Asset liability management for pension funds using multistage mixed-integer stochastic programming

Drijver, S.J.

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Chapter 2

ALM model

In the previous chapter, we have described what a pension fund is, and what ALM for pension funds is. In this chapter (the main part of) an ALM model is described. After describing the decision process and the characteristics of our model, scenarios and decisions are introduced. Then, the mathematical modeling is described in detail. This chapter focuses on the objective function and the constraints. Additional constraints will be discussed in detail in Chapter 3; these so-called risk constraints deal with the shortage after one year. The complete mathematical model is contained in Appendix A.

2.1 The ALM decision process

The goal of pension funds is to fulfill all obligations towards the participants. In this section we describe the decision process in the way it will be incorporated in the ALM model to be discussed.

In such models, the user has to fix a *planning horizon* which specifies the total number of years which are considered in the decision making process. The planning horizon is split into subperiods of one year. In every year, benefit payments are made, premiums are received, and changes in the status of the participants are recorded appropriately. At the end of the year, the board also knows the return of the asset portfolio. The value of the assets is determined using market prices at that moment. Moreover, at that time the fund makes an actual estimate of its liabilities. Once these two numbers are known, the level of the *funding ratio* (the ratio between the values of the assets and the liabilities) is determined. This funding ratio is an important performance measure: it indicates the actual financial position of the fund. It is compared to the development in the previous years to judge the effects of the actual strategy.

When all last year's information is revealed, the board looks forward: what are the expectations with respect to the future? For example, expectations with respect to future returns or developments of the inflation level may be adjusted.

Given the financial position of the fund at the end of a year, and possibly adjusted expectations, the board should make certain decisions. These (adjustments of) decisions aim at a sufficiently high future funding ratio, given the situation at the decision moment. One possible adjustment is to change the composition of the asset portfolio. For example, when the funding ratio increased last year to a relatively high level, the fund may consider to invest a larger fraction of its assets in asset classes with a high expected return (even though the associated risk may also be higher). In this case, the restrictions with respect to the composition of the asset portfolio should be kept in mind. Such restrictions may be imposed by the regulator, or may be established statutorily.

Another possible adjustment is to change the contribution rate. If the funding ratio decreased last year, the (board of the) pension fund may consider to increase the contribution rate. On the other hand, if the solvency of the fund increased last year, a decrease in the contribution rate may be considered. In the decision to change the level of the contribution rate, lower and upper bounds on this rate should be taken into account. Moreover, in determining the level of next year's contribution rate, it should be kept in mind that a rapidly changing contribution rate may be undesirable.

If at a decision moment the funding ratio is below a minimum required level, it is assumed that the sponsor has to pay a remedial contribution. This may for example be the case if the supervisor orders the board to undertake action on behalf of the participants of the fund. Such a remedial contribution in case of underfunding may also contractually be determined between the fund and the sponsor.

At the end of a year, also the level of last year's inflation is known. Given this level, and the solvency position of the fund at that moment, the board may decide to adjust future benefit payments entirely (full indexation), only partially, or not at all. This decision immediately influences the level of the benefit payments of the current old aged.

The decisions the board of a fund has to make are influenced by the interests of different parties involved in the decision making process. Moreover, decisions should be made such that unfavorable circumstances will be avoided as much as possible in the future. Decisions have to satisfy constraints on the level of the contribution rate, on the composition of the asset portfolio, on the values of the liabilities and benefit payments, and on an upper bound on the expected next year's shortage.

In determining which decisions to make, most recent information with respect to uncertain future circumstances will be taken into account. It is also kept in mind that in future years adjustments of the decisions can be made, when new information is revealed.

Optimization with respect to the decision variables in models which take into account multiple decision moments and uncertainty is called *multistage stochastic programming*. To formulate such models, *scenario trees* have to be used. A tree gives a collection of possible future developments of uncertain elements, like returns on assets and inflation.

Unfavorable circumstances, which the board would like to avoid, are for example large changes in the contribution rate for active participants, remedial contributions and not giving full indexation. These decisions are undesirable, but they are not ruled out: they may be necessary to avoid an even worse event: *underfunding*. Constraints which allow unfavorable events are called *soft constraints*, as opposed to *hard constraints*, which always have to be satisfied. To make these soft constraints meaningful, so-called *penalty parameters* are introduced. These parameters serve to penalize undesirable events. The penalty costs incurred by such undesirable events are balanced in the objective function with 'real' funding costs, as will become clear in the description of our ALM model.

As will also become clear later in this chapter, the ALM model described in this thesis is a *multistage mixed-integer stochastic program*. The integer variables appear into the model as indicators of the unfavorable events mentioned above, needed for a correct introduction of the penalty parameters in the model. Moreover, the integer variables are used to model flexible risk measures: the sponsor is obliged to make a remedial contribution if the funding ratio is below a minimum required level in a number of consecutive years.

Before we describe the ALM model mathematically, we will describe the characteristics of the model in more detail in the next section.

2.2 Characteristics of the ALM model

The ALM model described in this thesis is an optimization model, and therefore it is formulated as a set of constraints and an objective function. It describes the decision process the pension fund has to deal with. The user, which is assumed to be the board of a pension fund can (and should) specify certain preferences.

In our ALM model, some constraints serve for a correct bookkeeping. In these constraints, cash inflows and outflows are registered. Besides, constraints on the portfolio mix are present in the ALM model. These constraints deal with the composition of the asset portfolio: the fraction of the assets invested in each asset class has to satisfy lower and upper bounds specified by the board or regulator.

The model also contains constraints which deal with underfunding and a possible remedial contribution by the sponsor of the fund. Both underfunding and remedial contributions are penalized by means of fixed penalty costs. These (fictitious) fixed penalty costs are incorporated into the model to express the undesirability of certain events.

In our ALM model, we use the following policy rules. If the funding ratio is below a prespecified level, fixed penalty costs are incurred. If this observed underfunding means that this ratio is below the minimum required level in a (prespecified) number of consecutive years, we assume (as one of the decision rules) that the sponsor is forced to restore the funding ratio. If the funding ratio falls even below a predefined level (which is lower than the threshold value considered in defining underfunding), an immediate contribution from the sponsor is required. Next to these rules, we also introduce the policy rule that the marginal costs associated with large remedial contributions (which are payments above a certain fraction of the total pensionable salaries) are higher than the corresponding costs associated with low ones. Moreover, in our ALM model the sponsor is only willing to make a remedial contribution if the funding ratio is below a minimum required level.

The model described in this thesis also uses policy rules with respect to the contribution rate. It is assumed that the level of the contribution rate, which is expressed as a fraction of the total pensionable salaries, has to satisfy lower and upper bounds. Because a highly volatile contribution rate is not appreciated (at least by participants in case of large increases and by the supervisor in case of large decreases), large increases and decreases are penalized (although they are allowed if the lower and upper bounds are satisfied). In case of a remedial contribution, a minimum level of the contribution rate is required.

Our ALM model also takes into account the indexation of pension rights. The decision whether or not to increase the pension rights of participants of the fund for increases in wages or prices will be an outcome of the model. Since not giving full indexation is undesirable, (fixed) penalty costs will be imposed in this case. However, all contractually determined minimum benefit payments have to be made in time.

Because the ability to fulfill obligations is a central issue in asset liability management, additional constraints which deal with the *risk of underfunding* are considered. These constraints impose restrictions on the contribution rates and compositions of asset portfolios, such that the expected next year's shortage is sufficiently small. These constraints will be discussed in Chapter 3.

In our model also *overfunding* is considered. This is a situation in which the pension fund has a large surplus. In case of overfunding, the board of the fund may consider to transfer money back to the sponsor. Such a *restitution* is forced if overfunding is present in a prespecified number of consecutive years.

We assume that the board of the pension fund under consideration makes decisions, while keeping in mind the long-run desire to stay (or become) solvable. To do so, a target level of the funding ratio at the horizon of our decision model is introduced. Surplusses and shortages with respect to this level at the horizon are rewarded and penalized respectively.

Before we describe these characteristics of our ALM model in more detail, we first introduce scenarios. In addition, we explain what the decision variables in our ALM model are.

2.3 Scenarios and decisions

We assume that the ALM model has a horizon T years from now. The resulting years are denoted by an index t, where time 0 is the current time. By year t ($t = 1, \ldots, T$), we mean the span of time $[t-1, t)$.

At each time $t \in \mathcal{T}_0 := \{0, 1, \ldots, T-1\}$, the pension fund is allowed to make decisions, based on the actual knowledge of parameters. For example, given that last year's returns on the different asset classes are known, the fund may change its asset mix. Time t is assumed to be the end of the financial year t. We assume that a financial year coincides with a calendar year.

One way of modeling uncertainty of parameters in an optimization model is through a large but finite number S of scenarios. Each scenario represents a possible realization of all random parameters in the model. To be specific, let ω_t represent the vector of random parameters whose values are revealed in year t. Then, the set of all scenarios is the set of all realizations $(\omega_1^s, \ldots, \omega_T^s)$, $s \in \mathcal{S} := \{1, 2, \ldots, S\}$, of $(\omega_1, \ldots, \omega_T)$. Scenario s has probability p^s, where $p^s > 0$ and $\sum_{s=1}^S p^s = 1$. Since in a dynamic model information on the actual value of the random parameters is revealed in stages, a suitable representation of the set of scenarios is given by a

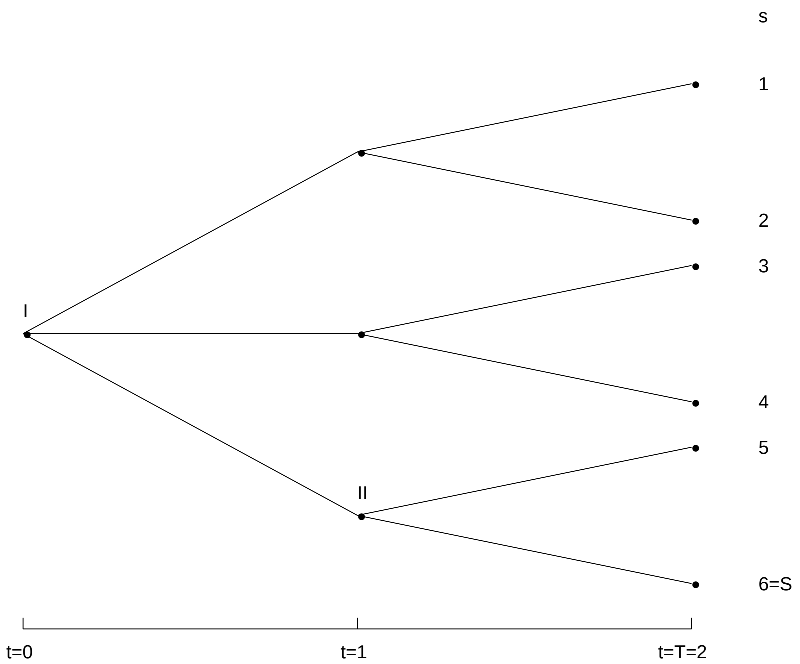

Figure 2.1: Example of a scenario tree.

scenario tree. In Figure 2.1 an example of a scenario tree is presented; in this tree, $T = 2$ and $S = 6$. Each path from $t = 0$ to $t = T$ represents one scenario. Any node of the tree, corresponding to time t, symbolizes a possible *state* at time t, represented by the observed values of $\omega_1, \ldots, \omega_t$. The *branches* directly to the right of it symbolize the various values of ω_{t+1}, given the realization of $\omega_1, \ldots, \omega_t$. Obviously, all scenarios passing this node have the same history in the years $1, \ldots, t$.

Now, we will explain the representation of states in the scenario tree. We denote the number of branches from each state at time t to states at time $t + 1$ by $branch_t$. The structure of the scenario tree is completely determined by the values of $branch_t$, $t = 0, \ldots, T - 1$. Given these values, the total number of scenarios, S, is known. This number is given by

$$S := \Pi_{t=0}^{T-1} branch_t,$$

and gives the total number of end-nodes at time T. Every end-node corresponds to one path starting at time 0, hence, with one scenario.

We now describe how the S scenarios in the tree are numbered, given the structure of the tree. Every branching from a parent node (also called predecessor) to the $branch_t$ successors gets a certain order. This order is determined arbitrary. As a result, every end-node (and therefore also every scenario) is uniquely indicated by the series of branching indices $(i_0, i_1, \ldots, i_{T-1})$, with $i_t \in \{1, \ldots, branch_t\}$, $t = 0, \ldots, T - 1$, and where i_t is the index of the branch at the node at time t. Now, the scenarios are numbered $1, \ldots, S$, corresponding to the lexicographical ordering of the successive branchings. In Table 2.1 the relation between the scenario-index and the branching structure is given.

Scenario index s	branching indices $(i_0, \ldots, i_{T-1})$
1	$(1, 1, \ldots, 1)$
2	$(1, 1, \ldots, 2)$
$\vdots$	$\vdots$
$branch_{T-1}$	$(1, 1, \ldots, 1, branch_{T-1})$
$branch_{T-1} + 1$	$(1, 1, \ldots, 2, 1)$
$\vdots$	$\vdots$
$S - 1$	$(branch_0, branch_1, \ldots, branch_{T-2}, branch_{T-1} - 1)$
S	$(branch_0, branch_1, \ldots, branch_{T-2}, branch_{T-1})$

Table 2.1: Relation between the scenario-index and the branching structure.

Example 2.1

We will explain the above introduced notation by means of the scenario tree which is depicted in Figure 2.1. In this tree, we have $branch_0 = 3$, $branch_1 = 2$, $T = 2$, and $S = 6$. In Table 2.2, for every scenario $s = 1, \ldots, S$ its series of indices (i_0, i_1) are given. □

Scenario index s	branching (i_0, i_1)
1	$(1, 1)$
2	$(1, 2)$
3	$(2, 1)$
4	$(2, 2)$
5	$(3, 1)$
6	$(3, 2)$

Table 2.2: Relation between the scenario-index and the branching structure in Example 2.1.

At this point, we will describe how the states at time t are indicated. The node of scenario s at time t is uniquely determined, and can be denoted by (t, s), $t \in \mathcal{T}$, $s \in \mathcal{S} = \{1, \ldots, S\}$. Often, this notation is sufficient, but sometimes it is not. Especially when for all nodes at time t something has to be calculated, duplicate work can be avoided if one only considers different nodes. Indeed, two scenarios have the same node at time t, exactly when they share the same path from time 0 to t. Therefore, at time t ($t \in \mathcal{T}$), there are

$$\Pi_{q=0}^{t-1} branch_q$$

different nodes, each with a different history before time t. When we choose for every node the scenario with the lowest scenario index, we obtain the set $\mathcal{S}_t$ for $t = 0, \ldots, T$:

$$\mathcal{S}_t = \{s \in \mathcal{S} : s' \in \mathcal{S}', s' < s \Rightarrow (t, s') \neq (t, s)\}.$$

For example, $\mathcal{S}_0 = \{1\}$, $\mathcal{S}_T = \mathcal{S}$, and $|\mathcal{S}_t| = \Pi_{q=0}^{t-1} branch_q$.

From each of the nodes (t, s), $s \in \mathcal{S}_t$, where t is fixed, Ξ_t different scenarios develop, where

$$\Xi_t = \frac{S}{\Pi_{q=0}^{t-1} branch_q} = \Pi_{q=t}^{T-1} branch_q.$$

For example, $\Xi_0 = S$, and $\Xi_T = 1$ (empty products are by definition equal to 1). The parameter Ξ_t has a clear interpretation: it defines the cardinality of the bundle of scenarios through any node at time t. The set of scenarios which develop via (t, s) are denoted by $\mathcal{K}_t^s$. It holds that $|\mathcal{K}_t^s| = \Xi_t$.

More generally, a representation of all nodes at time $q \in \{t + 1, \ldots, T\}$ of scenarios with the same history up to and including time t, is given by

$$\{(q, s') : s' \in \mathcal{K}_t^s(q)\},$$

with $\mathcal{K}_t^s(q) = \mathcal{K}_t^s \cap \mathcal{S}_q$. Indeed, for $q = T$ it holds that $\mathcal{K}_t^s(T) = \mathcal{K}_t^s$.

Example 2.2

This example makes use of the scenario tree depicted in Figure 2.1, and is intended to clarify the notation introduced above. For the tree under consideration, we obtain $\Xi_0 = 6$, $\Xi_1 = 2$, and $\Xi_2 = 1$. Moreover, we have the following sets: $\mathcal{S}_0 = \{1\}$, $\mathcal{S}_1 = \{1, 3, 5\}$, $\mathcal{S}_2 = \{1, 2, 3, 4, 5, 6\}$.

In the node $(t, s) := (0, 1)$, represented by I in Figure 2.1, the set $\mathcal{K}_0^1$ is given by $\{1, 2, 3, 4, 5, 6\}$, since all nodes $(2, s')$, $s' \in \mathcal{K}_0^1$ can be reached from state I. Moreover, $\mathcal{K}_0^1(1) := \{1, 2, 3, 4, 5, 6\} \cap \{1, 3, 5\} = \{1, 3, 5\}$. We see that this gives the unique set of successors of node I, with the lowest scenario indices.

In the node $(t, s) = (1, 5)$, represented by II, we obtain $\mathcal{K}_1^5 = \{5, 6\}$, since only nodes $(2, 5)$ and $(2, 6)$ are accessible from this state. From node $(1, 5)$ only the nodes $\mathcal{K}_1^5(2) = \{1, 2, 3, 4, 5, 6\} \cap \{5, 6\} = \{5, 6\}$ are accessible. $\Box$

We will now introduce the random parameters. For $t \in \mathcal{T}_1 := \{1, 2, \ldots, T\}$, we define the realizations in scenario $s \in \mathcal{S}$ by

$$\omega_t^s = (r_{1t}^s, r_{2t}^s, \ldots, r_{Nt}^s, w_t^s, \underline{L}_t^s, \overline{L}_t^s, \underline{B}_t^s, \overline{B}_t^s, \gamma_t^s, W_t^s),$$

where

r^s_{jt} = return (expressed as a fraction) on asset class j in year t in scenario s, $j = 1, \ldots, N$,
w^s_t = change (expressed as a fraction) in the general wage level in year t in scenario s,
$\underline{L}^s_t$ = lower bound on the value of the liabilities at time t in scenario s,
$\overline{L}^s_t$ = upper bound on the value of the liabilities at time t in scenario s,
$\underline{B}^s_t$ = lower bound on the value of the benefit payments at time t in scenario s,
$\overline{B}^s_t$ = upper bound on the value of the benefit payments at time t in scenario s,
γ^s_t = discount factor associated with cash flows at time t in scenario s,
W^s_t = total level of the pensionable wages of the active participants in year t in scenario s.

All financial quantities, except r^s_{jt} and w^s_t, are denoted in million euros, and N denotes the total number of asset classes.

In the description of the ALM decision process in Section 2.1, we have seen that the board of the pension fund has to make decisions at time 0 ('now'), based on the actual knowledge of the fund, and on given expectations with respect to uncertain future developments, like returns on assets and inflation. Once new information is revealed (i.e., realizations of the uncertain parameters become available), the fund will make new decisions, based on this information, and possible adjusted expectations.

Because decisions are made in every node of the scenario tree, *decision variables* are related to this tree, too. Basically, a decision at time t may depend on the observed part of the scenario at that time, but not on unknown values of parameters of future years. That is, for each possible history (i.e. for each node at time t in the scenario tree) there is precisely one vector of decision variables representing the decisions at hand.

However, in the model formulation it is convenient to introduce a complete set of decision variables for each scenario separately. Therefore, so-called *nonanticipativity* or *information constraints* have to be added, in order to guarantee that decisions do not depend on values of random parameters that will be revealed in later years. Denoting the vector of decision variables at time t in scenario s by x^s_t, the nonanticipativity constraints imply $x^s_t = x^q_t$ if scenarios s and q coincide up to and including year t. The decision vector x^s_t is defined as follows:

$$x^s_t = (XI^s_{1t}, \ldots, XI^s_{Nt}, XD^s_{1t}, \ldots, XD^s_{Nt}, c^s_t, L^s_t, B^s_t, Z^s_t, DZ^s_t, V^s_t),$$

where

XI^s_{jt}	=	value of assets in class j bought at time t in scenario s, $j = 1, \ldots, N$,
XD^s_{jt}	=	value of assets in class j sold at time t in scenario s, $j = 1, \ldots, N$,
c^s_t	=	contribution rate for year $t+1$ in scenario s,
L^s_t	=	value of the liabilities at time t in scenario s,
B^s_t	=	value of the benefit payments at time t in scenario s,
Z^s_t	=	remedial contribution by the sponsor at time t in scenario s,
DZ^s_t	=	direct cash flow by the sponsor, because of a funding ratio which is (far) too low at time t in scenario s,
V^s_t	=	restitution to the sponsor at time t in scenario s.

At the time horizon $t = T$, only the decisions L^s_T, B^s_T, Z^s_T, DZ^s_T, and V_T occur. The precise meaning of the decision variables will become clear in the next sections. We stress here that some other decisions introduced in Section 2.2 follow from the values of the decision variables presented above. For example, the degree of indexation in a state is a result of the value of the liabilities in that state.

The following additional variables are important too. For each $t \in \mathcal{T}_1$ and each scenario $s \in \mathcal{S}$ we have:

A^s_t	=	total asset value at time t in scenario s,
X^s_{jt}	=	value of investments in asset class j, at the beginning of year t in scenario s.

These are state variables. They are determined by the parameters and the decision variables, but from an optimization point of view they are decision variables too, if one includes their definitions as constraints in the model, as we shall do. Next, we have to explain in more detail what we mean by 'time t' in the definition of A^s_t. We assume that at the end of year t, i.e., just before time t, the contribution of year t comes in (although it is common that contributions are paid monthly to the fund) and the benefit obligations of year t are paid. At the same time, the revenues of the assets of year t are revealed. At that time, the board of the fund also has to make a decision with respect to the level of the indexation. After this decision is made, the value of the liabilities is determined, and one knows whether underfunding is present or not. In case of underfunding, possibly a remedial contribution from the sponsor Z^s_t or DZ^s_t is made. In case of overfunding, a restitution is considered.

In Figure 2.2 we have depicted the decisions at time t graphically. Given the decisions at the previous decision moment (at time $t-1$), and the observed realization of the stochastic parameters, A^s_t is known after the benefit payments of year t are made. Given this asset value, decisions are made with respect to the values of the liabilities. A^s_t and L^s_t together determine also the level of the funding ratio of the fund. As a result, decisions with respect to remedial contributions and restitutions have to be made. Finally, the asset portfolio is rebalanced for the next year, and also the level of next year's contribution rate is determined.

In the next subsections, accounting and policy constraints and the objective function of the ALM model are discussed. Constraints which impose a restriction on next year's expected shortage are considered in the next chapter. In appendix A, the mathematical formulation of all constraints and the objective function of our

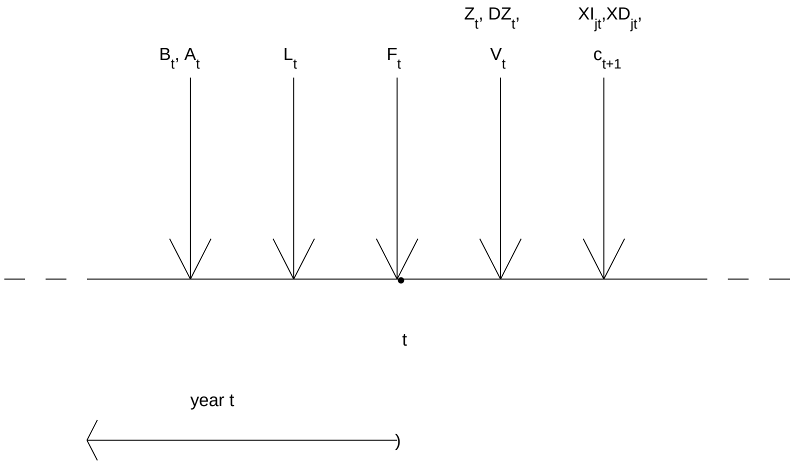

Figure 2.2: The order in which decisions and calculations are made at time t. Note that, since L_t is a decision variable in our model, the board of the fund can decide if rights are indexed or not (or only partially) at time t.

ALM model is given, too.

2.4 Accounting and policy constraints

In the previous section we defined the scenarios and the decision variables of our ALM model. Now we introduce constraints which deal with bookkeeping. Also, some policy constraints will be introduced in this section. Some other constraints were already mentioned before: the nonanticipativity constraints and the definitions of the state variables. In addition, nonnegativity is required for buying and selling assets, for cash flows from the sponsor, and restitutions:

$$XI_{jt}^s \geq 0, \quad XD_{jt}^s \geq 0, \quad j = 1, \ldots, N, t \in \mathcal{T}_0, s \in \mathcal{S},$$

$$Z_t^s \geq 0, \quad DZ_t^s \geq 0, \quad V_t^s \geq 0, \quad t \in \mathcal{T}, s \in \mathcal{S}.$$

The total value of the assets at time t in scenario s is given by the value of the asset portfolio, increased with contributions by active participants, and corrected for benefit payments which were paid in year t:

$$A_t^s = \sum_{j=1}^{N} (1 + r_{jt}^s) X_{jt}^s + c_t^s W_t^s - B_t^s, \quad t \in \mathcal{T}_1, s \in \mathcal{S}, \tag{2.1}$$

where W_t^s denotes the total level of the pensionable salaries of the active participants in year t in scenario s. The value of the investments in asset class j, at the beginning of year $t+1$ in scenario s, is recursively defined by

$$X_{j,t+1}^s = (1 + r_{jt}^s)X_{jt}^s + XI_{jt}^s - XD_{jt}^s, \quad t \in \mathcal{T}_0. \tag{2.2}$$

After a possible remedial contribution by the sponsor of the fund at time t, Z_t, a direct cash flow from the sponsor because of a funding ratio which is far too low, DZ_t, or a restitution to the sponsor, V_t, the asset allocation has to be made, such that all assets are allocated, and transaction costs are taken into account appropriately:

$$\sum_{j=1}^{N} X_{j,t+1}^s = A_t^s + Z_t^s + DZ_t^s - V_t^s - \sum_{j=1}^{N} k_j (XI_{jt}^s + XD_{jt}^s), \qquad s \in \mathcal{S}, t \in \mathcal{T}_0, \tag{2.3}$$

where k_j denotes the proportional transaction costs for asset class j. These transaction costs, arising from the adjustment of the asset portfolio at time t, do not affect A_t^s, but they do influence the new asset portfolio. Equation (2.3) states that all assets have to be invested, and that transaction costs are considered, and can be interpreted as a *cash balance equation* for cash flows.

Constraint (2.3) is an accounting constraint, since incoming and outgoing cash flows are recorded appropriately. Similar constraints appear in all known ALM models (although Dert [24] does not take into account transactions costs), see for example the ALM models presented in Consigli and Dempster [17], Dert [24], and Kouwenberg [55].

In Table 2.3 an overview of incoming and outgoing cash flows at time t in scenario s is given. Recall that we assume that all cash inflows and outflows in year t are recorded at time t.

Incoming cash flows	Outgoing cash flows
$c_t^s W_t^s$	B_t^s
Z_t^s	V_t^s
DZ_t^s	$\sum_{j=1}^{N}(1+k_j)XI_{jt}^s$
$\sum_{j=1}^{N}(1-k_j)XD_{jt}^s$	

Table 2.3: Incoming and outgoing cash flows at time t in scenario s.

In Lemma 2.1 it is shown that definitions (2.1), (2.2), and accounting constraints (2.3) together imply that the cash inflow equals the cash outflow in state (t, s).

Lemma 2.1 *Constraints (2.1), (2.2), and (2.3) imply that for each state* (t, s)*, the cash inflow equals the cash outflow.*

Proof

From equality (2.2), we have:

$$\sum_{j=1}^{N} X_{j,t+1}^s = \sum_{j=1}^{N}(1 + r_{jt}^s)X_{jt}^s + \sum_{j=1}^{N} XI_{jt}^s - \sum_{j=1}^{N} XD_{jt}^s. \tag{2.4}$$

Substituting definition (2.1) in (2.3) gives

$$\sum_{j=1}^{N} X^s_{j,t+1} = \sum_{j=1}^{N}(1+r^s_{jt})X^s_{jt} + c^s_t W^s_t - B^s_t + Z^s_t + DZ^s_t - V^s_t - \sum_{j=1}^{N} k_j(XI^s_{jt} + XD^s_{jt}). \tag{2.5}$$

Because the right-hand sides of (2.4) and (2.5) must be equal, we obtain after rearranging terms

$$c^s_t W^s_t + Z^s_t + DZ^s_t + \sum_{j=1}^{N}(1-k_j)XD^s_{jt} = B^s_t + V^s_t + \sum_{j=1}^{N}(1+k_j)XI^s_{jt}. \tag{2.6}$$

On the left-hand side of (2.6), we have the cash inflows at time t, whereas the right-hand side of 2.6 represents the cash outflows at that time. These coincide with those presented in Table 2.3. □

Next to the equalities and inequalities presented above, there are also fund-dependent lower and upper bounds on the asset mix:

$$\underline{f}_j \sum_{i=1}^{N} X^s_{it} \leq X^s_{jt} \leq \overline{f}_j \sum_{i=1}^{N} X^s_{it} \quad j = 1, \ldots, N, t \in \mathcal{T}_1, s \in \mathcal{S},$$

where $\underline{f}_j$ and $\overline{f}_j$ are parameters that specify lower and upper bounds on the value of asset class j, as a fraction of the total assets.

Instead of fixed lower and upper bounds, these bounds may be time dependent. Given the current portfolio (just before time 0), the bounds may be functions of the current fractions and time. However, we use fixed values for $\underline{f}_j$ and $\overline{f}_j$ for every time $t \in \mathcal{T}_0$ and scenario $s \in \mathcal{S}$ in our ALM model.

For the initial asset portfolio, the following constraints are added:

$$X^s_{j0} = X_{j0} + XI^s_{j0} - XD^s_{j0} - k_j(XI^s_{j0} + XD^s_{j0}), \quad j = 1, \ldots, N, \quad s \in \mathcal{S},$$

where X_{j0} is the initial investment in asset class j, just before possible changes at time 0 can be made.

In the next sections, we describe some important extensions to the constraints mentioned above. These extensions are made to make the model flexible so that it can accommodate the policies of the pension fund.

2.5 Cash flows from the sponsor in case of financial distress

Pension funds want to avoid underfunding, because this implies that it cannot be guaranteed that all future benefit payments can be done. Formally, underfunding means that the funding ratio is less than 1. We will use this concept in a more general way, by saying that at time t a fund faces *underfunding with respect to the level* v (for some positive number v) if the funding ratio is less than v at that time ($A_t < vL_t$). In our ALM model, we distinguish various levels for underfunding, each with its own purpose. In this section we introduce two levels, θ and α ($\theta < \alpha$). They play a role in the policy rules for remedial payments of the sponsor to the fund.

In the circular [74], the PVK requires a minimum level of the funding ratio of 1.05. Therefore, we set $\alpha = 1.05$ in the numerical experiments presented in Chapter 6. As soon as the funding ratio is less then 1.05, the PVK requires a scheme how the board will tackle the problem to restore the funding ratio. In this study, we do not require such an immediate intervention. Only if the funding ratio falls even below the level θ, the sponsor should make a remedial payment immediately. The numerical value of θ may for example be 1 or 0.95. This value may also be the result of negotiations between the sponsor and the fund, or it may be prescribed by the supervisor.

If the sponsor has to make an immediate payment to the fund because the funding ratio is less than θ, this payment should at least be equal to the amount of the shortage with respect to the level θ. This immediate payment, which is denoted by DZ_t^s, should prevent that the financial position of the fund will erode completely.

If at any time the level of the funding ratio is at least α, then there is no direct financial distress, and a remedial contribution of the sponsor is not needed. In our model, it is forbidden in these circumstances.

If at any time the funding ratio is at least θ, but less than α, in our model a remedial payment of the sponsor is allowed, but not obliged. But the sponsor is obliged to restore the funding ratio to the level α if this ratio is below the minimum required level α in a consecutive years, where a is a parameter. For $a > 1$ and $\theta < \alpha$, we see that we introduced flexibility into our model. Requiring a remedial contribution as soon as the funding ratio is below α (i.e. $a = 1$) may lead to solutions which are very expensive. It is quite possible, that such a radical interference is not really necessary. For instance, if there is a quick recovery of the financial markets after a correction, it may not be necessary to have a remedial contribution from the sponsor to the fund. In this case, the total cost of funding is reduced.

Dert [24] formulated an ALM model in which a remedial contribution has to be made as soon as the funding ratio drops below a certain threshold value. In other ALM models, *shortages* are not even modeled (as in Consigli and Dempster [17]), or they are only recorded (as is done in Kouwenberg [55]).

On the basis of four possible future developments of the funding ratio, which are presented in Figure 2.3, we clarify these policy rules. In these examples, we make the assumption that the sponsor of the fund has to make a remedial contribution if the funding ratio is below α in two consecutive years ($a = 2$).

In case I, a remedial contribution has to be made at time 3. Moreover, we also see an advantage of the flexible modeling. If the sponsor was obliged to make a remedial contribution as soon as the funding ratio drops below the level α, the sponsor should have paid more at time 2.

In case II, the sponsor does not need not to restore the funding ratio, because underfunding is recorded only once. Here, the sponsor is allowed to make a remedial contribution (just as in case I on the second decision moment). Whether or not

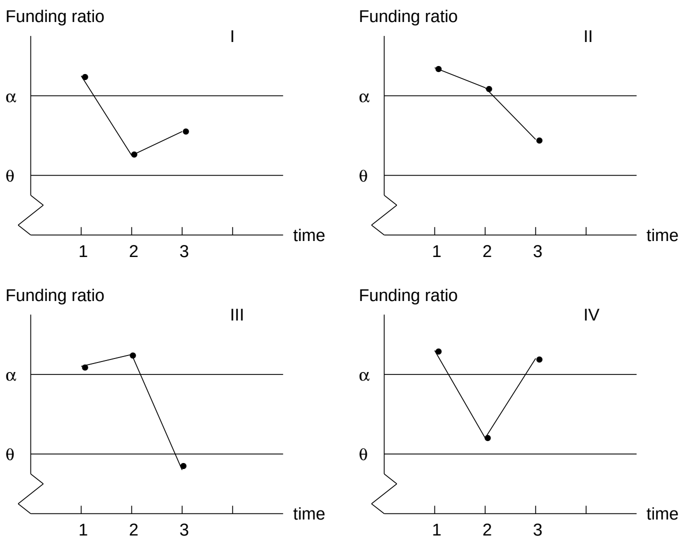

Figure 2.3: Four possible developments of the funding ratio, presented to clarify the decision rules with respect to underfunding in our ALM model.

a remedial contribution will be made at time 3, depends on future developments in the scenarios. If for example with very large probability a remedial contribution has to be made at time 4, the sponsor may prefer to pay at time 3. The reason of this will become clear in the next few pages.

Case III may be the result of a stock market crash. For the pension fund under consideration, the funding ratio drops below the level θ at time 3. Consequently, the sponsor has to interfere immediately. The funding ratio should at least be restored to the level θ.

Case IV emphasizes the advantage of the flexible modeling. After a decrease of the funding ratio under the critical level α, a recovery of the financial position of the fund occurs. Because the sponsor is not obliged to react at time 2, a feasible outcome is that no remedial contribution is made at all. The cause of the increase of the funding ratio may be favorable developments of financial markets, but also interventions by the board of the fund: at time 2, the asset portfolio may be changed, and/or the contribution rate may be increased.

This new modeling is important, since requiring a sufficiently high funding ratio at each balance date may be a too stringent perception of risk. Moreover, as a result of discussions in the beginning of the twenty-first century between the PVK and pension funds, resulted in the fact that the supervisor judges the solvency position of a fund partly on the basis of the funding ratio in successive years, see [74]. Another advantage of this modeling is already mentioned above: it may lead to a lower total cost of funding.

We model the payment of a remedial contribution after a consecutive years of underfunding as mixed-integer restrictions. We introduce binary variables u_t^s and

z_t^s to indicate underfunding and a remedial contribution respectively:

$$u_t^s = \begin{cases} 1 & \text{if } A_t^s < \alpha L_t^s \\ 0 & \text{otherwise,} \end{cases}$$

$$z_t^s = \begin{cases} 1 & \text{if } Z_t^s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

From now on, we have $s \in \mathcal{S}$ and $t \in \mathcal{T}$, unless otherwise mentioned. The binary decision variables u_t^s and z_t^s get the correct values, because of the following 'definition inequalities':

$$A_t^s - \alpha L_t^s \geq -M u_t^s \tag{2.7}$$

$$A_t^s - \alpha L_t^s \leq M(1 - u_t^s) - \frac{1}{M} \tag{2.8}$$

$$Z_t^s \geq M(z_t^s - 1) - (A_t^s - \alpha L_t^s) \tag{2.9}$$

$$Z_t^s \leq M z_t^s \tag{2.10}$$

$$Z_t^s \geq 0$$

$$u_t^s \in \{0, 1\}, \quad z_t^s \in \{0, 1\}.$$

In these inequalities, M is a sufficiently large number ('big M'). Inequalities (2.7) and (2.8), together with the requirement $u_t^s \in \{0, 1\}$ provide the correct value for the binary decision variable u_t^s. If the funding ratio is below α, u_t^s is forced to become 1. Otherwise, this binary variable gets the value 0.

Inequalities (2.9) and (2.10), and the requirement $z_t^s \in \{0, 1\}$ force that in case of a remedial contribution $Z_t^s > 0$, z_t^s becomes 1, and otherwise it becomes 0. That is, although these two inequalities do not rule out that z_t^s becomes 1 if $Z_t^s = 0$, $z_t^s = 0$ is preferred if one considers the objective function (see Section 2.9). Moreover, if z_t^s is 1, the level of the remedial contribution is at least equal to the amount of underfunding. This forces the sponsor to restore the funding ratio at least to its minimum required level α.

The rules that a remedial contribution is only allowed in case of underfunding, and if at the last a decision moments the funding ratio is below α, a remedial contribution has to be made, also force to hold the following conditions:

$$z_t^s \leq u_t^s$$

$$z_t^s \geq \sum_{i=t-a+1}^{t} u_i^s - a + 1.$$

Here, for any $i \leq 0$, u_i^s is a given parameter, not depending on s, that indicates whether the funding ratio in year i was sufficiently high ($u_i^s = 0$) or not ($u_i^s = 1$).

If the sponsor of the fund has to pay a remedial contribution at time t, we still count year t as a year of underfunding. As a result, it is possible that the sponsor has to pay remedial contributions in successive years. This modeling makes sense, because of the weak financial position of the fund: even after a remedial contribution, the financial position is on the border of acceptability. Moreover, this way of modeling is also convenient from a mathematical point of view.

Drijver et al. [27] describe a more general way to model that the sponsor has to restore the funding ratio after some periods in which the funding ratio is too low. They model that the sponsor has to restore the funding ratio if in a of the last b periods ($b \geq a$) underfunding is present:

$$bz_t^s \geq \sum_{i=(t-b)^+ +1}^{t} u_i^s - a + 1.$$

For more details about this modeling, we refer to Drijver et al. [27].

Since the sponsor is generally not willing (and possibly not even able) to pay extremely large remedial contributions, a soft upper bound is given on this amount. This upper bound is defined as a fraction τ of the total level of the pensionable salaries W_t^s. Remedial contributions above τW_t^s are allowed, although the amount above this threshold, denoted by ZI_t^s, is penalized harder in the objective function. The decision variable $ZI_t^s = (Z_t^s - \tau W_t^s)^+$, is defined as follows in the ALM model:

$$ZI_t^s \geq Z_t^s - \tau W_t^s$$

$$ZI_t^s \geq 0.$$

Moreover, fixed penalty costs are incurred if the fund has to deal with underfunding and/or a remedial contribution is made, since these events are highly undesirable. As we have seen in the years 2002 and 2003, this makes sense: as soon as a fund announces that its financial position is weak, and possibly the sponsor of the fund has to make a remedial contribution, this is a hot issue in newspapers and magazines. All interested parties are far from happy: the supervisor because the fund is insolvent, the old aged because their benefit payments may not be indexed, the active participants because of a possible increase in the contribution rate, and the sponsor because it may have to pay a relatively large amount to the fund.

Figure 2.4 shows the relationship between the level of a remedial contribution to be paid by the sponsor of the fund, and the corresponding penalty. The fixed penalty costs due to this payment, is denoted by λ_z. The level of the remedial contribution is penalized by a factor ζ_Z. In addition, the level of the remedial contribution above τW_t, is penalized further by ζ_{ZI}.

The immediate cash flows from the sponsor to the fund in case of a shortage with respect to the level θ are modeled as follows:

$$DZ_t^s \geq \theta L_t^s - A_t^s - Mz_t^s \tag{2.11}$$

$$DZ_t^s \geq 0.$$

If $Z_t^s > 0$, and as a result, $z_t^s = 1$, the funding ratio is already restored to the level α. In this case, no additional immediate payment is required. The term $-Mz_t^s$ in (2.11) prevents a positive cash flow DZ_t^s in case of $Z_t^s > 0$. However, if $Z_t^s = z_t^s = 0$, and if $A_t^s < \theta L_t^s$, the sponsor is forced to pay at least $\theta L_t^s - A_t^s$.

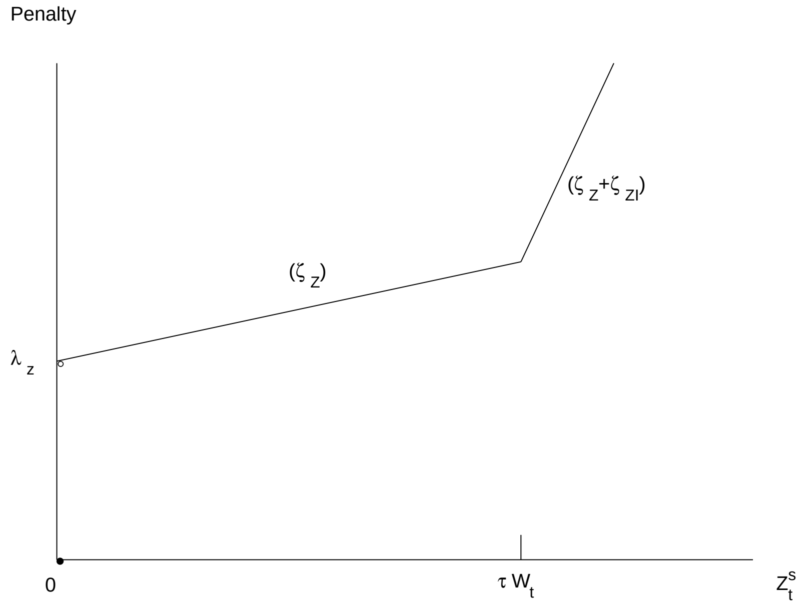

Figure 2.4: Relationship between the level of a remedial contribution by the sponsor, Z, and the corresponding penalty.

The user of the model has to specify the numerical value of ζ_{DZ} in relation with the numerical values of λ_z, ζ_Z, and ζ_{ZI} carefully. Indeed, if ζ_{DZ} would be chosen too low (relative to λ_z, ζ_Z, and ζ_{ZI}), optimization may result in states (t, s) with $DZ_t^s > 0$, and $Z_t^s = 0$, because then fixed penalty costs λ_z will be avoided. The value of ζ_{DZ} should be set sufficiently high to represent that such an immediate payment is extremely undesirable. Moreover, the numerical specification of the penalty parameters should be made, such that a correct representation is found of the wishes of the board of the fund under consideration (although this may not be easy), and also what is for example contractually determined between the sponsor and the pension fund.

2.6 Contribution rate

To finance the pension fund, employers and employees make on a regular basis payments to the fund. These payments are a percentage of a part of the total wages of the participants. This part is called the pensionable salary, and is denoted by W_t^s for year t. The board of the pension fund has to determine the level of the percentage of these pensionable salaries. For time t and scenario s, the level of this percentage is denoted by c_t^s. This level is determined at the beginning of every year. As a result, for year $t+1$, this level is specified at time t.

In determining next year's level of the contribution rate, the board of a fund not only has to take into account bounds on this level, it also considers the wish of a relatively stable contribution rate through time. Moreover, from the perspective of the sponsor it is reasonable to assume that the level of the contribution rate should be sufficiently high if the sponsor of the fund makes a remedial contribution, so that also active participants contribute to a better financial position of the fund. Details with respect to these policy rules and the mathematical formulations are discussed in this section.

Although the board can determine the level of the contribution rate, this level is bounded. Even if the funding ratio is so low, that not all future benefit payments can be guaranteed, c_t^s cannot increase indefinitely. This upper bound on c_t^s is denoted by $\overline{c}$, and is assumed to be time independent. Also a time independent lower bound on c_t^s, which is denoted by $\underline{c}$, exists. The numerical specification of $\underline{c}$ and $\overline{c}$ may be fund dependent, and may be stated in the fund's pension regulation. We assume that the following conditions hold: $0 \leq \underline{c} < \overline{c} < 1$.

The lower and upper bounds on the contribution rate c_t^s give rise to the following constraint in our ALM model:

$$\underline{c} \leq c_t^s \leq \overline{c}. \tag{2.12}$$

Not only the level of the contribution rate is important, but also its stability, since too much variability is undesirable.

We can model this refinement as

$$-\eta \leq c_t^s - c_{t-1}^s \leq \rho, \tag{2.13}$$

where $c_t^s - c_{t-1}^s$ represents the change in the contribution rate in two consecutive years (t and $t-1$) and η and ρ are fixed bounds for decreases and increases of the contribution rates. In the ALM model presented in Kouwenberg [55], these 'hard' constraints are used.

The numerical values for η and ρ also fund dependent. In addition, these values are expected to be not too low. On the other hand, if the values of η and ρ are large, (2.13) loses its meaning in modeling the undesirability of a contribution rate which changes rapidly.

However, if the funding ratio is relatively low for a number of years, it may be preferable to increase the contribution rate rather than to ask for a remedial contribution. This may lead to an increase in the contribution rate which exceeds ρ. Hence, it is better to specify (2.13) as a goal constraint: increases (decreases) greater than ρ (η) are allowed (although constraint (2.12) still has to be satisfied), but they are penalized in the objective function. Now, η and ρ denote the maximum decrease and increase in the contribution rate in two consecutive years, such that no penalties are incurred due to a rapidly changing contribution rate.

We model this in a linear programming formulation by the introduction of additional decision variables ci_t^s, representing the amount by which the increase in the contribution rate exceeds ρ at time t compared with the contribution rate at time $t-1$. The second inequality of (2.13) is replaced by

$$ci_t^s \geq c_t^s - c_{t-1}^s - \rho$$

$$ci_t^s \geq 0.$$

In the objective function, we penalize $ci_t^s = (c_t^s - c_{t-1}^s - \rho)^+$, which is positive if $c_t^s > c_{t-1}^s + \rho$.

We can use an analogous reasoning when the funding ratio is relatively high for a number of successive years. In this situation it may be preferable to lower the contribution rate in two successive years by more than η. This may be more desirable than, for example, making a restitution to the sponsor, because the board of the fund would like that the active participants profit from the financial prosperity of the fund.

We can model this by the introduction of additional decision variables cd_t^s, representing the amount by which the decrease of the contribution rate in year t exceeds η, as compared with the contribution rate at time $t-1$:

$$cd_t^s \geq c_{t-1}^s - c_t^s - \eta$$

$$cd_t^s \geq 0.$$

In the objective function, we penalize $cd_t^s = (c_{t-1}^s - c_t^s - \eta)^+$, which is positive if $c_t^s < c_{t-1}^s - \eta$.

We penalize ci_t^s and cd_t^s by positive parameters ζ_{ci} and ζ_{cd} (usually $\zeta_{cd} \leq \zeta_{ci}$), whereas no penalty is imposed if (2.13) is satisfied. Figure 2.5 shows an example of such a penalty function. In stochastic programming, this structure, that models

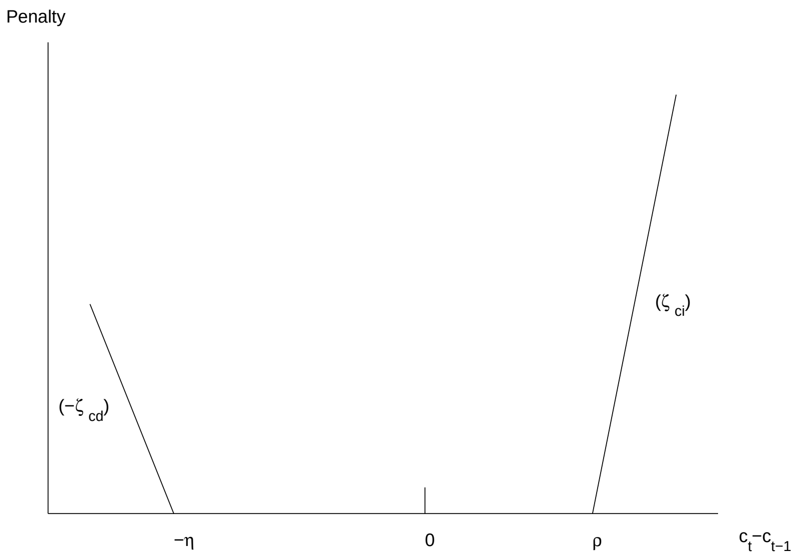

Figure 2.5: Penalty costs associated with a change in the contribution rate at time t, as compared with the contribution rate at time $t-1$.

piecewise linear increasing costs for shortages and surpluses, is known as multiple simple recourse. For details on multiple simple recourse, see Klein Haneveld [53] and Van der Vlerk [91]. We have never seen this structure in ALM models in the literature before.

The formulation described above makes sense in practise: relatively large increases in contribution rates are faced with a lot of resistance by active participants. As a result, if large deviations of the current level of the contribution rate could be avoided, this would be the best alternative for the parties involved in the decision making process. On the other hand, if large increases are needed, because otherwise the financial position would become too weak, such an increase can be considered.

Note that the above introduced formulation is also very flexible: if managers do not want to penalize large deviations in the contribution rate, they can choose to either specify $\zeta_{cd} = \zeta_{ci} = 0$, or to set $\eta = \rho \geq \overline{c} - \underline{c}$.

Contribution rate in case of a remedial contribution

As we have seen before, we included in our model the policy rule, that the sponsor of the fund has to restore the funding ratio if there is a shortage with respect to the level α in a number of consecutive years. It seems reasonable that if such a remedial contribution is made, the sponsor requires that the active participants also provide a sufficient contribution to the financing of the fund in the form of a sufficiently high contribution rate. Therefore, we will include this condition in a policy rule of our model, as follows. We denote the minimum required contribution rate in case of a remedial contribution by c^*. The numerical value of c^* may for example be the fund's actuarial premium. The constraint, which serves the rule mentioned above, is given by:

$$c_t^s - c^* \geq M(z_t^s - 1). \tag{2.14}$$

In this case, big M may be taken equal to $\overline{c}-\underline{c}$. If the sponsor of the fund has to make a remedial contribution, and, as a result, $z_t^s = 1$, (2.14) leads to the requirement $c_t^s \geq c^*$. On the other hand, if $z_t^s = 0$, no additional requirement with respect to the level of the contribution rate is made. Note that if one does not want to model this relationship between c_t^s and Z_t^s, one can simply eliminate this rule by choosing $c^* = \underline{c}$, since $c_t^s \geq \underline{c}$ always has to be satisfied.

Of course, there are a variety of alternative formulations for requirements with respect to the level of the contribution rate if the sponsor of the fund makes a remedial contribution. A few alternatives are presented below.

- Require that the contribution rate does not decrease if $z_t^s = 1$. This can be accomplished by the constraint

  $$c_t^s - c_{t-1}^s \geq M(z_t^s - 1).$$

  The disadvantage of this modeling is, however, that even if the sponsor of the fund makes a remedial contribution, it is allowed that the contribution rate may still be very low.

- Penalize the deviation of the contribution rate from its upper bound $\overline{c}$. This can be accomplished by the constraint

$$\overline{c} - c_t^s - cdu_t^s \leq M(1 - z_t^s), \tag{2.15}$$

and penalize the nonnegative decision variable cdu_t^s in the objective function. If the sponsor has to make a remedial contribution, (2.15) results in

$$cdu_t^s \geq \overline{c} - c_t^s,$$

and, as a result, the deviation of the contribution rate from its upper bound can be penalized. On the other hand, if $z_t^s = 0$, constraint (2.15) is non-binding.

A disadvantage of this modeling is that it may be difficult to find an appropriate value for the penalty parameter associated with cdu_t^s.

Which constraint(s) are added to the ALM model may depend on the contribution policy of the fund. Because we think formulation (2.14) is the most important for real world practice, we have chosen to include this constraint in our ALM model.

2.7 Indexation

Indexation of liabilities is the adjustment of the built-up rights to increases in prices or wages in a certain year. As we have seen in Chapter 1, pension funds are not obliged to adjust benefit payments. Generally, every year again, the board decides whether or not to increase pension rights. This decision may of course depend on the financial position of the fund, and also on the level of the increase in prices or wages. Therefore, this decision is made after the realization of the development in prices or wages is known. Typically, pension funds adjust the benefit payments for increases in the price or wage level fully if the solvency of the fund is sufficient. Moreover, if pension rights will be indexed only partially, or not at all, this leads to great dissatisfaction, especially of retired people. In this section, we describe how we have modeled indexation. Considering indexation as a decision instrument in an optimization model is new in the financial literature. In the remainder of this thesis, we use increases in the general wage level as the base to index rights.

Mathematical formulation

The basic idea of incorporating indexation as decisions in a linear programming structure is relatively easy. Consider the liabilities in state (t, s), L_t^s, not as a parameter, but as a decision variable, that may vary within a certain range. The bounds on this range are denoted by $\underline{L}_t^s$ and $\overline{L}_t^s$, and they are parameters in the scenario tree. The upper bound $\overline{L}_t^s$ represents the value of the liabilities if in all years $0, \ldots, t$ full indexation is given to the participants of the fund. The lower bound $\underline{L}_t^s$ represents the nominal value of the liabilities in state (t, s). This means that from year

0 to year t the benefit payments are not adjusted at all for increases in the general wage level in those years. So we add the following constraint to our ALM model:

$$\underline{L}_t^s \leq L_t^s \leq \overline{L}_t^s. \tag{2.16}$$

Of course, more constraints are needed to model the indexation policy of a fund in a proper way. For instance, constraint (2.16) does not prevent $L_t^s = \underline{L}_t^s$ in all states (t, s), even if $L_t^s = \overline{L}_t^s$ would result in a sufficiently high funding ratio. In our model, we assume that the board of a pension fund has the following indexation goals:

- It strives to index liabilities with respect to last year's increase in the general wage level.
- If in a certain year the pension rights are not fully compensated for increases in the general wage level, it strives to give this compensation in a later year.

These goals are incorporated in our model by introducing penalties if they are not reached.

Incorporating indexation into the model, implies that the board of a pension fund gets an additional instrument to indicate what to do in case of financial distress. After all, if the funding ratio is sufficiently high, the board generally gives full compensation for increases in the wage level. Optimization of the model will also lead to this solution, because penalty costs are avoided in this case, However, the question remains what to do in case of less desirable financial circumstances. In that case, underfunding may only be avoided by giving up full indexation. This decision may also be influenced by the power of retired people, or by the financial soundness of the sponsor. In our ALM model, this balance of decisions will be the result of the numerical specifications of the penalty parameters.

Now, we will describe the constraints which are added to our model, which serves for a penalty in case the interests described above are not satisfied. We first describe the penalty associated with not giving full indexation in all years up to the current year. Adding the term

$$\sum_{s=1}^{S}\sum_{t=0}^{T} p_t^s \gamma_t^s (\zeta_L \left(L_t^s - \overline{L}_t^s\right)^-) \tag{2.17}$$

to the objective function, the total deviation is penalized. Here, the parameter ζ_L is a penalty parameter, and its numerical value should be specified by the user of the model.

We also want to include a (fixed) penalty if liabilities are not fully indexed. We have chosen this formulation, because the decision not to give this compensation is very undesirable and causes much commotion among interested parties.

To include such a penalty into a linear programming framework, we not only need to know the numerical value of w_t^s, but also what the change in the liabilities from time $t-1$ to time t is. In state (t, s) liabilities are fully indexed with respect to last years' increase in the general wage level w_t^s, if the following condition holds:

$$L_t^s \geq (1 + \varphi_t^s)(1 + w_t^s) L_{t-1}^s,$$

where

$$\varphi_t^s := \frac{\underline{L}_t^s - \underline{L}_{t-1}^s}{\underline{L}_{t-1}^s}$$

denotes the relative change in the nominal liabilities. Indeed, $\underline{L}_t^s$ generally differs from $\underline{L}_{t-1}^s$, for example because of changes in the age distribution of the participants of a specific pension fund. Note that φ_t^s is data in the scenario tree, since the lower bounds $\underline{L}_t^s$ and $\underline{L}_{t-1}^s$ are data, too. Because of the definition of φ_t^s, $L_t^s = (1+\varphi_t^s)L_{t-1}^s$ gives the value of the liabilities if the benefit rights are not indexed (but also not deteriorates). This number is multiplied by $(1+w_t^s)$ to get the value of the liabilities, such that future rights are indexed with respect to last years' increase in the general wage level.

Now, we introduce the '*degree of change of indexing*' in year t in scenario s. We denote it by I_t^s, and it is given by

$$I_t^s := \frac{L_t^s}{(1+\varphi_t^s)(1+w_t^s)L_{t-1}^s}.$$

The values of I_t^s can be found after the ALM model is solved, since then the values of L_t^s are known in all states (t,s). If I_t^s gets the value 1, liabilities are fully indexed, whereas values less than 1 indicate that benefit payments are only partially adjusted, or not adjusted at all. Note that a value of I_t^s greater than 1 is also possible. This may happen if in at least one year $q < t$ future benefit rights are not fully indexed. If at time t the benefit payments are also adjusted for increases in w_q^s, the numerical value of I_t^s exceeds 1. Note that I_t^s is not introduced as a decision variable, since then nonlinearities would have been introduced.

To include fixed penalty costs if the fund does not correct future benefit payments for w_t^s, we need binary decision variables. They indicate whether pension rights are fully indexed or not. These binary decision variables are denoted by m_t^s, and are defined by

$$m_t^s = \begin{cases} 1 & \text{if } L_t^s < (1+\varphi_t^s)(1+w_t^s)L_{t-1}^s \quad \text{i.e. } (I_t^s < 1) \\ 0 & \text{otherwise.} \end{cases}$$

This means that m_t^s gets the value 1 if the participants of the fund do not receive full compensation for last year's increase in the general wage level, and 0 if this compensation is given.

To find the correct values for the decision variable $m_t^s \in \{0,1\}$, the following constraints are added to the set of restrictions of our ALM model:

$$L_t^s - (1+\varphi_t^s)(1+w_t^s)L_{t-1}^s \geq -Mm_t^s,$$

$$L_t^s - (1+\varphi_t^s)(1+w_t^s)L_{t-1}^s \leq M(1-m_t^s) - \frac{1}{M}.$$

To penalize m_t^s, the following term is added to the objective function:

$$\sum_{s=1}^{S}\sum_{t=1}^{T} p_t^s \gamma_t^s (\lambda_m m_t^s),$$

where λ_m denotes the fixed costs associated with not giving full compensation. The numerical value of this penalty parameter has to be specified by the user of the model.

Benefit payments

The degree of indexing not only influences the value of the liabilities, but also the level of next year's benefit payments. If pension rights are never indexed, only nominal levels of these payments are made. On the other hand, if always full compensation is given, the nominal benefit payments are adjusted appropriately to reflect this policy. In this case, the benefit payments in state (t, s) are given by

$$\overline{B}_t^s = \underline{B}_t^s \Pi_{q=1}^t (1 + w_q^s),$$

where $\underline{B}_t^s$ denotes the nominal expected benefit payment in state (t, s). In our model, both $\underline{B}_t^s$ and $\overline{B}_t^s$ appear as parameters. They are the lower and upper bound, respectively, of the actual benefit payment B_t^s in year t and scenario s. The model adopts the following rule to find the value of B_t^s. The nominal expected benefit payment is increased (in the same proportion as) the fraction of the wage inflation participants of the fund received last year. These fractions are found by the values of the liabilities. Formally, this relationship between the benefit payments and the value of the liabilities is state (t, s) is given by

$$B_t^s = \underline{B}_t^s + \frac{L_{t-1}^s - \underline{L}_{t-1}^s}{\overline{L}_{t-1}^s - \underline{L}_{t-1}^s}(\overline{B}_t^s - \underline{B}_t^s). \tag{2.18}$$

Equality (2.18) is added to the set of constraints of our ALM model.

Implications of the modeling

Modeling indexing as described above has the following implications. First of all, if in a certain year t only partial compensation (or no compensation at all) is given, it is still possible that in the future compensation with respect to wage inflation of year t is given. However, the value of the binary decision variable m_t^s in state (t, s) remains 1, since at that moment, full compensation was not given.

On the other hand, it is also possible that once indexing is given, this decision is turned back at a later decision moment if the financial position of the fund is weakened. However, this does not change the values of the binary decision variable m in previous years.

2.8 Restitutions

In Chapter 1 we have seen that because of favorable developments of financial markets, the funding ratio may increase rapidly. In such a case, the question arises for the board of a fund, what to do with large surplusses. One possibility is that money is transferred back to the sponsor of the fund. These *restitutions* may also be established contractually: the sponsor makes a remedial contribution in case of underfunding, and benefits from large surplusses of the fund too. Recall that in our ALM model a restitution at time t in scenario s is denoted by V_t^s.

Our ALM model adopts the following rules for restitutions to the sponsor of the fund, in terms of two policy parameters β and b:

- A restitution to the sponsor can only occur if the funding ratio is greater than β, where β is a fixed level ($\beta \gg \alpha$).
- If a restitution is made, it should at least be equal to the amount of the surplus with respect to the level β.
- If in b consecutive years a surplus with respect to the level β is recorded, a restitution to the sponsor has to be made, where b is a fixed number.
- If a restitution has to be made, this payment is made as a lump-sum.
- A restitution in state (t, s) is only allowed if full indexing is given for increases in the general wage level in all previous years, i.e. if $L_t^s = \overline{L}_t^s$.

These rules are formulated as linear constraints in the decision variables, after adding the following binary decision variables:

$$o_t^s = \begin{cases} 1 & \text{if } A_t^s > \beta L_t^s \\ 0 & \text{otherwise,} \end{cases}$$

$$v_t^s = \begin{cases} 1 & \text{if } V_t^s > 0 \\ 0 & \text{otherwise.} \end{cases}$$

These binary variables get the correct values, because of the following 'definition inequalities':

$$\beta L_t^s - A_t^s \geq -M o_t^s$$

$$\beta L_t^s - A_t^s \leq M(1 - o_t^s) - \frac{1}{M}$$

$$V_t^s \geq M(v_t^s - 1) - (\beta L_t^s - A_t^s) \tag{2.19}$$

$$V_t^s \leq M v_t^s,$$

where $o_t^s \in \{0, 1\}$, $v_t^s \in \{0, 1\}$ and $V_t^s \geq 0$ for all $s \in \mathcal{S}$, and $t \in \mathcal{T}$.

Inequality (2.19) also forces the restitution to satisfy the second rule. The first and third rule are forced to hold by the conditions

$$v_t^s \leq o_t^s$$

$$v_t^s \geq \sum_{i=t-b+1}^{t} o_i^s - b + 1.$$

Here o_i^s ($i = 1 - b, \ldots, 0$) is a given parameter, not depending on s.

We still have to model the last rule, that a restitution is only allowed if full compensation is given for the increase in the general wage level in all previous years. In other words, a restitution in state (t, s) is only allowed if, next to conditions regarding the level of the funding ratio, $L_t^s = \overline{L}_t^s$ is satisfied.

To model this rule, an additional binary variable is needed to indicate whether $L_t^s = \overline{L}_t^s$ or not. This binary variable, denoted by l_t^s, is defined as follows:

$$l_t^s = \begin{cases} 1 & \text{if } L_t^s < \overline{L}_t^s \\ 0 & \text{otherwise.} \end{cases}$$

This binary variable gets the correct value by means of the following inequalities, together with the requirement $l_t^s \in \{0,1\}$:

$$\overline{L}_t^s - L_t^s \leq M l_t^s \tag{2.20}$$

$$\overline{L}_t^s - L_t^s \geq M l_t^s - \frac{1}{M}. \tag{2.21}$$

The rule under consideration is modeled by adding the following inequality to the set of constraints:

$$V_t^s \leq M(1 - l_t^s).$$

If the binary decision variable l_t^s gets the value 0, i.e. if full compensation for increases in the general wage level is given up to and including year t, a restitution to the sponsor is allowed (or even forced because of other constraints). On the other hand, if l_t^s gets the value 1, no restitution will be made. Note that we could also have introduced fixed penalty costs associated with $l_t^s = 1$. However, from discussions with advisors of pension funds we concluded that not indexing liabilities is considered as the most important indicator.

2.9 Objective function

A pension fund wants to minimize the total cost of funding, i.e., the contributions made by the active participants of the fund and the remedial contributions. It also wants to avoid 'undesirable events'. Therefore, in our ALM model, we do not only include the funding costs, but also penalty costs and rewards in the objective function.

Fixed penalty costs due to underfunding, a remedial contribution, and a deterioration of indexation were introduced, because these events are highly undesirable. Moreover, large increases and decreases in the contribution rate in two consecutive years, the level of a remedial contribution, and deviations of the value of the liabilities from its upper bound are also penalized. On the other hand, a fixed reward is given for overfunding with respect to the level β and also for restitutions to the sponsor of the fund. In addition, the level of a restitution is rewarded, too.

At the horizon, surplusses and shortages with respect to the level Λ are rewarded and penalized, respectively. The parameter Λ serves as a minimum desired level of the funding ratio after T years.

The objective of the board of the fund is to minimize the total expected discounted costs (including penalty costs). As a result, the probability and the discount factor in each state (t, s), denoted by p_t^s and γ_t^s respectively, appear in the objective function.

All these components together constitute the objective function

$$
\begin{aligned}
&\sum_{s=1}^{S}\Big[\sum_{t=0}^{T} p_t^s\gamma_t(c_t^s W_t + Z_t^s) && \text{funding costs}\\
&\text{penalties:} && \\
&+\sum_{t=1}^{T} p_t^s\gamma_t^s(\zeta_{ci} ci_t^s + \zeta_{cd} cd_t^s)W_t && \text{change in contribution rate}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\lambda_u u_t^s) && \text{underfunding}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\lambda_z z_t^s + (\zeta_Z - 1)Z_t^s + \zeta_{ZI} ZI_t^s) && \text{remedial contribution}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\zeta_{DZ} DZ_t^s) && \text{cash payment}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\lambda_m m_t^s) && \text{deterioration of indexation}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\zeta_L (L_t^s - \overline{L}_t^s)^-) && \text{no full indexation}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\lambda_o o_t^s) && \text{overfunding}\\
&+\sum_{t=0}^{T} p_t^s\gamma_t^s(\lambda_v v_t^s + \zeta_V V_t^s) && \text{restitution}\\
&+p_T^s\gamma_T^s(\zeta_{\Lambda d}(A_T^s - \Lambda L_T^s)^- + \zeta_{\Lambda i}(A_T^s - \Lambda L_T^s)^+)\Big]. && \text{horizon}
\end{aligned}
$$

At first sight, the above presented objective function may not seem appropriate to be used in a linear program, due to the presence of $(A_T^s - \Lambda L_T^s)^-$, $(A_T^s - \Lambda L_T^s)^+$ and $(L_t^s - \overline{L}_t^s)^-$. However, these terms can be taken into account in a linear programming framework. To do so, we replace $(A_T^s - \Lambda L_T^s)^-$ and $(A_T^s - \Lambda L_T^s)^+$ by nonnegative decision variables $Sho\Lambda_T^s$ and $Sur\Lambda_T^s$. Moreover, we add the following constraint to the set of restrictions:

$$A_T^s - \Lambda L_T^s = Sur\Lambda_T^s - Sho\Lambda_T^s \qquad s \in \mathcal{S}. \tag{2.22}$$

The requirement $\zeta_{\Lambda d} > -\zeta_{\Lambda i}$ has to be made, in order to obtain a bounded solution. In a similar way, the term $(L_t^s - \overline{L}_t^s)^-$ can be incorporated in a linear program.

In this chapter we have presented a large part of our ALM model. We have argued that indicators are useful. They are for example needed in appropriately modeling mid-term risks of pension funds. The introduction of indicators has consequences in an optimization model. From a computational point of view, such models become extremely difficult for reasonably sized problems. However, if one uses insights into the problem, one may obtain heuristic solutions to such ALM problems. This will be the subject of Chapter 4. Although considering mid-term risk is important in ALM problems, restrictions may also be imposed on next year's solvability of the fund. This type of constraint will be discussed in the next chapter.