfolio r_{3t}^s.

5. Find lower and upper bounds on the value of the liabilities, $\underline{L}_t^s$ and $\overline{L}_t^s$, and of the benefit payments, $\underline{B}_t^s$ and $\overline{B}_t^s$, and find discount rates γ_t^s.

5.2 Returns on the bank account and changes in the general wage level

It is to be expected that r_{4t} and w_t, which specify the return on the bank account and the change in the general wage level respectively, are both integrated processes of the first order, denoted by I(1). Moreover, they are correlated. These expectations will be discussed in the next paragraph.

The developments in the general wage level in the next few years may be the result of negotiations in a given year. Moreover, if there are negotiations about the wage level in one sector, the outcomes of these negotiations tend to be used in other sectors. We think that the stochastic process r_{4t} is also an integrated process of the first order, because it is to be expected that information regarding the short-term interest rate in a given year also has predictive power regarding this rate one year later. This follows for example from the fact that this level depends on the phase of the economy in the business cycle. These cycles generally last more than one year, see for example Mankiw [65]. As soon as the changes in the general wage level are high (low), the inflation rate tends to be high (low), which results in a high (low) nominal interest rate, since the nominal return on a bank account is equal to the sum of the real return and the inflation. On the other hand, if the interest rate is high, the inflation is likely to be high, and it is reasonable to expect that employees want to be compensated for the high price level, so they ask for higher wages.

As a result, we expect that the stochastic processes r_{4t} and w_t are not only I(1)-processes, but that they are even cointegrated. In order to validate this assumption, we studied the returns on the bank account and the changes in the general wage level in The Netherlands from 1983 to 2002. These data were derived from the Dutch central bureau of statistics, CBS [16]. After applying Dickey-Fuller tests [25], we conclude with 95 percent confidence that the returns on a bank account and the changes in the general wage level are indeed integrated processes of the first order.

Because of the arguments given above, we believe that these two stochastic processes can be described by an error-correction model. Error-correction models were first introduced into the econometric literature by Sargan [84], and were popularized by Davidson et al. [21]. The main characteristics of error-correction models are the notion of an equilibrium long-run relationship and the introduction of past disequilibriums as explanatory variables in the dynamic behavior of current variables. Granger and Weiss [37], have demonstrated that if two variables are integrated of order 1, and are cointegrated, they can be modeled as having been generated by an error-correction model. The error-correction model, used to describe the dependencies between r^s_{4t} and w^s_t, is presented in Appendix 5.A.1.

In order to test the hypothesis that the returns on the bank account and the changes in the general wage level are cointegrated indeed, we applied the Johansen cointegration test [46] on the same data set as described above. With 99 percent confidence, we conclude that the returns on a bank account and the change in the general wage level are cointegrated.

Generating r^s_{4t} and w^s_t with the error-correction model

We have seen that autoregressive terms are important in modeling the stochastic processes for the short-term risk-free interest rate and the change in the general wage level. This is the reason why we generate r^s_{4t} and w^s_t in a forward manner. That is, we generate the values for these stochastic processes from time 1 to time T. Note that at time 0 the risk-free interest and the (change in last year's) general wage level are known.

Basically, the error-correction model generates numerical values for r^s_{4t} and w^s_t by first estimating the parameters and then generating values for the error terms. Given state (t, s), we apply a stratified sampling procedure to find normally distributed error terms associated with $r^{s'}_{4,t+1}$ and $w^{s'}_{t+1}$, denoted by $\epsilon^{s'}_{4,t+1}$ and $\epsilon^{s'}_{w,t+1}$, $s' \in \mathcal{K}^s_t(t+1)$. Stratification is suitable, since if one samples randomly from a normal distribution, and the number of realizations is relatively low, the sample estimation approximates the underlying distribution relatively poorly.

To find numerical values for $\epsilon^{s'}_{4,t+1}$ and $\epsilon^{s'}_{w,t+1}$, such that they are stratified sampled from a normal distribution, we first find $branch_t$ points, which are all different, have equal probabilities, and approximate a uniform $[0, 1]$-distribution. These $branch_t$ points describe a discrete uniform distribution with equidistant values and leads to correct values for the mean and variance. They are given by

$$e_i = (i - \frac{branch_t + 1}{2})\sqrt{\frac{1}{branch_t^2 - 1}} + \frac{1}{2}, \quad i = 1, \ldots, branch_t. \tag{5.1}$$

The inverse transform method is used to transform the uniform error terms (5.1) into normally distributed ones.

Because $\epsilon^{s'}_{4,t+1}$ and $\epsilon^{s'}_{w,t+1}$ are assumed to be independent in the error-correction model, we generate numerical values of these error terms also independently. This is done by allocating the error terms $\epsilon^{s'}_{4,t+1}$ and $\epsilon^{s'}_{w,t+1}$ randomly to the nodes $(t + 1, s')$, $s' \in \mathcal{K}^s_t(t+1)$. Given the error terms, we use the error-correction model to generate numerical values for $r^{s'}_{4,t+1}$ and $w^{s'}_{t+1}$.

To obtain scenarios which are consistent with financial theory, we also enforce lower and upper bounds on r^s_{4t} and w^s_t. Moreover, we also enforce a lower bound on the increase in the general wage level. We use historical data to find numerical values for these bounds.

5.3 Returns on bonds, stocks, and real estate

In this section, we describe how returns are generated for bonds, stocks and real estate. Before the procedure is described to generate returns for each of these three asset classes, we first give a list of properties we would like the returns to satisfy:

- Returns on the bond portfolio are consistent with observed yield curves.
- If developments of variances of historical data of returns on stocks and real estate are best described by means of a GARCH specification, then the same specification is also used when generating future returns.
- The probability of excess returns of stock returns over bond returns and returns on real estate over bond returns are consistent with historically observed values, to obtain consistency with the method to value liabilities that will be described in Section 5.4.
- In our model we make the assumption that returns on stocks and real estate follow lognormal distributions, because this assumption is also made frequently in financial theory (see for example the Black-Scholes option pricing model [5]).
- If autoregressive terms are observed in historical time series for the returns, these are also taken into account.
- Lower and upper bounds on the returns are considered, which are consistent with observed market prices.
- The scenario tree is arbitrage free.

Given these wishes, we now describe how the returns are generated for bonds, stocks, and real estate, respectively.

5.3.1 Bond returns

In many ALM models in the financial literature, bond returns are generated by simply drawing from a return distribution. This is for example the case if a VAR model is used. Next to the disadvantages of using VAR models mentioned in Section 5.1, the use of these models may also lead to implied very low (or even negative) yields. This is made clear by means of the following example.

Example 5.1

Consider a zero-coupon, non-callable bond with maturity ten years. If the current price of the bond, with principal €$1,000$, is €675.56, the implied 10-year yield equals 4%.

Assume that in one scenario the return on this bond is 9% per year. This implies that after 5 years, the price of the bond equals $(1.09)^5$€675.56 =€$1,039.44$. This means that one is willing to pay €$1,039.44$ to receive €$1,000$ five years from then! In other words, a negative implied yield is observed. □

To avoid such unrealistic returns, we would like to find bond returns, which are consistent with observed yields. For that reason we will specify a yield curve in every state of the scenario tree. These yield curves are used to discount future coupon and principal payments in order to obtain market values of the bond portfolio. These market values are used in the specification of bond returns. The advantages of this approach are twofold: one can specify any current bond portfolio, and find consistent future returns, using observed market prices. Moreover, these curves are used in Section 5.4 to value the liabilities, using observed market prices. We use the following equation, which is derived from Haugen [39], to define a yield curve in state (t, s):

$$y_t^s(q) = (a_1 + a_2 q)e^{-a_3 q} + a_{4t}^s, \quad q = 0, 1, 2, \ldots \tag{5.2}$$

In equation (5.2), $y_t^s(q)$ denotes the yield corresponding to a risk-free zero-coupon bond maturing q years from time t, given the current state (t, s). Moreover, coefficient a_{4t}^s is the yield on bonds with the longest terms to maturity, and a_1 is the difference between the yield on bonds with the longest and shortest terms to maturity. This can easily be seen by considering $q = \infty$ and $q = 0$, respectively. The other two coefficients, a_2 and a_3, control the shape of the curve between the shortest en longest maturities.

For simplicity, we assume only parallel shifts in the yield curve. As a result, the coefficients a_1, a_2 and a_3 in (5.2) do not depend on t and s; they have to be estimated only once (at time 0). In other states, only a_{4t}^s will be adjusted, so that the new yield curve is consistent with expected one-year returns on the bank account.

At time 0, the yield curve (5.2) is estimated. To do so, we used data on March 1, 2002, of yields implied by Dutch government bonds with maturity 10 years. These data were derived from Datastream [20]. Numerical values for the coefficients in (5.2) are found by using a nonlinear regression routine. In this routine, numerical values for the four coefficients in (5.2) are found which minimizes the sum of the squared vertical distances from the curve. As a result, the yield curve at time 0 is specified.

Now we explain how yield curves are found in all other states (t, s), $t \in \mathcal{T}_1$, $s \in \mathcal{S}$. Since we assume that only parallel shifts in the yield curve occur, the estimates $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are used in every state to specify the shape of the yield curve. As a result, only the values of $\hat{a}_{4t}^s$ have to be found in all states (t, s), $t \in \mathcal{T}_1$, $s \in \mathcal{S}$. This can be accomplished by asking for consistency with the values of the risk-free interest rates. Recall that these are already present in the scenario tree. This consistency can be obtained by considering the following definition:

$$\mathbb{E}_{(t,s)}[r_{4,t+1}^{s'}] = y_t^s(1), \quad t \in \mathcal{T}_1, s \in \mathcal{S}, s' \in \mathcal{K}_t^s(t+1). \tag{5.3}$$

Equation (5.3) has to be satisfied, since both the left-hand side and the right-hand side define the expected next year's risk-free rate. Given equation (5.3), it is relatively easy to find updated values of $\hat{a}_{4t}^s$, given that its value in its predecessor is already known:

$$\hat{a}_{4t}^s = \hat{a}_{4,t-1}^s - y_{t-1}^s(1) + \mathbb{E}_{(t,s)}[r_{4,t+1}^{s'}], \quad t \in \mathcal{T}_1, s' \in \mathcal{K}_t^s(t+1).$$

Therefore, we consider the scenario tree in an increasing order of time. As a result, a parallel shift in the yield curve arises, and all term structures in the scenario tree are consistent with the expected returns on the bank account.

Generating bond returns

We make the following assumptions in generating bond returns:

- The pension fund under consideration only invests in risk-free, non-callable bonds.
- The coupon and principal payments of the current bond portfolio are known (and therefore, also the duration of the portfolio).
- Each year, the board of the fund adjusts the bond portfolio, so that the duration at the beginning of each year is the same.

We would stress here that each of these assumptions may be relaxed, although relaxing some of these assumptions may lead to additionally required data.

Given the yield curve in a state, we use the implied discount rates to find the current price of the bond portfolio. The payoffs of the bond portfolio, i.e. coupon and principal payments, together with the change in the price of the bond portfolio, define the return on this portfolio. The mathematical details of these calculations are given in Appendix 5.A.2.

5.3.2 Stock returns

Before we describe how stock returns are generated, we first discuss mean returns, variances, autoregressive terms and bounds on returns. These issues are considered, because we would like to obtain scenarios which are in accordance with historically observed characteristics.

Mean stock returns

The considered period of observation is crucial in estimating mean stock returns, and can lead to large differences. In Table 5.2 mean yearly stock returns, used in some financial models in the literature, are presented.

Author(s)	Mean return (in %)
Boender et al. [7]	10.7
Cariño and Turner [15]	11.0
Dert [24]	8.6
Ibbotson and Sinquefield [43]	10.5
Kouwenberg [55]	10.2
Rudolf and Zimmermann [83]	7.1

Table 5.2: Mean yearly stock returns used in financial models.

For most of the mean returns presented in Table 5.2, the considered period started either in 1956 or in 1926 and ended in the late 1990s. To obtain an estimate of mean stock returns based on many observations, we use the results presented in Siegel [88]. He found a yearly return of 6.7 percent in the period 1802-1992. This figure is based on research by Schwert [87], Schiller [85], and data from the Center for Research in Stock Prices, CRSP [19].

We use the returns on the broadly diversified MSCI World-index to include even more recent data in the estimate of mean stock returns. In the period 1993-2002, the mean return on the MSCI World-index was 9.0%. Taking into account both periods, we decided to specify the mean return as 6.8%.

Variances

Variances of stock returns are not stable over time, see for example Bollerslev et al. [11] and French et al. [34]. They conclude that the stochastic processes for the variances of the returns, are best described by a GARCH(1,1)-model. We have tested this condition (against the alternative to include higher moments) using historical values of the returns on the MSCI World-index in the period 1970-2002. We concluded that a GARCH(1,1) representation is indeed the best one.

ARCH models were introduced in the econometric literature by Engle [31] and generalized by Bollerslev [9], which led to the introduction of GARCH models. These models are widely used in various branches of econometrics, especially in financial time series analysis. Details about the GARCH(1,1) model, used to describe time varying variances, are presented in Appendix 5.A.3.

Autoregressive terms

As already mentioned before, in the literature it is customary to model the relationships between the stochastic parameters as a first-order autoregressive model. In these models, the autoregressive components for the returns on stocks are omitted. This is done to avoid predictability of asset returns. However, Lo and MacKinlay [61] found positive correlation in daily, weekly, and monthly index returns. If yearly returns are considered, as we do in our ALM model, Fama and Frech [33] and Poterba and Summers [79] found negative serial correlation in index returns.

To find an appropriate description for the development of the stock returns, we follow the analysis made by Campbell et al. [13] to test the presence of autocorrelation in the lognormal stock returns. Assume that the returns on stocks are described by the following model:

$$\tilde{r}_{1t} = \nu_1 \tilde{r}_{1,t-1} + \epsilon_{1t}, \tag{5.4}$$

where the disturbance terms ϵ_{1t} are assumed to be independent and identically distributed (IID), and follow a normal distribution with mean 0.

We tested the null hypothesis that yearly stock returns are IID (which implies $\nu_1 = 0$). The estimate of ν_1 is 0.056. We used again yearly returns on the MSCI World-index from 1970 to 2002. The Ljung-Box Q-statistic [60], and the corresponding probability are 0.2186 and 0.607 respectively. We conclude that the first order autoregressive term is not needed in the description of stock returns.

Bounds on stock returns

Based on historical observations, one may expect a higher return on a broadly diversified stock portfolio than on a corresponding bond portfolio. As a result, the *ex-ante risk premium*, the difference between the returns on a stock and bond portfolio, is strictly positive. This risk premium may vary through time, and may for example be influenced by the level of the interest rate. If this rate is very low, one may not expect large positive returns. On the other hand, a low interest rate may lead to many investment opportunities by companies, resulting in more economic activity. Therefore, one may expect a relatively high return on stocks. However, even though the ex-ante risk premium may vary, it is reasonable to assume that an upper bound on this premium exists.

We want to generate stock returns in such a way that the implied internal rate of return on the stock portfolio, based on the growth model developed by Gordon [36], does not violate lower and upper bounds on the ex-ante risk premium. This implies lower and upper bounds on stock returns. The mathematical details about the Gordon growth model and the derivation of the bounds on stock returns are presented in Appendix 5.A.4.

Given time t, the lower and upper bounds on the stock returns imply that a mean reverting component is introduced, taking into account the whole history from time 0 to time t. This mean reverting effect depends on the lower and upper bound on the ex-ante risk premium. Depending on these values, these bounds may prevent that scenarios are present in which stock returns are extremely high or low every year.

Generating stock returns

We assume that the stock return in each state (t, s), denoted by r_{1t}^s, follows a lognormal distribution. This is consistent with the assumption underlying many pricing models of derivatives based on stock prices. This is for example the case in the Black-Scholes option pricing model, see Black and Scholes [5]. In the description below, we distinguish r_{1t} and $\tilde{r}_{1t}$. The first is the so-called *simple net return*, while $\tilde{r}_{1t}$ denotes the *continuously compounded return*, or the *log return*. The values of r_{1t} and $\tilde{r}_{1t}$ are related by means of the following equation:

$$\tilde{r}_{1t} := \log(1 + r_{1t}).$$

Given this assumption, we need numerical values for next year's mean return and variance to generate returns. However, we cannot directly use the numerical values for the mean and variance of the simple net returns, denoted by μ_1 and σ_1^2 respectively. It is well known that given the estimates of μ_1 and σ_1^2, the mean and variance of the continuously compounded returns are given by

$$\tilde{\mu}_1 = \log\Big(\frac{\mu_1 + 1}{\sqrt{1 + \frac{\sigma_1}{\mu_1+1}}}\Big), \tag{5.5}$$

$$\tilde{\sigma}_1 = \log\Big(1 + (\frac{\sigma_1}{\mu_1 + 1})^2\Big), \tag{5.6}$$

see for example Campbell et al. [13].

We assume that the mean return is the same for every year (although this may later be adjusted if returns are truncated), and the variances are given by the GARCH(1,1) specification. Given the expected mean return and variance for each state (t, s),

$t \in \mathcal{T}_0$, $s \in \mathcal{S}_t$ in the scenario tree, given by (5.5) and (5.6), we apply the same procedure as described in Section 5.2 to find numerical values which approximate a normal distribution. Given state (t, s), this gives us $branch_t$ values for $\tilde{r}_{1,t+1}^{s'}$, $t \in \mathcal{T}_0$, $s' \in \mathcal{K}_t^s(t+1)$. These $branch_t$ values are transformed to obtain simple net returns by means of the formula

$$r_{1,t+1}^{s'} = e^{\tilde{r}_{1,t+1}^{s'}} - 1. \tag{5.7}$$

Finally, we check whether the returns (5.7) satisfy their lower and upper bounds. See for the formulas of these bounds Appendix 5.A.4. If these bounds are not satisfied, the returns are truncated (and therefore the probability distribution is adjusted), so that these bounds are satisfied.

Probability of outperformance of stocks over bonds

As a result of the calculations above, we have, for any state (t, s), a set of precisely $branch_t$ values for the simple net return of stocks in the next year. Together with the (equal) conditional probabilities they represent the marginal distribution of $r_{1,t+1}$ given (t, s). Moreover, already in Section 5.3.1 a similar marginal distribution of the bond returns $r_{2,t+1}$ was derived. The question comes up: how to join these marginal distributions to a joint distribution? An easy way to do so is by assuming independence: then random assignments will be appropriate. But there are good reasons to assume that interdependencies between the returns of stocks and bonds are realistic. In the literature, special attention is paid to the probability that stock returns outperform bond returns, depending on the length t of the time period, see for example Bernstein [2], who concludes that in the long run, stocks are fundamentally less risky than bonds. It is argued, that this probability of outperformance increases with t. Indeed, according to H.A. Klein Haneveld [51] this probability is even equal to 1 if broadly diversified stock and bond portfolios are considered (as we do) over periods longer than 20 years.

We use a heuristic way to assign stock returns to nodes in the tree, taking into account the probability of outperformance. We would like to minimize the value of

$$\sum_{t=1}^{T} |P_t(r_1 \geq r_2) - P_t^*(r_1 \geq r_2)|, \tag{5.8}$$

where, for any t, $P_t(r_1 \geq r_2)$ denotes the probability of outperformance. It is calculated as

$$P_t(r_1 \geq r_2) = \sum_{s \in \mathcal{K}_0^s(t)} p_t^s \delta_{1t}^s$$

with

$$\delta_{1t}^s := \begin{cases} 1 & \text{if } \Pi_{q=1}^t (1 + r_{1q}^s) \geq \Pi_{q=1}^t (1 + r_{2q}^s) \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

Moreover, $P_t^*(r_1 \geq r_2)$ denotes the historical probability of outperformance of stock returns over bond returns over a period of t years.

We do not want to violate the following constraints in the minimization of 5.8, since these are more important in our opinion:

- The mean values and the variances of the stock returns over all sets of successors may not be changed.
- Bond returns may not be altered, because of the relationship with the yields.

Now we will describe the heuristic we use to assign stock returns to the nodes of the tree. We start with a random assignment. Then, we try to find a lower value of (5.8), given the current realizations of r^s_{1t} and r^s_{2t}. We do this by considering an interchange of two stock returns in two nodes with the same predecessor. Here comes the choice of the equal conditional probabilities into play. This choice for the conditional probabilities allows for an interchange of the returns on the stock portfolio, such that the conditions listed above are still satisfied: the marginal distributions of the stock returns remain unaltered in this case.

We consider the scenario tree from $t = 0$ to time $T - 1$. If $|P_{t+1}(r_1 \geq r_2) - P^*_{t+1}(r_1 \geq r_2)|$ is larger than $\frac{1}{|\mathcal{S}_t|}$, we consider the interchange of the stock returns in two states $(t+1, s^{'})$, $(t+1, s^{''})$, with $s^{'}, s^{''} \in \mathcal{K}^s_t(t+1)$. If it leads to a lower value of (5.8), this interchange is made. Note that given an interchange of two returns, all implications for future years have to be considered in the definition of δ^s_{1t} in (5.9).

In this way, all states before the horizon are considered. If an improvement is found, i.e. a lower value of (5.8) is obtained, the corresponding adjustments in the scenario tree are made, otherwise not.

5.3.3 Returns on real estate

Returns on the real estate portfolio are generated in the same way as returns on the stock portfolio. They are also assumed to be lognormally distributed. The variance of the continuously compounded returns are described by a GARCH(1,1)-process. Numerical values for the GARCH(1,1) specification for a world index for real estate can be found in Appendix 5.A.3.

We also tested whether an autoregressive term is present in the historical yearly returns on real estate. If one assumes that (5.4) describes the returns (with index 3 instead of 1), the estimate for ν_3 gets the value 0.076 for the given data. The corresponding Q-statistic and probability are given by 0.3744 and 0.541, respectively. We conclude that the first-order autoregressive term is not needed in generating returns on the real estate portfolio.

Given the specification of the mean and variance for the returns, we truncate the returns if the lower or upper bounds, as found by using the Gordon growth model [36], are not satisfied. Moreover, these returns are adjusted, so that they represent a lognormal distribution. Finally, the interdependencies with the bond returns are specified in a heuristic way to obtain realized probabilities of outperformance which are close to empirically observed ones.

5.3.4 No arbitrage

A very important concept in financial models, is the *no arbitrage condition*, see e.g. Pliska [78]. The existence of arbitrage in the data of portfolio models means that, without risk, money can be made from nothing (so that so-called *money machines* or *free lunches* are possible). That is, if arbitrage is present, it is possible to construct portfolios that finance themselves and give a sure win.

Arbitrage is a theoretical concept, which should be considered in financial optimization models in order to avoid spurious outcomes, because optimization models exploit these opportunities if they exist. Indeed, as Klaassen [50] shows, if scenarios are not arbitrage free, stochastic programming makes use of the arbitrage opportunities, leading to portfolios that are biased to spurious profit opportunities in a nonrealistic way. It is obvious that realistic scenarios for returns on investments should be arbitrage free. In the remainder of this section, we describe the arbitrage concept. This concept is not especially related to ALM. Therefore, we discuss arbitrage in a more general context. In particular we may consider an investment of zero.

We first consider arbitrage opportunities in a one-period sense. Given state (t, s), arbitrage opportunities are present if:

- the total amount invested in state (t, s) is equal to zero,
- in each state $(t+1, s')$, $s' \in \mathcal{K}_t^s(t+1)$ the value of the portfolio is nonnegative, and
- the expected next year's value of the portfolio is strictly positive. Therefore, in at least one state at time $t+1$ with (t, s) as parent, the value of the portfolio is strictly positive.

Dert [24] formulated a linear program to test for arbitrage opportunities. Another way to check for arbitrage opportunities is introduced by Harrison and Kreps [38]. There are no arbitrage opportunities if and only if numbers $\pi_{t+1}^{s'} > 0$ exist for the set of child nodes for each state (t, s), $t \in \mathcal{T}_0$ and $s \in \mathcal{S}_t$, such that the following system has a solution:

$$\sum_{s' \in \mathcal{K}_t^s(t+1)} (1 + r_{j,t+1}^{s'})\pi_{t+1}^{s'} = \sum_{s' \in \mathcal{K}_t^s(t+1)} (1 + r_{1,t+1}^{s'})\pi_{t+1}^{s'} \qquad j = 2, \dots, N, \tag{5.10}$$

$$\sum_{s' \in \mathcal{K}_t^s(t+1)} \pi_{t+1}^{s'} = 1, \tag{5.11}$$

$$\pi_{t+1}^{s'} > 0. \tag{5.12}$$

Such $\pi_{t+1}^{s'}$, $s' \in \mathcal{K}_t^s(t+1)$ are called *risk neutral probabilities*. If the returns $r_{j,t+1}^{s'}$, $j = 1, \dots, N$, $s' \in \mathcal{K}_t^s(t+1)$ are given, one can test whether arbitrage opportunities exist by solving a linear program that maximizes the value of ϵ while satisfying constraints (5.10) and (5.11), and replacing constraint (5.12) by

$$\pi_{t+1}^{s'} \geq \epsilon.$$

Given the values of $r_{j,t+1}^{s'}$, there are N linear equalities in $branch_t$ unknown probabilities and $branch_t$ positivity constraints in order to rule out arbitrage opportunities in year $t+1$.

In the context of our ALM model, arbitrage implies that current investments could be extended for free. To be sure that no arbitrage opportunities exist in the multiperiod sense, one has to consider the whole scenario tree at once. Given a scenario tree which is not arbitrage free, it is very difficult to improve the realizations of the stochastic variables, so that the tree becomes arbitrage free. Because we do not expect that we could implement testing for arbitrage opportunities in reasonable time, we ignored this item in the numerical realization of scenario trees.

Fortunately, if the number of branches increases, whereas the number of asset classes remains the same, the probability that arbitrage opportunities are present decreases. This can be seen in model (5.10), (5.11), and (5.12), because more decision variables $\pi_{t+1}^{s'}$ are present in this case. Dert [24] also shows numerically that the probability that arbitrage opportunities are present decreases rapidly if the number of scenarios increases.

5.4 Liabilities, benefit payments, discount rates, and wages

In this section we describe how we obtain numerical values for $\underline{L}_t^s$, $\overline{L}_t^s$, $\underline{B}_t^s$, $\overline{B}_t^s$, γ_t^s, and W_t^s. First, we start with answering the question how we get values for the lower and upper bounds on the initial value of the liabilities, based on observed market prices. Then we will describe how we get values for the lower bounds $\underline{L}_t^s$ for all remaining states (t, s), $t \in \mathcal{T}_1$, $s \in \mathcal{S}$.

After it is made clear how the values for $\underline{L}_t^s$ are generated, we describe how values for $\overline{L}_t^s$ are found. Finally, we concentrate on the stochastic parameters $\underline{B}_t^s$ and $\overline{B}_t^s$, the lower and upper bounds on the benefit payments, the discount rates γ_t^s, and the level of the wages W_t^s.

Initial value of the liabilities

The initial value of the liabilities is by definition the present value of the nominal expected future benefit payments of the current built-up rights. A large Dutch pension fund provided us data with respect to these nominal expected benefit payments, and also data with respect to developments of the built-up rights in the next years. An example of these expected nominal future benefit payments is given in Figure 5.1. From this figure we see that it is expected that in the near future more people will retire, and as a result, expected future benefit payments will increase. In the years thereafter, only a fraction of the people survive that year and therefore, we get the (long) tail to the right. As will become clear in this chapter, the data we obtained will be transformed.

The major issue to determine the current value of the liabilities, is the choice of the discount factors to be used. Of course, the board of a pension fund cannot determine these discount factors themselves; instead, prescriptions with respect to these rates by the supervisor have to be satisfied. In The Netherlands, pension funds used to value their liabilities using a (prescribed) fixed discount rate, that is, one rate for all future years. However, as already mentioned in Section 1.2, the Dutch supervisor of pension funds wants that the liabilities are valued, such that a good judgement can be make about the financial position of a fund, see the discussion

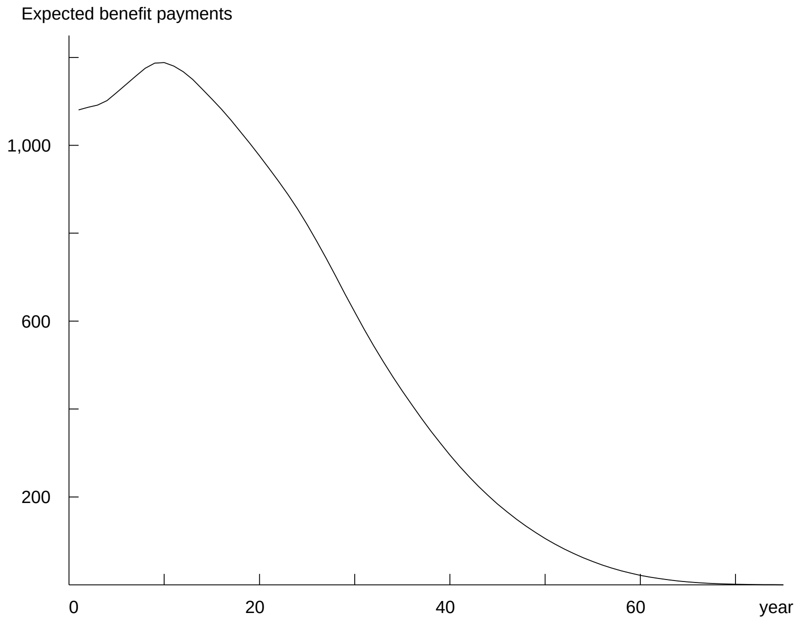

Figure 5.1: Expected future benefit payments, based on the current built-up rights (of a large Dutch pension fund), contained in the liabilities of year 0.

paper [75]. Since the assets are valued based on observed market prices, it is therefore better to use market values to value the liabilities too. In the discussion paper [75], the supervisor of the pension funds in The Netherlands writes that a market value for the liabilities should be found.

Although the concept of finding a market value of the liabilities seems easy (the fund ‘only’ has to discount future cash flows), it is far from trivial to answer the question which discount factors should be used to value the liabilities. It is important that these discount factors are based on the prices in the capital markets at the moment one wants to find the market value. One possibility to discount future benefit payments is to use risk free rates. However, H.A. Klein Haneveld [51] suggests not only to use risk free rates, but also to use internal rates of returns on stocks and bonds in the discount factors. He gives two reasons why to use the internal rates:

- Time diversification. This means that above average returns tend to offset below average returns over long horizons, see Kritzman [56].

- Returns on stocks and bonds. As we have seen in Section 5.3.2, Bernstein [2] concluded that in the long run, stocks are fundamentally less risky than bonds.

Therefore, H.A. Klein Haneveld [51] suggests to take into account the maturity of the expected future benefit payments to decide if returns on stocks should be considered into the discount factors. We also used this approach to discount future benefit payments. Now, we will describe how we implemented this approach.

In the discussion how returns on bonds are generated, we have already introduced yield curves. Moreover, we used the *internal rate of return* of the stock portfolio in generating stock returns by considering the Gordon growth model (see Appendix 5.3.2). Next to the yield curve, we introduce two additional curves: the *stock spot curve*, and the *pension spot curve*. These two curves are based on H.A. Klein Haneveld [51]. One point on the stock spot curve is the internal rate of return of stocks. We assume that this curve and the yield curve run parallel to each other. The difference between these two curves is the so-called *equity risk premium*.

The third curve is the pension spot curve. This is a combination of the previously defined two spot curves. The pension spot curve is used to discount expected future benefit payments. Because returns of a broadly diversified stock portfolio always outperformed a broadly diversified bond portfolio if the considered period was longer than 20 years (see H.A. Klein Haneveld [51]), we use the stock spot curve to discount cash flows with a maturity of 20 years or more. Because investments in stocks with shorter maturities are exposed to more risk, a combination between stock and bond investments is considered to discount cash flows with maturities less than 20 years. For simplicity, we let the fraction of stocks increase linearly in the first 20 years to obtain discount rates. In state (t, s), the discount rate q years from year t on the pension spot curve, denoted by $PSC_t^s(q)$, is defined as follows:

$$PSC_t^s(q) := \begin{cases} y_t^s(q) + \frac{q}{20} earp_t^s & \text{if } 0 \leq q \leq 20 \\ y_t^s(q) + earp_t^s & \text{if } q \geq 20, \end{cases}$$

where $earp_t^s$ denotes the *ex-ante equity risk premium*. The numerical values of the ex-ante risk premiums vary from state to state. These values depend on previously realized returns, the growth rate and the levels of the dividends, and the yield curve. In Appendix 5.A.4, the formulas of these ex-ante risk premiums are given. A typical example of these three curves, is given in Figure 5.2.

The discount rates used in our model, which are denoted by γ_t^s, are the values on the pension spot curve as determined at time 0. We use these discount rates to get consistency in computing financial cash flows at all times, just as in valuing liabilities.

Generating lower bounds on the value of the liabilities

At time 0, the lower bound on the value of the liabilities, denoted by $\underline{L}_0^1$, is given by

$$\underline{L}_0^1 := \sum_{q=0}^{\infty} \frac{B_0^*(q)}{\Pi_{q'=1}^q (1 + PSC_0^1(q'))},$$

where $B_0^*(q)$ is the current expected benefit payment over q years. In Figure 5.1 the values of $B_0^*(q)$ are presented. Moreover, $PSC_0^1(q)$ denotes the pension spot curve at time 0, for all maturities $q \geq 1$.

Numerical values for $\underline{L}_t^s$ for all states (t, s) with $t \geq 1$, are found in a similar way. Future values $B_t^*(q)$, which denote the conditional expected benefit payments in year $t+q$, given time t, were obtained from a large Dutch pension fund for all future

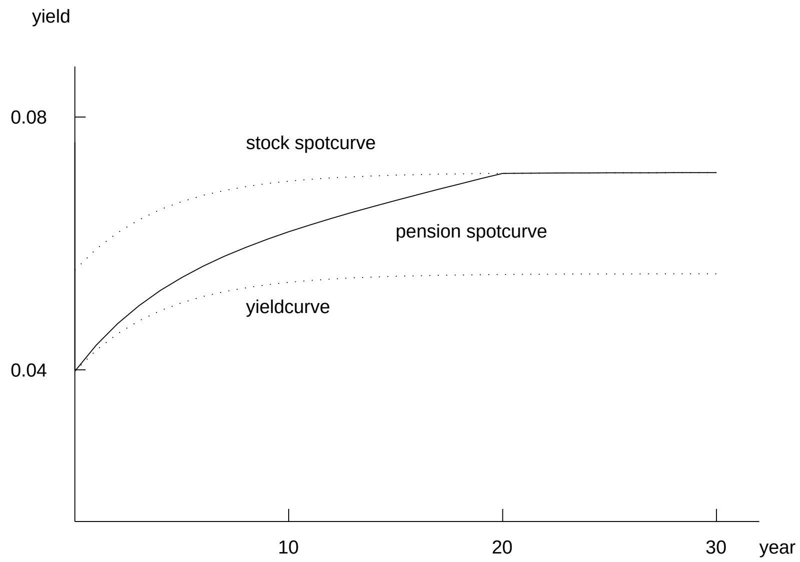

Figure 5.2: The pension spot curve, the curve used to discount future cash flows, is composed of the yield curve and the stock spot curve.

times $t \in \mathcal{T}_1$. In addition, we have calculated the pension spot curves $PSC_t^s(q)$ in every state of the scenario tree. Appropriate discounting gives numerical values for $\underline{L}_t^s$ for all (t, s):

$$\underline{L}_t^s := \sum_{q=t+1}^{\infty} \frac{B_t^*(q)}{\Pi_{q'=1}^{q}(1 + PSC_0^1(q'))}.$$

Upper bound on the value of the liabilities

The upper bound on the value of the liabilities in state (t, s) is found by increasing the expected future benefit payments with observed accumulated increases in the general wage level:

$$\overline{L}_t^s = \underline{L}_t^s \Pi_{q=0}^t (1 + w_q^s) \qquad t \in \mathcal{T}, s \in \mathcal{S}.$$

We see that the upper bound on the value of the liabilities is found by always giving full indexation to the participants of the fund.

Lower and upper bound on the benefit payments

The lower bound $\underline{B}_t^s$ on the benefit payments in state (t, s), is equal to $B_t^*(0)$, the nominal benefit payment. The corresponding upper bounds, $\overline{B}_t^s$, are equal to their lower bound, increased with the observed cumulative increase in the general wage level.

Wages

We obtained data for the expected level of the pensionable wages of the active participants for the next years. In our ALM model, it is assumed that this level is the same for all scenarios at time t. Ideally, one would like these levels to be state dependent, so that the interactions between the values of the assets and liabilities are reflected even better. However, obtaining appropriate values for W_t^s would require information with respect tot the development of the pensionable wages in each state of the scenario tree. These developments require the introduction of business scenarios, because the total pensionable wages depends for example on the number of employees of the company. This number is correlated with other realizations in the scenario tree, because they all depend on the state of the economy. In the absence of the additionally required data to calculate W_t^s we let the values of W_t^s only depend on t and not on s, so that for each state at time t the same value is used.

5.5 Future research

In this chapter, we have seen that scenarios are generated by considering dependencies between stochastic parameters. Moreover, both empirical and theoretical reasons were given to justify the used approaches. These have a number of desirable properties. For example, returns on the bond portfolio were generated, while keeping in mind that the implied yields do not become negative.

Although this seems nice at first sight, we will give here some suggestions for improvements. Because of a time constraint, we were not able to implement these refined ideas yet.

First of all, relationships between returns on stocks, bonds, and real estate should be considered in more detail. Although these relationships are considered by means of probabilities of outperformance (and this approach is consistent with the way the liabilities are valued), one may explicitly use empirically observed correlations between the returns. An alternative way to consider relationships between returns, which also makes use of historical data, is to apply principal component analysis. Another alternative is to consult an expert, who may choose 'good' combinations from a number of alternatives. One can also use a heuristic to match moments, so that also for example the skewness and kurtosis of the marginal distributions are taken into account, see Høyland et al. [42].

In the generation of bond returns, some assumptions were made for simplicity. In order to find possible future bond returns for pension funds, some of these assumptions may be relaxed. For example, one should include the possibility to invest not only in risk-free non-callable bonds. Moreover, nonparallel shifts of the yield curve should be considered.

Another important issue is the no arbitrage concept. Although the probability of arbitrage opportunities decreases if the tree becomes more bushy, it would be nice to generate returns such that it is guaranteed that these opportunities are excluded.

5.A Appendix: Mathematical details

In this appendix, we discuss mathematical backgrounds regarding concepts underlying our scenario generator. We first describe the error-correction model which we use to model the relationship between the returns on the bank account and changes in the general wage level. Then, we will consider bond returns. In Section 5.A.3, we discuss GARCH(1,1)-models. Finally, we describe how we find bounds on stock returns and the implications on the ex-ante risk premium.

5.A.1 Error-correction model

In this section, we describe the error-correction model which is used to model the interdependency between the returns on the bank account and the change in the general wage level (see Section 5.2).

The cointegrating equation is described by

$$w_t = \chi r_{4t}, \tag{5.13}$$

and can be interpreted as a long-run equilibrium relationship between the short-term risk-free interest rate and the change in the general wage level. The parameter χ gives the proportionality between r_{4t} and w_t in the long run. The error-correction model is given by

$$\Delta r_{4t} = \vartheta_1 (w_{t-1} - \chi r_{4,t-1}) + \epsilon_{4t} \tag{5.14}$$

$$\Delta w_t = \vartheta_2 (w_{t-1} - \chi r_{4,t-1}) + \epsilon_{wt}, \tag{5.15}$$

with $\epsilon_{4t} \sim \mathcal{N}(0, \sigma^2_{\epsilon 4})$ and $\epsilon_{wt} \sim \mathcal{N}(0, \sigma^2_{\epsilon w})$, and $\epsilon_{4t}, \epsilon_{wt}$ are all mutually independent. In equalities (5.14) and (5.15), Δ denotes a change, that is $\Delta r_{4t} := r_{4t} - r_{4,t-1}$, and $\Delta w^s_t := w^s_t - w^s_{t-1}$. In addition, the parameters ϑ_1 and ϑ_2 are measures for the speed of adjustment.

The error-correction model described above forces r_{4t} and w_t to converge in the long run to their cointegrating relationship, while allowing a wide range of short-run (and randomly) dynamics. Therefore, deviation from the long-run equilibrium is corrected gradually through a series of partial short-term adjustments.

We use risk free interest rates from 1981 to 2001 from Datastream [20], and data of changes in the general wage level of the same years from the website of the Dutch central bureau of statistics, CBS [16] to estimate the equations (5.13), (5.14) and (5.15). To do so, we use the quasi-maximum likelihood method, as described by Bollerslev et al. [10]. The numerical values are presented in Table 5.3. These parameter estimates imply the following. The estimated value of χ means that there is a positive relationship between the short-term risk-free interest rate and the change in the general wage level. This is also what we expected, because both r_{4t} and w_t are influenced by the inflation rate.

The negative signs of ϑ_1 and ϑ_2 are also consistent with economic theory. If r_{4t} is above the long-run relationship (5.13), relatively high wage increases lead to a relatively high value of the inflation. As a result, the interest rate will increase, since this rate consists of the inflation and a real part. Therefore, it is to be expected that r_{4t} will increase. If r_{4t} is above the long-run relationship, w_t is of course below

parameter	estimation	t-value
χ	1.320	6.38
ϑ_1	-0.368	-2.83
ϑ_2	-0.390	-2.49
σ_4^2	0.01323	
σ_w^2	0.0204	

Table 5.3: Estimated parameter values of the error-correction model.

this relationship. In this case, the government may call for a wage restraint, since further high wage increases will harm the economy.

On the other hand, if r_{4t} is below the long-run relationship (5.13), low inflation may result in a decreasing risk-free interest rate. This may for example be the case if economic growth is very low (or even negative): the central bank will decrease the risk-free interest rate to stimulate the economy. Moreover, w_t is below the long-run relationship with r_{4t}. In this case, it is to be expected that unions claim higher wages.

5.A.2 Bond returns

In this section we describe the details about generating bond returns (see Section 5.3.1).

Price of a bond portfolio

We assume that future payoffs, i.e. coupon payments and principal payments of the bond portfolio at time 0 are known. Moreover, we assume that these payments are made with certainty in the future. Therefore, we assume that these are risk-free and that non-callable bonds are considered.

Given state (t, s), portfolio coupon and principal payments, which are made q years ahead, are denoted by $C_t^s(q)$ and $PrB_t^s(q)$, respectively. The initial values, i.e. at time zero, have to be specified by the user. These values describe the initial bond portfolio of the pension fund under consideration.

The value of the bond portfolio at time t in scenario s, denoted by PB_t^s, is (by definition) given by

$$PB_t^s = \sum_{q=1}^{\infty} \frac{C_t^s(q) + PrB_t^s(q)}{\Pi_{q'=1}^{q}(1 + y_t^s(q'))}.$$

This gives us the price of the bond portfolio in each state (t, s), given the future stream of coupon and principal payments.

As time passes, the duration (the expected mean term) of the bond portfolio decreases. To avoid this, new zero-coupon bonds with maturity strictly greater than the desired duration are bought. The number of newly acquired bonds is chosen, so that the duration is exactly equal to its desired level.

Generating bond returns

We have explained how the price of the bond portfolio changes, given the desired level of the duration and the movements in the yield curve. Now, we are able to generate bond returns. By definition, the return on the bond portfolio in state (t, s) is given by

$$r_{2t}^s = \frac{C_{t-1}^s(1) + PrB_{t-1}^s(1) + PB_t^s}{PB_{t-1}^s} - 1.$$

That is, the return is based on the coupon and principal payments made in state (t, s), and on the current and previous (market) values of the bond portfolio.

If one would like to add also more risky bonds to the bond portfolio, one may change the corresponding payments in expected payments and increase the discount rates appropriately.

5.A.3 GARCH(1,1) models

A GARCH(1,1) model for the continuously compounded returns on stocks, as discussed in Section 5.3.2, are defined as follows:

$$\tilde{r}_{1t} = \tilde{\mu}_1 + \varrho_{1,t+1} \tag{5.16}$$

$$\varrho_{1,t+1} = \tilde{\sigma}_{1t}\varsigma_{1,t+1}, \tag{5.17}$$

$$\tilde{\sigma}_{1t}^2 = d_{11} + h_{12}\tilde{\sigma}_{1,t-1}^2 + h_{11}\varrho_{1t}^2 \tag{5.18}$$

where $\varsigma_{1,t+1} \sim \mathcal{N}(0,1)$ is assumed to be standard normally distributed. As a result, $\varrho_{1,t+1} \sim \mathcal{N}(0, \tilde{\sigma}_{1t}^2)$. Moreover, the $\tilde{}$-sign denotes that continuously compounded returns are used.

In this GARCH(1,1) model, $\varrho_{1,t+1}$ can be interpreted as an innovation: it has mean zero, conditional on time t information. In addition, $\tilde{\sigma}_{1t}^2$ is the time t conditional variance of $\varrho_{1,t+1}$ or, equivalently, the conditional expectation of $\varrho_{1,t+1}^2$. This means that (5.18) can be rewritten as follows:

$$\tilde{\sigma}_{1t}^2 = d_{11} + (h_{11} + h_{12})\tilde{\sigma}_{1,t-1}^2 + h_{11}(\varrho_{1t}^2 - \tilde{\sigma}_{1,t-1}^2). \tag{5.19}$$

In equality (5.19), $(\varrho_{1t}^2 - \tilde{\sigma}_{1,t-1}^2)$ has mean zero, conditional on time $t-1$ information, and can be thought of as the shock to volatility. The extent to which a volatility shock this year feeds through into next year's volatility is given by h_{11}, while $(h_{11} + h_{12})$ measures the rate at which this effect dies out over time.

Estimating the parameters of the GARCH(1,1) model for the returns on the MSCI World-index in the period 1967-2002 using the maximum likelihood method, gives the following results. $\tilde{\mu}_1 = 0.075$, $d_{11} = 0.013$, $h_{11} = 0.251$, $h_{12} = 0.533$.

We also estimated the parameters of the GARCH(1,1)-model for returns on real estate. We used the returns of a world index for real estate from 1970 to 2002. These data were derived from Datastream [20]. The estimates of the parameter values are $\tilde{\mu}_3 = 0.068$, $d_{31} = 0.012$, $h_{31} = 0.267$, and $h_{32} = 0.487$.

5.A.4 Bounds on stock returns

The Gordon growth model [36] mentioned in Section 5.3.2 gives the relation between the price of a stock(portfolio), the dividend payments of this stock(portfolio), and a growth rate of the dividend payments. It is assumed that the discount rates used to discount future dividend payments, is the same for all time periods. The price of a stock(portfolio) at time t, denoted by PS_t, is determined by its future stream of dividends:

$$PS_t = \mathbb{E}_t[\sum_{q=1}^{\infty} \frac{D_{t+q}}{(1+R_1)^q}], \tag{5.20}$$

where $\mathbb{E}_t$ denotes the conditional expectation given the state at time t (e.g. given D_t), and D_{t+q} denotes the dividend payment q years ahead, and $R_1 = \mathbb{E}_t[r_{1q}]$, $q \geq t+1$ so R_1 denotes the assumed constant expected stock return, also called *internal rate of return*.

Since the dividends are assumed to grow at a constant rate g_1 (which is less than R_1, a prerequisite to keep the stock price finite), the following relationship holds:

$$\mathbb{E}_t[D_{t+q}] = (1+g_1)\mathbb{E}_t[D_{t+q-1}] = (1+g_1)^q D_t. \tag{5.21}$$

Substituting (5.21) into (5.20), the well-known Gordon growth model is obtained:

$$PS_t = \frac{(1+g_1)D_t}{R_1 - g_1}. \tag{5.22}$$

Although the growth rate of stock dividends may fluctuate much for individual stocks, this figure is rather stable for stock portfolios, see Jagannathan et al. [45]. This growth rate was approximately 5 percent per year in the period 1927-1999. The reason why many companies prefer to pay stable, but increasing dividends (even though profits fluctuate significantly), is that the company seems financially sound, see Smith [90].

We will explain now what implication the Gordon growth model has on the stock returns in the scenario tree. For that reason, (5.22) is reformulated as follows:

$$R_1 = \frac{(1+g_1)D_t}{PS_t^s} + g_1. \tag{5.23}$$

Given the numerical values for PS_0, D_0 and g_1, we can find R_1. As explained in the beginning of this section, the ex-ante risk premium is assumed to be positive, and finite. The lower and upper bounds on this ex-ante risk premium are denoted by $\underline{earp}$ and $\overline{earp}$, respectively, and are assumed to be time independent. We require

$$r_{4t}^s + \underline{earp} \leq R_1 \leq r_{4t}^s + \overline{earp}. \tag{5.24}$$

Therefore, the implied internal rate of return of the stock portfolio is related to the one-year risk-free interest rate, r_{4t}^s, and bounds on the ex-ante risk premium.

Now, we explain how bounds on the returns on the stock portfolio are derived. The price of the stock portfolio in state (t, s) is by definition given by

$$PS_t^s = (1 + r_{1t}^s)PS_{t-1}^s. \tag{5.25}$$

Because of (5.23) and (5.24), we obtain the upper bound on PS_t^s as follows:

$$\frac{(1+g_1)D_t}{PS_t^s} \geq r_{4t}^s + \underline{earp} - g_1,$$

or

$$PS_t^s \leq \frac{(1+g_1)D_t}{r_{4t}^s + \underline{earp}} - g_1.$$

Given definition (5.25), we obtain the upper bound on r_{1t}^s:

$$(1 + r_{1t}^s)PS_{t-1}^s \leq \frac{(1+g_1)D_t}{r_{4t}^s + \underline{earp} - g_1},$$

$$r_{1t}^s \leq \frac{(1+g_1)D_t}{(r_{4t}^s + \underline{earp} - g_1)PS_{t-1}^s} - 1, \tag{5.26}$$

where PS_{t-1}^s is known at time t.

Analogously, one can find the lower bound on the stock returns:

$$r_{1t}^s \geq \frac{(1+g)D_t}{(r_{4t}^s + \overline{earp} - g)PS_{t-1}^s} - 1. \tag{5.27}$$

If a return on the stock portfolio does not satisfy (5.26) or (5.27), in our scenario generator we truncate it, so that these bounds are forced. This means that the distribution of the stock returns is adjusted.

Ex-ante risk premium

Given the Gordon growth model, we would like to find its implications on the ex-ante risk premium in state (t,s), denoted by $earp_t^s$. By definition, we have

$$earp_t^s := R_{1t}^s - y_t^s(1), \tag{5.28}$$

where R_{1t}^s denotes the internal rate of return of the stock portfolio in state (t,s). Using the definition of R_{1t}^s, as given by (5.23), we immediately obtain the formula for $earp_t^s$:

$$earp_t^s = \frac{(1+g_1)D_t}{PS_t^s} + g_1 - y_t^s(1).$$

From this equation we see that its value depends on PS_t^s and, as a result, on returns r_{1q}^s, $q = 1, \ldots, t-1$. In addition, $earp_t^s$ also depends on g_1, D_t and the expected next year's return on the bank account.