Modular sequent calculi for classical modal logics

Gilbert, David; Maffezioli, Paolo

Published in:
Studia Logica

DOI:
10.1007/s11225-014-9556-1

Publication date:
2015

David R. Gilbert
Paolo Maffezioli

Modular Sequent Calculi for Classical Modal Logics

Abstract. This paper develops sequent calculi for several classical modal logics. Utilizing a polymodal translation of the standard modal language, we are able to establish a base system for the minimal classical modal logic **E** from which we generate extensions (to include **M**, **C**, and **N**) in a modular manner. Our systems admit contraction and cut admissibility, and allow a systematic proof-search procedure of formal derivations.

Keywords: Classical modal logics, Neighborhood semantics, Labelled sequent systems.

1. Introduction

Since their inception (brought about by Montague [14] and Scott [21], independently), neighborhood frames have become a standard semantic resource for logicians and philosophers wishing to work with modal logics weaker than those characterized by relational frames. However, while the domain of application of these so-called "classical" modal logics and their neighborhood semantics grows ever wider, and the model theory of neighborhood structures becomes better understood (e.g. [9]), there remains a noticeable absence in the development of the proof theory for these systems. This paper takes some first steps towards filling this void.

In particular, we introduce a labelled sequent system (such systems for normal modal logics can be found in [3] or [15], amongst others) for the minimal classical modal logic (often referred to as **E** [1]), as well as several extensions. Labelled sequent systems are variants of standard sequent calculi that encode semantic information into the syntax of the rules (in a manner parallel to the prefixes of some tableau systems [2]). These semantic markers allow us to bridge the gap more easily between the extant model theoretic results and the purely syntactic approaches familiar to proof theorists.

There has been some prior work in this area. For example, Hansen [8] and Governatori and Luppi [6] both provide tableau systems for monotonic modal logics. In addition, Indrzejczak [10,11] provides sequent systems for the minimal classical modal logic **E**, **N**, and the basic monotonic logic **M**, as well as several extensions (specifically those obtained by adding $D, T, B, 4$,

Presented by **Andrzej Indrzejczak**; *Received* October 25, 2013

Studia Logica (2015) 103: 175–217
DOI: 10.1007/s11225-014-9556-1

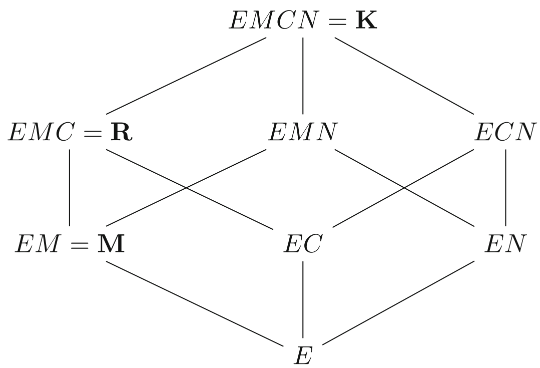

Figure 1. Map of classical modal logics, from Chellas [1]

and 5). These studies have helped to provide a clearer picture of proof systems for monotonic modal logics. However, what is still largely absent from the literature are unified proof-theoretic accounts of the family of logics situated between **E** and **K** (See Fig. 1). One exception to this is the work by Lavendhomme and Lucas [13], in which sequent systems are given for **E**, **M**, **C**, and **N**, as well as several combinations. One drawback of their approach is that one cannot easily move between logics; such moves are necessarily accompanied by changes to the cores of the underlying systems.

An attractive feature of Hilbert-style axiomatizations of modal logics is that they are *modular*: axioms can be added or subtracted in a component-wise fashion, generating a well-behaved lattice of logics with a common base. Maintaining this modularity is challenging when developing Gentzen systems for these logics.[1]

Importantly, our approach allows us to preserve this feature: specific frame conditions are each associated with a set of rules, and various logics can then be obtained via the addition of these sets to a common core governing the logical connectives and modal operators. Furthermore, we show that, by extending the method of [15] to non-normal modal logics, these

[1]For example, the ordinary sequent calculus for the logic **R**, as presented in [13], is not obtained by simply adding to the rule for **M** the rules for **C**. Rather, a new set of rules is given. While these rules represent the synthesis of the two different systems very nicely (a strength of their approach), the proof-theoretic apparatus already developed for **M** and **C** can not be immediately implemented. This has the consequence that proving cut-elimination and other admissibility results for the resulting system is not immediate. One of the advantages of modular systems is that cut-elimination for an extension of a system for which cut-elimination has already been proven reduces to demonstrating that the new rules continue to permit the elimination of cut.

rules can be added in a cumulative manner without requiring us to forgo cut elimination (or other desirable properties of sequent systems).

Because labelled systems have been developed primarily in the context of logics characterized by relational semantics, it is not immediately obvious how to adapt this method when one is working with structures described by a relatively rich set-theoretical language. Unlike in the relational case, where representing properties of binary relations is often more or less straightforward, it is unclear how one ought to represent, in the labelled system, the richer structural information present when dealing with neighborhood frames. We address this problem by making use of the fact that one can simulate a classical modal logic by way of a normal polymodal logic (for details, see [12] or [4]). This allows us straightforwardly to borrow much of the basic proof-theoretic apparatus already developed for normal modal logics. Furthermore, working in the translated language facilitates a clearer understanding of the set-theoretic frame conditions from a purely relational perspective. However, the translation only helps establish a starting point, and leaves open the important question of how best to achieve a uniform approach incorporating the intermediate logics between **E** and **K**. The majority of the paper is concerned with answering this question.

In order to do so, we make use of so-called *systems of rules*, which allow labelled systems to be developed for theories described by *generalized geometric formulas* (this has been introduced very recently in [18]). We show that the neighborhood frame conditions (understood relationally) for the intermediate logics correspond to instances of this type, and so can be characterized by these types of rule systems.

In the next section we provide a brief overview of classical modal logics as they are often presented (for example in Chellas [1]). We also discuss a translation from the language of propositional modal logic into a multi-modal language, due to [12], that we will utilize in constructing our sequent systems. Section 3 introduces the system for **E**, which will serve as the groundwork for the rest of the paper. Sections 4, 5, and 6 extend the approach to cover **N**, **C**, and **M**. Lastly, we examine an approach to some completeness results in Sect. 8.

2. Semantics and Axiomatics for Classical Modal Logics

2.1. Syntax

Given a countable set of propositional variables Var, the wff of the basic propositional modal language, $\mathcal{L}_1$, are generated by:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

where $\top, \vee, \rightarrow, \leftrightarrow$, and $\Diamond$ can be defined as usual. It is most common to provide a Hilbert-style axiomatization of classical modal logics. We mostly follow Chellas [1] in his naming conventions. **E** is the smallest set of $\mathcal{L}_1$ formulas containing all propositional tautologies and closed under modus ponens and the following rule:

$$(RE)\ \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

As usual, we can extend **E** by adding various rules and axiom schemes. Of relevance to this paper is the rule

$$(RM)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

and the axioms:

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$
C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$
N $\Box\top$

It is worth pointing out that the smallest *monotonic* modal logic, **M**, can be described in two, equivalent ways: (i) the smallest set of formulas containing all propositional tautologies and closed under modus ponens and the rule RM; (2) the smallest set of formulas that contains all propositional tautologies, all instances of the axiom scheme M, and is closed under modus ponens and RE. One obtains **C** from **E** by adding the C scheme, and **N** by adding N, as expected. **R** is the logic obtained from **E** by ensuring M and C are both included, and one can generate **K**, the smallest normal modal logic, by adding M, C, and N to **E** (see [1] for details). When φ is a theorem of **E**, for example, we will sometimes write this as $\vdash_{\mathbf{E}} \varphi$.

2.2. Semantics

The most common semantic approach to classical modal logics are by way of *neighborhood frames* (also called "minimal frames" in [1]).

DEFINITION 1. (*Neighborhood Frame*). A neighborhood frame is a pair $\langle W, n\rangle$ where W is a non-empty set of states and $n : W \rightarrow \wp(\wp(W))$ is a neighborhood function assigning, to each state, a set of propositions (where a proposition is understood as a subset of W). A neighborhood model is a triple $\langle W, n, V\rangle$ where W and n are as before, and $V : Var \rightarrow \wp(W)$ is a valuation function.

Truth at a state in a model for formulas can be defined recursively:

$M, w \models p$	iff	$w \in V(p)$
$M, w \models \neg\varphi$	iff	it is not the case that $M, w \models \varphi$ (or $M, w \not\models \varphi$)
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \Box\varphi$	iff	$[\![\varphi]\!] \in n(w)$

where $[\![\varphi]\!] := \{w \in W \mid M, w \models \varphi\}$ (the *truth-set* of φ). A formula φ is said to be *globally true* in a model M just in case $M, w \models \varphi$ for all $w \in W$. This will be denoted $M \models \varphi$. A formula is *valid* in a frame iff it is globally true in all models based on the frame. Lastly, a formula is *valid with respect to a class of frames* iff it is valid on each frame in the class.

In order to provide frame characterization theorems for the classical modal logics we are interested in, we make use of the following definitions. Given a set W, $\mathcal{F} \subseteq \wp(W)$ is said to

- be *supplemented* if for $X, Y \in \wp(W)$, $X \in \mathcal{F}$ and $X \subseteq Y$ implies $Y \in \mathcal{F}$;
- be *closed under (binary) intersections* if for $X, Y \in \wp(W)$, $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ implies $(X \cap Y) \in \mathcal{F}$;
- *contain the unit* if $W \in \mathcal{F}$;
- *contain the core* if $\bigcap \mathcal{F} \in \mathcal{F}$.

$\mathcal{F}$ is said to be a *quasi-filter* if it is supplemented and closed under intersections. If it also contains the unit then we simply call it a *filter*. If $\mathcal{F}$ is supplemented and contains the core then it is said to be *augmented*. (Note that being augmented implies being a filter, but not vice versa.) This terminology can be extended to neighborhood frames in the obvious manner. For example, if $n(w)$ is augmented for all $w \in W$ in some frame, then we say that the frame itself is augmented. The following theorem summarizes some well-known characterization results (once again, interested readers should consult [1] for all details).

THEOREM 2.

- **E** *is sound and complete with respect to the class of all neighborhood frames.*
- **M** *is sound and complete with respect to the class of all supplemented neighborhood frames.*
- **C** *is sound and complete with respect to the class of neighborhood frames closed under (binary) intersections.*

- **N** *is sound and complete with respect to the class of neighborhood frames containing the unit.*
- **R**(= **EMC**) *is sound and complete with respect to the class of quasi-filter neighborhood frames.*
- **K** *is sound and complete with respect to the class of filters.*
- **K** *is sound and complete with respect to the class of augmented neighborhood frames.*

2.3. A Multi-Modal Translation of Classical Modal Logics

In normal modal logics the $\Box$ modality has a strongly universal character: $\Box\varphi$ is true at some state w in a model just in case φ holds at all states accessible from w.[2] When dealing with classical modal logics interpreted over neighborhood frames, however, the nature of $\Box$ is slightly more complex, and its universal characteristics are embedded in a distinctively existential context: $\Box\varphi$ is true at a world w iff *there is some* set in $n(w)$ that is the truth-set of φ (and the universal aspects are involved in the explication of "truth-set").[3]

Work by Kracht and Wolter [12] and Gasquet and Herzig [4] illuminates the situation most clearly. They demonstrate how non-normal modal logics can be simulated in a multi-modal setting with three normal modalities, thus elucidating the implicit components of the semantic condition we gave above. For the purposes of this paper, and the project of constructing sequent systems for classical modal logics, the significant upshot of working in the multi-modal setting is that the modal operators are all normal, and therefore allow us to utilize standard proof-theoretic machinery for normal modal logics in our analysis of non-normal modal logics.

Specifically, we will now work in a language, $\mathcal{L}_5$, containing three unary modalities, $\Box_N$, $\Box_\ni$, and $\Box_{\not\ni}$, two nullary modalities σ and τ, and the usual propositional connectives and abbreviations. Then, given a formula φ in $\mathcal{L}_1$, we can translate it into the $\mathcal{L}_5$ formula φ^*, defined recursively by:

$$\begin{array}{rcl} p^* & := & p \\ (\neg\varphi)^* & := & \neg\varphi^* \\ (\varphi \wedge \psi)^* & := & (\varphi^* \wedge \psi^*) \\ (\Box\varphi)^* & := & \Diamond_N(\Box_\ni\varphi^* \wedge \Box_{\not\ni}\neg\varphi^*) \end{array}$$

[2]This is made even clearer when one considers the standard translation of modal logic into first-order logic in which $\Box$ corresponds explicitly to $\forall$.

[3]This distinction has led some authors, for example Hansen [7], to use ∇ as the primitive modal symbol in neighborhood contexts.

Notice that this makes explicit the relationship between the existential and universal components of the standard semantic account for neighborhood models. The rough intuitive understanding of this translation is that for $\Box\varphi$ to be true at w, there must be *some* set accessible from w containing *all* states at which φ is true, and omitting *all* states at which φ is false (i.e., this set accessible from w is the truth-set of φ).

We also describe a semantic translation to accompany the syntactic one. Specifically, given a neighborhood frame $F = \langle W, n\rangle$ we can define[4] the frame $F^\circ = \langle W^\circ, R_N, R_\ni, R_{\not\ni}, R_\sigma, R_\tau\rangle$ as follows[5]:

$$\begin{array}{rcl} W^\circ & := & W \cup \wp(W) \\ R_N & := & \{\langle w, a\rangle \in W \times \wp(W) \mid a \in n(w)\} \\ R_\ni & := & \{\langle a, w\rangle \in \wp(W) \times W \mid w \in a\} \\ R_{\not\ni} & := & \{\langle a, w\rangle \in \wp(W) \times W \mid w \notin a\} \\ R_\sigma & := & W \\ R_\tau & := & \wp(W) \end{array}$$

Semantically, truth at a world is defined as usual; the unary modalities, which are normal, are treated as such. In the case of the nullary modalities, this is also handled as usual. For example, for σ we have:

$$M^\circ, x \models \sigma \quad \text{iff} \quad x \in R_\sigma.$$

One can then obtain the following correspondence result connecting neighborhood frames with these relational structures.

Theorem 3. ([12]) *Given any neighborhood model $M = \langle F, V\rangle$, and any $\mathcal{L}_1$-formula φ,*

$$\langle F, V\rangle, w \models \varphi \textit{ iff } \langle F^\circ, V\rangle, w \models \varphi^*$$

[4]The definition of W° as $W \cup \wp(W)$ is in many senses not ideal. For example, by defining it this way rather than as $W \cup \bigcup_{w\in W} n(w)$, we actually prevent certain completeness results from going through, as discussed in Sect. 8, below. The trade-off, however, is a more elegant set of sequent rules for **M**. Given that our primary focus in this paper is the construction of elegant, and well-behaved, sequent systems, we accept, for now, the less parsimonious domain.

[5]Technically, this definition is not careful enough. Formally, we require the intersection of W and $\wp(W)$ to be empty, which, depending on what we take to be our W, will not necessarily be the case. There are several solutions to this. One relatively simple one would be to define W° as $W \sqcup \wp(W)$ and then adjust the relational definitions accordingly. For example, we would then have that $R_N := \{\langle\langle w, 0\rangle, \langle a, 1\rangle\rangle \in W \sqcup \wp(W) \mid a \in n(w)\}$. For simplicity's sake, throughout this paper we will use the simpler formulation, but with these issues in mind.

for all $w \in F$. *Utilizing* σ *we can be more general:*

$$\langle F, V\rangle \models \varphi \textit{ iff } \langle F^\circ, V\rangle \models \sigma \to \varphi^*.$$

The importance of σ becomes especially apparent when attention is turned to the system **N**, and all extensions thereof. In these cases, a mechanism is required to differentiate between the two sorts of elements W° comprises, as translations of some theorems of **N** will not be valid in the class of converted frames (specifically, they will be falsified at the τ states due to the construction of the R_N relation).

Notationally, given a translated frame F°, we will let M° refer to a model based on F°. Also, given a class $\mathfrak{F}$ of neighborhood frames, we can let $\mathfrak{F}^\circ$ denote the class of all the F° for $F \in \mathfrak{F}$.

3. Labelled Sequent Systems for Classical Modal Logics

A key feature of labelled systems is that each appearance of a formula φ in a sequent is accompanied by a label, e.g. $w : \varphi$, intended to represent the semantic notion of φ being true at w. We will call these label/formula pairs, *labelled formulas.* Moreover, labels may occur in the absence of formulas, expressing relational properties, such as $w_1 R_N w_2$ (called *relational atoms*). A labelled sequent $\Gamma \Rightarrow \Delta$ consists of two multisets (lists without order) of labelled formulas and relational atoms separated by the symbol $\Rightarrow$. As usual, this is intuitively understood as saying that if everything in Γ holds, something in Δ will also.

In this section we work entirely in the language $\mathcal{L}_5$.[6] For reading convenience, we will use lower case letters from the beginning of the Latin alphabet $(a, b, c, ...)$ to denote states that can be helpfully thought of as sets. We

[6] The reason for working in the extended language $\mathcal{L}_5$ rather than directly with $\mathcal{L}_1$ is our desire for systems that are simultaneously well-behaved and modular. As has already been remarked upon, this is very difficult to maintain when working in $\mathcal{L}_1$. On the other hand, one might consider constructing a labelled system that more directly mirrors the conditions placed on neighborhood models in the semantics. However, this also poses problems, and might not take place in a much simpler language. For one, some sort of typing relation (similar to our R_σ and R_τ) may still be required, as one will want to explicitly reference relations that hold between sets rather than worlds. To simplify this, one can build these conditions into the logical rules in some cases. However, doing so may well subvert the goal of modularity, as changing systems would then entail changing logical rules. The approach we adopt has the advantage of allowing us to deal with three normal modal operators, the basic, and unchanging, logical rules of which compose the core of all the systems. We do, however, mention in the conclusion a simpler method that can be adopted if one is working just with monotonic logics.

$$w : p, \Gamma \Rightarrow \Delta, w : p$$

$$a : \tau, \Gamma \Rightarrow \Delta, a : \tau \qquad\qquad x : \sigma, \Gamma \Rightarrow \Delta, x : \sigma$$

$$\frac{\Gamma \Rightarrow \Delta, w : \varphi}{w : \neg\varphi, \Gamma \Rightarrow \Delta}\ L\neg \qquad\qquad \frac{w : \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg\varphi}\ R\neg$$

$$\frac{w : \varphi, w : \psi, \Gamma \Rightarrow \Delta}{w : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \Delta, w : \varphi \quad \Gamma \Rightarrow \Delta, w : \psi}{\Gamma \Rightarrow \Delta, w : \varphi \wedge \psi}\ R\wedge$$

$$\frac{wNa, a : \varphi, \Gamma \Rightarrow \Delta}{w : \Diamond_N\varphi, \Gamma \Rightarrow \Delta}\ L\Diamond_N \qquad\qquad \frac{wNa, \Gamma \Rightarrow \Delta, w : \Diamond_N\varphi, a : \varphi}{wNa, \Gamma \Rightarrow \Delta, w : \Diamond_N\varphi}\ R\Diamond_N$$

$$\frac{x : \varphi, a : \Box_{\ni}\varphi, a \ni x, \Gamma \Rightarrow \Delta}{a : \Box_{\ni}\varphi, a \ni x, \Gamma \Rightarrow \Delta}\ L\Box_{\ni} \qquad\qquad \frac{a \ni x, \Gamma \Rightarrow \Delta, x : \varphi}{\Gamma \Rightarrow \Delta, a : \Box_{\ni}\varphi}\ R\Box_{\ni}$$

$$\frac{x : \varphi, a : \Box_{\not\ni}\varphi, a \not\ni x, \Gamma \Rightarrow \Delta}{a : \Box_{\not\ni}\varphi, a \not\ni x, \Gamma \Rightarrow \Delta}\ L\Box_{\not\ni} \qquad\qquad \frac{a \not\ni x, \Gamma \Rightarrow \Delta, x : \varphi}{\Gamma \Rightarrow \Delta, a : \Box_{\not\ni}\varphi}\ R\Box_{\not\ni}$$

$$\frac{x : \sigma, \sigma(x), \Gamma \Rightarrow \Delta}{\sigma(x), \Gamma \Rightarrow \Delta}\ L_\sigma 1 \qquad\qquad \frac{\sigma(x), x : \sigma, \Gamma \Rightarrow \Delta}{x : \sigma, \Gamma \Rightarrow \Delta}\ L_\sigma 2$$

$$\frac{a : \tau, \tau(a), \Gamma \Rightarrow \Delta}{\tau(a), \Gamma \Rightarrow \Delta}\ L_\tau 1 \qquad\qquad \frac{\tau(a), a : \tau, \Gamma \Rightarrow \Delta}{a : \tau, \Gamma \Rightarrow \Delta}\ L_\tau 2$$

Figure 2. The basic core of the labelled sequent system **GE**. $L\Diamond_N$, $R\Box_{\ni}$, and $R\Box_{\not\ni}$ satisfy the eigenvariable condition

use lower case letters from the end of the alphabet $(w, x, y, z \dots)$ for the other states (i.e. those states that are convenient to think of as points in the original neighborhood frames). Moreover, when it is unclear which kind of state we might be working with, or when it doesn't matter, we will often just use $x, y, z \dots$. We reiterate that this is simply a notational convention to aid intuitive readings of the sequents, and could be eliminated with no consequence. In addition, for the relational atoms, we will use a simplified notation for our sequents: wNa for wR_Na; $a \ni w$ for $aR_{\ni}w$; and $a \not\ni w$ for $aR_{\not\ni}w$. Also, we will let $\sigma(w)$ and $\tau(w)$ stand for $R_\sigma(w)$ and $R_\tau(w)$. In addition to making the sequents easier to read, these conventions help one keep separated the syntactic entities composing our sequents and the semantic concepts to which they are meant to correspond.

The basic rules covering initial sequents, propositional connectives, and the modal operators are described in Fig. 2 and explicitly follow [15], with

$$\frac{}{\sigma(x), \tau(x), \Gamma \Rightarrow \Delta}\ Excl_{\sigma\tau} \qquad \frac{\sigma(x), \tau(a), a \ni x, \Gamma \Rightarrow \Delta}{a \ni x, \Gamma \Rightarrow \Delta}\ Typ_{\ni}$$

$$\frac{\sigma(x), \tau(a), a \not\ni x, \Gamma \Rightarrow \Delta}{a \not\ni x, \Gamma \Rightarrow \Delta}\ Typ_{\not\ni} \qquad \frac{}{a \ni x, a \not\ni x, \Gamma \Rightarrow \Delta}\ Excl_{\ni\not\ni}$$

Figure 3. The frame-related sequent rules of **GE**

the exception of the rules concerning the nullary modalities. However, once one examines the labelled systems for normal modal logics it is easy to see how to adapt the same method in nullary cases.

Those unfamiliar with labelled systems will still immediately recognize the rules for the logical connectives, as these are the same as in the unlabelled case, except that labelled formulas appear now where before one would simply have formulas. One actually utilizes these labels, along with the relational atoms, in the case of the modal rules, and one will recognize in these rules a structure analogous to that found in the semantic clauses for the modal operators.

Some rules come with a condition on the labels involved. For example, the rules $L\Diamond_N$, $R\Box_{\ni}$ and $R\Box_{\not\ni}$ must meet the so-called *eigenvariable* condition, i.e. the label a in $L\Diamond_N$ (or x in $R\Box_{\ni}$ and $R\Box_{\not\ni}$) must not occur in the conclusion. In other words, a (or x) does not occur in Γ or Δ. This condition is analogous to the variable conditions one finds in, for example, universal introduction or existential elimination rules in natural deduction systems for first-order logic. Also, in the rules $R\Diamond_N$, $L\Box_{\ni}$ and $L\Box_{\not\ni}$ the formulas in the conclusion are repeated in the premise. We call such rules *cumulative*.[7]

Lastly, while we will prove that we can consider arbitrary initial sequents (Lemma 5), we take as primitive only the ones in which labelled variables $w : p$ or nullary modalities $x : \sigma/a : \tau$ are principal.

With our inclusion of the nullary modalities we also require rules to govern their behavior. These basically act as type restrictions. For example, the semantic meaning of $Excl_{\sigma\tau}$ (short for "exclusion") is that nothing can be both in R_σ and R_τ. These rules are given in Fig. 3.[8]

[7]This property plays an important role in proving that contraction is admissible.

[8]When expressed in a suitable first-order language, all of these type restrictions take the form of *universal axiom schemas* [19], and are thus straightforwardly expressible with labelled sequents in such a way that all structural properties will be preserved. We will discuss generalizations of this form in Sect. 4 below.

GE is the system composed of the logical and type rules that have been presented. (The "G" in the system name stands for "Gentzen".)

We will demonstrate in the next section that all the usual structural rules are admissible in **GE**. Since our sequents are made up of labelled formulas as well as relational atoms, we need to consider structural rules that have both as principal. However, as relational atoms occur only in the antecedent of sequents (the left-hand side of the arrow), we can, without loss of generality, ignore the right-hand rules where these are principal.

3.1. Admissibility of the Structural Rules and Cut in GE

When establishing whether a sequent is derivable in a system[9], it is better not to have the structural rules explicitly included as their presence makes systematic proof search problematic. For example, consider the cut rule. To check whether a given sequent $\Gamma \Rightarrow \Delta$ is derivable in a system, one should be able to decompose $\Gamma \Rightarrow \Delta$ into simpler sequents via applications of the system's rules. If cut is taken as primitive, then one encounters (at each stage of the derivation) the problem of determining whether $\Gamma' \Rightarrow \Delta'$ is the result of a cut on the sequents $\Gamma' \Rightarrow \Delta', w : \varphi$ and $w : \varphi, \Gamma' \Rightarrow \Delta'$ for an arbitrary new formula $w : \varphi$. The cut rule, by eliminating formulas, constitutes a serious obstacle to the systematic search.

On the other hand, structural rules are practically convenient, especially for shortening derivations. For instance, using cut, it is easy to see that the rule of modus ponens is present in **GE**, and by using weakening one immediately obtains necessitation for $\Box_{\ni}$, $\Box_{\not\ni}$ and $\Diamond_N$. Therefore, although none of the structural rules of Fig. 4 are considered primitive for **GE**, we prove that they are all admissible (i.e. the set of derivable sequents would not change were one to add the rules to the system).

Moreover, the proof of cut admissibility we shall give is constructive: we effectively show how to find a derivation of the conclusion of cut from all derivations of its premises. This corresponds to Gentzen's celebrated theorem (*Hauptsatz*) for systems **LJ** and **LK** of intuitionistic and classical predicate logic, respectively [5]. In contrast with Gentzen's original systems, ours also admit weakening and contraction. Notice that the latter can be as problematic as cut for proof search: reading contraction bottom-up (from

[9] Derivations can helpfully be understood if read from the root to the leaves as an attempt to make all the formulas in the antecedent true and all the formulas in the consequent false. According to such refutative reading, which is common to semantic tableaux, the initial sequents correspond to the impossibility of making the same formula simultaneously true and false.

$$\frac{\Gamma \Rightarrow \Delta}{w : \varphi, \Gamma \Rightarrow \Delta}\ \text{W} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \varphi}\ \text{W} \qquad \frac{\Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta}\ \text{W}$$

$$\frac{w : \varphi, w : \varphi, \Gamma \Rightarrow \Delta}{w : \varphi, \Gamma \Rightarrow \Delta}\ \text{C} \qquad \frac{\Gamma \Rightarrow \Delta, w : \varphi, w : \varphi}{\Gamma \Rightarrow \Delta, w : \varphi}\ \text{C} \qquad \frac{wRv, wRv, \Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta}\ \text{C}$$

$$\frac{\Gamma \Rightarrow \Delta, w : \varphi \quad w : \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \text{CUT}$$

Figure 4. Structural rules and cut.
(R stands in for N, $\ni$ and $\not\ni$.)

the conclusion to the premise), formulas in the antecedent can be multiplied with no end, and so proof search will be irremediably lost when contraction is primitive. In turn, the admissibility of contraction requires that the logical rules are invertible, and so we prove this as well.

Weakening and contraction are, in fact, shown to be *height-preservingly* admissible, i.e. the derivation height of the conclusion is bounded by the derivation height of the premise. Moreover, all the logical rules are also invertible in a height-preserving manner.

Definition 4. (*Derivation*). The notions of derivation and derivation height are defined inductively as follows:

- If $\mathcal{D}$ is an initial sequent or an instance of a zero-premise rule, then $\mathcal{D}$ is a derivation and its height is 0;
- If $\mathcal{D}_1, \dots, \mathcal{D}_n$ are derivations and R is an inference rule, then

$$\frac{\mathcal{D}_1, \dots, \mathcal{D}_n}{\mathcal{D}}\ R$$

is a derivation and its height is $max(h(\mathcal{D}_1), \dots h(\mathcal{D}_n)) + 1$.

We write $\mathcal{D} \vdash \Gamma \Rightarrow \Delta$ to indicate a derivation $\mathcal{D}$ whose lowest sequent is $\Gamma \Rightarrow \Delta$. We can also write $\vdash \Gamma \Rightarrow \Delta$ to mean that there is a $\mathcal{D}$ such that $\mathcal{D} \vdash \Gamma \Rightarrow \Delta$, i.e. $\Gamma \Rightarrow \Delta$ is derivable. If the height of $\mathcal{D} \vdash \Gamma \Rightarrow \Delta$ is n we can write $\mathcal{D} \vdash^n \Gamma \Rightarrow \Delta$. Finally, $\vdash_{\mathbf{S}} \Gamma \Rightarrow \Delta$ means that $\Gamma \Rightarrow \Delta$ is derivable in a particular system, $\mathbf{S}$.

In order to prove the desired admissibility results, some preliminary lemmas are needed. First, though initial sequents are given above as involving only atomic formulas, one can demonstrate that this can be extended to incorporate arbitrary formulas, i.e. $\vdash w : \varphi, \Gamma \Rightarrow \Delta, w : \varphi$ for an arbitrary

φ. Defining the rules such that initial sequents only contain atomic formulas allows for height-preserving invertibility of the logical rules, which is needed in the proof of contraction admissibility. However, without arbitrary initial sequents we would not be able to prove the correspondence results with axiomatic systems in full generality. Therefore the proof that the choice of atomic initial sequents is not restrictive is important for the rest of the paper.

Secondly, and unsurprisingly, the specific labels used in a derivation do not affect the height of the derivation. To be more precise, given a multiset Γ, let $\Gamma[u/u']$ denote the multiset exactly like Γ except all occurrences of u' have been replaced with u.

LEMMA 5. *Substitution of labels is height-preservingly admissible in* **GE***:*

$$\vdash^n_{\mathbf{GE}} \Gamma \Rightarrow \Delta \quad \textit{implies} \quad \vdash^n_{\mathbf{GE}} \Gamma[u/u'] \Rightarrow \Delta[u/u'].$$

PROOF. The proof is by induction on n and follows the pattern of Lemma 4.3. in [15]. ■

The proof of the admissibility results also follows from [15]. It is the syntactic translation of formulas from $\mathcal{L}_1$ to $\mathcal{L}_5$ that facilitates this. For, while we are addressing the problem of creating proof systems for non-normal modal logics, the translation allows us to do so by way of (several) normal operators. Thus, for the basic system **GE**, most of the machinery is already in place. The only novel portions at this stage concern the nullary modalities, but their inclusion is unproblematic and straightforward. As one would expect, when we create extensions of **GE** this picture becomes more complicated, but this is all addressed below.

THEOREM 6. *The following hold in* **GE***:*

(i) *Weakening is hp-admissible;*

(ii) *All the rules are hp-invertible;*

(iii) *Contraction is hp-admissible;*

(iv) *Cut is admissible.*

3.2. Soundness and Correspondence

We can demonstrate that the theorems of **E** are included in those of **GE**.

THEOREM 7. *If* $\vdash_{\mathbf{E}} \varphi$ *then* $\vdash_{\mathbf{GE}} \Rightarrow w : \sigma \rightarrow \varphi^*$.

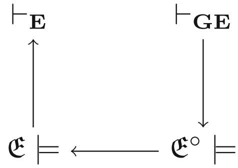

Figure 5. The route to correspondence

PROOF. Since **E** has no axioms besides those of classical propositional logic, this simply involves proving the admissibility of modus ponens and the translated version of *RE*.

The admissibility of modus ponens follows from the admissibility of cut and the invertibility of the logical rules. For *RE* we just have to prove that $\Rightarrow w : \sigma \to (\Diamond_N(\Box_\ni \varphi \wedge \Box_{\not\ni} \neg\varphi) \leftrightarrow \Diamond_N(\Box_\ni \psi \wedge \Box_{\not\ni} \neg\psi))$ can be derived from $\Rightarrow w : \sigma \to (\varphi \leftrightarrow \psi)$.

Using theorem 6, we can construct the relevant derivation by starting with the conclusion and "working upwards" by applying the rules in all possible ways until we reach the premise. It is a straightforward exercise to construct such a derivation. ■

Obviously, while nice, this result is not very significant unless we also have the reverse inclusion. That is, in order for **GE** to be successful as a sequent calculus system corresponding to **E**, we must have that the translations of non-theorems of **E** are not provable in **GE**. We will subsequently demonstrate this, though our proof will take a slightly circuitous route—rather than proving the result directly, as above, we will first prove a soundness result for **GE** that will be used, along with the completeness of **E**, to obtain the desired proof. Graphically, the process can be represented as in Fig. 5.

Our approach to proving the first step, along with our notation, is that of Negri's [16], generalized in the obvious manner.

DEFINITION 8. (*Interpretation*). Let F° be any frame in the class $\mathfrak{E}^\circ$ (where $\mathfrak{E}$ is the class of all neighborhood frames). Let U be the set of labels used in **GE** derivations. An interpretation is a function $i : U \to W^\circ$, assigning to each label an element of W°. Given two such functions i and j, let $i \sim_x j$ denote that i and j are identical except, possibly, with respect to the label x.

DEFINITION 9. (*Truth of a Sequent*). A sequent $\Gamma \Rightarrow \Delta$ is true with respect to an interpretation function in a model M° if

whenever $M^\circ, i(s) \models \varphi$, $i(t)R_N i(u)$, $i(v)R_\ni i(x)$, and $i(y)R_{\not\ni} i(z)$

for all labelled formulas $s : \varphi$ and relational atoms tNu, $v \ni x$, and $y \not\ni z$ in Γ,

$$\text{then } M^\circ, i(r) \models \psi$$

for some $r : \psi$ in Δ.

We will say that a sequent is F°-valid if it is true with respect to every interpretation function and every model based on the frame F°. It is $\mathfrak{E}^\circ$-valid when it is F°-valid for every $F^\circ \in \mathfrak{E}^\circ$. We will also refer to a labelled formula $w : \varphi$ being true (and hence valid) with respect to an interpretation function i and a model M° when $M^\circ, i(w) \models \varphi$.

Lemma 10. *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in* **GE** *then it is $\mathfrak{E}^\circ$-valid.*

Proof. This is proved via an induction on the **GE**-derivation of $\Gamma \Rightarrow \Delta$. Though this result follows almost immediately from the proof in [16], we will include some cases here simply to provide a flavor of how it works, and to lay the groundwork for when we will have to supplement the proof in subsequent sections.

When $\Gamma \Rightarrow \Delta$ is an initial sequent the result is immediate since $w : p$ (for some label and propositional variable) occurs on both sides of the $\Rightarrow$. Or, in the case of the nullary modalities, we have $w : \sigma$, for example, on both sides. For the $Excl_{\sigma\tau}$ rule, the antecedent of the conditional in the definition of a true sequent can never be satisfied.

Jumping to the cases for $\Diamond_N$, consider first the possibility that $\Gamma \Rightarrow \Delta$ is the conclusion of $L\Diamond_N$, with the premise $wNa, a : \varphi, \Gamma' \Rightarrow \Delta'$. Take M° to be an arbitrary model and i to be an arbitrary interpretation function rendering $w : \Diamond_N\varphi$, as well as every formula in Γ', true for M°. In particular, this means that $M^\circ, i(w) \models \Diamond_N\varphi$, from which it follows that there is some $u \in W^\circ$ such that $i(w)R_N u$ and $M^\circ, u \models \varphi$. Let j be an interpretation function s.t. $i \sim_a j$ and $j(a) = u$. Clearly, then, $j(w)R_N j(a)$ and $M^\circ, j(a) \models \varphi$. Since a does not appear in Γ', and since i validated all formulas in Γ', so does j. Then, by the induction hypothesis, j must validate some formula in Δ', implying, since a does not appear in Δ', that i must also validate some formula in Δ'. Since i was arbitrary, we have proven that $\Gamma \Rightarrow \Delta$ must be $\mathfrak{E}^\circ$-valid.

In the case of $R\Diamond_N$, $\Gamma \Rightarrow \Delta$ is of the form $wNa, \Gamma' \Rightarrow \Delta', w : \Diamond_N\varphi$. Once again, take i to be an arbitrary interpretation function making all the formulas in Γ true for arbitrary M°. From the induction hypothesis, then, we have that i also satisfies one of the formulas in Δ' or $w : \Diamond_N\varphi$ or $a : \varphi$. In either of the first two cases we are done, as these formulas also occur in Δ. So we need only consider the case of $a : \varphi$. But we have that

$i(w) R_N i(a)$ and $M^\circ, i(a) \models \varphi$ for each relevant M°. But then, by definition, $M^\circ, i(w) \models \Diamond_N \varphi$, as desired.

The rules governing σ and τ, though unique to our account, are dealt with straightforwardly. ■

Before continuing, we point out that the proof of Lemma 10 actually proves something slightly stronger than simply that the rules preserve validity. In particular, if we consider a rule

$$\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$$

what we actually prove is that if $\Gamma \Rightarrow \Delta$ is not valid, that is, if there is some model and an interpretation i such everything in Γ is satisfied but nothing in Δ is, then there is an interpretation i' that agrees with i on the interpretation of all the labels in Γ and Δ (but perhaps not on other labels) and invalidates $\Gamma' \Rightarrow \Delta'$. This fact will prove helpful in establishing soundness results in subsequent sections, especially when we begin dealing with systems of rules.

This result enables us now to prove the reverse direction of theorem 7. We can sum up much of our progress thus far in the following theorem.

Theorem 11. *Let φ be an arbitrary well-formed formula in $\mathcal{L}_1$ and φ^* its $\mathcal{L}_5$ translation. The following are all equivalent:*

(i) $\vdash_{\mathbf{GE}} \Rightarrow w : \sigma \rightarrow \varphi^*$

(ii) *the sequent* $\Rightarrow w : \sigma \rightarrow \varphi^*$ *is* $\mathfrak{E}^\circ$*-valid*

(iii) φ *is* $\mathfrak{E}$*-valid*

(iv) $\vdash_{\mathbf{E}} \varphi$

Proof. $(i) \Longrightarrow (ii)$ follows from the soundness result for **GE**, which was proved in lemma 10.

$(ii) \Longrightarrow (iii)$: If $\Rightarrow w : \sigma \rightarrow \varphi^*$ is a $\mathfrak{E}^\circ$-valid sequent then it must be globally true for every interpretation function on each model based on an $\mathfrak{E}^\circ$-frame. But this (properly) includes all those models with a "neighborhood appropriate" valuation function. Thus, from theorem 3, it must be that φ is globally true on every neighborhood model corresponding to a model with a frame in $\mathfrak{E}^\circ$. However, this is all neighborhood models. Thus φ is valid with respect to the class of all neighborhood frames.

$(iii) \Longrightarrow (iv)$ is a consequence of the completeness of **E**.

$(iv) \Longrightarrow (i)$ is just theorem 7. ■

4. The System GN

4.1. Geometric and Generalized Geometric Formulas

Obtaining the system **GE** is almost an automatic consequence of combining the translation between $\mathcal{L}_1$ and $\mathcal{L}_5$ and labelled sequent systems for normal modal logics. However, this was not our goal, simply the first step. What we actually want is a modular approach to a family of classical modal logics, so we have to be able to extend **GE** appropriately. We will start this project with the simple case of **N**.[10]

Recall that **N** is sound and complete with respect to the class of neighborhood frames containing the unit. In the context of relational frames resulting from neighborhood frames, this condition can be intuitively expressed (in an appropriate first-order language) as:

$$\forall w(\sigma(w) \to \exists a(wNa \wedge \forall x(\sigma(x) \to a \ni x)))$$

Work by Negri and von Plato [20] have made clear how to create cut-free labelled sequent calculi for *geometric theories* by converting their axioms into rules. Geometric theories are first-order theories described by axioms of the form $\forall x_1, ..., x_n(P \to Q)$ where P and Q do not contain $\forall$ or $\to$. Equivalently, one can think of these as conjunctions of formulas of the form:

$$\text{(Geo)} \quad \forall \overline{x}(P_1 \wedge \cdots \wedge P_m \to \exists \overline{y} M_1 \vee \cdots \vee \exists \overline{y} M_n)$$

where each P is an atomic formula (in the first-order language), and each M_i is the conjunction of atomic formulas $Q_{i_1}, \ldots, Q_{i_j}$. Unfortunately, our formula expressing inclusion of the unit is not geometric, and so a rule is not straightforwardly extractable using the established method. Fortunately, however, very recent work focused on extending the purview of labelled sequent systems [18] has made progress that is pertinent to our aims here. Specifically, it has been shown that while there may not be a (well-behaved) rule capturing the condition we are attempting to codify, there may be a *system of rules*. A system of rules, as the name implies, is a collection of rules that bear a specific order relation to each other determining their sequence of use in a proof. For example, we might have a system $\{R_1, R_2\}$ where, in addition to any sort of variable conditions that might be enforced in R_1 and R_2 individually, the order in which they are applied may also be stipulated; the condition might state that in the course of a derivation, R_2 must always be applied above an application of R_1.

[10] We begin with **N** mainly because it is straightforward, and serves as something of an introduction to the methods we will use for **C** and **M**.

More concretely, given the definition of a geometric formula, as above, we can now recursively define *generalized geometric formulas* [18].

Definition 12. (*Generalized Geometric Formulas).* The most basic generalized geometric formulas are the conjunctions of formulas of the form

$$\forall\overline{x}(P_1 \wedge \cdots \wedge P_m \rightarrow \exists\overline{y}M_1 \vee \cdots \vee \exists\overline{y}M_n)$$

i.e. the geometric formulas mentioned above (as axioms of geometric theories). These we will call GF_0 formulas.

At stage GF_{n+1} we have the formulas:

$$\forall\overline{x}(P_1 \wedge \cdots \wedge P_m \rightarrow (\exists\overline{y}\bigwedge GF_{i_1} \vee \cdots \vee \exists\overline{y}\bigwedge GF_{i_k}))$$

for $i_1, ..., i_k \leq n$ (here $\bigwedge GF_j$ denotes the conjunction of GF_j formulas).

Given this definition, one can begin to unearth the systems of rules to which they correspond. First, geometric formulas correspond to the following rule scheme:

$$\frac{\overline{Q}_1[\overline{z}_1/\overline{y}], \overline{P}, \Gamma \Rightarrow \Delta \ldots \overline{Q}_n[\overline{z}_n/\overline{y}], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; GR_0$$

where $\overline{Q}_i[\overline{z}_i/\overline{y}]$ denotes $Q_{i_1}[\overline{z}_i/\overline{y}], \ldots, Q_{i_j}[\overline{z}_i/\overline{y}]$ such that when $k \neq l$, $\overline{z}_k$ and $\overline{z}_l$ are disjoint. Because generalized geometric formulas contain nested occurrences of further generalized geometric formulas (grounded, ultimately, in geometric implications), we can expect the rules corresponding to these formulas to contain an analogous cascade of nested rules based on the general scheme for a geometric rule where the ultimate conclusion is the antecedent conjunction of the outermost generalized geometric formula, and each inner antecedent conjunction forms the conclusion of its own system. A simple example should help illuminate this.

Consider the generic GF_1 formula:

$$\forall x(Ux \rightarrow (\exists y(Sy \wedge \forall z(Rxz \rightarrow Ryz)) \vee \exists w(Sw \wedge \forall z(Twz \rightarrow Txz))))$$

This gives rise to the following system of rules:

$$\frac{Ryz, \Gamma' \Rightarrow \Delta'}{Rxz, \Gamma' \Rightarrow \Delta'} \; R_2 \qquad \frac{Txz, \Gamma'' \Rightarrow \Delta''}{Twz, \Gamma'' \Rightarrow \Delta''} \; R_3$$

$$\frac{\begin{array}{c}\vdots\\ Sy, \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\vdots\\ Sw, \Gamma \Rightarrow \Delta\end{array}}{Ux, \Gamma \Rightarrow \Delta} \; R_1$$

equipped not only with the variable conditions that y and w are new (i.e. do not occur in Γ or Δ), but also the order conditions that R_2 and R_3 are always applied above R_1 in a derivation (and on the appropriate branch), though they need not appear immediately above R_1. In the above tree, the vertical ellipses represent (potentially empty) derivations between $Rxz, \Gamma' \Rightarrow \Delta'$ (and $Twz, \Gamma'' \Rightarrow \Delta''$) and $Sy, \Gamma \Rightarrow \Delta$ (and $Sw, \Gamma \Rightarrow \Delta$). Thus, Γ' and Δ', for example, need not have any specific relationship (e.g. be substitution instances) to Γ and Δ beyond whatever is dictated by the intermediate deductions. Graphically, this example makes clear the role of the nesting and the relationship with the GR_0 scheme.

A more thorough discussion of systems of rules in general far exceeds the purview of this paper, and interested readers should consult [18]. For our purposes, we already have enough machinery to make sense of the above frame condition capturing "containing the unit". It is easily seen to be a GF_1 formula, and therefore gives rise to the system:

$$\cfrac{\cfrac{a \ni x, \sigma(x), \Gamma' \Rightarrow \Delta'}{\sigma(x), \Gamma' \Rightarrow \Delta'}\, N_2 \qquad \vdots}{} \qquad \cfrac{wNa, \sigma(w), \Gamma \Rightarrow \Delta}{\sigma(w), \Gamma \Rightarrow \Delta}\, N_1$$

with the variable condition that a not occur in Γ or Δ, and the order condition that rule N_2 must be applied above (though not necessarily immediately above) the rule N_1. Let **GN** be **GE** supplemented with the system $\{N_1, N_2\}$.

4.2. Admissibility of the Structural Rules and Cut in GN

We can prove that adding the system $\{N_1, N_2\}$ preserves the structural properties of the sequent systems considered so far. Using systems of rules creates new complications in the proofs of hp-admissibility of the structural rules. Specifically, we must show that the rules are *system-preserving* (this terminology is taken from [18])—in the stages of the inductive proofs corresponding to the system, we have to show that in the process of converting derivations we do not interfere with the applicability of the system, for example by violating eigenvariable conditions.

First, lemma 5 can easily be adapted for **GN**.

LEMMA 13. $\vdash^n_{\mathbf{GN}} \Gamma \Rightarrow \Delta$ *implies* $\vdash^n_{\mathbf{GN}} \Gamma[u/u'] \Rightarrow \Delta[u/u']$.

PROOF. The proof is exactly the same as that of lemma 5, supplemented with the new case for when $\Gamma \Rightarrow \Delta$ is of the form $\sigma(w), \Gamma' \Rightarrow \Delta$ and concluded via application of the system $\{N_1, N_2\}$.

The most interesting subcase concerns substitutions that would violate the eigenvariable condition in N_1, and which require us to make another substitution. In this case we consider the substitution $[a/w]$. From the inductive hypothesis we first obtain $wNb, \sigma(w), \Gamma' \Rightarrow \Delta$ from $wNa, \sigma(w), \Gamma' \Rightarrow \Delta$ (the substitution is vacuous everywhere but in wNa, since a is the eigenvariable). Then we again invoke the IH with the desired substitution of w with a and obtain $aNb, \sigma(a), \Gamma'[a/w] \Rightarrow \Delta[a/w]$, from which, by way of the system $\{N_1, N_2\}$, we can conclude $\sigma(a), \Gamma'[a/w] \Rightarrow \Delta[a/w]$. For all the other subcases one application of IH suffices. ■

The substitution result now allows us to prove the admissibility of the structural rules.

Theorem 14. *The following hold in* **GN***:*

(i) *Weakening is hp-admissible;*

(ii) *All the rules are hp-invertible;*

(iii) *Contraction is hp-admissible;*

(iv) *Cut is admissible.*

Proof. All the proofs straightforwardly extend those of theorem 6 (which, again, can be found in more detail in [15]).

Concerning (i), we need to show that weakening does not change the derivation height or the applicability of the rule system $\{N_1, N_2\}$. If the premise of weakening is initial, the conclusion is also initial. Otherwise, we assume hp-admissibility of weakening for derivations with height k (IH) and prove that the claim holds also for those with $k + 1$. We analyze the case where the formula produced by weakening is of the form $a : \varphi$, and the premise has been derived by an application of the system, i.e. it is of the form $\sigma(w), \Gamma' \Rightarrow \Delta$. This derivation (on the left, below) can be converted into one (on the right) with the same conclusion, but where weakening is applied on a derivation of lower height (and hence admissible by IH). Notice that in this case we require the ability to substitute a with a completely new b (which is height preserving by lemma 13).

$$\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma'' \Rightarrow \Delta'}{\sigma(x), \Gamma'' \Rightarrow \Delta'}\,N_2}{\vdots \atop wNa, \sigma(w), \Gamma' \Rightarrow \Delta}}{\dfrac{\sigma(w), \Gamma' \Rightarrow \Delta}{a : \varphi, \sigma(w), \Gamma' \Rightarrow \Delta}\,\mathrm{W}}\,N_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma'' \Rightarrow \Delta'}{\sigma(x), \Gamma'' \Rightarrow \Delta'}\,N_2}{\vdots \atop wNa, \sigma(w), \Gamma' \Rightarrow \Delta}}{wNb, \sigma(w), \Gamma' \Rightarrow \Delta}\,b/a}{\dfrac{a : \varphi, wNb, \sigma(w), \Gamma' \Rightarrow \Delta}{a : \varphi, \sigma(w), \Gamma' \Rightarrow \Delta}\,N_1}\,\mathrm{W}$$

(*ii*) Notice that the presence of the system $\{N_1, N_2\}$ does not impair the hp-invertibility of logical rules since the system involves only relational atoms. The proof then follows the scheme (set out in [20]) for geometric rule schemes.

(*iii*) We only show the case of hp-admissibility of left contraction with respect to relational atoms. If the premise of contraction is an initial sequent or the conclusion of a zero-premise rule, then so is its conclusion. Otherwise, when the premise of contraction is the conclusion of some rule R, we distinguish two cases: (a) neither of the contracted formulas is principal in R; (b) at least one of the contracted formulas is principal in R. In the first case, both occurrences of the contracted formula are also found in the premise(s) of R, to which the inductive hypothesis can be applied, followed by R. For (b), IH is not necessarily applicable, and so one must first make use of the hp-invertibility of the logical rules before applying IH and then R. For example, suppose the contracted formula is a relational atom of the form $\sigma(w)$ and is principal in $\{N_1, N_2\}$. Then the derivation on the left can be converted into the one on the right with the same conclusion, but where the application of contraction is hp-admissible (by IH).

$$
\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma' \Rightarrow \Delta'}{\sigma(x), \Gamma' \Rightarrow \Delta'}\; N_2 \quad \vdots}{\dfrac{wNa, \sigma(w), \sigma(w), \Gamma \Rightarrow \Delta}{\sigma(w), \sigma(w), \Gamma \Rightarrow \Delta}\; N_1}}{\sigma(w), \Gamma \Rightarrow \Delta}\; \mathrm{C}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma' \Rightarrow \Delta'}{\sigma(x), \Gamma' \Rightarrow \Delta'}\; N_2 \quad \vdots}{\dfrac{wNa, \sigma(w), \sigma(w), \Gamma \Rightarrow \Delta}{wNa, \sigma(w), \Gamma \Rightarrow \Delta}\; \mathrm{C}}}{\sigma(w), \Gamma \Rightarrow \Delta}\; N_1
$$

(*iv*) In the case of cut, it will either be the case that at least one of the premises of cut is initial or the conclusion of a zero-premise rule, or that neither of the premises is initial or the conclusion of a zero-premise rule. In the first case, the proof of cut-admissibility is unchanged. In the second, we consider four exhaustive subcases. Taking R_l and R_r to be the potential inference rules concluding the premises of cut, we have:

(a) the cut formula is principal in both R_r and R_l;

(b) the cut formula is principal in R_r but not in R_l;

(c) the cut formula is principal in R_l but not in R_r;

(d) the cut formula is principal in neither R_r nor R_l.

The only new cases are those in which either R_l or R_r is the system $\{N_1, N_2\}$. Since all the formulas active in an application of $\{N_1, N_2\}$ are relational atoms, the first subcase cannot actually arise. For (b), suppose that R_l is the system $\{N_1, N_2\}$ (therefore, Γ is of the form $\sigma(w), \Gamma''$)

$$\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma''' \Rightarrow \Delta''}{\sigma(x), \Gamma''' \Rightarrow \Delta''}\ N_2 \quad \vdots \quad wNa, \sigma(w), \Gamma'' \Rightarrow \Delta, w : \varphi}{\sigma(w), \Gamma'' \Rightarrow \Delta, w : \varphi}\ N_1 \qquad w : \varphi, \Gamma' \Rightarrow \Delta'}{\sigma(w), \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}$$

The application of cut can now be permuted upward with $\{N_1, N_2\}$ in the usual way, provided that the eigenvariable a is replaced with a new b that does not occur in the derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \ni x, \sigma(x), \Gamma''' \Rightarrow \Delta''}{\sigma(x), \Gamma''' \Rightarrow \Delta''}\ N_2 \quad \vdots \quad wNa, \sigma(w), \Gamma'' \Rightarrow \Delta, w : \varphi}{wNb, \sigma(w), \Gamma'' \Rightarrow \Delta, w : \varphi}\ b/a \qquad w : \varphi, \Gamma' \Rightarrow \Delta'}{wNb, \sigma(w), \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}}{\sigma(w), \Gamma'', \Gamma' \Rightarrow \Delta', \Delta}\ N_1$$

The last two cases are completely analogous. ■

4.3. Soundness and Correspondence

All theorems **N**, when translated and made conditional on σ, are derivable in **GN**.

THEOREM 15. *If* $\vdash_{\mathbf{N}} \varphi$ *then* $\vdash_{\mathbf{GN}} \Rightarrow w : \sigma \rightarrow \varphi^*$.

PROOF. The proof reduces to the derivation of (the translation of) $\Box\top$ (axiom N). Specifically, when the translation is the consequent of an implication where σ is the antecedent: $\Rightarrow w : \sigma \rightarrow (\Diamond_N(\Box_{\ni}\top \wedge \Box_{\not\ni}\neg\top))$. We take $\top$ to be an abbreviation of any propositional tautology, and so $\Rightarrow w : \top$ is derivable. The full derivation is given below. Note that the application of the system $\{N_1, N_2\}$ satisfies both the eigenvariable condition and the order condition.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \ni x, wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, x : \top}{wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, a : \Box_{\ni}\top}\, R\Box_{\ni} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{a \ni x, \tau(a), \sigma(x), a \not\ni x, wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, x : \neg\top}\, Excl_{\ni\not\ni}}{\tau(a), \sigma(x), a \not\ni x, wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, x : \neg\top}\, N_2}{a \not\ni x, wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, x : \neg\top}\, Typ_{\not\ni}}{wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, a : \Box_{\not\ni}\neg\top}\, R\Box_{\not\ni}}{wNa, \sigma(w), w : \sigma \Rightarrow w : \chi, a : \Box_{\ni}\top \wedge \Box_{\not\ni}\neg\top}\, R\wedge}{wNa, \sigma(w), w : \sigma \Rightarrow w : \chi}\, R\Diamond_N}{\sigma(w), w : \sigma \Rightarrow w : \chi}\, N_1}{w : \sigma \Rightarrow w : \underbrace{\Diamond_N(\Box_{\ni}\top \wedge \Box_{\not\ni}\neg\top)}_{\chi}}\, L_{\sigma}2}{}
$$

■

Let $\mathfrak{N}$ be the class of neighborhood frames containing the unit. (And so $\mathfrak{N}^{\circ}$ denotes the class of relational frames arising from the conversion of frames in $\mathfrak{N}$.)

Lemma 16. *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* **GN** *then it is* $\mathfrak{N}^{\circ}$*-valid.*

Proof. We need only consider the case corresponding to our new system of rules.

So assume that $\Gamma \Rightarrow \Delta$ is of the form $\sigma(w), \Gamma' \Rightarrow \Delta'$, and that it is not valid. Then there is some interpretation i and model M° (based on a frame in $\mathfrak{N}^{\circ}$) s.t. $M^{\circ}, i(w) \models \sigma$. Then $i(w) \in R_{\sigma}$. Since M° is based on a frame in $\mathfrak{N}^{\circ}$, we have that $i(w) R_N R_{\sigma}$. Take $j \sim_a i$ to be such that $j(a) = R_{\sigma}$. Then $j(w) R_N j(a)$, and, since a does not occur in Γ' or Δ', $wNa, \sigma(w), \Gamma' \Rightarrow \Delta'$ is invalidated by j.

This, however, is just the first step. What we have shown so far is that if somehow an invalidity is "introduced", as it were, into our proof, it was not done so by N_1. But in order to show that the system of rules is good, we also have to show that N_2 is also well behaved. (The pre-established soundness for all the other rules allows us to be sure that if a non-validity was concluded on the basis of a valid premise of a rule, then the culprit must be one of the rules in our new system $\{N_1, N_2\}$.)

So we have shown that if i invalidates the conclusion of our system, then there is a j that invalidates the premise of N_1. We now show that the rule N_2 preserves validity with respect to all interpretations that agree with j for all the labels $wNa, \sigma(w), \Gamma' \Rightarrow \Delta'$, and so could have introduced the invalidity in question.

Now, taking k to be any interpretation function agreeing with j on all labels in $wNa, \sigma(w), \Gamma' \Rightarrow \Delta'$ and assume that k invalidates $\sigma(x), \Gamma'' \Rightarrow \Delta''$. That is, in particular, $M^{\circ}, k(x) \models \sigma$. This means that $k(x) \in R_{\sigma} = k(a) = j(a)$, and so if $\sigma(x), \Gamma'' \Rightarrow \Delta''$ is invalidated by k, so is $a \ni x, \sigma(x), \Gamma'' \Rightarrow \Delta''$. ■

Unsurprisingly, this gives way, via the same methods, to another classification theorem.

THEOREM 17. *Let φ be an arbitrary well-formed formula in $\mathcal{L}_1$ and φ^* its $\mathcal{L}_5$ translation. The following are all equivalent:*

(i) $\vdash_{\mathbf{GN}} \Rightarrow w : \sigma \rightarrow \varphi^*$

(ii) *the sequent* $\Rightarrow w : \sigma \rightarrow \varphi^*$ *is* $\mathfrak{N}^\circ$*-valid*

(iii) φ *is* $\mathfrak{N}$*-valid*

(iv) $\vdash_{\mathbf{N}} \varphi$

PROOF. This is proved as in theorem 11. ■

5. The System GC

We move now to **C**. We start by considering the condition on neighborhood frames that characterizes **C**, closure under (binary) intersections:

$$\text{if } X \in n(w) \text{ and } Y \in n(w) \text{ then } X \cap Y \in n(w)$$

If we unpack this definition, and express it in a suitable first-order language resembling the one used in our sequent system (ignoring, for the time being, type-based predicates), we obtain the following:

$$\forall a, b \forall w((wNa \wedge wNb) \rightarrow \exists c(wNc \wedge \forall x((c \ni x \rightarrow a \ni x \wedge b \ni x) \wedge (c \not\ni x \rightarrow a \not\ni x \vee b \not\ni x))))$$

Yet again, this is not geometric. However, if we rewrite the condition slightly, we can easily notice that it is a GF_1 formula:

$$\forall a, b \forall w((wNa \wedge wNb) \rightarrow \exists c(wNc \wedge \forall x(c \ni x \rightarrow a \ni x) \wedge \forall x(c \ni x \rightarrow b \ni x) \wedge \forall x(c \not\ni x \rightarrow a \not\ni x \vee b \not\ni x)))$$

which can then be transformed into the following system:

$$\frac{a \ni x, b \ni x, c \ni x, \Gamma' \Rightarrow \Delta'}{c \ni x, \Gamma' \Rightarrow \Delta'} \, C_{\ni} \qquad \frac{a \not\ni x, c \not\ni x, \Gamma'' \Rightarrow \Delta'' \quad b \not\ni x, c \not\ni x, \Gamma'' \Rightarrow \Delta''}{c \not\ni x, \Gamma'' \Rightarrow \Delta''} \, C_{\not\ni}$$

$$\frac{\begin{matrix} \ddots \;\; \therefore \\ wNc, wNa, wNb, \Gamma \Rightarrow \Delta \end{matrix}}{wNa, wNb, \Gamma \Rightarrow \Delta} \, C_1$$

with the usual variable condition that c not occur in Γ or Δ, as well as the order condition that $C_{\ni}$ and $C_{\not\ni}$ must appear above, though not necessarily immediately above, C_1 in the course of a derivation. No order is imposed between $C_{\ni}$ and $C_{\not\ni}$ (and, indeed, they could occur on separate branches).

Let **GC** denote the sequent system **GE** supplemented with the new system of rules just described.

5.1. Admissibility of the Structural Rules and Cut in GC

Since the frame condition for **C** is also an instance of a generalized geometric scheme, the admissibility of the structural rules and cut in **GC** follows, to a large extent, the pattern of the proof for **GN**.

LEMMA 18. $\vdash^n_{\mathbf{GC}} \Gamma \Rightarrow \Delta$ *implies* $\vdash^n_{\mathbf{GC}} \Gamma[u/u'] \Rightarrow \Delta[u/u']$.

PROOF. The proof is completely analogous to the one given in lemma 13. ∎

THEOREM 19. *In* **GC** *the following holds:*

(i) *Weakening is hp-admissible;*

(ii) *All the logical rules are hp-invertible;*

(iii) *Contraction is hp-admissible;*

(iv) *Cut is admissible.*

PROOF. The proof of all the claims can be obtained easily in the same manner as in theorem 14. ∎

5.2. Soundness and Correspondence

We are now in a position to prove our correspondence result between **C** and **GC**.

THEOREM 20. *If* $\vdash_{\mathbf{C}} \varphi$ *then* $\vdash_{\mathbf{GC}} \Rightarrow w : \sigma \rightarrow \varphi^*$*.*

PROOF. This reduces to showing that **GC** can derive $w : \sigma, w : \Diamond_N(\Box_{\ni}\varphi \wedge \Box_{\not\ni}\neg\varphi), w : \Diamond_N(\Box_{\ni}\psi \wedge \Box_{\not\ni}\neg\psi) \Rightarrow w : \Diamond_N(\Box_{\ni}(\varphi \wedge \psi) \wedge \Box_{\not\ni}\neg(\varphi \wedge \psi))$. This can be done via a proof-search procedure, which we omit. The derivation requires an essential application of the system $\{C_1, C_{\ni}, C_{\not\ni}\}$. ∎

In order to prove the converse we will proceed as in the previous sections, and first prove a soundness result for our sequent system. Take $\mathfrak{C}$ to be the class of neighborhood frames closed under (binary) intersections.

LEMMA 21. *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* **GC** *then it is* $\mathfrak{C}^{\circ}$*-valid.*

PROOF. Once again, this proof just requires the addition of a case in the inductive proof of Lemma 10 above corresponding to our new system of rules $\{C_1, C_{\ni}, C_{\not\ni}\}$.

So assume $\Gamma \Rightarrow \Delta$ is the result of the application of our system. In particular, this means that $\Gamma \Rightarrow \Delta$ must be the immediate result of the use of C_1 and is of the form $wNa, wNb, \Gamma' \Rightarrow \Delta'$.

Let i be an arbitrary interpretation function rendering the formulas in Γ true with respect to some M° based on a frame in $\mathfrak{C}^\circ$ complying with the relational atoms, but such that i makes all the elements of $\Delta = \Delta'$ false. That is, in particular, we take i and M° to be such that $i(w)R_N i(a)$ and $i(w)R_N i(b)$. Thus, in M, the model of which M° is a conversion, $i(a) \in n(i(w))$ and $i(b) \in n(i(w))$. Because M° is presumed to be in $\mathfrak{C}^\circ$, the frame upon which M is based is closed under intersections. Therefore, there exists a set $\mathfrak{c} = i(a) \cap i(b)$ such that $\mathfrak{c} \in n(i(w))$ and so $i(w)R_N\mathfrak{c}$.

Now let $j \sim_c i$ be an interpretation that agrees with i on everything except possibly c, and take $j(c) = \mathfrak{c}$. Then we have $j(w)R_N j(c)$. Also, since ‘c’ is absent from Γ' and Δ', j invalidates $wNc, wNa, wNb, \Gamma' \Rightarrow \Delta'$.

It remains to show that the two rules $C_\ni$ and $C_{\not\ni}$ preserve validity with respect to all interpretations that agree with j for all the labels $wNc, wNa, wNb, \Gamma' \Rightarrow \Delta'$, and so neither of them could have introduced the invalidity in question.

Consider $C_\ni$ first, and assume that some j' (agreeing with j on all the relevant variables) invalidates $c \ni x, \Gamma'' \Rightarrow \Delta''$. In particular, then, $j'(c)R_\ni j'(x)$ and j' satisfies all the elements of Γ'' but none of Δ''. But $j'(c) = j(c) = \mathfrak{c} = j(a) \cap j(b) = j'(a) \cap j'(b)$. Therefore, $j'(a)R_\ni j'(x)$ and $j'(b)R_\ni j'(x)$. But this means, since j' satisfies all elements of Γ'' but none of Δ'', that j' invalidates $a \ni x, b \ni x, c \ni x, \Gamma'' \Rightarrow \Delta''$, as desired.

For $C_{\not\ni}$ things work just the same. If j' invalidates $c \not\ni x, \Gamma'' \Rightarrow \Delta''$ then $j'(c)R_{\not\ni} j'(x)$ and so it must be that either $j'(a)R_{\not\ni} j'(x)$ or $j'(b)R_{\not\ni} j'(x)$. Either way, one of the premises is invalidated. ■

THEOREM 22. *Let φ be an arbitrary well-formed formula in $\mathcal{L}_1$ and φ^* its $\mathcal{L}_5$ translation. The following are all equivalent:*

(i) $\vdash_{\mathbf{GC}} \Rightarrow w : \sigma \rightarrow \varphi^*$

(ii) *the sequent* $\Rightarrow w : \sigma \rightarrow \varphi^*$ *is* $\mathfrak{C}^\circ$*-valid*

(iii) φ *is* $\mathfrak{C}$*-valid*

(iv) $\vdash_{\mathbf{C}} \varphi$

PROOF. This proof follows exactly the same pattern as theorems 11 and 17. ■

6. The System GM

In section 2.2 we remarked that the logic **M** is characterized by supplemented neighborhood frames. Recall that the supplemented frames are those in which, for all $w \in W$, $n(w)$ is closed under supersets. More symbolically:

$$\text{if } X \in n(w) \text{ and } X \subseteq Y \text{ then } Y \in n(w).$$

Translating, once again, into a first-order language in order to make this condition more explicit, one can see that the property of being supplemented is expressed by

$$\forall a, b(wNa \wedge \forall x(a \ni x \rightarrow b \ni x) \rightarrow wNb)$$

This is not a geometric formula. However, it can be seen to be equivalent to one. Recognizing the equivalence of $\neg(b \ni x)$ and $b \not\ni x \vee \tau(x) \vee \sigma(b)$ allows us to rewrite it as

$$\forall a, b(wNa \rightarrow \exists x((a \ni x \wedge b \not\ni x) \vee (a \ni x \wedge \tau(x)) \vee (a \ni x \wedge \sigma(b)) \vee wNb))$$

which is geometric and so gives us the following rule:

$$\frac{a \ni x, b \not\ni x, wNa, \Gamma \Rightarrow \Delta \quad a \ni x, \tau(x), wNa, \Gamma \Rightarrow \Delta \quad a \ni x, \sigma(b), wNa, \Gamma \Rightarrow \Delta \quad wNb, wNa, \Gamma \Rightarrow \Delta}{wNa, \Gamma \Rightarrow \Delta}\ {\scriptstyle Mon}$$

with x satisfying the eigenvariable condition. Notice that the rule can be simplified by ignoring the second premise, which is derivable by way of $Type_{\ni}$ and $Excl_{\sigma\tau}$. Moreover, the third premise can be removed if we make explicit that the only bs under consideration are τs (i.e. by paying attention to the type predicates we ignored in the above formulation). This is easily achieved by stipulating $\tau(b)$ in the antecedent of the conclusion. This then renders the third premise derivable via $Excl_{\sigma\tau}$, and so it can be removed. Our resulting monotonicity rule is then:

$$\frac{a \ni x, b \not\ni x, wNa, \tau(b), \Gamma \Rightarrow \Delta \quad wNb, wNa, \tau(b), \Gamma \Rightarrow \Delta}{wNa, \tau(b), \Gamma \Rightarrow \Delta}\ {\scriptstyle Mon}$$

Interestingly, and unlike in the previous sections, simply adding this rule to **GE** will not suffice to obtain a sequent calculus for **M**. Unlike with the frame conditions for **N** and **C**, supplementation, in practice, involves extra-logical presuppositions. Specifically, there is often a step involving the assertion of the existence of the truth set of some formula (with respect to the *model* in question).

In an attempt to draw the problem out, consider the standard soundness proofs for **M** and **C** (found, for example, in [1]). For **C**, from the assumption that $\Box\varphi$ and $\Box\psi$ are true at a world, we can infer, from the semantic clause for $\Box$, that both $[\![\varphi]\!]$ and $[\![\psi]\!]$ are neighborhoods of that world. Then the frame condition (closure under intersections) tells us that the intersection must also be a neighborhood, which one can then straightforwardly show to be equal to $[\![\varphi \wedge \psi]\!]$, as desired. At no stage in this are we required to independently identify, or select, any sets—everything we need is, in a sense, given to us.

On the other hand, consider the rule RM. Showing that RM preserves validity usually proceeds by assuming $\varphi \to \psi$ is a theorem and that $\Box\varphi$ is true at an arbitrary world. This means that $[\![\varphi]\!]$ is a neighborhood of that world. So far so good. But now we are stuck. The supplementation clause does not "provide" us with any other set that we also know to be a neighborhood. Rather, it says that if we have another set, one that is also a superset of $[\![\varphi]\!]$, then this set will also be a neighborhood of the world. It is at this stage that we make use of something more. In particular, we make use of the fact that we know the set $[\![\psi]\!]$ exists. Once we have this, we can then use the fact that $\varphi \to \psi$ is a theorem to reason that $[\![\varphi]\!] \subseteq [\![\psi]\!]$, at which point the frame condition enters the picture to ensure that $[\![\psi]\!]$ is a neighborhood of our world.

The difference in these two examples is the amount of reasoning that takes place between invoking our semantic clause and our frame condition. In the case of **C**, the frame condition immediately follows the semantic clause. For **M**, this is not so—we have to do some extra work (obtain the relevant truth set).

It is this extra work that is missing from the rule Mon. Mon is, and was designed to be, a sequent rule corresponding to a frame condition. But as we have just seen, the frame condition requires input, and this input comes from somewhere other than just the rules governing the logical connectives and modal operators. So this means that we need a sequent rule that fills this gap, and makes explicit these extra assumptions. Specifically, we need a rule that tells us that there are sets corresponding to formulas.

Semi-formally, the intuitive condition we have in mind is:

$$\forall x \forall \varphi (x : \varphi \to \exists b(\tau(b) \wedge \forall y(b \ni y \to y : \varphi \wedge y : \varphi \to b \ni y)))$$

which says that if φ is true at x then there is some set b, the elements of which are exactly the worlds that makes φ true. This can actually be converted into a system of rules:

$$\dfrac{\Gamma \Rightarrow \Delta, x : \varphi \qquad \begin{array}{c} \dfrac{y : \varphi, b \ni y, \Gamma' \Rightarrow \Delta'}{b \ni y, \Gamma' \Rightarrow \Delta'}\, {\scriptstyle TS_1} \quad \dfrac{b \ni y, y : \varphi, \Gamma'' \Rightarrow \Delta''}{y : \varphi, \Gamma'' \Rightarrow \Delta''}\, {\scriptstyle TS_2} \\ \ddots \; \cdot \; \cdot^{\cdot^{\cdot}} \\ \tau(b), \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta}\, {\scriptstyle TS}$$

where b is an eigenvariable that does not occur in Γ or Δ. Also, φ must be a subformula of some ψ that *does* occur in Γ or Δ. Without such a subformula restriction, we would be introducing the same sort of indeterminacy that we take great care to avoid by eliminating cut. Intuitively, the system expresses the fact that for any formula it is possible to define its truth-set.[11]

The exact form of this rule might at first glance seem strange, and one might expect $x : \varphi$ to appear in the consequence, rather than in the consequent of the left-hand premise. However, these are semantically equivalent (and corresponds to treating a conditional as a disjunction with an added negation) and facilitates an easier cut-elimination proof.

6.1. Admissibility of the Structural Rules and Cut in GM

Let **GM** be the system obtained from **GE** by adding the monotonicity rule Mon and the system $\{TS, TS_1, TS_2\}$. We now prove that **GM** admits the structural rules and cut elimination. Since the rule Mon follows the generalized geometric rule scheme (specifically, it is a geometric formula), the proofs below are similar to those already given for systems **GN** and **GC**. What is peculiar to **GM** is the system $\{TS, TS_1, TS_2\}$, which will be analyzed in detail.

Lemma 23. $\vdash^n_{\mathbf{GM}} \Gamma \Rightarrow \Delta$ *implies* $\vdash^n_{\mathbf{GM}} \Gamma[u/u'] \Rightarrow \Delta[u/u']$.

Proof. We only deal with the case when $\Gamma \Rightarrow \Delta$ is the conclusion of the system $\{TS, TS_1, TS_2\}$, and consider the substitution $[b/x]$ (causing a conflict with the eigenvariable condition). We must prove that the sequent $\Gamma[b/x] \Rightarrow \Delta[b/x]$ is derivable. First, by IH we can replace every occurrence of b in $\Gamma \Rightarrow \Delta, x : \varphi$ and $\tau(b), \Gamma \Rightarrow \Delta$ with a completely new c. Since b occurs only where it is displayed (because it is the eigenvariable of the system), the substitution is vacuous in $\Gamma \Rightarrow \Delta, x : \varphi$, whereas it gives $\tau(c), \Gamma \Rightarrow \Delta$ from $\tau(b), \Gamma \Rightarrow \Delta$. Now, by IH again, we can apply the desired substitution

[11] Note that if we had defined F° differently, such that its domain only contained W and the neighborhoods of F, we would have to add a branch to this rule that allows for the possibility that the truth set of φ is not actually to be found in the domain.

$[b/x]$ obtaining $\Gamma[b/x] \Rightarrow \Delta[b/x], b : \varphi$ and $\tau(c), \Gamma[b/x] \Rightarrow \Delta[b/x]$. Finally, an application of $\{TS, TS_1, TS_2\}$ gives $\Gamma[b/x] \Rightarrow \Delta[b/x]$. ■

Theorem 24. *We have the following in* **GM***:*

(i) *Weakening is hp-admissible;*

(ii) *All the logical rules are hp-invertible;*

(iii) *Contraction is hp-admissible;*

(iv) *Cut is admissible.*

Proof. (i) The hp-admissibility of weakening follows the same pattern as Theorems 14 and 19. Here we only explicitly consider the case where the premise of weakening is a conclusion of the system $\{TS, TS_1, TS_2\}$. Thus, we need to prove that the sequent $b : \varphi, \Gamma \Rightarrow \Delta$ is derivable from $\Gamma \Rightarrow \Delta$, for some labelled formula $b : \varphi$. (The proof for relational atoms is the same.)

$$\dfrac{\Gamma \Rightarrow \Delta, x : \psi \qquad \dfrac{y : \psi, b \ni y, \Gamma' \Rightarrow \Delta'}{b \ni y, \Gamma' \Rightarrow \Delta'}\ TS_1 \quad \dfrac{b \ni y, y : \psi, \Gamma'' \Rightarrow \Delta''}{y : \psi, \Gamma'' \Rightarrow \Delta''}\ TS_2 \quad \cdots \quad \tau(b), \Gamma \Rightarrow \Delta}{\dfrac{\Gamma \Rightarrow \Delta}{b : \varphi, \Gamma \Rightarrow \Delta}\ \mathrm{W}}\ TS$$

The system $\{TS, TS_1, TS_2\}$ can be permuted below weakening in order obtain a derivation of $b : \varphi, \Gamma \Rightarrow \Delta$ where the application of weakening is admissible by IH. As usual, we first replace the eigenvariable b of the system in $\Gamma \Rightarrow \Delta, x : \psi$ and $\tau(b), \Gamma \Rightarrow \Delta$ with a new c. Because of the variable condition of $\{TS, TS_1, TS_2\}$, the substitution is vacuous in the former and yields $\tau(c), \Gamma \Rightarrow \Delta$ from the latter. Then we apply IH and obtain $b : \varphi, \Gamma \Rightarrow \Delta, x : \psi$ and $b : \varphi, \tau(b), \Gamma \Rightarrow \Delta$. Hence, via $\{TS, TS_1, TS_2\}$, one obtains $\tau(b), \Gamma \Rightarrow \Delta$ as desired.

(ii) It is easy to see that hp-invertibility of logical rules can be preserved in **GM**. For the case involving the system $\{TS, TS_1, TS_2\}$, the proof is completely analogous to the one for geometric rule schemes.

(iii) The proof for contraction follows the standard derivation transformation. Suppose the premise of contraction $w : \varphi, w : \varphi, \Gamma \Rightarrow \Delta$ has been derived by an application of the system $\{TS, TS_1, TS_2\}$. That is, the following derivation obtains

$$
\dfrac{\dfrac{w:\varphi, w:\varphi, \Gamma \Rightarrow \Delta, x:\psi \qquad \begin{matrix} \dfrac{y:\psi, b \ni y, \Gamma' \Rightarrow \Delta'}{b \ni y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle TS_1} \quad \dfrac{b \ni y, y:\psi, \Gamma'' \Rightarrow \Delta''}{y:\psi, \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle TS_2} \\ \ddots \cdot \\ \tau(b), w:\varphi, w:\varphi, \Gamma \Rightarrow \Delta \end{matrix}}{w:\varphi, w:\varphi, \Gamma \Rightarrow \Delta}\,{\scriptstyle TS}}{w:\varphi, \Gamma \Rightarrow \Delta}\,{\scriptstyle \mathrm{C}}
$$

A derivation of the same conclusion is obtained by a downward permutation of the system $\{TS, TS_1, TS_2\}$ with respect to contraction.

(iv) Finally, cut admissibility follows, to a large extent, the proofs of Theorems 14 and 19.

When at least one of the premises of cut is initial or the conclusion of a zero-premise rule, things remain unchanged. If both the right and the left premises of cut are concluded by some inference rules (R_r and R_l, respectively) then consider first the case where the cut formula $w:\varphi$ is principal in R_r but not in R_l and R_l is an application of the system $\{TS, TS_1, TS_2\}$.

$$
\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w:\varphi, x:\psi \qquad \begin{matrix} \dfrac{y:\psi, b \ni y, \Gamma'' \Rightarrow \Delta''}{b \ni y, \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle TS_1} \quad \dfrac{b \ni y, y:\psi, \Gamma''' \Rightarrow \Delta'''}{y:\psi, \Gamma''' \Rightarrow \Delta'''}\,{\scriptstyle TS_2} \\ \ddots \cdot \\ \tau(b), \Gamma \Rightarrow \Delta, w:\varphi \end{matrix}}{\Gamma \Rightarrow \Delta, w:\varphi}\,{\scriptstyle TS} \qquad w:\varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,{\scriptstyle \mathrm{CUT}}
$$

The above derivation is transformed into one with two cuts, each on lower derivations, and hence admissible by IH. One must first replace the eigenvariable b of the system $\{TS, TS_1, TS_2\}$ with a new c.

$$
\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w:\varphi, x:\psi}{\Gamma \Rightarrow \Delta, w:\varphi, x:\psi}\,{\scriptstyle c/b} \qquad w:\varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta, x:\psi}\,{\scriptstyle \mathrm{CUT}} \qquad \dfrac{\dfrac{\begin{matrix} \dfrac{y:\psi, b \ni y, \Gamma'' \Rightarrow \Delta''}{b \ni y, \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle TS_1} \quad \dfrac{b \ni y, y:\psi, \Gamma''' \Rightarrow \Delta'''}{y:\psi, \Gamma''' \Rightarrow \Delta'''}\,{\scriptstyle TS_2} \\ \ddots \cdot \\ \tau(b), \Gamma \Rightarrow \Delta, w:\varphi \end{matrix}}{\tau(c), \Gamma \Rightarrow \Delta, w:\varphi}\,{\scriptstyle c/b} \qquad w:\varphi, \Gamma' \Rightarrow \Delta'}{\tau(c), \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle \mathrm{CUT}}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,{\scriptstyle TS}
$$

The same derivation transformation applies when the cut formula is principal in R_l but not in R_r and when $w:\varphi$ is principal in neither. As in theorem 14, the cut formula cannot be principle in this case on both sides. ■

6.2. Soundness and Correspondence

We are now in a position to prove our final correspondence result, between **M** and **GM**. First, **GM** proves all of the translated versions of the theorems of **M**.

THEOREM 25. *If* $\vdash_{\mathbf{M}} \varphi$ *then* $\vdash_{\mathbf{GM}} \Rightarrow w:\sigma \to \varphi^*$.

Proof. We simply prove that the corresponding sequent version of the rule RM is admissible in **GM**. That is, one has to show that $\Rightarrow w : \sigma \to (\Diamond_N(\Box_\ni \varphi \wedge \Box_{\not\ni} \neg\varphi) \to \Diamond_N(\Box_\ni \psi \wedge \Box_{\not\ni} \neg\psi))$ follows from $\Rightarrow w : \sigma \to (\varphi \to \psi)$. It is relatively straightforward to construct the deduction for this. ■

We can also prove the reverse direction. As before, we require a soundness result for our sequent system. Let $\mathfrak{M}$ denote the class of supplemented neighborhood frames.

Lemma 26. *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in* **GM** *then it is $\mathfrak{M}^\circ$-valid.*

Proof. Beginning with the Mon rule, take $wNa, b : \tau, \Gamma \Rightarrow \Delta$ to be invalid. That is, for some interpretation i and model M° (based on a frame in $\mathfrak{M}^\circ$), $i(w)R_N i(a)$ and $M^\circ, i(b) \models \tau$. Therefore $i(b) \in R_\tau$ and either $i(a) \subseteq i(b)$ or $i(a) \nsubseteq i(b)$. In the first case, since M° is based on a supplemented frame we would have that $i(w)R_N i(b)$ and thus i would in fact invalidate $wNb, wNa, b : \tau, \Gamma \Rightarrow \Delta$ as well.

In the second case, where $i(a) \nsubseteq i(b)$, it would have to be that there is some state z s.t. $z \in i(a)$ but $z \notin i(b)$. Taking $j \sim_x i$ to be such that $j(x) = z$, we then have that $j(a)R_\ni j(x)$, $j(b)R_{\not\ni} j(x)$, $j(w)R_N j(a)$, and $M^\circ, j(b) \models \tau$. But, since x doesn't occur in Γ or Δ, this means that $a \ni x, b \not\ni x, wNa, b : \tau, \Gamma \Rightarrow \Delta$ would be invalidated by j.

Moving to the system $\{TS, TS_1, TS_2\}$, we proceed in a fashion analogous to the approach in the Proof of Theorem 21. Assume that $\Gamma \Rightarrow \Delta$ is invalidated by an interpretation function i in some M°. Then either $M^\circ, i(x) \models \varphi$ or not. In the second case, then, $\Gamma \Rightarrow \Delta, x : \varphi$ is not valid.

Otherwise, let j be an interpretation function agreeing with i on all labels except, possibly, b such that $j(b) = [\![\varphi]\!]$. We then have that $M^\circ, j(b) \models \tau$, and so the invalidity of $\Gamma \Rightarrow \Delta$ implies the invalidity of $b : \tau, \Gamma \Rightarrow \Delta$ (so long as b is not in Δ).

For TS_1, letting k be an interpretation agreeing with j on all labels in $b : \tau, x : \varphi, \Gamma \Rightarrow \Delta$, we have that if $b \ni y, \Gamma' \Rightarrow \Delta'$ is invalidated by k, in which case $k(y) \in k(b)$, then $M^\circ, k(y) \models \varphi$ (since $k(b)$ is the truth set of φ in M°) and so $y : \varphi, b \ni y, \Gamma' \Rightarrow \Delta'$ is also invalidated by k. For TS_2, taking k once again to be an "extension" of j, if it is the case that $y : \varphi, \Gamma'' \Rightarrow \Delta''$ is invalidated by k, in which case $M^\circ, k(y) \models \varphi$, then $k(y) \in [\![\varphi]\!] = k(b)$, and so $b \ni y, y : \varphi, \Gamma'' \Rightarrow \Delta''$ is also invalidated by k. ■

Theorem 27. *Let φ be an arbitrary well-formed formula in $\mathcal{L}_1$ and φ^* its $\mathcal{L}_5$ translation. The following are all equivalent:*

(i) $\vdash_{\mathbf{GM}} \Rightarrow w : \sigma \rightarrow \varphi^*$

(ii) *the sequent* $\Rightarrow w : \sigma \rightarrow \varphi^*$ *is* $\mathfrak{M}^\circ$*-valid*

(iii) φ *is* $\mathfrak{M}$*-valid*

(iv) $\vdash_{\mathbf{M}} \varphi$

Proof. $(i) \Longrightarrow (ii)$ follows from the soundness result for **GM**, which was proved in lemma 26.

$(ii) \Longrightarrow (iii)$: If $\Rightarrow w : \sigma \rightarrow \varphi^*$ is a $\mathfrak{M}^\circ$-valid sequent then it must be globally true for every interpretation function on each model based on an $\mathfrak{M}^\circ$-frame—every frame that is the translation of a supplemented neighborhood frame. But this (properly) includes all those models with a "neighborhood appropriate" valuation function. Thus, from theorem 3, it must be that φ is globally true on every neighborhood model corresponding to a model with a frame in $\mathfrak{M}^\circ$. However, this is all supplemented neighborhood models, so φ is valid with respect to the class of all supplemented neighborhood frames.

$(iii) \Longrightarrow (iv)$ is a consequence of the completeness of **M**.

$(iv) \Longrightarrow (i)$ is just theorem 25. ■

7. Combining Systems

In the sections above, we have provided sequent calculi for **E**, **C**, **N**, and **M**. However, the systems that have been given can easily and straightforwardly be combined, providing sequent systems for a wider family of logics.

For example, we can consider the logic $\mathbf{R} = \mathbf{EMC}$. Let us call **GR** the system obtained by adding to **GE** the rule Mon and the systems $\{TS, TS_1, TS_2\}$ and $\{C_1, C_\ni, C_{\not\ni}\}$. As expected, we easily obtain a result that is analogous to theorems 24 and 14.

Theorem 28. *We have the following in* **GR***:*

(i) *Weakening is hp-admissible;*

(ii) *All the logical rules are hp-invertible;*

(iii) *Contraction is hp-admissible;*

(iv) *Cut is admissible.*

Proof. Combining Mon and $\{C_1, C_\ni, C_{\not\ni}\}$ is unproblematic with respect to cut-elimination and the admissibility of the structural rules as they both follow the generalized geometric rule scheme. It is easy to see that in the presence of the system $\{TS, TS_1, TS_2\}$, the standard procedure of Theorem 24 suffices to obtain the desired result. ■

Furthermore, the soundness and correspondence theorems are also now straightforward.

THEOREM 29. *If* $\vdash_{\mathbf{R}} \varphi$ *then* $\vdash_{\mathbf{GR}} \Rightarrow w : \sigma \to \varphi^*$.

PROOF. This is a result of having the relevant translations of the axioms and rules in the **GR** system. ∎

LEMMA 30. *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* **GR** *then it is* $\mathfrak{R}^\circ$*-valid. That is, it is valid on the class of frames obtained by translating neighborhood frames that are quasi-filters.*

PROOF. This result, however, is an immediate consequence of lemmas 21 and 26: we know that the **GC** rules preserve validity on frames translated from neighborhood frames closed under intersection and that the **GM** rules preserve validity on translated supplemented frames. ∎

THEOREM 31. *Let* φ *be an arbitrary well-formed formula in* $\mathcal{L}_1$ *and* φ^* *its* $\mathcal{L}_5$ *translation. The following are all equivalent:*

(i) $\vdash_{\mathbf{GR}} \Rightarrow w : \sigma \to \varphi^*$

(ii) *the sequent* $\Rightarrow w : \sigma \to \varphi^*$ *is* $\mathfrak{R}^\circ$*-valid*

(iii) φ *is* $\mathfrak{M}$*-valid*

(iv) $\vdash_{\mathbf{R}} \varphi$

PROOF. This proof works in exactly the same manner as the others. ∎

This same technique works for other combinations of the base logics as well, rendering a unified account of the usual classical modal logics.

8. Completeness

The main purpose of this paper was to provide sequent systems for a family of classical modal logics in a modular manner. However, while this was the main focus, there are also peripheral questions that arise naturally, specifically surrounding the models in $\mathfrak{E}^\circ$. To what extent can these be investigated from a proof-theoretic perspective? Can one establish completeness theorems? This section explores such questions. It turns out that completeness proofs for **GE**, **GN**, and **GC** are forthcoming, but involve certain trade-offs.

To begin with, however, we must alter the definition of the semantic translation given in section 2.3 above. There, recall, W° was defined as

$W \cup \wp(W)$. This will no longer suffice. Instead, we must adopt the following definitions[12] of W° and R_τ:

$$\begin{aligned} W^\circ &:= W \cup \bigcup_{w \in W} n(w) \\ R_\tau &:= \bigcup_{w \in W} n(w) \end{aligned}$$

Changing the definition in this way actually has very little effect on the results presented thus far. The one change that must be made is an alteration of the truth set system of rules in **GM**. Specifically, in order for the rule to be sound we have to explicitly consider the case that a truth set for a particular formula might not appear in W° because it is not a neighborhood of any state in F. If we do this, the system then becomes the somewhat ungainly:

$$\dfrac{\Gamma \Rightarrow \Delta, x : \varphi \qquad \dfrac{\dfrac{y : \varphi, b \ni y, \Gamma' \Rightarrow \Delta'}{b \ni y, \Gamma' \Rightarrow \Delta'}\ TS_1 \quad \dfrac{b \ni y, y : \varphi, \Gamma'' \Rightarrow \Delta''}{\sigma(y), y : \varphi, \Gamma'' \Rightarrow \Delta''}\ TS_2}{\tau(b), \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{c \ni z, \tau(c), \Gamma''' \Rightarrow \Delta''', z : \varphi}{\tau(c), \Gamma''' \Rightarrow \Delta'''}\ TS_3}{x : \top, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\ TS$$

(where z also satisfies the eigenvariable condition.) Were we to adopt the more parsimonious domain of the translated models, nothing in **GE**, **GN**, or **GC** would change. For **GM**, the soundness proof would have to be altered as well as the proof of the admissibility of structural rules. These, however, go through. So, as we remarked earlier, the decision to work with the bloated domains is motivated entirely by the desire to create the most elegant proof system possible.

However, if one desires completeness proofs to go along with the proof systems, this elegance must be sacrificed. Once the restricted domain has been adopted, we must also add the rules noted in Fig. 6.

In [9], a first-order axiomatization (in a two-sorted language) is provided that is shown to be complete with respect to the class of translated neighborhood structures (where the translated structures are first-order, rather than relational models). Their axiom $\forall u \exists x (xNu)$ corresponds to the Nbr

[12]Again, some care must be taken to ensure that the union of W and $\bigcup_{w \in W} n(w)$ is disjoint. This can be handled in a variety of ways, including by taking the disjoint union as we did above. Once more, to rein in an already complicated notation, we will continue as if the simple definition is sufficient, with the caveat that in certain circumstance extra steps must be taken.

$$\frac{\sigma(x), \Gamma \Rightarrow \Delta \quad \tau(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Dom$$

$$\frac{\sigma(x), \tau(a), xNa, \Gamma \Rightarrow \Delta}{xNa, \Gamma \Rightarrow \Delta} \; Typ_N$$

$$\frac{a \ni x, \sigma(x), \tau(a), \Gamma \Rightarrow \Delta \quad a \not\ni x, \sigma(x), \tau(a), \Gamma \Rightarrow \Delta}{\sigma(x), \tau(a), \Gamma \Rightarrow \Delta} \; Incl$$

$$\frac{wNa, \tau(a), \Gamma \Rightarrow \Delta}{\tau(a), \Gamma \Rightarrow \Delta} \; Nbr$$

Figure 6. Additional type and frame conditions. The w in the Nbr rule is an eigenvariable

$$\frac{a \ni x, b \not\ni x, \Gamma \Rightarrow \Delta, a \approx b \quad b \ni x, a \not\ni x, \Gamma \Rightarrow \Delta, a \approx b}{\Gamma \Rightarrow \Delta, a \approx b} \; R\approx$$

$$\frac{a \approx b, \tau(a), \tau(b), \Gamma \Rightarrow \Delta \quad \tau(a), \tau(b), \Gamma \Rightarrow \Delta, a \approx b}{\tau(a), \tau(b), \Gamma \Rightarrow \Delta} \; \approx/\not\approx$$

$$\frac{wNb, a \approx b, wNa, \Gamma \Rightarrow \Delta}{a \approx b, wNa, \Gamma \Rightarrow \Delta} \; N\approx$$

Figure 7. Similarity sequents. The x in $R\approx$ is an eigenvariable

rule. Their other axiom, $\forall u, v(\forall x(uEx \leftrightarrow vEx)) \rightarrow u = v)$, and the role of equality in general, can be sufficiently captured by the rules noted in Fig. 7.

Let us call **GE** combined with these two new sets of rules $\mathbf{GE}^+$. We can now prove a completeness theorem.

LEMMA 32. *If a sequent* $\Gamma \Rightarrow \Delta$ *is* $\mathfrak{E}^\circ$*-valid then it is derivable in* $\mathbf{GE}^+$.

PROOF. The beginning part of the proof follows the method in [16], which is standard. The general strategy is to systematically, and in reverse, attempt to build a proof-tree for $\Gamma \Rightarrow \Delta$ by applying all possible rules at each step. If this process does not terminate, meaning $\Gamma \Rightarrow \Delta$ is not provable, we show that a counter-model can be constructed from the infinite, failed proof-tree, and so it is not valid either. However, this counter model is just a relational model $\langle W, R_N, R_\ni, R_{\not\ni}, R_\sigma, R_\tau \rangle$. We will then need to show that this model corresponds to the translation of some neighborhood frame. It is this second part that occupies most of our attention.

However, to begin, one must first construct the proof-tree for $\Gamma \Rightarrow \Delta$. It is built in stages.

At stage 0, we set $\Gamma \Rightarrow \Delta$ as the root of the tree.

For all stages $n > 0$, we have two cases.

Case 1: all of the leaves of the tree are initial sequents. If so, we are done, and the construction terminates resulting in a proof of $\Gamma \Rightarrow \Delta$.

Case 2: otherwise, we have topsequents that are not initial and so sequent rules have to be applied to these. At each stage n where this is the case, we choose a particular rule, and apply it to every topsequent for which it is applicable. For all of the other topsequents, the ones for which the chosen rule does not apply, we simply repeat the sequent above it, growing the depth of that branch by one. In order to control the growth of the tree, one imposes an order according to which the rules are applied. The specific order one uses is not essential. What is important is that at each stage exactly one rule is chosen and applied as many times as possible, and that the rules are applied in some sort of cyclical fashion. We will omit the details here, as they require too much space and can easily be found in [16].

If the process ends, then all of the topsequents in the construction are initial. In this case, the tree constitutes a proof of $\Gamma \Rightarrow \Delta$. Otherwise, if the process does not terminate, the tree will be infinite. Because the tree is finitely branching, by König's lemma we must have at least one infinitely long branch rooted in $\Gamma \Rightarrow \Delta$: $\Gamma \Rightarrow \Delta$, $\Gamma_1 \Rightarrow \Delta_1$, $\Gamma_2 \Rightarrow \Delta_2$, ... Letting $\Gamma_0 \Rightarrow \Delta_0$ stand for $\Gamma \Rightarrow \Delta$, we can create the following two multisets:

$$\mathbf{\Gamma} := \bigcup_{i \geq 0} \Gamma_i \quad \mathbf{\Delta} := \bigcup_{i \geq 0} \Delta_i$$

and can now define a model that will simultaneously satisfy all the formulas in $\mathbf{\Gamma}$ and none in $\mathbf{\Delta}$.

Let the domain of our frame, W^{GE}, be the collection of all labels occurring in $\mathbf{\Gamma}$.

We will define the relations as follows:

$$\begin{array}{lcl} R_N^{GE} & := & \{\langle w, a\rangle \in W^{GE} \times W^{GE} \mid wNa \text{ appears in } \mathbf{\Gamma}\} \\ R_{\ni}^{GE} & := & \{\langle a, w\rangle \in W^{GE} \times W^{GE} \mid a \ni w \text{ appears in } \mathbf{\Gamma}\} \\ R_{\not\ni}^{GE} & := & \{\langle a, w\rangle \in W^{GE} \times W^{GE} \mid a \not\ni w \text{ appears in } \mathbf{\Gamma}\} \\ R_{\sigma}^{GE} & := & \{w \in W^{GE} \mid \sigma(w) \text{ appears in } \mathbf{\Gamma}\} \\ R_{\tau}^{GE} & := & \{a \in W^{GE} \mid \tau(a) \text{ appears in } \mathbf{\Gamma}\} \end{array}$$

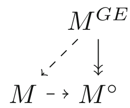

Figure 8. Completeness

Given this frame, F^{GE}, we can construct the model M^{GE} by defining an appropriate valuation function. Specifically, for all propositional variables p, take

$$V^{GE}(p) := \{w \in W^{GE} \mid w : p \text{ appears in } \mathbf{\Gamma}\}$$

Note that this definition restricts V^{GE} such that if $w : p \in \mathbf{\Delta}$ then $w \notin V^{GE}(p)$. This is because atomic formulas are always preserved in the construction process, and if $w : p$ had occurred at any stage on both the right and left hand side, the sequent would have been initial, and the branch would not have been infinite.

We can now prove, via structural induction on formulas, that for all $w \in W^{GE}$, and all $\mathcal{L}_5$ formulas,

$$\text{if } w : \varphi \in \mathbf{\Gamma} \text{ then } M^{GE}, w \models \varphi$$
$$\text{and if } w : \varphi \in \mathbf{\Delta} \text{ then } M^{GE}, w \not\models \varphi$$

This proof is also standard, and so we leave it out.

At this stage we now have a model invalidating invalid sequents. However, to complete the proof of the theorem we still have to demonstrate that a *frame contained in* $\mathfrak{E}^\circ$ invalidates invalid sequents. To do this we will show that we can construct from M^{GE} a neighborhood model that we can then translate, and show that this translation and M^{GE} satisfy exactly the same formulas. Diagrammatically, the situation looks something like in Fig. 8.

So, to this end, consider the frame $F = \langle W, n \rangle$ defined as follows. First, set

$$W := R_\sigma$$

Then, for each $a \in W^{GE}$ s.t. $a \in R_\tau^{GE}$, define the set

$$s(a) := \{x \in W^{GE} \mid \langle a, x \rangle \in R_\ni^{GE}\}.$$

We can now define the neighborhood function n for each $w \in W$:

$$n(w) := \{s(a) \in \wp(W) \mid \langle w, a \rangle \in R_N^{GE}\}$$

It must be demonstrated that this is a good definition. Specifically, we must show that if $s(a) = s(b)$ then $\langle w, a \rangle \in R_N^{GE}$ iff $\langle w, b \rangle \in R_N^{GE}$ for all w.

First, notice that if $s(a) = s(b)$ then it must be that $a \approx b \in \mathbf{\Gamma}$. $a \approx b$ must either be in $\mathbf{\Gamma}$ or $\mathbf{\Delta}$ (from $\approx$ / $\not\approx$). Assuming that it is in $\mathbf{\Delta}$, we would have, from the $R\approx$ rule, that there is some $x \in R_\sigma^{GE}$ for which either $a \not\ni x \in \mathbf{\Gamma}$ and $b \ni x \in \mathbf{\Gamma}$ or $a \ni x \in \mathbf{\Gamma}$ and $b \not\ni x \in \mathbf{\Gamma}$. In either case, and relying on the $Excl_{\ni\not\ni}$ rule (to ensure, for example in the first case, that $b \ni x \notin \mathbf{\Gamma}$), we would have that $s(a) \neq s(b)$.

So, assuming that $s(a) = s(b)$ and $s(a) \in n(w)$, we have that $a \approx b \in \mathbf{\Gamma}$ and $\langle w, a\rangle \in R_N^{GE}$, which just means that $wNa \in \mathbf{\Gamma}$. But the presence of the $N\approx$ rule immediately ensures that $wNb \in \mathbf{\Gamma}$ as well, as desired.

We now have a neighborhood frame F. Define, in the usual way, F°. Consider the following function $m : W^{GE} \to W^\circ$:

$$m(u) = \begin{cases} u & u \in R_\sigma^{GE} \\ s(u) & u \in R_\tau^{GE} \end{cases}$$

(Note that the Dom and $Excl_{\sigma\tau}$ rules ensure that all elements of W^{GE} are either in R_σ^{GE} or in the domain of s, and not both.)

We claim that m is a bounded morphism between F^{GE} and F°.

Note first that $R_\sigma^{GE} = W = R_\sigma$.

For R_τ^{GE}, we have that if $u \in R_\tau^{GE}$ then $\tau(u) \in \mathbf{\Gamma}$, and so there is some v for which $vNu \in \mathbf{\Gamma}$ (from the Nbr rule). Then $s(u) \in n(v)$, which means that $s(u) \in R_\tau$. The other direction just follows this argument in reverse.

Assume that $\langle w, a\rangle \in R_N^{GE}$. By definition, it must be that wNa appears in $\mathbf{\Gamma}$. Then, because of the rules Typ_N and $Excl_{\sigma\tau}$, $\sigma(w) \in \mathbf{\Gamma}$ and $\sigma(a) \notin \mathbf{\Gamma}$ (and $\tau(a) \in \mathbf{\Gamma}$), so $w \in R_\sigma^{GE}$ but $a \notin R_\sigma^{GE}$. This means that $m(w) = w$ and $m(a) = s(a)$. Also, by definition, $s(a) \in n(w)$, so $\langle w, s(a)\rangle \in R_N$, i.e. $\langle m(w), m(a)\rangle \in R_N$.

For the *back* condition, assume that $\langle m(w), a'\rangle \in R_N$. From the definition of n we know that it must be that $w \in W$, and thus $w \in R_\sigma$, and that $a' \in n(w)$. But this means that $a' = s(a)(= m(a))$ for some a with $\langle w, a\rangle \in R_N^{GE}$.

Moving on to the $R_\ni$ case, assume that $\langle a, w\rangle \in R_\ni^{GE}$. By definition we then have that $a \ni w$ appears in $\mathbf{\Gamma}$. From $Typ_\ni$ and $Excl_{\sigma\tau}$, $\sigma(w) \in \mathbf{\Gamma}$ and $\sigma(a) \notin \mathbf{\Gamma}$ (and $\tau(a) \in \mathbf{\Gamma}$). This means that $m(w) = w$, $m(a) = s(a)$, and $w \in s(a)$. Also, it must be that $vNa \in \mathbf{\Gamma}$ for some $v \in R_\sigma^{GE}$ (from Nbr). Thus $s(a) \in n(v)$ and so $s(a) \in W^\circ$ with $\langle s(a), w\rangle \in R_\ni$. But this is just $\langle m(a), m(w)\rangle \ni R_\ni$.

When $\langle m(a), w'\rangle \in R_\ni$, $w' \in m(a)$, by definition. Clearly, then, $a \notin R_\sigma^{GE}$ so $m(a) = s(a)$. Also, we have that $w' \in W$, so $w' \in R_\sigma^{GE}$ which means that $m(w') = w'$. So $w' \in s(a)$, which can only be the case if $\langle a, w'\rangle \in R_\ni^{GE}$.

Finally, consider $R^{GE}_{\not\ni}$. Let $\langle a, w\rangle \in R^{GE}_{\not\ni}$. By definition $a \not\ni w$ appears in $\mathbf{\Gamma}$. Also, $\sigma(w) \in \mathbf{\Gamma}$ and $\sigma(a) \notin \mathbf{\Gamma}$. Thus $m(w) = w$ and $m(a) = s(a)$. Also, from $Excl_{\ni\not\ni}$, $w \notin s(a)$. Since it must be that for some $v \in R^{GE}_{\sigma}$, $vNa \in \mathbf{\Gamma}$, $s(a) \in n(v)$ and so $s(a) \in W^{\circ}$. Obviously, also, $w \in W^{\circ}$ and $\langle s(a), w\rangle \in R_{\not\ni}$, which means $\langle m(a), m(w)\rangle \in R_{\not\ni}$.

Lastly, take $\langle m(a), w\rangle \in R_{\not\ni}$. This means that $w \notin m(a) = s(a)$ for $w \in W$ and $m(a) \in n(v)$ for some $v \in W$. Therefore, $\langle v, a\rangle \in R^{GE}_{N}$. Because $w \notin s(a)$, $\langle a, w\rangle \notin R^{GE}_{\ni}$, which in turn implies that $a \ni w \notin \mathbf{\Gamma}$. Then, by the *Incl* rule, it has to be that $a \not\ni w \in \mathbf{\Gamma}$, and so $\langle a, w\rangle \in R^{GE}_{\not\ni}$.

So we have that m is a bounded morphism from F^{GE} to F°. Furthermore, m is surjective. Obviously all the members of R_{σ} are mapped onto by the elements of R^{GE}_{σ}, as in this case m is just the identity function. For elements of R_{τ}, consider some $A \in R_{\tau}$. Then there exists some $w \in W$ s.t. $A \in n(w)$. Then $A = s(a)$ for some a s.t. $\langle w, a\rangle \in R^{GE}_{N}$. But then this $a \in R^{GE}_{\tau}$ and $m(a) = s(a)$.

Therefore, $F^{GE} \twoheadrightarrow F^{\circ}$ (and so if $F^{GE} \models \varphi$ then $F^{\circ} \models \varphi$).

Lastly, define $M^{\circ} = \langle F^{\circ}, V\rangle$ where $V(p) = \{m(w) \mid w \in V^{GE}(p)\}$. Clearly, then, w and $m(w)$ satisfy the same propositional variables, so $M^{GE} \twoheadrightarrow M^{\circ}$.

Finally then, suppose that the sequent $\Gamma \Rightarrow \Delta$ is not provable in $\mathbf{GE}^{+}$. Then we can construct, out of an attempted proof search, the model M^{GE}, as above, such that for all $w : \varphi \in \mathbf{\Gamma}$, $M^{GE} \models \varphi$ but for all $w : \varphi \in \mathbf{\Delta}$ $M^{GE} \not\models \varphi$. But, because $M^{GE} \twoheadrightarrow M^{\circ}$, this means that it is also the case that for all $w : \varphi \in \mathbf{\Gamma}$, $M^{\circ} \models \varphi$ but for all $w : \varphi \in \mathbf{\Delta}$ $M^{\circ} \not\models \varphi$. Thus, it must be that $\Gamma \Rightarrow \Delta$ is not $\mathfrak{E}^{\circ}$-valid. ■

We can easily extend this result for $\mathbf{GC}^{+}$ and $\mathbf{GN}^{+}$ as well.

THEOREM 33. *If a sequent* $\Gamma \Rightarrow \Delta$ *is* $\mathfrak{C}^{\circ}$*-valid then it is derivable in* $\mathbf{GC}^{+}$.

PROOF. First, we must supplement the construction phase of the Proof of Theorem 32 with the new rules of **GC**. After that, all we need to demonstrate is that the neighborhood model we construct from M^{GC} is closed under intersections. That is, for each $a, b \in R^{GC}_{\tau}$, and each $w \in W$, if $s(a) \in n(w)$ and $s(b) \in n(w)$, then $s(a) \cap s(b) \in n(w)$.

First, if $s(a) \in n(w)$ and $s(b) \in n(w)$ then, by the definition of the model, it must be that $wNa, wNb \in \mathbf{\Gamma}$. Then, since C_1 must have been applied, there is a c for which $wNc \in \mathbf{\Gamma}$. We claim that $s(c) = s(a) \cap s(b)$. In the first direction, taking x to be any element of W, if $x \in s(c)$ then $c \ni x \in \mathbf{\Gamma}$ and so, from $C_{\ni}$ we have that $a \ni x \in \mathbf{\Gamma}$ and $b \ni x \in \mathbf{\Gamma}$. Therefore $x \in s(a)$ and $x \in s(b)$.

Otherwise, if $x \notin s(c)$, from the *Incl* rule it must be that either $c \ni x \in \mathbf{\Gamma}$ or $c \not\ni x \in \mathbf{\Gamma}$. The $Excl_{\ni\not\ni}$ rule eliminates the first possibility, and so $c \not\ni x \in \mathbf{\Gamma}$. But then, from $C_{\not\ni}$ we must have either $a \not\ni x \in \mathbf{\Gamma}$ or $b \not\ni x \in \mathbf{\Gamma}$. This ensures that either $x \notin s(a)$ or $x \notin s(b)$. ■

THEOREM 34. *If a sequent* $\Gamma \Rightarrow \Delta$ *is* $\mathfrak{N}^\circ$*-valid then it is derivable in* $\mathbf{GN}^+$*.*

PROOF. As before, we assume the construction of M^{GN} has been appropriately supplemented with the system of rules $\{N_1, N_2\}$. Now we just need $W \in n(w)$ for all $w \in W$. If $w \in W$ then $w : \sigma \in \mathbf{\Gamma}$. Thus, from N_1 it is also the case that there is some a for which $wNa \in \mathbf{\Gamma}$. Also, by construction of $\mathbf{\Gamma}$, for every other $x \in R^{GN}_\sigma$ we have that $a \ni x \in \mathbf{\Gamma}$, from N_2, and so $x \in s(a)$. As this includes every $x \in W$, we have that $s(a) = W$. ■

Achieving such a result for $\mathbf{GM}^+$ is less straightforward (because we cannot simply translate M as we do above, but must now take into account a supplementation of M), and we leave this for future work.

9. Conclusion

Using the approach of systems of rules for generalized geometric formulas, we have demonstrated that a modular sequent calculus is possible for classical modal logics. Specifically, we gave systems for **E**, **N**, **C**, and **M**, and our approach allows for these to be combined modularly, and straightforwardly. Several questions remain open, however. The most obvious is: why stop with classical modal logics? To what extent can normal modal logics be incorporated into this picture? Nested modalities will make the syntactic translations more cumbersome, but nothing, in principle, seems to stand in the way of such extensions. We intend to investigate these possibilities in future work.

Another important aspect that is left for future work concerns decision methods for our logics. It might happen that the attempted derivation of a sequent produces an infinite branch (as in our completeness proof). Thus, this does not serve as an effective method for obtaining counter-models from failed proof-searches. In order to obtain a decision method based on the construction of counter-models, we would need to prove that we can always construct a finite counter-model from the wreckage of a failed proof-search. A standard method involves involves imposing certain saturation conditions. Interestingly, this has been recently achieved for various labelled systems in [17].

Lastly, if one is only interested in working with monotonic modal logics, one can utilize a simpler translation and, therefore, logical system.[13] Very briefly, one makes use of a bimodal logic, containing the operators $\Diamond_N \varphi$ and $\Box_\ni \varphi$. The relative translation clause (corresponding to the monotonic semantic condition) then becomes just

$$(\Box\varphi)^* \quad := \quad \Diamond_N \Box_\ni \varphi^*$$

and no rules need to be added to the core logical system. The result is very neat, and we think that such a system can be well utilized in various domains in which monotonic modal logics are studied. We leave this for future work.

Acknowledgments. We would like to thank Barteld Kooi, Sara Negri, Louwe Kuijer, and two anonymous referees for very helpful feedback and comments.

References

[1] Chellas, B., *Modal Logic: An Introduction*. Cambridge University Press, Cambridge, 1980.

[2] Fitting, M., Proof theory for modal logic, in P. Blackburn, J. Van Benthem, and F. Wolter, (eds.), *Handbook of Modal Logic*. Elsevier, Amsterdam, 2007.

[3] Gabbay, D., *Labelled Deductive Systems: Volume 1*. Clarendon Press, Oxford, 1996.

[4] Gasquet, O., and A. Herzig, From classical to normal modal logics, in H. Wansing, (ed.), *Proof Theory of Modal Logic*. Kluwer, 1996, pp. 293–311.

[5] Gentzen, G., Investigations into logical deduction, in M. Szabo, (ed.), *The Collected Papers of Gerhard Gentzen*. North-Holland, 1969, pp. 68–131.

[6] Governatori, L., and A. Luppi, Labelled tableaux for non-normal modal logics, in *Sixth Conference of the Italian Association for Artificial Intelligence, AI*IA '99*, 2000, pp. 119–130.

[7] Hansen, H. H., *Monotonic modal logics*. Master's thesis, University of Amsterdam, 2003.

[8] Hansen, H. H., *Tableau games for coalition logic and alternating-time temporal logic—theory and implementation*. Master's thesis, University of Amsterdam, 2004.

[9] Hansen, H. H., C. Kupke, and E. Pacuit, Neighborhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science* 5:1–38, 2009.

[10] Indrzejczak A., Sequent calculi for monotonic modal logics. *Bulletin of the Section of Logic* 34(3):151–164, 2005.

[11] Indrzejczak A., Admissibility of cut in congruent modal logics. *Logic and Logical Philosophy* 21:189–203, 2011.

[12] Kracht, M., and F. Wolter, Normal monomodal logics can simulate all others. *The Journal of Symbolic Logic* 64:99–138, 1999.

[13] Lavendhomme, R., and T. Lucas, Sequent calculi and decision procedures for weak modal systems. *Studia Logica* 66:121–145, 2000

[13] Details on this translation can be found in [7,12], and [4].

[14] Montague, R., Universal grammar. *Theoria* 36:373–398, 1970.

[15] Negri, S., Proof analysis in modal logic. *Journal of Philosophical Logic* 34:507–544, 2005.

[16] Negri, S., Kripke completeness revisited, in G. Primiero and S. Rahman, (eds.), *Acts of Knowledge—History, Philosophy and Logic*. College Publications, 2009, pp. 242–282.

[17] Negri, S., Proofs and countermodels in non-classical logics. *Logica Universalis* 8:25–60, 2014.

[18] Negri, S., Proof analysis beyond geometric theories: from rule systems to systems of rules. *Journal of Logic and Computation*. (in press).

[19] Negri, S., and J. von Plato, *Structural Proof Theory*. Cambridge University Press, Cambridge, 2001.

[20] Negri, S., and J. von Plato, *Proof Analysis: A Contribution to Hilbert's Last Problem*. Cambridge University Press, Cambridge, 2011.

[21] Scott, D., Advice on modal logic, in K. Lambert, (ed.), *Philosophical Problems in Logic: Some Recent Developments*. Reidel, Dordrecht, 1970.

D. R. Gilbert and P. Maffezioli
University of Groningen
Groningen
Netherlands
d.r.gilbert@rug.nl