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Technical Notes and Correspondence

Self-Triggered Network Coordination Over Noisy Communication Channels

Mingming Shi, Pietro Tesi, and Claudio De Persis

***Abstract*—This paper deals with the coordination problems over noisy communication channels. We consider a scenario where the communication between network nodes is corrupted by unknown-but-bounded noise. We introduce a novel coordination scheme which ensures: 1) boundedness of the state trajectories and 2) a linear map from the noise to the nodes disagreement value. The proposed scheme does not require any global information on the network parameters and/or the operating environment (the noise characteristics). Moreover, network nodes can sample at independent rates and in an aperiodic manner.**

***Index Terms*—Communication noise, cooperative control, network analysis and control, self-triggered control, sensor networks.**

I. Introduction

Distributed coordination is one of the most active research areas in control engineering [1]. To achieve coordination, the network units (nodes) collect and process data from neighbouring nodes. In practice, a main issue is that data exchange is often carried out through digital communication channels. Thus, coordination algorithms should take into account that the data flow can only occur at finite rates and that the communication medium can introduce issues such as packet loss, transmission delay, and noise. The goal of this paper is to study coordination algorithms in the presence of communication noise, which is one of the major issues that arise in problems involving data exchange. We shall focus on *consensus* [1] algorithms since consensus is the prototypical problem in distributed coordination.

Literature review—Even if one neglects network issues such as finite transmission rate, dropouts, and delay, developing noise-robust consensus algorithms is a very challenging task. The intuitive reason is that consensus algorithms usually rely on Laplacian dynamics. Since the Laplacian matrix has an eigenvalue at zero, communication noise can cause the state of the nodes to diverge. This means that even if consensus is achieved, the consensus value need not be bounded, in which case convergence may become useless.

Most of the research works in this area assume that noise has specific statistical properties, for example that it is *white* [2], [3], *Brownian*-like [4], or *martingale* [5], [6]. In contrast, only few research works have approached the problem where, due to uncertain channel characteristics, one can only regard noise as a bounded signal (*unknown-but-bounded*). Arguably, the lack of noise statistical properties makes it much more difficult to ensure state boundedness since one cannot rely on features such as *zero-mean* or *stationarity*. In [7], Kingston *et al.* consider a Kalman-based coordination scheme and show that the node disagreement satisfies input-to-state stability properties, but no results are given regarding boundedness of the state trajectories. In [8], Shi and Johansson study robust and integral robust consensus with respect to L_∞ and L_1 norms of the noise function, but again the analysis only involves the disagreement variable and no results are given regarding state boundedness. This is also the case in [9] where Garulli and Giannitrapani consider discrete consensus under bounded measurement noise, and in [10], where Franceschelli *et al.* propose discontinuous interaction rules to mitigate the effect of disturbances on the node disagreement. A framework more close to ours is presented in [11]. There, Bauso *et al.* propose a coordination scheme that guarantees *approximate* consensus along with boundedness of the state trajectories, but an upper bound on the magnitude of the noise is required to be known.

Summary of contributions—In this paper, we consider a novel coordination algorithm that can handle *unknown-but-bounded* noise without requiring the knowledge of a noise upper bound. We propose a *state-dependent* coordination scheme where each node dynamically adjusts its update rule based on the magnitude of its own value. This approach can be regarded as a coarse dynamic quantization strategy, which updates the quantization based on the state of the nodes [12]. We show that this approach prevents state divergence and ensures, in the noiseless case, a maximum consensus error for the worst case over the initial vector of states, which is reminiscent of *normalized* consensus metrics [13], [14]. As for the noisy case, this approach ensures that both disagreement and state variables scale linearly with the magnitude of the noise.

From a technical viewpoint, our approach employs a *self-triggered* control scheme [15]. Each node uses a local clock to decide its update times. At each update time, the node polls its neighbors, collects the data, and determines whether it is necessary to modify its controls along with its next update time. Similar to *event-triggered* control [16], [17], *self-triggered* control [18]–[21] features the remarkable property that the

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communication among nodes occurs only at discrete time instants. Moreover, the nodes can sample independently and aperiodically. Thus, the proposed approach is also appealing from a practical point of view (as we will show, including the case where the data exchange encounters delays).

The proposed self-triggered algorithm shares similarities with several pairwise gossip or multi-gossip approaches with randomized [13], [22] and deterministic [23] protocols, which also account for aperiodic communication. However, to the best of our knowledge, gossiping has not been considered in connection with unknown-but-bounded noise, even in the most recent literature [24], [25].

A preliminary version of the manuscript appeared in [26]. Compared with the latter, this paper provides complete proofs of the results, a thorough discussion of the proposed method, and extensive numerical results for large-scale networks. Furthermore, the presence of delays in the communication is considered in a new section.

A. Notation

We assume to have a set of nodes $I := \{1, 2, \ldots, n\}$ connected over an undirected graph $G := (I, E)$, where $E \subseteq I \times I$ is the set of edges (links). We denote by L the Laplacian matrix of G. For each node $i \in I$, we denote by $\mathcal{N}_i$ the set of its neighbors and by d_i the cardinality of $\mathcal{N}_i$. Given n scalar-valued variables $v_1, v_2, \ldots, v_n$, we define $v := \mathrm{col}(v_1, v_2, \ldots, v_n)$. Given a vector $v \in \mathbb{R}^n$, $|v|$ denotes its Euclidean norm and $|v|_\infty$ its infinity norm. Given a signal s mapping $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^n$, we let $|s|_\infty := \sup_{t \in \mathbb{R}_{\geq 0}} |s(t)|_\infty$ and say that s is *bounded* if $|s|_\infty$ is finite.

II. Framework and Outline of the Main Results

We consider a network of n dynamical systems interconnected over an undirected graph G. Each node obeys

$$\begin{aligned} \dot{x}_i &= u_i \\ z_i &= x_i + w_i \end{aligned} \tag{1}$$

where $i \in I$; $x_i \in \mathbb{R}$ is the state; $u_i \in \mathbb{R}$ is the control input; and $z_i \in \mathbb{R}$ is the output where $w_i \in \mathbb{R}$ is a bounded signal, which models communication noise. As it will become clear in the sequel, one can replace the second of (1) with $z_{ij} = x_i + w_{ij}$, where $i \in I$ and $j \in \mathcal{N}_i$, so that each neighbor of node i receives different noisy data. We will not pursue this model in order to keep the notation as streamlined as possible.

According to the usual notion of consensus [1], the network nodes should converge, asymptotically or in a finite time, to an equilibrium point where all the nodes have the same value lying somewhere between the minimum and maximum of their initial values. In the presence of noise, however, convergence to an exact common value is generally impossible to achieve. As outlined hereafter, the main contribution of this paper is a new coordination scheme that ensures *practical* (*approximate*) consensus, namely convergence to a set whose radius depends on the noise amplitude.

A. Outline of the Main Results

Let

$$r := \max\{\varepsilon, \varepsilon\chi_0\} + \left(\frac{\varepsilon}{3} + 3d_{\max}\right)|w|_\infty \tag{2}$$

where $\varepsilon \in (0, 1)$ is a design parameter which specifies the desired accuracy level for consensus, $\chi_0 := |x(0)|_\infty$, and $d_{\max} := |d|_\infty$. In the general case where data are noisy, that is when $|w|_\infty \neq 0$, our scheme ensures that the network state x remains bounded and, in a finite time, remains confined in the set

$$\mathcal{D} := \{x \in \mathbb{R}^n : |\sum_{j \in \mathcal{N}_i}(x_j - x_i)| < r, \ \forall i \in I\}. \tag{3}$$

Moreover, if $|w|_\infty < \varepsilon/(2d_{\max})$, which includes as a special case the noise-free case, then all the network nodes also remain between the minimum and the maximum of their initial values, and converge in a finite time to a point belonging to $\mathcal{D}$. This result is reminiscent of *normalized* consensus [13], [14]. In fact, the result establishes that in the ideal case when $|w|_\infty = 0$ each node reaches in finite time a local average satisfying

$$\frac{|\sum_{j \in \mathcal{N}_i}(x_j - x_i)|}{\chi_0} < \varepsilon \tag{4}$$

when $\chi_0 \geq 1$. Thus, it ensures a (prespecified) maximum error ε for the *worst case* over the initial vector of states, which is indeed a form of normalized consensus [13], [14]. If $\chi_0 < 1$, then the tolerance simply reduces to ε. As for the noisy case, the result establishes that the consensus error scales linearly with respect to the magnitude of the noise.

From an implementation point of view, the proposed scheme enjoys the following features:

1) no knowledge of χ_0 is required;
2) no knowledge of $|w|_\infty$ is required;
3) the control action is distributed;
4) the communication between the network nodes occurs only at discrete time instants. Moreover, the network nodes can sample independently and in an aperiodic manner.

These features indicate that the implementation does not require any global information on the network parameters and/or the operating environment (the noise). In particular, the last feature renders the proposed scheme applicable when coordination must be implemented on digital communication networks.

All the derivations will be carried out assuming that there is no communication delay, which is briefly addressed in the Appendix. The analysis shows that, in practice, delays have the same effect as an additional noise source. For this reason, also numerical simulations will be restricted to the delay-free case.

III. Coordination Algorithm

The main feature of the proposed coordination algorithm lies in the use of adaptive consensus thresholds. Each network node is equipped with a local variable

$$\varepsilon_i(t) := \begin{cases} \varepsilon|x_i(t)| & \text{if } |x_i(t)| \geq 1 \\ \varepsilon & \text{otherwise} \end{cases} \tag{5}$$

that specifies the threshold used to assess whether or not consensus is achieved. In contrast with previous self-triggered schemes [15], [27], this threshold is *adaptive* as it scales dynamically with the state magnitude. It is exactly this feature that ensures robustness against communication noise.

We now describe the control action and communication protocol. For each $i \in I$, let $\{t_k^i\}_{k \in \mathbb{N}_0}$ with $t_0^i = 0$ be the sequence of time instants at which node i collects data from its neighbors. At these time instants, the node updates its control action and determines when the next update will be triggered. For each $i \in I$, let

$$\mathrm{ave}_i^w(t) := \sum_{j \in \mathcal{N}_i}(z_j(t) - x_i(t)) \tag{6}$$

denote the local noisy average.

The control action makes use of a quantized sign function. The control signals take values in the set $\mathcal{U} := \{-1, 0, +1\}$, and the specific quantizer of choice is $\text{sign}_\alpha : \mathbb{R} \to \mathcal{U}$, $\alpha > 0$, which is given by

$$\text{sign}_\alpha(z) := \begin{cases} \text{sign}(z) & \text{if } |z| \geq \alpha \\ 0 & \text{otherwise} \end{cases}. \quad (7)$$

The control action for $t \in [t_k^i, t_{k+1}^i[$ is given by

$$u_i(t) = \text{sign}_{\varepsilon_i(t_k^i)}\left(\text{ave}_i^w(t_k^i)\right). \quad (8)$$

The triggering times are given by $t_{k+1}^i = t_k^i + \Delta_k^i$, where

$$\Delta_k^i := \begin{cases} \frac{|\text{ave}_i^w(t_k^i)|}{4d_i} & \text{if } |\text{ave}_i^w(t_k^i)| \geq \varepsilon_i(t_k^i) \\ \frac{\varepsilon}{4d_i} & \text{otherwise} \end{cases}. \quad (9)$$

By construction, the inter-sampling times are bounded away from zero. This guarantees the existence of a unique Carathèodory solution for the state trajectories.

Remark 1: In the noise-free case, the control law (8) is an approximation of the pure (nonquantized) sign function which yields "max–min" consensus [28], that is, convergence to the center of the interval containing the node's initial values. Specifically, in the noise-free case, the scheme reduces to the one in [28] when $\varepsilon_i \equiv 0$ and the flow of information among nodes is continuous. We refer the reader to Section VI for further discussions on this point. ■

IV. Main Results

We start by showing that the proposed coordination scheme ensures boundedness of the state trajectories.

A. Boundedness of the State Trajectories

Let $\overline{x} := \max_{i\in I} x_i(0)$, $\underline{x} := \min_{i\in I} x_i(0)$, and

$$\gamma := \left(\frac{1}{3} + \frac{4}{3}\frac{d_{\max}}{\varepsilon}\right)|w|_\infty. \quad (10)$$

Theorem 1: Consider a network of n dynamical systems as in (1), which are interconnected over an undirected connected graph G. Let each local control input be generated in accordance with (5)–(9). Then, for every initial condition, the state x satisfies

$$\max_{i\in I} x_i(t) \leq \begin{cases} \overline{x} & \text{if } |\overline{x}| \geq \gamma \\ \gamma & \text{otherwise} \end{cases} \quad (11)$$

$$\min_{i\in I} x_i(t) \geq \begin{cases} \underline{x} & \text{if } |\underline{x}| \geq \gamma \\ -\gamma & \text{otherwise} \end{cases} \quad (12)$$

for every $t \in \mathbb{R}_{\geq 0}$.

Proof: Let $\text{ave}_i(t) := \sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t))$ denote the noiseless average. We will only prove the result regarding $\max_{i\in I} x_i(t)$ since the other result can be proved in an analogous manner. Notice that $\text{ave}_i^w(t) = \text{ave}_i(t) + \phi_i(t)$ for all $t \in \mathbb{R}_{\geq 0}$ and all $i \in I$, where we defined $\phi_i(t) := \sum_{j\in\mathcal{N}_i} w_j(t)$. Clearly, $|\phi_i(t)| \leq d_{\max}|w|_\infty$ for all $t \in \mathbb{R}_{\geq 0}$ and all $i \in I$.

Case 1: $|\overline{x}| \geq \gamma$. We show that there is no node that can exceed $\overline{x}$. Suppose that there exists a time t_* such that $\max_{i\in I} x_i(t_*) = \overline{x}$ and $u_i(t_*) > 0$, with i being the index of the node exceeding $\overline{x}$ for the first time (clearly, more than one node could exceed $\overline{x}$ at the same time but this does not affect the analysis). Let t_k^i be the last sampling instant not greater than t_*, which implies $x_s(t_k^i) \leq \overline{x}$ for all $s \in I$. Notice that t_k^i is well defined even if t_* occurs in the first intersampling interval of node i since $x_s(0) \leq \overline{x}$ for all $s \in I$.

Subcase 1: $x_i(t_k^i) > \overline{x} - \frac{1}{3}|w|_\infty$. The condition for x_i to grow is $\text{ave}_i^w(t_k^i) = \text{ave}_i(t_k^i) + \phi_i(t_k^i) \geq \varepsilon_i(t_k^i)$. Since $x_s(t_k^i) \leq \overline{x}$ for all $s \in I$, we have

$$\begin{aligned} \text{ave}_i(t_k^i) &\leq d_i\overline{x} - d_i x_i(t_k^i) \leq d_i\left(\overline{x} - (\overline{x} - \frac{1}{3}|w|_\infty)\right) \\ &\leq \frac{1}{3}d_{\max}|w|_\infty. \end{aligned} \quad (13)$$

Hence, in order for x_i to grow we must have $\frac{4}{3}d_{\max}|w|_\infty \geq \varepsilon_i(t_k^i)$. This leads to a contradiction. To see this, note that if $|x_i(t_k^i)| \geq 1$, then $\varepsilon_i(t_k^i) = \varepsilon|x_i(t_k^i)|$. Moreover, $|x_i(t_k^i)| > |\overline{x}| - \frac{1}{3}|w|_\infty$. Thus, we must have $\frac{4}{3}d_{\max}|w|_\infty > \varepsilon(|\overline{x}| - \frac{1}{3}|w|_\infty)$. This implies $|\overline{x}| < \gamma$, which leads to a contradiction. If instead $|x_i(t_k^i)| < 1$, then $\varepsilon_i(t_k^i) = \varepsilon$ and we must have $\frac{4}{3}d_{\max}|w|_\infty \geq \varepsilon$. This leads again to a contradiction since, by hypothesis, $\gamma \leq |\overline{x}|$ and $|\overline{x}| < 1 + \frac{1}{3}|w|_\infty$, which would imply $\frac{4}{3}d_{\max}|w|_\infty < \varepsilon$.

Subcase 2: $x_i(t_k^i) \leq \overline{x} - \frac{1}{3}|w|_\infty$. By construction, x_i can grow at most up to

$$\begin{aligned} &x_i(t_k^i) + \frac{1}{4d_i}(\text{ave}_i(t_k^i) + \phi_i(t_k^i)) \\ &= \frac{3}{4}x_i(t_k^i) + \frac{1}{4d_i}\sum_{j\in\mathcal{N}_i}(x_j(t_k^i) + w_j(t_k^i)) \\ &\leq \frac{3}{4}x_i(t_k^i) + \frac{1}{4}(\overline{x} + |w|_\infty) \end{aligned} \quad (14)$$

where the inequality follows because $x_s(t_k^i) \leq \overline{x}$ for all $s \in I$. Since $x_i(t_k^i) \leq \overline{x} - \frac{1}{3}|w|_\infty$, we conclude that x_i can grow at most up to $\overline{x}$, leading to a contradiction.

Case 2. $|\overline{x}| < \gamma$: The proof of this case is exactly the same as that for the previous case with $\overline{x}$ replaced by γ. ■

B. Properties on the Consensus Value

Theorem 2: Consider a network of n dynamical systems as in (1), which are interconnected over an undirected connected graph G. Let each local control input be generated in accordance with (5)–(9). Then, for every initial condition, the network state x enters in a finite time the set $\mathcal{D}$ in (3) and remains there forever. Moreover, x converges in a finite time to a point belonging to the set $\mathcal{D}$ in (3) when the noise converge to zero.

We prove two technical results which are instrumental for the proof of Theorem 2. The first result relates ε_i and r.

Lemma 1: Consider the same assumptions and conditions as in Theorem 2. For any $i \in I$, it holds that

$$\varepsilon_i(t_k^i) \leq r - \frac{5}{3}d_{\max}|w|_\infty \quad (15)$$

for every $k \in \mathbb{N}_0$.

Proof: By Theorem 1, $|x_i(t_k^i)| \leq \chi_0 + \gamma$. Hence

$$\begin{aligned} \varepsilon_i(t_k^i) &\leq \max\{\varepsilon, \varepsilon(\chi_0 + \gamma)\} \leq \max\{\varepsilon, \varepsilon\chi_0\} + \varepsilon\gamma \\ &= r - \frac{5}{3}d_{\max}|w|_\infty \end{aligned} \quad (16)$$

where the equality holds by the definitions of r and γ. ■

The second result shows that the average preserves the sign as long as its absolute value remains large enough compared with r.

Lemma 2: Consider the same assumptions and conditions as in Theorem 2. Pick any $i \in I$ and any $M \in \mathbb{N}_0$.

If $|\,\text{ave}_i(t^i_{k+m})| \geq r$ for $m = 0, 1, \ldots, M$, then for $m = 1, 2, \ldots, M+1$

$$\text{sign}(\text{ave}_i(t^i_{k+m})) = \text{sign}(\text{ave}_i(t^i_k)). \tag{17}$$

Proof: Assume without loss of generality that $\text{ave}_i(t^i_k) \geq r$, the other case being analogous. From Lemma 1, we have

$$\text{ave}^w_i(t^i_k) \geq \text{ave}_i(t^i_k) - d_{\max}|w|_\infty \geq r - d_{\max}|w|_\infty \geq \varepsilon_i(t^i_k). \tag{18}$$

Hence, $u_i(t^i_k) = 1$. Moreover

$$\begin{aligned}\text{ave}_i(t) &\geq \text{ave}_i(t^i_k) - 2d_i(t - t^i_k) \geq \text{ave}_i(t^i_k) - \frac{1}{2}\,\text{ave}^w_i(t^i_k)\\ &= \frac{1}{2}\,\text{ave}_i(t^i_k) - \frac{1}{2}\phi_i(t^i_k) \geq \frac{1}{2}r - \frac{1}{2}d_{\max}|w|_\infty\\ &\geq \frac{1}{2}\max\{\varepsilon, \varepsilon\chi_0\}\end{aligned} \tag{19}$$

for all $t \in [t^i_k, t^i_{k+1}]$. We conclude that $\text{ave}_i(t^i_{k+1}) > 0$. Thus, ave_i preserves its sign. ■

Proof of Theorem 2: We only consider the case $|w|_\infty \geq \varepsilon/(2d_{\max})$ since the other case follows from Theorem 3. To begin with, we introduce three sets into which we partition the set of switching times of each node i. For each $i \in I$, let

$$\begin{aligned}\mathscr{S}_{i1} &:= \left\{t^i_k : |\,\text{ave}^w_i(t^i_k)| \geq \varepsilon_i(t^i_k) \wedge |\,\text{ave}_i(t^i_k)| \geq r\right\}\\ \mathscr{S}_{i2} &:= \left\{t^i_k : |\,\text{ave}^w_i(t^i_k)| \geq \varepsilon_i(t^i_k) \wedge |\,\text{ave}_i(t^i_k)| < r\right\}\\ \mathscr{S}_{i3} &:= \left\{t^i_k : |\,\text{ave}^w_i(t^i_k)| < \varepsilon_i(t^i_k)\right\}.\end{aligned} \tag{20}$$

Clearly, $t^i_k \in \mathscr{S}_{i1} \cup \mathscr{S}_{i2} \cup \mathscr{S}_{i3}$ for every $k \in \mathbb{N}_0$.

Pick any $i \in I$, and assume by contradiction that there exists a time t_* such that $|\,\text{ave}_i(t^i_k)| \geq r$ for all $t^i_k \geq t_*$. In view of Lemma 1, u_i is never zero from t_* onward since the condition above yields $|\,\text{ave}^w_i(t^i_k)| \geq r - d_{\max}|w|_\infty \geq \varepsilon_i(t^i_k)$. Lemma 2 also implies that $\text{sign}(\text{ave}_i(t^i_{k+m})) = \text{sign}(\text{ave}_i(t^i_k))$ for every m. Thus, either $u_i(t) = 1$ for all $t^i_k \geq t_*$ or $u_i = -1$ for all $t^i_k \geq t_*$. This would imply that x_i diverges, violating the state boundedness property of Theorem 1. Hence, there exists a time instant t^i_k such that $|\,\text{ave}_i(t^i_k)| < r$. This implies that $t^i_k \notin \mathscr{S}_{i1}$, or equivalently, that $t^i_k \in \mathscr{S}_{i2} \cup \mathscr{S}_{i3}$. It remains to show that transitions from $\mathscr{S}_{i2}$ and $\mathscr{S}_{i3}$ to $\mathscr{S}_{i1}$ are not possible.

Case 1: $t^i_k \in \mathscr{S}_{i2}$. In this case, $u_i(t^i_k) = \{-1, 1\}$. Suppose $u_i(t^i_k) = 1$, the other case being analogous. Then $\text{ave}_i(t) \leq \text{ave}_i(t^i_k) < r$ for all $t \in [t^i_k, t^i_{k+1}]$ where the first inequality follows because $u_i(t^i_k) = 1$ while the second inequality follows because $t^i_k \in \mathscr{S}_{i2}$ by hypothesis. In addition, condition $u_i(t^i_k) = 1$ implies $\text{ave}_i(t^i_k) \geq \varepsilon_i(t^i_k) - \phi_i(t^i_k)$. Thus

$$\begin{aligned}\text{ave}_i(t) &\geq \text{ave}_i(t^i_k) - 2d_i(t - t^i_k) \geq \frac{1}{2}\,\text{ave}_i(t^i_k) - \frac{1}{2}d_{\max}|w|_\infty\\ &\geq \frac{1}{2}\varepsilon_i(t^i_k) - d_{\max}|w|_\infty > -r\end{aligned} \tag{21}$$

for all $t \in [t^i_k, t^i_{k+1}]$. Thus, $|\,\text{ave}_i(t^i_{k+1})| < r$, which implies that $t^i_{k+1} \notin \mathscr{S}_{i1}$.

Case 2: $t^i_k \in \mathscr{S}_{i3}$. In this case $u_i(t) = 0$ for all $t \in [t^i_k, t^i_{k+1}]$ and $t^i_{k+1} - t^i_k = \varepsilon/(4d_i)$. Hence,

$$\begin{aligned}|\,\text{ave}_i(t)| &\leq |\,\text{ave}_i(t^i_k)| + d_i(t - t^i_k) < \varepsilon_i(t^i_k) + d_{\max}|w|_\infty + \frac{\varepsilon}{4}\\ &< \varepsilon_i(t^i_k) + \frac{3}{2}d_{\max}|w|_\infty < r\end{aligned} \tag{22}$$

for all $t \in [t^i_k, t^i_{k+1}]$, where the third inequality follows from $\varepsilon \leq 2d_{\max}|w|_\infty$ and the fourth from Lemma 1. Hence, $t^i_{k+1} \notin \mathscr{S}_{i1}$.

Hence, we conclude that $t^i_\ell \in \mathscr{S}_{i2} \cup \mathscr{S}_{i3}$ for all $\ell \geq k$. Moreover, the previous arguments show that $|\,\text{ave}_i(t)| < r$ for all $t \in [t^i_\ell, t^i_{\ell+1}]$, for all $\ell \geq k$, which guarantees that x remains forever inside $\mathcal{D}$. Finally, if w converges to zero then there exists a finite instant t_* such that $\varepsilon > 2d_{\max}\sup_{t\geq t_*}|w(t)|$, and the convergence result follows along the same lines as in Theorem 3. ■

C. Low-Magnitude Noise

For noise of a low-magnitude, one can establish a stronger, point convergence result.

Theorem 3: Consider a network of n dynamical systems as in (1), which are interconnected over an undirected connected graph G. Let each local control input be generated in accordance with (5)–(9). Suppose that $|w|_\infty < \varepsilon/(2d_{\max})$. Then, for every initial condition, the state x converges in a finite time to a point belonging to the set $\mathcal{D}$ in (3). Moreover, $\max_{i\in I} x_i(t) \leq \overline{x}$ and $\min_{i\in I} x_i(t) \geq \underline{x}$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof: We first prove the last property. We only show that $\max_{i\in I} x_i(t) \leq \overline{x}$ for all $t \in \mathbb{R}_{\geq 0}$. Suppose that there exists a time t_* such that $\max_{i\in I} x_i(t_*) = \overline{x}$ and $u_i(t_*) > 0$, with i the index of the first node exceeding $\overline{x}$ (clearly, more than one node could exceed $\overline{x}$ at the same time but this does not affect the analysis). Let t^i_k be the last sampling instant not greater than t_*, which implies $x_s(t^i_k) \leq \overline{x}$ for all $s \in I$. Note that t^i_k is well defined even if t_* occurs during the first intersampling interval of node i since $x_s(0) \leq \overline{x}$ for all $s \in I$. Clearly, $|\,\text{ave}^w_i(t^i_k)| = \text{ave}^w_i(t^i_k) \geq \varepsilon_i(t^i_k)$. By (9)

$$\begin{aligned}\Delta^i_k &= \frac{1}{4d_i}|\,\text{ave}^w_i(t^i_k)| = \frac{1}{4d_i}\sum_{j\in\mathcal{N}_i}\left(x_j(t^i_k) - x_i(t^i_k) + w_j(t^i_k)\right)\\ &\leq \frac{1}{4}(\overline{x} - x_i(t^i_k) + |w|_\infty)\end{aligned} \tag{23}$$

where the inequality follows from the fact that $x_s(t^i_k) \leq \overline{x}$ for all $s \in I$. By hypothesis, t^i_k is the last sampling instant not greater than t_*. Since the control input is constant over $[t^i_k, t^i_{k+1}]$ and because x_i must exceed $\overline{x}$, we must have $\overline{x} < x_i(t^i_{k+1})$. Hence

$$\overline{x} - x_i(t^i_k) < \Delta^i_k \leq \frac{1}{4}(\overline{x} - x_i(t^i_k) + |w|_\infty). \tag{24}$$

This inequality is possible only when $\overline{x} - x_i(t^i_k) < \frac{1}{3}|w|_\infty$. However, this implies

$$\begin{aligned}\text{ave}^w_i(t^i_k) &= \sum_{j\in\mathcal{N}_i}\left(x_j(t^i_k) - x_i(t^i_k) + w_j(t^i_k)\right)\\ &\leq d_{\max}(\overline{x} - x_i(t^i_k) + |w|_\infty) < \varepsilon\end{aligned} \tag{25}$$

since $2d_{\max}|w|_\infty < \varepsilon$ by hypothesis. Hence, $\text{ave}^w_i(t^i_k) < \varepsilon_i(t^i_k)$ which leads to a contradiction.

We now focus on convergence. Let $V = \frac{1}{2}x^\top Lx$ where L is the graph Laplacian matrix, and consider the evolution of V along the solutions to (1). By letting $t^i_k = \max\{t^i_h \leq t, h \in \mathbb{N}_0\}$ and recalling the definition of the quantized sign function, we have

$$\dot{V}(x(t)) = u^\top(t)Lx(t) = -\sum_{i=1}^{n} \text{ave}_i(t)\,\text{sign}_{\varepsilon_i(t_k^i)}(\text{ave}_i^w(t_k^i))$$
$$= -\sum_{i:|\text{ave}_i^w(t_k^i)|\geq\varepsilon_i(t_k^i)} \text{ave}_i(t)\,\text{sign}(\text{ave}_i^w(t_k^i)). \quad (26)$$

Observe now that if $\text{ave}_i^w(t_k^i) \geq \varepsilon_i(t_k^i)$ then $\text{sign}(\text{ave}_i^w(t_k^i)) = 1$. Moreover

$$\text{ave}_i(t) \geq \text{ave}_i(t_k^i) - 2d_i(t - t_k^i) \geq \text{ave}_i(t_k^i) - \frac{1}{2}\text{ave}_i^w(t_k^i)$$
$$= \frac{1}{2}\text{ave}_i^w(t_k^i) - \phi_i(t_k^i) \geq \frac{1}{2}\varepsilon - d_{\max}|w|_\infty \quad (27)$$

for all $t \in [t_k^i, t_{k+1}^i]$. In a similar way, if $\text{ave}_i^w(t_k^i) \leq -\varepsilon_i(t_k^i)$ then $\text{sign}(\text{ave}_i^w(t_k^i)) = -1$, and

$$\text{ave}_i(t) \leq \text{ave}_i(t_k^i) + 2d_i(t - t_k^i) \leq \text{ave}_i(t_k^i) + \frac{1}{2}|\text{ave}_i^w(t_k^i)|$$
$$\leq -\frac{1}{2}\varepsilon + d_{\max}|w|_\infty \quad (28)$$

for all $t \in [t_k^i, t_{k+1}^i]$. This gives

$$\dot{V}(x(t)) \leq -\sum_{i:|\text{ave}_i^w(t_k^i)|\geq\varepsilon_i(t_k^i)} \left(\frac{1}{2}\varepsilon - d_{\max}|w|_\infty\right) \quad (29)$$

for all $t \geq 0$. Since $\varepsilon > 2d_{\max}|w|_\infty$, there exists a finite time T' after which each node satisfies $|\text{ave}_i^w(t_k^i)| < \varepsilon_i(t_k^i)$ for every $t_k^i \geq T'$, otherwise V would take on negative values. In addition, since x remains within the initial envelope then $|\text{ave}_i^w(t)| \leq d_i(2\chi_0 + |w|_\infty)$ for all $t \in \mathbb{R}_{\geq 0}$. Thus, $\Delta_k^i \leq \max\{\varepsilon, (2\chi_0 + |w|_\infty)\}/4 := \bar{\Delta}$ for every $k \in \mathbb{N}_0$. This shows that all the controls eventually become zero not later than $T := T' + \bar{\Delta}$, which implies that $x_i(t) = x_i(T)$ and $\text{ave}_i(t) = \text{ave}_i(T)$ for all $t \geq T$. Moreover, since x remains within the initial envelope, we also have $\varepsilon_i(t) \leq \max\{\varepsilon, \varepsilon\chi_0\}$ for all $t \in \mathbb{R}_{\geq 0}$. Taking any $t_k^i \geq T$, we then have

$$|\text{ave}_i(t)| = |\text{ave}_i(t_k^i)| \leq |\text{ave}_i^w(t_k^i)| + d_{\max}|w|_\infty$$
$$\leq \max\{\varepsilon, \varepsilon\chi_0\} + d_{\max}|w|_\infty. \quad (30)$$

The proof is concluded by noting that the right side of (30) is upper bounded by r. ■

V. Discussion

In this section, we further discuss a number of properties related to the proposed coordination scheme.

A. Adaptive Thresholds and Sign Function

The main problem when dealing with communication noise is that the graph Laplacian matrix has an eigenvalue at zero. This may cause the state to drift when the noise has nonzero mean. In this paper, drifting is prevented by resorting to local adaptive thresholds

$$\varepsilon_i(t) := \begin{cases} \varepsilon|x_i(t)| & \text{if } |x_i(t)| \geq 1 \\ \varepsilon & \text{otherwise} \end{cases}. \quad (31)$$

These adaptive thresholds scale with the magnitude of the data and this feature is essential to guarantee that any drifting will eventually stop. Specifically, recall that the local control action is given by

$$u_i(t) = \text{sign}_{\varepsilon_i(t_k^i)}(\text{ave}_i^w(t_k^i))$$
$$= \text{sign}_{\varepsilon_i(t_k^i)}\Big(\text{ave}_i(t_k^i) + \sum_{j\in\mathcal{N}_i} w_j(t_k^i)\Big). \quad (32)$$

Suppose that x_i starts drifting, for example, growing ($u_i \equiv 1$). Since $u_i \equiv 1$, $\text{ave}_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i)$ cannot grow, so that ave_i^w must remain bounded. Hence, adapting the threshold of the sign function to the magnitude of x_i eventually forces ε_i to become larger than ave_i^w. In contrast, a pure constant ε need not counteract the drifting of x_i since ave_i^w may persistently remain larger than ε.

Another interesting feature of the proposed scheme lies in the sign function. When the level of disagreement is large compared with the noise magnitude, for example, during the initial phase of coordination, then $\text{ave}_i^w \approx \text{ave}_i$. In this situation, the sign function ensures that the control action will be the same as in the noiseless case. In other terms, the noise will affect coordination only when nodes are sufficiently close to consensus. We will exemplify this feature in Section VI. The sign function also permits to save communication resources, which is one of the main issues when coordination is carried out through digital networks. As noted before, when ave_i is large compared with the noise magnitude, then $\text{ave}_i^w \approx \text{ave}_i$ and the control action behaves as in the noiseless case. In the proposed scheme, condition $\text{ave}_i^w \approx \text{ave}_i$ is implemented as $|\text{ave}_i^w| \geq \varepsilon_i$. When $|\text{ave}_i^w| \geq \varepsilon_i$, then Δ_k^i increases with ave_i^w with the idea that large values of ave_i^w correspond to a situation where the disagreement is large so that there is no need for frequent control variations. The situation is different when $|\text{ave}_i^w| < \varepsilon_i$. In this case, it may happen that ave_i^w is significantly different from ave_i. Moreover, $|\text{ave}_i^w| < \varepsilon_i$ also implies that the level of disagreement is small compared with the data magnitude. Thus, if $|\text{ave}_i^w| < \varepsilon_i$ then Δ_k^i is decreased to $\varepsilon/(4d_i)$ with the idea that control variations should be made more frequent so as to counteract the effect of noise and maintain a small level of disagreement. Clearly, in this situation, Δ_k^i may become small if ε is chosen small, and the latter is desired to ensure a small level of disagreement. As discussed in the next section, there is often no need to pick ε very small in order to secure a small level of disagreement, which means that communications need not be frequent even when the nodes are within the consensus region.

B. Node-To-Node Error

Our coordination scheme guarantees that in the noise-free case all the nodes converge in a finite time to a point satisfying

$$\left|\sum_{j\in\mathcal{N}_i}(x_j - x_i)\right| < \max\{\varepsilon, \varepsilon\chi_0\},\ \forall i \in I. \quad (33)$$

The parameter ε plays a crucial role for consensus. On one side, it is desirable to choose $\varepsilon \ll 1$ so as to guarantee a small level of disagreement. On the other hand, a very small value of ε can render the coordination scheme very sensitive to noise. Moreover, as noted before, small values of ε can induce large communication rates since ε determines the smallest intersampling time of each node. It is the term $|\sum_{j\in\mathcal{N}_i}(x_j - x_i)|$ that somehow makes this tradeoff less critical.

At first glance, it seems indeed more natural to search for coordination schemes that guarantee a property like

$$|x_j - x_i| < \max\{\varepsilon, \varepsilon\chi_0\} \quad \forall i, j \in I \quad (34)$$

or *node-to-node* error. In fact, the latter guarantees that the disagreement is small for every pair of nodes (not necessarily connected), while (33) only ensures that the disagreement is small locally (for its neighbourhood). Actually, in many cases of practical interest it turns out that a bound r on the local averages implies a bound on the node-to-node error which is strictly smaller than r. In this situation, working with (33) is advantageous compared with (34) since this guarantees a small node-to-node error without requiring to choose ε too small. In turn, this moderates the noise sensitivity and the number of communications. As discussed next, this happens when the network connectivity is sufficiently large.

Consider the same setting as in Theorem 2, and let T denote the time after which the network state remains confined in $\mathcal{D}$. Pick any fixed time instant $t \geq T$ and let x_M and x_m denote the network nodes taking on the maximum and minimum values, respectively. The indices M and m may change with time but we consider a fixed time t. Let $\alpha := x_M(t) - x_m(t)$ with $\alpha > 0$ (the case $\alpha = 0$ is not interesting because the network would be at perfect consensus). By Theorem 2, $|\,\text{ave}_i(t)| < r$ for all $i \in I$. We now relate α and r. First notice that

$$\begin{aligned}\text{ave}_M &= d_M(x_m - x_M) + \sum_{j\in\mathcal{N}_M}(x_j - x_m)\\ &= -\,d_M\alpha + \sum_{j\in\mathcal{N}_M}(x_j - x_m)\end{aligned} \tag{35}$$

where we omitted the time argument for brevity. Decompose $\mathcal{N}_M = (\mathcal{N}_M \setminus \mathcal{N}_m) \cup (\mathcal{N}_M \cap \mathcal{N}_m)$. Since $x_j - x_m \leq \alpha$ for all $j \in I$, we obtain

$$\sum_{j\in(\mathcal{N}_M\setminus\mathcal{N}_m)}(x_j - x_m) \leq \delta\alpha \tag{36}$$

where

$$\delta := \begin{cases} |\mathcal{N}_M \setminus \mathcal{N}_m| - 1 & \text{if } m \in \mathcal{N}_M \\ |\mathcal{N}_M \setminus \mathcal{N}_m| & \text{otherwise}\end{cases}. \tag{37}$$

Moreover,

$$\sum_{j\in(\mathcal{N}_M\cap\mathcal{N}_m)}(x_j - x_m) < \mu \tag{38}$$

where

$$\mu := \begin{cases} r - \alpha & \text{if } M \in \mathcal{N}_m \\ r & \text{otherwise}\end{cases}. \tag{39}$$

In fact, $\sum_{j\in Q}(x_j - x_m) < r$ for every set $Q \subseteq \mathcal{N}_m$ because $|\,\text{ave}_m\,| < r$ and since $x_m = \min_{i\in I} x_i$. In addition, if $M \in \mathcal{N}_m$, we then have $(\mathcal{N}_M \cap \mathcal{N}_m) \subseteq (\mathcal{N}_m \setminus \{M\})$, which implies $\mu = r - \alpha$. Since $|\,\text{ave}_M\,| < r$, we have

$$-r < \text{ave}_M < -(d_M - \delta)\alpha + \mu \tag{40}$$

which implies

$$\alpha < (r + \mu)\frac{1}{d_M - \delta} \tag{41}$$

assuming $d_M - \delta > 0$.

The quantity $d_M - \delta$ represents the number of neighbors that are common to x_M and x_m. Since $\mu \leq r$, it is sufficient that $d_M - \delta \geq 2$ to guarantee that $\alpha < r$. Even more, α may become significantly smaller than r for large values of $d_M - \delta$. Consider, for example, *complete* graphs. In this case, $d_M = n - 1$, $\delta = 0$, and $\mu = r - \alpha$. Hence, $\alpha < 2r/n$. Since $n \geq 2$, we always have $\alpha < r$. Moreover, recalling that $r = \max\{\varepsilon, \varepsilon\chi_0\} + \left(\frac{\varepsilon}{2} + 3d_{\max}\right)|w|_\infty$, one sees that in the noiseless case α actually decreases with n whenever the initial conditions do not depend on the network size, and remains bounded irrespective of w with a maximum noise amplification factor of six. These considerations apply in general since (41) does not depend on the network topology. In fact, (41) suggests that working with (33) can be advantageous compared with (34) whenever the network connectivity is sufficiently large. We will further substantiate this analysis in Section VI through numerical simulations.

VI. Numerical Simulations

We consider Erdös–Rényi (ER) and random geometric (RG) graphs [29]. The former is obtained from the n-dimensional complete graph by retaining each edge with probability p (independently). The latter is obtained by considering a random uniform deployment of n points in a two-dimensional Euclidian space. Denoting by s_i the position of node i, a link between i and k exists if and only if $|s_i - s_k| \leq R$ where R denotes the communication range, assumed identical for every node. We run Monte Carlo simulations with $N_{\text{trials}} = 1000$ trials. For each trial, we generate an ER (RG) graph of 100 nodes. Graphs which are not connected are not taken into account. For the ER graph we consider a link probability $p = 0.08$, while for the RG graph we consider a random deployment over a region of 1 km × 1 km with node communication range $R = 160$ m. For each trial, the node's initial values are taken randomly within $[-2, 2]$, and the noise is taken as a random number within $[-0.2, 0.2]$. The sensitivity parameter is $\varepsilon = 0.1$ for all the trials.

Let $\{t_s\}_{s\in\mathbb{N}_0}$ be the sequence of time instants at which one of the node samples, i.e., $t_s = t_k^i$ for some $i \in I$ and $k \in \mathbb{N}_0$. Given a simulation horizon H, this sequence will range from t_0 up to t_S where S is the largest integer such that $t_S \leq H$. The *asymptotic* behavior of the nodes is defined as the behavior of the nodes over the time interval $[t_{S-W+1}, t_{S-W+2}, \ldots, t_S]$, where W is a positive integer that is selected so as to satisfy $W \gg 1$ and $W \ll S$. The reason for this choice is twofold: 1) since the network nodes need not converge, it makes little sense to consider only the value of the nodes at the final step t_S. In this respect, $W \ll S$ makes it possible to evaluate the network behavior for a sufficiently large number of samples and 2) we aim at evaluating the network limiting behavior, that is, after the transient has vanished. Hence, $W \gg 1$ guarantees that initial samples are not taken into account. In the simulations, for each trial, we consider $H = 10^5$ and $W = 1000$. We consider three performance indices as follows.

1) *Asymptotic maximum local average:*

$$A_{MLA} := \frac{1}{N_{\text{trials}}}\sum_{k=1}^{N_{\text{trials}}}\left(\frac{1}{W}\sum_{s=S-W+1}^{S}\max_{i\in I}|\,\text{ave}_i(t_s)|\right)$$

Basically, for each of the trials, we compute the average of the largest value of the local averages over the time interval $[t_{S-W+1}, t_{S-W+2}, \ldots, t_S]$. Then, these values are averaged over the number of trials.

2) *Asymptotic maximum node-to-node distance:*

$$A_{MND} := \frac{1}{N_{\text{trials}}}\sum_{k=1}^{N_{\text{trials}}}\left(\frac{1}{W}\sum_{s=S-W+1}^{S}\max_{i,j\in I}|x_i(t_s) - x_j(t_s)|\right)$$

In this case, for each trial, we compute the average of the largest value of the node-to-node distances over the interval $[t_{S-W+1}, t_{S-W+2}, \ldots, t_S]$. As before, these values are then averaged over the number of trials.

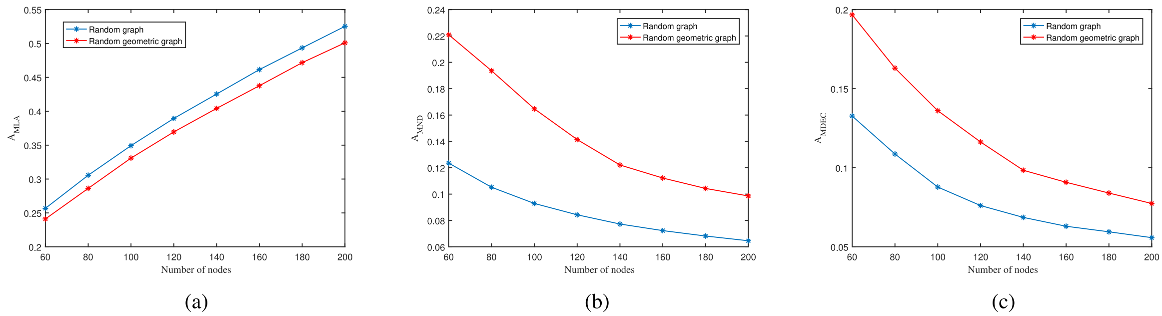

Fig. 1. Monte Carlo simulation results for the ER and RG graphs. (a) A_{MLA}. (b) A_{MND}. (c) A_{MDEC}.

3) *Asymptotic maximum distance from the expected convergence point:*

$$A_{MDEC} := \frac{1}{N_{\text{trials}}} \sum_{k=1}^{N_{\text{trials}}} \left(\frac{1}{W} \sum_{s=S-W+1}^{S} \max_{i \in I} |x_i(t_s) - x_*| \right)$$

where $x_* := \max_{i \in I} x(0) + \min_{i \in I} x(0)/2$. This index is similar to A_{MND}, with the exception that the node values are compared to the midpoint x_* of the maximum and minimum initial values of the nodes. This is because our algorithm is an approximation of the pure $\text{sign}(\text{ave}_i)$ consensus which is known to converge to x_* [28].

The results are reported in Fig. 1. Fig. 1(a) confirms the bound obtained in Theorem 2, showing that the local averages scale nicely with $d_{\max}$ [cf. (2)]. More interesting is the result in Fig. 1(b) which shows that the node-to-node error decreases as the number of nodes increases. This can be explained by observing that for both the graphs the expected number of common neighbors increases with n, which causes α in (41) to decrease in agreement with the comments made in Section V-B. In particular, for the ER graph the expected number of common neighbors between two network nodes is given by $(n-2)p^2$, while for the RG graph the probability that two nodes are connected is given by $\bar{p} = \pi R^2/|A| = 0.08$ where $|A|$ is the area of the deployment region, and the expected number of common neighbors between two connected nodes is approximately $0.58n\bar{p}$ [30]. This can explain why A_{MND} is smaller for the ER graph. Fig. 1(c) finally shows that the distance from the expected convergence point is indeed small and decreases with n. The latter property can be explained by noting that large values of n decrease the effect of ε (cf. Section V-B), which causes the quantized sign function to better approximate the pure $\text{sign}(\text{ave}_i)$ function.

We report in Fig. 2 the results of one of the trials for the ER graph. In this trial, we obtain $d_{\max} = 14$ which leads to $r = 8.8067$ and $\gamma = 37.2667$. The large theoretical bounds are due to the large value of $d_{\max}$. In practice, as reported in Fig. 2, the regulation performance is very high. In fact, the absolute value of the noiseless averages is eventually upper bounded by 0.5, which is much smaller than the theoretical bound given by r. We omit the simulation results of one trial for the RG graph since the figures are similar to the ones for the ER graph.

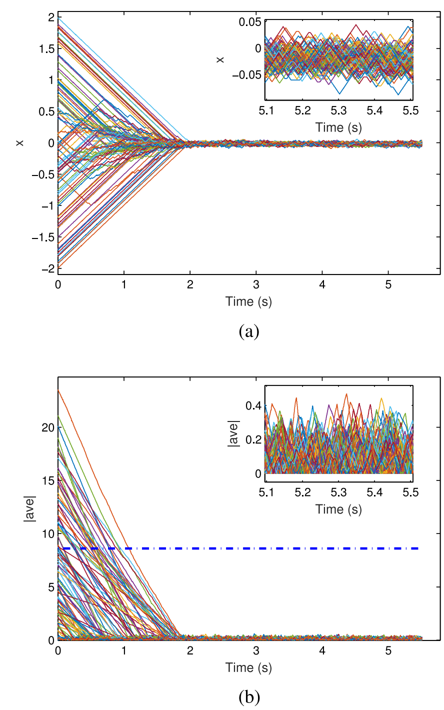

Fig. 2. Network behavior for one of the Monte Carlo trials relative to ER graphs. (a) State trajectory. (b) Absolute value of the noiseless averages.

VII. Conclusion

We proposed a novel self-triggered network coordination scheme that can handle unknown-but-bounded noise affecting network communication. The proposed scheme employs a dynamic, state-dependent, triggering policy and ternary controllers. It has been shown that the scheme can achieve finite-time practical consensus in both noiseless and noisy cases. In the latter situation, the node disagreement value scales nicely with the magnitude of the noise. An interesting feature of the proposed scheme is that the implementation does not require any global information about the network parameters and/or the operating environment. Moreover, the communication between nodes occurs only at discrete time instants, and nodes can sample independently and in an aperiodic manner. The last feature renders the proposed scheme applicable when coordination is through digital communication networks. An interesting outcome of this study is that the proposed scheme can guarantee a small node-to-node error without requiring to choose the consensus threshold too small. In turn, this can be beneficial for moderating the noise sensitivity as well as the number of communications. Investigating this point in more detail certainly represents an interesting venue for future research.

Appendix
Communication Delays

In this section, we briefly discuss how transmission delays can be taken into account.

For each $i \in I$, let $\{t_k^i\}_{k \in \mathbb{N}_0}$ with $t_0^i = 0$ be the sequence of time instants at which node i starts collecting data from its neighbors. Given a neighbor $j \in \mathcal{N}_i$, node i will receive information from j at a certain time $s_k^{ij} := t_k^i + \tau_k^{ij}$, where τ_k^{ij} represents the total delay in the communication between i and j. In general, τ_k^{ij} can be time varying (dependence on k) as well as link dependent (dependence on i and j). At s_k^{ij}, the information received by node i is given by $z_j(v_k^{ij})$ for some $v_k^{ij} \in [t_k^i, s_k^{ij}]$, which represents the time at which j transmits its value. At time $s_k^i := \max_{j\in\mathcal{N}_i} s_k^{ij} = t_k^i + \max_{j\in\mathcal{N}_i} \tau_k^{ij}$ node i will then have all the information needed to update its control action. Accordingly, $\{s_k^i\}_{k\in\mathbb{N}_0}$ will define the sequence of control updates.

The control action is given by

$$u_i(t) = \begin{cases} 0 & t \in [0, s_0^i[\\ \text{ave}_i^{w,\tau}(s_k^i) & t \in [s_k^i, s_{k+1}^i[\end{cases} \tag{42}$$

where $\text{ave}_i^{w,\tau}(s_k^i) := \sum_{j\in\mathcal{N}_i}(z_j(v_k^{ij}) - x_i(s_k^i))$. Before time s_0^i, node i has no information from the whole neighboring set so that its control is set to zero. On the other hand, $\text{ave}_i^{w,\tau}$ is the natural generalization of the control action considered in the delay-free case, where the additional superscript indicates the presence of delays.

The triggering instants are now given by $t_{k+1}^i = s_k^i + \Delta_k^i$, where

$$\Delta_k^i := \begin{cases} |\text{ave}_i^{w,\tau}(s_k^i)|/4d_i & \text{if } |\text{ave}_i^{w,\tau}(s_k^i)| \geq \varepsilon_i(s_k^i) \\ \varepsilon/4d_i & \text{otherwise} \end{cases} \tag{43}$$

which is also the natural generalization of the triggering policy of Section III. As before, by construction, the intersampling times are bounded away from zero. In addition, $s_k^i \geq t_k^i$ with equality holding if and only if delays are zero, and $t_{k+1}^i > s_k^i$.

Rewrite now

$$z_j(v_k^{ij}) = x_j(v_k^{ij}) + w_j(v_k^{ij}) = x_j(s_k^i) + \bar{w}_{ij}(s_k^i) \tag{44}$$

where $\bar{w}_{ij}(s_k^i) := w_j(v_k^{ij}) + x_j(v_k^{ij}) - x_j(s_k^i)$. Since the control action always belongs to $\{-1, 0, 1\}$ and since $s_k^i - v_k^{ij} \leq s_k^i - t_k^i \leq \max_{j\in\mathcal{N}_i} \tau_k^{ij}$, we are guaranteed that $|\bar{w}_{ij}(s_k^i)| \leq |w|_\infty + \tau_{\max}$, where $\tau_{\max} := \sup_{k\in\mathbb{N}_0} \max_{i\in I} \max_{j\in\mathcal{N}_i} \tau_k^{ij}$ represents the maximum delay that can occur over a communication channel. Hence

$$\begin{aligned} \text{ave}_i^{w,\tau}(s_k^i) &= \sum_{j\in\mathcal{N}_i}(x_j(s_k^i) - x_i(s_k^i)) + \sum_{j\in\mathcal{N}_i}\bar{w}_{ij}(s_k^i) \\ &= \text{ave}_i(s_k^i) + \sum_{j\in\mathcal{N}_i}\bar{w}_{ij}(s_k^i). \end{aligned} \tag{45}$$

This shows that the analysis for the case of delays can be approached as in the delay-free case by considering a different, possibly larger, noise contribution. In fact, it is simple to see that the analysis of Section IV carries over to the case of delays in a similar manner by replacing t_k^i with s_k^i and $|w|_\infty$ with $|w|_\infty + \tau_{\max}$. Due to the lack of space, details are omitted but can be found in [31].

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