Balancing for Nonlinear Differential-Algebraic Control Systems

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Published in:
Prodeedings of the American Control Conference 2024 (ACC2024)

DOI:
10.23919/ACC60939.2024.10644344

2024 American Control Conference (ACC)
July 8-12, 2024. Toronto, Canada

Balancing for nonlinear differential-algebraic control systems

Arijit Sarkar, Yu Kawano and Jacquelien M.A. Scherpen

Abstract— **In this paper, we develop a balancing theory for nonlinear differential-algebraic control systems. We exploit the maximally controlled invariant submanifold to define the controllability and observability functions and to provide a balanced realization. We also construct a reduced-order model based on truncation of states which also preserves the constraints associated with the original system. Finally, we illustrate the results with an example.**

I. Introduction

A. Differential-algebraic control systems

In several physical systems, the associated state variables are constrained, e.g. mechanical systems with holonomic and nonholonomic constraints [1], conservation laws such as Kirchoff's current and voltage law in electrical circuits [2], boundary conditions in chemical processes [3]. The mathematical model describing the dynamics of such systems also contains algebraic equations. Such systems are called *differential-algebraic systems*, *implicit systems*, *singular systems*, or *descriptor systems* in the literature. If external input signals are being introduced to control the dynamics of systems, then they are called *differential-algebraic control systems (DACSs)*.

B. Literature review on model reduction for differential-algebraic control systems

Balancing is a methodology being used to perform model reduction for large-scale complex dynamical systems. The notion of balanced truncation was introduced in the seminal work [4] for linear dynamical systems which is based on the simultaneous diagonalization of the controllability and observability Gramians, which are the solutions of respective Lyapunov equations. This approach of model reduction can preserve a few properties of the original system in the reduced order model e.g., stability, controllability and observability. Moreover, it can also provide an apriori error bound based on the Hankel singular values associated with the truncated states [5]. A thorough exposition of balanced truncation for linear systems is provided in [6]. Balanced truncation has also been extended to linear descriptor systems in [7], [8]. Afterwards, various other balanced truncation methods have been introduced preserving certain properties of the original systems such as passivity, and positivity [9], [10]. On the other hand, balancing had been introduced for nonlinear control systems in [11]. Since its inception, different types of balancing for nonlinear systems [12], [13], minimality considerations [14], association with Hankel operator [15] have been investigated. However, this has not been explored well for nonlinear DACSs apart from [16] under the *strangeness-free* assumption and considering the semi-explicit form of the DACS. It is worth mentioning that it is cumbersome to transform the differential-algebraic system into the semi-explicit form, and in the process, it modifies the original input for systems with (differentiation) index greater than 1. Moreover, in [16], the reduction procedure doesn't guarantee to preserve the constraints of the original system.

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C. Contribution

In this work, we propose controllability and observability functions associated with a nonlinear differential-algebraic control system with state algebraic constraints. We define the energy functions on the maximally controlled invariant submanifold associated with the system. Moreover, we exploit the invariant submanifold to come up with a balanced realization of the system. We also show that the truncation of states based on balanced realization also preserves the algebraic constraints associated with the system.

The rest of the paper is organized as follows. In Section II, we provide preliminaries about nonlinear DACSs. In Section III, we introduce the notion of Hankel operator for DACSs. In Section IV, we define the controllability and observability functions for nonlinear DACSs and a balanced realization, reduced order model based on that. In Section V, we provide an illustrative example to support the theoretical analysis. Finally, conclusions are drawn in Section VI.

Notation:

The set of real numbers is denoted by $\mathbb{R}$. An interval of the real line is denoted by $\mathcal{I}$. The dimension of a subspace (submanifold) M is denoted by $\dim(M)$. We denote the i^{th} component of a vector v by v_i. For a signal $x : \mathbb{R}_+ \to \mathbb{R}^n$, $\|x(t)\|_i$, $1 \leq i \leq \infty$ denotes its L_i-norm, e.g., $\|x(t)\|_2 := \int_0^\infty |x(t)|_2 dt$. The set of signals $x : \mathbb{R}_+ \to \mathbb{R}^n$ whose L_i-norm are bounded is denoted by $L_i^n[0, \infty)$. C^∞ denotes the space of smooth functions that are differentiable at all degrees.

II. Preliminaries

In this section, we summarize the basics of nonlinear differential-algebraic control systems. Let us consider a nonlinear differential-algebraic control system of the form

$$\Xi^u : \begin{cases} \dot{\overbrace{E(x)}} = E'(x)\dot{x} = f(x) + g(x)u, \\ y = h(x), \end{cases} \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is the generalized state, $u \in \mathbb{R}^m$ is the input vector and $y \in \mathbb{R}^p$ is the output vector. $E, f : X \to \mathbb{R}^n$, $g : X \to \mathbb{R}^{n\times m}$ and $h : X \to \mathbb{R}^p$ are $\mathcal{C}^\infty$ - smooth maps. For each $x \in X$, $E'(x) := \frac{\partial E}{\partial x}(x)$ induces a linear map from T_xX to $\mathbb{R}^n$, where T_xX is the tangent space at $x \in X$. As $X \subseteq \mathbb{R}^n$, $E'(x) \in \mathbb{R}^{n\times n}$. The autonomous system can be represented as follows

$$\Xi : \begin{cases} \dot{\overbrace{E(x)}} = E'(x)\dot{x} = f(x), \\ y = h(x), \end{cases} \tag{2}$$

Remark 1. *If a DACS is represented as* (1)*, then there exists a map $E : \mathbb{R}^n \to \mathbb{R}^n$ whose components are independent functions if and only if the co-distribution generated by the rows of $E'(x)$ is integrable (involutive).*

The condition mentioned in the above remark is satisfied in various applications, e.g., for mechanical systems with holonomic constraints, electrical circuits with nonlinear circuit elements (as considered in Section V), etc. It would be evident in Section IV that this particular representation of nonlinear DACS is essential for the controllability and observability functions we proposed in this paper. Before developing the balancing theory, we summarize some notions of DACSs.

Definition 1. *(Solution) For a nonlinear DACS* (1)*, $(x, u) : \mathcal{I} \to X \times \mathbb{R}^m$ defined on an open interval $\mathcal{I} \subseteq \mathbb{R}$ with $x(\cdot) \in \mathcal{C}^1(\mathcal{I})$ and $u(\cdot) \in \mathcal{C}^0(\mathcal{I})$ is called a solution to* (1) *for all $t \in \mathcal{I}$ if it satisfies $E'(x(t))\dot{x}(t) = f(x(t)) + g(x(t))u(t)$.*

A point $x_0 \in X$ is an admissible point of (1) if there exists at least one solution $(x(\cdot), u(\cdot))$ such that $x(t_0) = x_0$. The set of all admissible points is denoted by $\mathcal{S}_x$ and also referred to as *consistency space*.

Definition 2. *(Internal regularity) Ξ is locally internally regular (around $x_p \in M^*$) if there exists a neighbourhood $U \subseteq X$ of x_p such that for any point $x_o \in M^* \cap U$, there is only one maximal solution $x : \mathcal{I} \to M^* \cap U$ satisfying $x(t_0) = x_0$ for a certain $t_0 \in \mathcal{I}$.*

In this paper, we investigate a balanced realization on a controlled invariant submanifold as defined below.

Definition 3. *(Locally (controlled) invariant submanifold) Consider the DACS ((1)) (2). A smooth connected embedded submanifold $M \subset X$ is called a (locally) controlled invariant submanifold of* (1) *around a point $x_0 \in M$ if there exists a neighbourhood U of x_0, and for any $x_0 \in M \cap U$, there exists a solution $(x, u) : \mathcal{I} \to X \times \mathbb{R}^m$ such that $x(t_0) = x_0$ for a certain $t_0 \in \mathcal{I}$ and $x(t) \in M \cap U$ for all $t \in \mathcal{I}$.*

A locally (controlled) invariant submanifold M^* around a point $x_0 \in M^*$, called a locally maximally (controlled) invariant submanifold if there exists a neighbourhood U of x_0 such that for any other locally (controlled) invariant submanifold M containing x_0, we have $(M\cap U) \subseteq (M^*\cap U)$. It can be proven that the consistency space $\mathcal{S}_x$ is equal to M^* [17].

Assumption 1. *There exists an open neighbourhood V of $x_0 \in X$ such that $\dim E'(x)T_xM^* = \dim M^*$ and is constant for all $x \in M^* \cap V$ and $\dim(E'(x)T_xM^*) \leq \dim(\operatorname{Im} g(x))$ for all $x \in M \cap V$.*

Assumption 1 implies that the nonlinear system (1) is internally regular and has no constraints on the input signal.

Consider the DACS (1) which satisfies Assumption 1. Also, consider a point $x_0 \in X$ and U_0 be an open connected subset of X containing x_0. Set $M_0 = X$, $M_0^c = U_0$. Suppose that there is an open neighbourhood U_{k-1} of x_0 and sequence of smooth connected embedded submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} is constructed. Then we can define

$$M_k := \{x \in M_{k-1}^c | f(x) \in E'(x)T_xM_{k-1}^c + \operatorname{Im} g(x)\}. \tag{3}$$

There always exists a number $k^* \leq n$, the smallest integer such that $x_0 \notin M_{k^*+1}$ or k^* is the smallest integer such that $x_0 \in M_{k^*+1}^c$ and $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. In the latter case if $M^* = M_{k^*+1}^c$ satisfies Assumption 1 in a neighbourhood $U^* = U_{k+1}^*$ containing x_0, then there exists a transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ in U^* such that the locally maximally controlled invariant submanifold M^* can be represented as [17]

$$M^* = \{x|\bar{z}_1 = 0, \cdots, \bar{z}_{k^*} = 0\}, \tag{4}$$

where $z := \phi(x) = (z^*, \bar{z}_1, \cdots, \bar{z}_{k^*})$.

III. Hankel operator for nonlinear DACSs

Let us consider the nonlinear differential-algebraic system (1). By abuse of notation, the symbol $\Xi^u(u)$ is also used to denote the input-output operator induced by the system with $x(-\infty) = 0$. Suppose that the system is asymptotically stable in the neighbourhood of 0 and is L_2 input-output stable in the sense that $u \in L_2^m(-\infty, 0] \cup C^\infty$ implies $\Xi^u(u)$ restricted to $[0, \infty)$ is in $L_2^p[0, \infty)$. It is important to note that we only consider the inputs which are differentiable infinitely many times to avoid the otherwise encountered jump behaviour of the states.

The *Controllability operator* $\mathcal{C} : L_2^m[0, \infty) \cup C^\infty \to \mathbb{R}^n$ and the *Observability operator* $\mathcal{O} : \mathbb{R}^n \to L_2^p[0, \infty)$ are defined as follows

$$x_0 = \mathcal{C}(u) : \begin{cases} \dot{\overbrace{E(x)}} = f(x) + g(x)\mathcal{F}_-(u), \\ x_0 = x(0) \in M^*, \quad x(-\infty) = 0, \end{cases} \tag{5}$$

$$y = \mathcal{O}(x_0) : \begin{cases} \dot{\overbrace{E(x)}} = f(x), \quad x_0 = x(0) \in M^*, \\ y = h(x), \end{cases} \tag{6}$$

where $\mathcal{F}_- : L_2^m[0, \infty) \cup C^\infty \to L_2^m(-\infty, \infty) \cup C^\infty$ is the *time flipping operator* defined by

$$\mathcal{F}_-(u) := \begin{cases} u(-t) : t < 0, \\ 0 : t \geq 0. \end{cases}$$

Definition 4. *For any L_2-stable input-output system Ξ^u, the corresponding Hankel operator is*

$$\mathcal{H} : L_2^m[0,\infty) \to L_2^p[0,\infty) : u \to y = (\Xi^u \circ \mathcal{F}_-)(u).$$

Now, it is evident that $\mathcal{H} = \mathcal{O} \circ \mathcal{C}$. Let us consider that the Hankel operator is (Fréchet) differentiable. Now, let us define

$$\sigma_{\max}(c) := \sup_{||u||=c} \frac{||\mathcal{H}(u)||}{||u||}, \quad v_{\max}(c) := \arg \sup_{||u||=c} \frac{||\mathcal{H}(u)||}{||u||}.$$

Since, $v_{\max}$ is a critical point for $\frac{||\mathcal{H}(u)||}{||u||}$, $u = v_{\max}$ has to satisfy

$$\mathrm{d}\left(\frac{||\mathcal{H}(u)||}{||u||}\right)(\mathrm{d}u) = 0 \quad \text{s.t.} ||u|| = c. \tag{7}$$

This boils down to the following problem

$$\langle (\mathrm{d}\mathcal{H}(u))^* \circ \mathcal{H}(u) - (||\mathcal{H}||/||u||)^2\, u \,, \mathrm{d}u \rangle = 0 \;\forall \mathrm{d}u$$
$$\text{s.t.} \langle u \,, \mathrm{d}u \rangle = 0.$$

Equivalently, we can write it down as

$$(\mathrm{d}\mathcal{H}(u))^* \circ \mathcal{H}(u) = \lambda u, \quad \lambda \in \mathbb{R}\,. \tag{8}$$

The λ in (8) coincides with the square of the singular value σ^2 at the critical points of $(||\mathcal{H}(u)||/||u||)$.

IV. Model Order Reduction for Nonlinear DACSs

In this section, we introduce the controllability and observability functions for the nonlinear DACS (1). Further, we provide a balanced realization of the DACS. Finally, we propose a reduced-order model obtained by truncation of states of the balanced realization. It is important to mention that we define the energy functions as well as perform balancing in the locally maximally controlled invariant submanifold M^* to avoid jump solutions associated with inconsistent initial conditions. Moreover, it also helps us to perform model reduction in an index independent fashion and to preserve the constraints associated with the original system.

A. Controllability and Observability functions

The controllability function associated with (1) is defined as

$$L_c(x_0) := \min_{\substack{u \in L_2(-\infty,0),\\ x(-\infty)=0, x(0)=x_0 \in M^*}} \frac{1}{2}\int_{-\infty}^{0} ||u(t)||^2 dt.$$

The observability function is defined as

$$L_o(x_o) := \frac{1}{2}\int_0^\infty ||y(t)||^2 dt, x(0) = x_0 \in M^*, u(t) \equiv 0.$$

Theorem 1. *Assume $E(0) = 0$. Suppose, $\overset{\cdot}{\overbrace{E(x)}} = f(x)$ is asymptotically stable on a neighbourhood U of 0. Moreover, Assumption 1 is also satisfied in U. If for all $x \in M^* \cap U$, $\bar{L}_o(E(x))$ is the smooth solution to*

$$\frac{\partial \bar{L}_o(E(x))}{\partial E} f(x) + \frac{1}{2} h^\top(x) h(x) = 0, \bar{L}_o(0) = 0, \tag{9}$$

then it satisfies

$$L_o(x_0) = \bar{L}_o(E(x_0)). \tag{10}$$

Also, if $\bar{L}_c(E(x))$ is the smooth solution of

$$\frac{\partial \bar{L}_c(E(x))}{\partial E} f(x) + \frac{1}{2}\frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) g^\top(x) \frac{\partial^\top \bar{L}_c(E(x))}{\partial E} = 0,$$
$$\bar{L}_c(0) = 0, \tag{11}$$

such that $\overset{\cdot}{\overbrace{E(x)}} = -f(x) - g(x)g^\top(x)\frac{\partial^\top \bar{L}_c(E(x))}{\partial E}$ is asymptotically stable at 0 on $M^* \cap U$, then

$$L_c(x_0) = \bar{L}_c(E(x_0)). \tag{12}$$

Proof. Assume (9) has smooth solution $\bar{L}_o(E(x))$ on $M^* \cap U$. Then the following holds

$$L_o(x_0) = \frac{1}{2}\int_0^\infty ||y(t)||^2 dt = \frac{1}{2}\int_0^\infty h^\top(x(t)) h(x(t)) dt$$
$$= -\int_0^\infty \frac{\partial \bar{L}_o(E(x))}{\partial E} f(x(t)) dt = -\int_0^\infty \frac{\partial \bar{L}_o(E(x))}{\partial E} \overset{\cdot}{\overbrace{E(x)}} dt$$
$$= -\int_0^\infty \frac{\partial}{\partial t}\bar{L}_o(E(x)) dt = -\bar{L}_o(E(x(\infty))) + \bar{L}_o(E(x(0))).$$

Now, as $f(x)$ is asymptotically stable, $x(\infty) = 0$. This implies that $L_o(x_0) = \bar{L}_o(E(x_0))$.
Similarly, assume that $\bar{L}_c(E(x))$ is a positive solution of (11) on $M^* \cap U$. Then, $\frac{\partial \bar{L}_c(E(x))}{\partial E} f(x) = -\frac{1}{2}\frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) g^\top(x) \frac{\partial^\top \bar{L}_c(E(x))}{\partial E}$. Let us now consider an input u such that $x(0) = x_0 \in M^* \cap U$ and $x(-\infty) = 0$.

$$\frac{d}{dt}\bar{L}_c(E(x)) = \frac{\partial \bar{L}_c(E(x))}{\partial E}\overset{\cdot}{\overbrace{E(x)}}$$
$$= \frac{\partial \bar{L}_c(E(x))}{\partial E} f(x) + \frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) u$$
$$= -\frac{1}{2}\frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) g^\top(x) \frac{\partial^\top \bar{L}_c(E(x))}{\partial E} + \frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) u$$
$$= \frac{1}{2}u^\top u - \frac{1}{2}u^\top u + \frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) u$$
$$- \frac{1}{2}\frac{\partial \bar{L}_c(E(x))}{\partial E} g(x) g^\top(x) \frac{\partial^\top \bar{L}_c(E(x))}{\partial E}$$
$$= \frac{1}{2}||u||^2 - \frac{1}{2}\Big|\Big|u - g^\top(x)\frac{\partial^\top \bar{L}_c(E(x))}{\partial E}\Big|\Big|^2.$$

Now, integrating $\frac{d}{dt}\bar{L}_c(E(x))$ form $-\infty$ to 0 we have

$$\bar{L}_c(E(x_0)) = \int_{-\infty}^0 \frac{d}{dt}\bar{L}_c(E(x(t))) dt$$
$$= \frac{1}{2}\int_{-\infty}^0 ||u(t)||^2 dt$$
$$- \frac{1}{2}\int_{-\infty}^0 \Big|\Big|u(t) - g^\top(x(t))\frac{\partial^\top \bar{L}_c(E(x(t)))}{\partial E}\Big|\Big|^2 dt$$
$$\implies \bar{L}_c(E(x_0)) \le \frac{1}{2}\int_{-\infty}^0 ||u(t)||^2 dt, \forall x_0 \in M^* \cap U.$$

If we choose $u = g^\top(x)\frac{\partial^\top \bar{L}_c(E(x))}{\partial E}$, then $x(-\infty) = 0$ because of asymptotic stability of

$-f(x) - g(x)g^\top(x)\frac{\partial^\top \tilde{L}_c(E(x))}{\partial E(x)}$ on $M^* \cap U$. So, we can conclude that $L_c(x_0) = \tilde{L}_c(E(x_0))$. This completes the proof. □

Remark 2. *Linear Systems:*

Let us consider a linear DACS as follows

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{13}$$

Then the observability function of (13) *is given by* $L_o(x_0) = \frac{1}{2}x_0^\top E^\top Q E x_0$, *where* $Q \succeq 0$ *is the solution of the generalized Lyapunov equation*

$$E^\top QA + A^\top QE + C^\top C = 0. \tag{14}$$

On the other hand, the controllability function of (13) *is given by* $L_c(x)_0 = \frac{1}{2}x_0^\top P^\dagger x_0$, *where* $P^\dagger$ *is the unique solution of*

$$APE^\top + EPA^\top + BB^\top = 0, \quad PP^\dagger P = P. \tag{15}$$

Now, define $\tilde{L}_c(z) := L_c(\phi(x))$ and $\tilde{L}_o(z) := L_o(\phi(x))$, where $z = z^*, \bar{z}_1, \cdots, \bar{z}_{k^*}$ as in (4).

B. Balanced realization

We assume the following set of standard assumptions which is necessary to provide a balanced realization for the nonlinear DACS (1).

Assumption 2. *Assume the following holds*

- (2) *is locally internally regular in the neighbourhood* U *around 0.*
- 0 *is an asymptotically stable equilibrium of* $\dot{\overbrace{E(x)}} = f(x)$ *on* U.
- 0 *is an asymptotically stable equilibrium of* $\dot{\overbrace{E(x)}} = -f(x) - g(x)g^\top(x)\frac{\partial^\top \tilde{L}_c}{\partial E(x)}$ *on* U.
- (9) *and* (11) *have smooth solutions on* $M^* \cap U$.
- $\frac{\partial^2 \tilde{L}_c(0)}{\partial z^{*2}} \succ 0$, $\frac{\partial^2 \tilde{L}_o(0)}{\partial z^{*2}} \succ 0$ *and the eigenvalues of* $((\partial^2 \tilde{L}_c/\partial z^{*2})(0))^{-1}((\partial^2 \tilde{L}_o/\partial z^{*2})(0))$ *are distinct.*

Lemma 1. *[18] If there exists a neighbourhood* V *of* 0 *where the number of distinct eigenvalues of* $M(\bar{x})$ *is constant for* $\bar{x} \in V$, *then on* V *the eigenvalues* $\lambda_i(\bar{x}), i = 1, \cdots, n$ *are smooth fumctions of* $\bar{x}$, *as well as the associated eigenvectors.*

In what follows, we elucidate that an input-normal/output-diagonal realization of (1) can be obtained on M^*.

Theorem 2. *Consider the DACS* (1) *which satisfies Assumptions 1, 2 and assume that the condition of Lemma 1 is satisfied. Then, there exists a neighbourhood* U *such that on* $M^* \cap U$ *of* 0 *there exists a coordinate transformation* $\varphi : \mathbb{R}^n \to \mathbb{R}^n, x = \varphi(\tilde{x}), \varphi(0) = 0$, *such that* $L_c(x)$ *can be written in the following form in the new coordinates*

$$L_c(\varphi(\tilde{x}, 0)) = \frac{1}{2}\tilde{x}^\top \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}, \quad \tilde{x} \in W := \varphi^{-1}(U). \tag{16}$$

Moreover, L_o *is of the following form in the new coordinates*

$$L_o(\varphi(\tilde{x}, 0)) = \frac{1}{2}\tilde{x}^\top \begin{bmatrix} \tilde{\pi}_1(\tilde{x}) & & \mathbf{0} \\ & \ddots & \\ & & \tilde{\pi}_{n^*}(\tilde{x}) \\ \mathbf{0} & & \mathbf{0} \end{bmatrix} \tilde{x}, \tag{17}$$

where $\tilde{\pi}_1(\tilde{x}) \geq \tilde{\pi}_2(\tilde{x}) \geq \cdots \geq \tilde{\pi}_{n^*}(\tilde{x})$ *are smooth functions of* $\tilde{x}$.

Proof. From Section II, there exists a transformation $z = \phi(x)$ such that $z := (z^*, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{k^*})$ and for all $x \in M^*$, $\bar{z}_1 = \bar{z}_2 = \cdots = \bar{z}_{k^*}$. Since 0 is a minimum of controllability and observability functions, we have $\tilde{L}_c(0) = \tilde{L}_o = 0$ and $\frac{\partial \tilde{L}_c}{\partial z} = \frac{\partial \tilde{L}_o}{\partial z} = 0$. As $\frac{\partial^2 \tilde{L}_c}{\partial z^{*2}}(0) \succ 0$, we use the reduction lemma [19, Chapter 3], there exists a local diffeomorphism $\rho : \mathbb{R}^n \to \mathbb{R}^n$ such that $\bar{x} = \rho(z^*, \bar{z}_1, \cdots, \bar{z}_{k^*}), \rho(0) = 0$ and $(0, \cdots, 0, \bar{x}_{n^*+1}, \cdots, \bar{x}_n) = \rho(0, \cdots, 0, \bar{z}_1, \cdots, \bar{z}_{k^*})$. In these new coordinates,

$$L_c(\psi(\bar{x})) = \frac{1}{2}\bar{x}^\top \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \bar{x}, \tag{18}$$

and we can write $L_o(x)$ in the form below using Lemma 2.1 of [20] repeatedly

$$L_o(\psi(\bar{x})) = \frac{1}{2}\bar{x}^\top M(\bar{x})\bar{x}, \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0), \tag{19}$$

where $M(\bar{x})$ is a symmetric matrix whose elements are smooth functions of $\bar{x}$ and $\psi = \rho \circ \phi$. From Lemma 1, we know that there exists a transformation $\psi : \mathbb{R}^n \to \mathbb{R}^n, x = \psi(\bar{x}), \psi(0) = 0$, such that in $\bar{x}$ coordinates L_c and L_o have the forms as in (18) and (19). Let V is a neighbourhood where $M(\bar{x})$ is positive semi-definite and by the smoothness of eigenvalues and eigenvectors (using Lemma 1), $M(\bar{x})$ is diagonalizable. Now, as $M(\bar{x})$ is symmetric, we can write

$$M(\bar{x}) = T(\bar{x}) \begin{bmatrix} \Lambda_o(\bar{x}) & 0 \\ 0 & 0 \end{bmatrix} T^\top(\bar{x}),$$

where $\Lambda_o(\bar{x}) = \text{diag}\{\lambda_1(\bar{x}), \cdots, \lambda_{n^*}(\bar{x})\}$ are the positive eigenvalues of $M(\bar{x})$ and $T(\bar{x})$ is the corresponding matrix of eigenvectors with $T(\bar{x})$ satisfying $T^\top(x)T(x) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Now, if we define the new coordinates as $\kappa : \mathbb{R}^n \to \mathbb{R}^n, \bar{x} = \kappa(\tilde{x}) = T^\top(\bar{x})\tilde{x}$, we can get the following

$$\begin{aligned} L_c(\kappa^{-1}(\psi(\tilde{x}))) &= \frac{1}{2}\tilde{x}^\top \tilde{x} \\ L_o(\kappa^{-1}(\psi(\tilde{x}))) &= \frac{1}{2}\tilde{x}^\top \begin{bmatrix} \Lambda_o(\kappa(\tilde{x})) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tilde{x}, \end{aligned}$$

where $\tilde{x} \in W := \kappa^{-1}(V)$ with $\varphi = \kappa^{-1} \circ \psi, U = \varphi^{-1}(V)$ and $\lambda_i(\tilde{x}) = \lambda_i(\kappa(\bar{x})) = \tilde{\pi}_i(\tilde{x})$. This completes the proof. □

Finally, we can provide a realization in which the coordinate axes are decoupled in both controllability and observability functions and balanced for input-to-state as well as state-to-output behaviour of the differential-algebraic system.

Theorem 3. *Consider a DACS as in* (1) *which satisfies Assumptions 1, 2 and the linearized system at the origin is asymptotically stable. Then, there exists an open neighborhood* $W \subset \mathbb{R}^n$ *around the origin and a non-linear coordinate transformation* $x = \tilde{\Phi}(\tilde{x})$ *on* $M^* \cap W$ *which converts the* *system into the following realization*

$$L_c(\Phi(\tilde{x})) = \frac{1}{2}\tilde{x}^\top \begin{bmatrix} \tilde{\sigma}_1^{-1}(\tilde{x}_1) & & & \mathbf{0} \\ & \ddots & & \\ & & \tilde{\sigma}_{n^*}^{-1}(\tilde{x}_{n^*}) & \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix} \tilde{x}$$

$$L_o(\Phi(\tilde{x})) = \frac{1}{2}\tilde{x}^\top \begin{bmatrix} \tilde{\sigma}_1(\tilde{x}_1) & & & \mathbf{0} \\ & \ddots & & \\ & & \tilde{\sigma}_{n^*}(\tilde{x}_{n^*}) & \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix} \tilde{x}. \tag{20}$$

Proof. At first, via mathematical induction, as in Theorem 8 in [21], the following can be proved

$$\tilde{x}_i = 0 \iff \frac{\partial L_c(\Phi(\tilde{x}))}{\partial \tilde{x}_i} = 0 \iff \frac{\partial L_o(\Phi(\tilde{x}))}{\partial \tilde{x}_i} = 0 \tag{21}$$

for all $i \in \{1, 2, \cdots, n\}$ on $W \cap M^*$. It is worth-mentioning that (21) is satisfied as L_o and L_c are unique on M^*. Hence, we can obtain the system representation in coordinates $\breve{x}$ via a smooth coordinate transformation $x = \Phi(\breve{x})$ as follows

$$L_c(\Phi(\breve{x})) = \frac{1}{2}\breve{x}^\top \breve{x}$$

$$L_o(\Phi(\breve{x})) = \frac{1}{2}\breve{x}^\top \begin{bmatrix} \breve{\sigma}_1^2(\breve{x}_1) & & & \mathbf{0} \\ & \ddots & & \\ & & \breve{\sigma}_{n^*}^2(\breve{x}_{n^*}) & \\ \mathbf{0} & & & \mathbf{0} \end{bmatrix} \breve{x}. \tag{22}$$

Further, we can apply a coordinate transformation $\breve{x} = \Phi^{-1} \circ \tilde{\Phi}(\tilde{x}) := (\tilde{\phi}_1(\tilde{x}_1), \tilde{\phi}_2(\tilde{x}_2), \cdots, \tilde{\phi}_1(\tilde{x}_n))$ with $\tilde{x}_i = \tilde{\phi}_i^{-1}(\breve{x}_i) := \breve{x}_i\sqrt{\sigma_i(\breve{x}_i)}$ to the system in $\breve{x}$ coordinates. If we define $\tilde{\sigma}_i(\tilde{x}_i) := \sigma_i(\tilde{\phi}_i(\tilde{x}_i))$, in the $\tilde{x}$ coordinates we have L_c and L_o as in (20). This completes the proof. □

C. Balanced truncation

Now, we propose the balanced truncation procedure for nonlinear DACSs as described by (1) based on the balanced realization provided in the previous section.

Let us consider a nonlinear differential-algebraic system (1) which satisfies Assumptions 1 and 2. Then there exists a nonlinear coordinate transformation $x = \Phi(\tilde{x})$ which converts the system into a balanced realization for all $x \in M^*$ as depicted in Theorem 3. The system can be represented in $\tilde{x}$ coordinates as follows

$$\frac{d}{dt}\tilde{E}(\tilde{x}) = \tilde{E}'(\tilde{x})\frac{\partial \Phi}{\partial \tilde{x}}\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u, \quad y = \tilde{h}(\tilde{x}), \tag{23}$$

where

$$\tilde{E}(\tilde{x}) := E(\Phi(\tilde{x})) \quad \tilde{E}'(\tilde{x}) := E'(\Phi(\tilde{x}))$$
$$\tilde{f}(\tilde{x}) := f(\Phi(\tilde{x})) \quad \tilde{g}(\tilde{x}) := g(\Phi(\tilde{x})) \quad \tilde{h}(\tilde{x}) := h(\Phi(\tilde{x})).$$

Moreover, $\tilde{L}_c(\tilde{x}) = L_c(\Phi(\tilde{x}))$ and $\tilde{L}_o(\tilde{x}) = L_o(\Phi(\tilde{x}))$ as depicted in (20). The non-zero singular functions $\sigma_i(\tilde{x}_i)$ are ordered as

$$\max_{\pm c} \sigma_i(s) > \max_{\pm c} \sigma_{i+1}(s) \tag{24}$$

for all $i \in \{1, 2, \cdots, n^*\}$ in a neighbourhood around origin. Now, for a certain $r^* \in \{1, 2, \cdots, n^*\}$, we have

$$\max_{\pm c} \sigma_{r^*}(s) \gg \max_{\pm c} \sigma_{r^*+1}(s). \tag{25}$$

Then, we can split the coordinates into two parts as follows

$$\begin{aligned} &\tilde{x} = (\tilde{x}_a, \tilde{x}_b), \\ &\tilde{x}_a := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{r^*}, \tilde{x}_{n^*+1}, \cdots, \tilde{x}_n), \\ &\tilde{x}_b := (\tilde{x}_{r^*+1}, \tilde{x}_{r^*+2}, \cdots, \tilde{x}_{n^*}) \in \mathbb{R}^{n-r}, \\ &\tilde{E}(\tilde{x}) = \begin{bmatrix} \tilde{E}_a(\tilde{x}) \\ \tilde{E}_b(\tilde{x}) \end{bmatrix}, \tilde{f}(\tilde{x}) = \begin{bmatrix} \tilde{f}_a(\tilde{x}) \\ \tilde{f}_b(\tilde{x}) \end{bmatrix}, \\ &\tilde{g}(\tilde{x}) = \begin{bmatrix} \tilde{g}_a(\tilde{x}) \\ \tilde{g}_b(\tilde{x}) \end{bmatrix}, \end{aligned} \tag{26}$$

where $r = r^* + (n - n^*)$. Now, if we set $\tilde{x}_b = 0$, the reduced order model of the balanced realization is as follows

$$\Xi_r^u : \begin{cases} \dot{\overbrace{\hat{E}(\hat{x})}} = \hat{f}(\hat{x}) + \hat{g}(\hat{x})u, \\ \hat{y} = \hat{h}(\hat{x}), \end{cases} \tag{27}$$

where

$$\begin{aligned} &\hat{E}(\hat{x}) := \tilde{E}_a(\tilde{x}_a, 0), \hat{f}(\hat{x}) := \tilde{f}_a(\tilde{x}_a, 0), \\ &\hat{g}(\hat{x}) := \tilde{g}_a(\tilde{x}_a, 0), \hat{h}(\hat{x}) := \tilde{h}(\tilde{x}_a, 0). \end{aligned} \tag{28}$$

It can easily be proven that the controllability and observability functions of the reduced order model (27) are $\hat{L}_c(\hat{x}) = \tilde{L}_c(\tilde{x}_a, 0)$ and $\hat{L}_o(\hat{x}) = \tilde{L}_o(\tilde{x}_a, 0)$ respectively i.e. they also satisfy (11) and (9) respectively for the reduced-order model (27). Moreover, as we kept the coordinates $(\tilde{x}_{n^*+1}, \cdots, \tilde{x}_n)$ intact, the algebraic constraints associated with the original system (1) are preserved. This implies that the trajectories of the reduced order model reside on the maximally controlled invariant submanifold of Ξ_r^u.

V. Illustrative example

Consider a model of RLC circuit with a nonlinear resistor as in Fig. 1, given by the following differential equations

$$L_1\frac{di_1}{dt} = -v_2 + u, \quad C_2\frac{dv_2}{dt} = i_1 - G_1v_2 - i_4, \tag{29}$$

and the following nonlinear algebraic constraints

$$v_2 - v_3 = 0, \quad i_4 - G_2v_3^2 = 0, \tag{30}$$

where i_1 is the current through the inductor L_1, v_2 is the voltage across the capacitor C_2. G_1 represents the admittance of the linear resistor, G_2v_3 represents the voltage dependent admittance of the nonlinear resistor, u is the voltage input. Taking $x = \begin{bmatrix} i_1 & v_2 & v_3 & i_4 \end{bmatrix}^\top$ as the generalized state vector, we can represent the system as the following quadratic DACS

$$E\dot{x} = Ax + N(x \otimes x) + Bu, \quad y = Cx, \tag{31}$$

where

$$E = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -G_1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

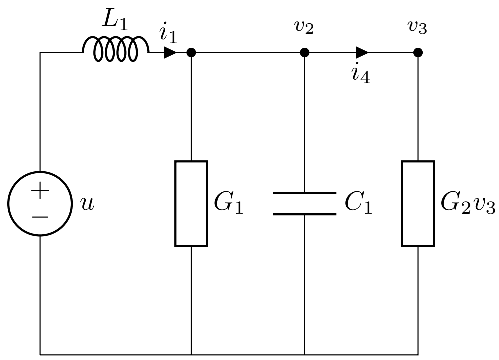

Fig. 1: RLC circuit with a nonlinear resistor

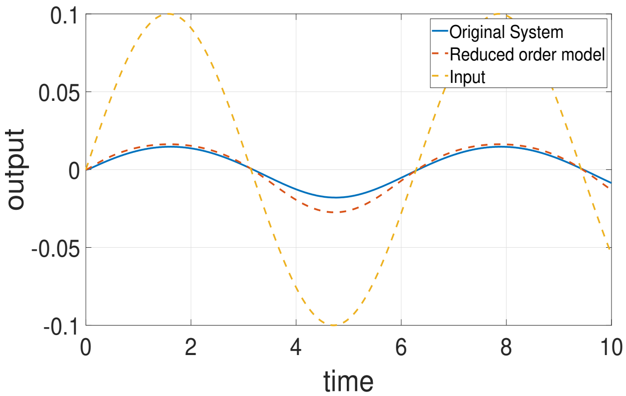

Fig. 2: Comparison of output trajectories of the original system and the reduced order model

and $N \in \mathbb{R}^{4\times 16}$ is a matrix whose $(4, 11)$ entry is $-G_2$ and all other entries are zero, and $\otimes$ represents the Kronecker product of vectors. The maximally controlled invariant submanifold for (31) can be represented as $M^* = \{x|\bar{z}_1 = 0, \bar{z}_2 = 0\}$, where $\bar{z}_1 = v_2 - v_3$ and $\bar{z}_2 = i_4 - G_2 v_3^2$. Via 4^{th} order approximation of the controllability and observability functions [22] and considering the 3^{rd} order approximation of the nonlinear transformation [23] , the approximated nonzero singular functions in (20) are as follows

$$\tilde{\sigma}_1(\tilde{x}_1) = 0.7623 - 20.2808\tilde{x}_1 + 1016.4\tilde{x}_1^2,$$
$$\tilde{\sigma}_2(\tilde{x}_2) = 0.2623 - 11.8973\tilde{x}_2 - 7333.7\tilde{x}_2^2.$$

Now, by setting $\tilde{x}_2 = 0$, we achieve the reduced order model which also preserves the algebraic constraints as $\tilde{x}_3$ and $\tilde{x}_4$ have not been removed from the original system. The output trajectory of the balanced truncated model is compared with the output trajectory of the original system in Fig. 2 inresponse to an input $u(t) = 0.1\sin(t)$. The reduced order model well approximates the original system.

VI. Conclusions

In this letter, we have defined the controllability and observability functions for a nonlinear differential algebraic control system on the locally maximally controlled invariant submanifold. Further, we have utilized them to come up with a balanced realization and a reduced order model via truncation that preserves the algebraic constraints of the original system.

Based on these results, nonlinear DACSs with state-input constraints are worth investigating in the future. Moreover, balancing for inconsistent initial conditions is another interesting direction of research to pursue in the future.

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