# Incremental versus Differential Approaches to Exponential Stability and Passivity

Kawano, Yu; Besselink, Bart

*Published in:*
IEEE Transactions on Automatic Control

*DOI:*
10.1109/TAC.2024.3385036

# Incremental Versus Differential Approaches to Exponential Stability and Passivity

Yu Kawano, *Member, IEEE*, and Bart Besselink, *Member, IEEE*

***Abstract*—There are two main Lyapunov approaches to incremental stability analysis. One is to use incremental Lyapunov functions directly, and the other is based on so-called Finsler–Lyapunov functions via contraction analysis. A system is incrementally exponentially stable if it admits either an incremental or Finsler–Lyapunov function, and the converse is also true when the Jacobian of the drift vector field satisfies a boundedness assumption. However, the direct relation between these Lyapunov functions is not very clear yet. In this article, we show that if one type of Lyapunov function is found, the other can directly be constructed from it without the boundedness assumption. As an application of our approach, we also show that an open system is incrementally passive if and only if it is differentially passive under a mild technical assumption.**

***Index Terms*—Contraction, incremental stability, Lyapunov functions, nonlinear systems, passivity.**

## I. Introduction

The notion of incremental stability [1] has been introduced to study the convergence between any pair of trajectories of a possibly nonlinear system. There are two major approaches to analyzing this property. One is to directly study the time evolution of the distance between two trajectories as in [1]. The other is to focus on the time evolution of an infinitesimal distance, which is referred to as contraction theory [2], [3], [4], see also [5]. Both approaches have been developed by tailoring Lyapunov stability theory, and the corresponding Lyapunov functions are referred to as an incremental Lyapunov function and Finsler (or differential) Lyapunov function, respectively. Incremental and/or differential approaches to stability analysis find widespread applications in topics ranging from synchronization [6], [7], observer design [8], [9], and controller design [10], [11] to model reduction [12], [13], [14], [15].

Similar to the extension of Lyapunov stability theory to incremental and differential versions, attempts at extending the notion of passivity [16], [17] have led to definitions of incremental passivity [18], [19] (called incremental positivity in the classical work [20]) and differential passivity [21], [22]. These definitions rely on so-called incremental and differential storage functions, respectively, which play a similar role as incremental and differential Lyapunov functions, see also [23]. These concepts are exploited for controller design in [24]. In this article, our goal is to show that the incremental and differential approaches are equivalent, both when studying incremental exponential stability and incremental/differential passivity.

*Literature Review:* Various works have investigated the connection between incremental and differential approaches for stability analysis. Andrieu et al. [9] showed that an incrementally exponentially stable (IES) system admits a quadratic-type Finsler–Lyapunov function under the assumption that the Jacobian of the drift vector field is bounded. Similar implications are found in [25] and [26], where Wu and Duan [26] considered incremental asymptotic stability and non-quadratic Finsler–Lyapunov functions. The result of [9] further implied that, under its boundedness assumption, there exists an incremental Lyapunov function for IES if and only if there exists a Finsler–Lyapunov function. A similar equivalence is shown for discrete-time systems [27] via IES analysis. However, neither work provides a direct construction of one type of Lyapunov function from the other. Still, Andrieu et al. [9] showed a construction of a standard (i.e., non-incremental) Lyapunov function at an equilibrium point from a Finsler–Lyapunov function. Ruffer et al. [28] studied the connection between incremental stability and convergent systems.

A similar idea is found in the scope of passivity (in fact, dissipativity) in [23], where an incremental storage function is constructed from a differential one under a positive definiteness assumption on the differential storage function. However, these papers do not consider constructing a Finsler–Lyapunov (or differential storage) function from an incremental one, although a related conjecture is found in [17] for passivity.

*Contribution:* In this article, we show that the existence of an incremental Lyapunov function for IES is equivalent to the existence of a Finsler–Lyapunov function, even if the Jacobian matrix does not satisfy a boundedness assumption. In particular, inspired by [9] and [23], we provide an explicit construction of an incremental Lyapunov function from a Finsler–Lyapunov one. Moreover, we present a converse construction, which has not been done in [9] and [23]. The application of our result to discrete-time systems also gives a direct relation between the discrete-time versions of an incremental and Finsler–Lyapunov functions without going through IES analysis. As another application of our result, we derive novel incremental Lyapunov functions for cooperative systems from Finsler-type Lyapunov functions [29]. Our incremental Lyapunov functions can be viewed as generalizations of the linear ones in [30].

Moving to open nonlinear dynamical systems, we further apply our method to show the equivalence between incremental and differential passivity under a mild technical assumption. Our construction of an incremental storage function from a differential one is more general than that of [23] in the sense that we do not require that the differential storage function is quadratic or positive definite. That is, our approach is applicable even if a geodesic does not exist. As mentioned above, the construction of a differential storage function from an incremental one is partly conjectured by [17]. We prove this conjecture by completing the construction. Our approach to connect incremental and differential analysis is not restricted to exponential stability and passivity and,

Manuscript received 11 November 2022; revised 19 February 2024; accepted 28 March 2024. Date of publication 4 April 2024; date of current version 29 August 2024. The work of Yu Kawano was supported by JSPS KAKENHI under Grant JP21H04875. Recommended by Associate Editor V. Andrieu. *(Corresponding author: Yu Kawano.)*

Yu Kawano is with the Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima 7398527, Japan (e-mail: ykawano@hiroshima-u.ac.jp).

Bart Besselink is with the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen 9700 AK, The Netherlands (e-mail: b.besselink@rug.nl).

Digital Object Identifier 10.1109/TAC.2024.3385036

thus, can be a tool to establish a direct bridge between incremental and differential approaches, that are often developed independently.

*Organization:* The rest of this article is organized as follows. We recall the definitions of incremental and Finsler–Lyapunov functions for IES analysis in Section II. In Section III, as the main result of this article, we show that the existence of these two types of Lyapunov functions is equivalent for continuous- or discrete-time systems. The proposed result is applied to IES analysis of cooperative systems in Section IV, whereas Section V builds on the main result to show that incremental and differential passivity are equivalent properties. Finally, Section VI concludes this article.

*Notation:* Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the field of real numbers and the set of nonnegative real numbers, respectively. Accordingly, $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote the $n$-dimensional real vector space and the set of $n$-dimensional real vectors with nonnegative components, respectively. For a vector $x \in \mathbb{R}^n$, its Euclidean norm is denoted by $|x| := (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$.

## II. Preliminaries

Consider the nonlinear system

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $C^2$ and, hence, locally Lipschitz. Let $\phi(t, x_0)$ denote a solution to the system (1) at time $t$ starting from $x_0 \in \mathbb{R}^n$ at $t = 0$, i.e., $x(t) = \phi(t, x_0)$. Such solution is guaranteed to exist and is unique for small $t$, e.g., [31, Thm. 3.1]. Throughout this article, we further assume that the system is forward complete, i.e., the solution exists for all $t \geq 0$. It is then also unique by local Lipschitz continuity of $f$.

*Standing Assumption 2.1:* For any $x_0 \in \mathbb{R}^n$, the solution $\phi(t, x_0)$ exists for all $t \geq 0$.

In this article, we are interested in studying incremental stability using Lyapunov theory. To do so, define the auxiliary system [1] consisting of the system (1) and its copy as follows:

$$\begin{cases} \dot{x} = f(x), & x(0) = x_0 \\ \dot{x}' = f(x'), & x'(0) = x'_0. \end{cases} \tag{2}$$

Now, incremental exponential stability [1], [3] can be defined as follows.

*Definition 2.2:* The system (1) is said to be IES if there exist $k, \lambda > 0$ such that, for the pair of solutions to (2) as follows:

$$|x(t) - x'(t)| \leq k e^{-\lambda t} |x_0 - x'_0| \tag{3}$$

for all $t \geq 0$ and all $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$. ◁

There are several approaches for studying IES. One is to use an incremental Lyapunov function, defined below.

*Definition 2.3:* A piecewise continuous function $V_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be an *incremental Lyapunov function* for the system (1) if there exist $\underline{c}$, $\overline{c}$, $\lambda > 0$ such that

$$\underline{c}|x_0 - x'_0|^2 \leq V_I(x_0, x'_0) \leq \overline{c}|x_0 - x'_0|^2 \tag{4a}$$

$$V_I(x(t), x'(t)) \leq e^{-\lambda t} V_I(x_0, x'_0) \tag{4b}$$

for any $t \geq 0$ and $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$. ◁

An equivalent representation of (4b) is as follows:

$$\limsup_{t \to 0+} \frac{V_I(x(t), x'(t)) - V_I(x_0, x'_0)}{t} \leq -\lambda V_I(x_0, x'_0).$$

Definition 2.3 is a slight modification of a Lyapunov function for incremental asymptotic (rather than exponential) stability in [1]. One can confirm that a system (1) is IES if it admits an incremental Lyapunov function. Also, by modifying a classical converse Lyapunov theorem, e.g., [31, Thm. 4.14], one can prove the converse in case the Jacobian $\partial f(x)/\partial x$ is *bounded* on $\mathbb{R}^n$. Here, we call the Jacobian bounded on $\mathbb{R}^n$ if there exists $c > 0$ such that $|\partial f(x)/\partial x| \leq c$ for all $x \in \mathbb{R}^n$, with $|\cdot|$ any matrix norm.

Another framework for studying incremental stability is given by contraction theory. In contraction theory, we use the variational system along the trajectory $x(\cdot)$ of the system (1), which reads as follows:

$$\dot{\delta x} = \frac{\partial f(x(t))}{\partial x} \delta x, \quad \delta x(0) = \delta x_0. \tag{5}$$

Using the variational system, we can introduce the so-called Finsler–Lyapunov function [3].

*Definition 2.4:* A piecewise continuous function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Finsler–Lyapunov function* for the system (1) if there exist $\underline{c}$, $\overline{c}$, and $\lambda > 0$ such that

$$\underline{c}|\delta x_0|^2 \leq V_D(x_0, \delta x_0) \leq \overline{c}|\delta x_0|^2 \tag{6a}$$

$$V_D\left(x(t), \delta x(t)\right) \leq e^{-\lambda t} V_D(x_0, \delta x_0) \tag{6b}$$

for any $t \geq 0$ and $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n$. ◁

We note that (6) does not imply that $V_D(x_0, \delta x_0)$ is quadratic with respect to $\delta x_0$. Similar to the incremental case, we can replace (6b) by the equivalent condition as follows:

$$\limsup_{t \to 0+} \frac{V_D(x(t), \delta x(t)) - V_D(x_0, \delta x_0)}{t} \leq -\lambda V_D(x_0, \delta x_0).$$

The original definition in [3] is given through this latter condition. According to [3, Thm. 1], a system (1) is IES if it admits a Finsler–Lyapunov function. The converse is also true if $\partial f(x)/\partial x$ is bounded on $\mathbb{R}^n$; see [8, Prop. 3].

Combining this with our earlier observation, we have that, in case $\partial f(x)/\partial x$ is bounded on $\mathbb{R}^n$, a system (1) admits an incremental Lyapunov function if and only if it does a Finsler–Lyapuonv function.

## III. Main Results

### A. Continuous-Time Systems

As mentioned in the previous section, the existence of an incremental Lyapunov function is equivalent to the existence of a Finsler–Lyapunov function, provided that $\partial f(x)/\partial x$ is bounded on $\mathbb{R}^n$. However, the main result of this article shows, first, that this boundedness assumption is redundant and, second, that a Finsler–Lyapunov function can be constructed from an incremental Lyapunov function and vice versa.

*Theorem 3.1:* A system (1) admits an incremental Lyapunov function if and only if it admits a Finsler–Lyapunov function.

*Proof:* (Finsler ⇒ Incremental) Following Definition 2.4, we denote the Finsler–Lyapunov function by $V_D$. Let $\Gamma(x, x')$ be the collection of class $C^1$ paths $\gamma : [0, 1] \to \mathbb{R}^n$ connecting $\gamma(0) = x$ and $\gamma(1) = x'$. Now, we aim to show that

$$V_I(x, x') := \inf_{\gamma \in \Gamma(x, x')} \int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds \tag{7}$$

is an incremental Lyapunov function for the system (1). We note that the integral in the right-hand side is well defined for any $\gamma \in \Gamma(x, x')$, since $V_D(\gamma(s), d\gamma(s)/ds)$ is a piecewise continuous function of $s \in [0, 1]$.

First, we show that (4a) holds for our candidate function (7). Substitution of $(x_0, \delta x_0) = (\gamma(s), d\gamma(s)/ds)$ into (6a) and integrating over

$s$ yields as follows:

$$\underline{c}\int_0^1 \left|\frac{d\gamma(s)}{ds}\right|^2 ds \leq \int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds \leq \bar{c}\int_0^1 \left|\frac{d\gamma(s)}{ds}\right|^2 ds.$$

Since taking the infimum preserves the inequalities, e.g., [32, Lemma 6.4.13], it follows from (7) that:

$$\underline{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1 \left|\frac{d\gamma(s)}{ds}\right|^2 ds \leq V_I(x,x') \leq \bar{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1 \left|\frac{d\gamma(s)}{ds}\right|^2 ds.$$

Since $\inf_{\gamma\in\Gamma(x,x')}\int_0^1 |d\gamma(s)/ds|^2 ds$ is the squared path length, the shortest path is $\gamma(s) = sx' + (1-s)x$. Thus, we have

$$\inf_{\gamma\in\Gamma(x,x')}\int_0^1 \left|\frac{d\gamma(s)}{ds}\right|^2 ds = |x-x'|^2$$

and indeed obtain (4a).

Next, we show that (4b) holds as well for (7). Let $\gamma \in \Gamma(x,x')$, $s \in (0,1)$, and consider the trajectory $\phi(\cdot,\gamma(s))$. Then, a direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\phi(t,\gamma(s))}{\partial s} &= \frac{\partial}{\partial s}\frac{d\phi(t,\gamma(s))}{dt}\\ &= \frac{\partial f(\phi(t,\gamma(s)))}{\partial s}\\ &= \frac{\partial f(\phi(t,\gamma(s)))}{\partial\phi}\frac{\partial\phi(t,\gamma(s))}{\partial s}.\end{aligned} \tag{8}$$

Hence, we have that $\partial\phi(\cdot,\gamma(s))/\partial s$ is a trajectory for the variational system (5) along $\phi(\cdot,\gamma(s))$. Stated differently as follows:

$$(x(t),\delta x(t)) = \left(\phi(t,\gamma(s)), \frac{\partial\phi(t,\gamma(s))}{\partial s}\right) \tag{9}$$

satisfies (1) and (5).

Now, let $t \geq 0$ and note that $\phi(t,\gamma(s)) \in \Gamma(x(t),x'(t))$, regardless of the choice for $\gamma \in \Gamma(x_0,x_0')$, as can be concluded from [33, Thm. 4.1 (p. 131)]. As a result, using the definition of $V_I$ in (7), we obtain the bound as follows:

$$\begin{aligned}V_I(x(t),x'(t)) &= \inf_{\bar\gamma\in\Gamma(x(t),x'(t))}\int_0^1 V_D\left(\bar\gamma(s), \frac{d\bar\gamma(s)}{ds}\right) ds\\ &\leq \int_0^1 V_D\left(\phi(t,\gamma(s)), \frac{\partial\phi(t,\gamma(s))}{\partial s}\right) ds.\end{aligned}$$

Noting $\phi(0,\gamma(s)) = \gamma(s)$, substituting (9) into (6b) yields as follows:

$$\begin{aligned}&V_D\left(\phi(t,\gamma(s)), \frac{\partial\phi(t,\gamma(s))}{\partial s}\right)\\ &\leq e^{-\lambda t}V_D\left(\phi(0,\gamma(s)), \frac{\partial\phi(0,\gamma(s))}{\partial s}\right)\\ &= e^{-\lambda t}V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right)\end{aligned}$$

and, consequently as follows:

$$V_I(x(t),x'(t)) \leq e^{-\lambda t}\int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds.$$

We recall that the choice for $\gamma \in \Gamma(x_0,x_0')$ was arbitrary. Therefore, from (7), we have as follows:

$$\begin{aligned}V_I(x(t),x'(t)) &\leq e^{-\lambda t}\inf_{\gamma\in\Gamma(x_0,x_0')}\int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds\\ &= e^{-\lambda t}V_I(x_0,x_0').\end{aligned}$$

This is nothing but (4b), confirming that $V_I$ in (7) is an incremental Lyapunov function for the system (1).

(Incremental $\Rightarrow$ Finsler) Let $V_I$ denote the incremental Lyapunov function as in Definition 2.3. To construct a candidate of a Finsler–Lyapunov function from $V_I$, introduce $\Gamma'(x,\delta x)$ as the collection of class $C^1$ paths $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $d\gamma(0)/ds = \delta x$ and define

$$V_D(x,\delta x) := \inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\frac{V_I(\gamma(0),\gamma(s))}{s^2}. \tag{10}$$

We claim that $V_D$ is a Finsler–Lyapunov function, i.e., it satisfies the properties of Definition 2.4.

First, we show that (6a) holds. Since taking the limit superior preserves the inequalities, e.g., [32, Lemma 6.4.13], it follows from (4a) with $(x_0,x_0') = (\gamma(s),\gamma(0))$ that:

$$\underline{c}\limsup_{s\to 0+}\frac{|\gamma(s)-\gamma(0)|^2}{s^2} \leq \limsup_{s\to 0+}\frac{V_I(\gamma(s),\gamma(0))}{s^2} \leq \bar{c}\limsup_{s\to 0+}\frac{|\gamma(s)-\gamma(0)|^2}{s^2}.$$

Noting that taking the infimum also preserves the inequalities gives as follows:

$$\underline{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\frac{|\gamma(s)-\gamma(0)|^2}{s^2} \leq V_D(x,\delta x) \leq \bar{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\frac{|\gamma(s)-\gamma(0)|^2}{s^2}$$

where (10) is used. For any $\gamma \in \Gamma'(x,\delta x)$, we have

$$\lim_{s\to 0+}\frac{|\gamma(s)-\gamma(0)|^2}{s^2} = \left(\lim_{s\to 0+}\left|\frac{\gamma(s)-\gamma(0)}{s}\right|\right)^2 = |\delta x|^2$$

such that the limit superior equals the (ordinary) limit. As a result, (6a) holds for $V_D$ in (10).

It remains to show that (6b) holds. To do so, pick any $\gamma \in \Gamma'(x_0,\delta x_0)$. From (8), we conclude that $\delta x(\cdot) = (\partial\phi(\cdot,\gamma(s))/\partial s)|_{s=0}$ is the solution to the variational system (5) starting from $\delta x_0$ along $x(\cdot) = \phi(\cdot,\gamma(0)) = \phi(\cdot,x_0)$. In particular, this means that

$$\left(\phi(t,x_0), \left.\frac{\partial\phi(t,\gamma(s))}{\partial s}\right|_{s=0}\right) \in \Gamma'(x(t),\delta x(t)).$$

As a result, it follows from (10) that:

$$\begin{aligned}V_D(x(t),\delta x(t)) &= \inf_{\bar\gamma\in\Gamma'(x(t),\delta x(t))}\limsup_{s\to 0+}\frac{V_I(\bar\gamma(0),\bar\gamma(s))}{s^2}\\ &\leq \limsup_{s\to 0+}\frac{V_I(\phi(t,\gamma(0)),\phi(t,\gamma(s)))}{s^2}\end{aligned}$$

which can be further bounded using (4b) to obtain as follows:

$$V_D(x(t),\delta x(t)) \leq e^{-\lambda t}\limsup_{s\to 0+}\frac{V_I(\gamma(0),\gamma(s))}{s^2}.$$

Recalling that this holds for any $\gamma \in \Gamma'(x_0,\delta x_0)$, we can use (10) to conclude that (6b) holds. Hence, $V_D$ in (10) is an incremental Lyapunov function. ∎

The main contribution of Theorem 3.1 is that it gives an explicit construction of an incremental Lyapunov function from a Finsler–Lyapunov function and vice versa.

*Remark 3.2:* The construction of the Finsler–Lyapunov function in (10) relies on taking a limit superior. However, this can be replaced by an ordinary limit when $V_I$ is such that the latter exists. As a simple example, for $V_I(x,x') = (x-x')^\top M(x-x')$ for some positive definite symmetric $M$, we readily observe that $V_D(x,\delta x) = \delta x^\top M\,\delta x$. ◁

*Remark 3.3:* From the proof, one notices that the statement of Theorem 3.1 also holds if $e^{-\lambda t}$ in (4b) and (6b) is replaced with an arbitrary function $p(t)$. ◁

*Remark 3.4:* A similar approach to constructing an incremental Lyapunov function from a Finsler–Lyapunov function can be found in [9, Prop. 4] for global exponential stability analysis at an equilibrium point. Our result can be viewed as its extension to incremental stability. Moreover, we reveal that the converse is also true, whereas [9] does not proceed with converse analysis. For each direction of the proof, we use a different representation of paths. Both are standard in Riemannian and Finsler geometry, e.g., see [34]. ◁

*Remark 3.5:* In the proof of Theorem 3.1, we do not assume that a shortest path exists for the construction of an incremental Lyapunov function in (7). Namely, we do not require that there exists $\gamma^* \in \Gamma(x,x')$ such that

$$V_I(x,x') = \int_0^1 V_D\left(\gamma^*(s), \frac{d\gamma^*(s)}{ds}\right) ds.$$

A similar remark holds for the construction of a Finsler–Lyapunov function in (10). ◁

### B. Discrete-Time Systems

In this section, we establish a counterpart of Theorem 3.1 for the discrete-time system as follows:

$$z(t+1) = g(z(t)), \quad z(0) = z_0 \tag{11}$$

where $g$ is of class $C^1$. The variational system of (11) along $z(\cdot)$ is as follows:

$$\delta z(t+1) = \frac{\partial g(z(t))}{\partial z}\delta z, \quad \delta z(0) = \delta z_0. \tag{12}$$

As a modification of Theorem 3.1, we can show the following equivalence.

*Corollary 3.6:* For a discrete-time system (11), the following two statements are equivalent:

1) there exists a piecewise continuous function $V_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $p : \mathbb{R} \to \mathbb{R}$, and $\underline{c}, \overline{c} > 0$ such that (4a) holds and
$$V_I(z(t), z'(t)) \le p(t) V_I(z_0, z_0')$$
for any $t \ge 0$ and $(z_0, z_0') \in \mathbb{R}^n \times \mathbb{R}^n$;
2) there exists a piecewise continuous function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $p : \mathbb{R} \to \mathbb{R}$, and $\underline{c}, \overline{c} > 0$ such that (6a) holds and
$$V_D(z(t), \delta z(t)) \le p(t) V_D(z_0, \delta z_0)$$
for any $t \ge 0$ and $(z_0, \delta z_0) \in \mathbb{R}^n \times \mathbb{R}^n$.

*Proof:* The proof follows directly from that of Theorem 3.1 after noting that an equality similar to (8) holds for the discrete-time system (11). To show this, let $\psi(t, z_0)$ denote the solution to the system (11) at time $t$ starting from $z_0 \in \mathbb{R}^n$ at $t = 0$. For $\gamma \in \Gamma(z, z')$, it follows that:

$$\begin{aligned}\frac{\partial \psi(t+1, \gamma(s))}{\partial s} &= \frac{\partial g(\psi(t, \gamma(s)))}{\partial s} \\ &= \frac{\partial g(\psi(t, \gamma(s)))}{\partial \psi}\frac{\partial \psi(t, \gamma(s))}{\partial s}.\end{aligned}$$

Therefore, it suffices to replace $e^{-\lambda t}$ with $p(t)$ in the proof of Theorem 3.1. ■

For IES analysis, the case where $p(t) = \lambda \in [0,1)$ and $t = 1$, i.e., $z(1) = g(z_0)$ is important. In this case, a similar statement can be found in [27, Thm. 11] through IES analysis. Our proof, however, provides a direct connection between incremental and Finsler–Lyapunov functions.

## IV. Applications to Cooperative Systems

In this section, we tailor Theorem 3.1 for the analysis of cooperative systems. The system (1) is said to be *cooperative* [35], [36] if its auxiliary system (2) satisfies the following:

$$x_0' - x_0 \in \mathbb{R}^n_+ \quad \Rightarrow \quad x'(t) - x(t) \in \mathbb{R}^n_+ \quad \forall t \ge 0$$

or, equivalently (see [37, Thm. 1]), its variational system (5) satisfies, for all $x_0 \in \mathbb{R}^n$ as follows:

$$\delta x_0 \in \mathbb{R}^n_+ \quad \Rightarrow \quad \delta x(t) \in \mathbb{R}^n_+ \quad \forall t \ge 0.$$

For cooperative systems, IES can be studied using Finsler–Lyapunov functions that are either sum-separable or max-separable.

*Proposition 4.1:* A cooperative system (1) is IES if either of the following conditions hold:

1) there exists a piecewise class $C^1$ function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\underline{c}$, $\overline{c}$, and $\lambda > 0$ such that
$$\underline{c}\sum_{i=1}^n \delta x_{0,i} \le V_D(x_0, \delta x_0) \le \overline{c}\sum_{i=1}^n \delta x_{0,i} \tag{13}$$
and (6b) holds for all $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n_+$;
2) there exists a piecewise class $C^1$ function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\underline{c}$, $\overline{c}$, and $\lambda > 0$ such that
$$\underline{c}\max_{i=1,\dots,n}\{\delta x_{0,i}\} \le V_D(x_0, \delta x_0) \le \overline{c}\max_{i=1,\dots,n}\{\delta x_{0,i}\} \tag{14}$$
and (6b) holds for all $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n_+$.

The converse is also true for item i) [respectively, item ii)] if $\partial f(x)/\partial x$ is bounded on $\mathbb{R}^n$, and the system is forward [respectively, backward] complete on $\mathbb{R}^n$.

*Proof:* The statement of item i) is shown in [29, Thm. 4.3] for the case when $V_D$ is of the form as follows:

$$V_D(x, \delta x) = \sum_{i=1}^n v_i(x)\delta x_i$$

with $\underline{c} \le v_i(x) \le \overline{c}$, $i = 1, \dots, n$. Since this $V_D(x, \delta x)$ satisfies (13) for any $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n_+$, the converse statement is a direct consequence of [29, Thm. 4.3]. For the sufficiency, (6b) and (13) yield as follows:

$$\sum_{i=1}^n \delta x_i(t) \le \frac{\overline{c}}{\underline{c}} e^{-\lambda t}\sum_{i=1}^n \delta x_{0,i}$$

for any $t \ge 0$ and $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n_+$. The rest of the proof follows from [29, Thm. 4.3]. Item ii) can be proven similarly on the basis of [29, Thm. 4.3]. ■

An important feature of Proposition 4.1 is that IES can be verified by checking the conditions for $\delta x$ on $\mathbb{R}^n_+$ instead of the whole $\mathbb{R}^n$.

By a slight modification of Theorem 3.1, we can obtain an incremental Lyapunov version of the first statement in Proposition 4.1. Here, we emphasize that this equivalence between existence of a Finsler–Lyapunov function and an incremental Lyapunov function does not require the boundedness of $\partial f(x)/\partial x$ on $\mathbb{R}^n$.

*Corollary 4.2:* Item i) of Proposition 4.1 holds for a piecewise continuous function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if and only if there exist a piecewise continuous function $V_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\underline{c}, \overline{c}, \lambda > 0$ such that

$$\underline{c}\sum_{i=1}^{n}(x'_{0,i} - x_{0,i}) \leq V_I(x_0, x'_0) \leq \overline{c}\sum_{i=1}^{n}(x'_{0,i} - x_{0,i}) \tag{15}$$

and (4b) hold for all $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $x'_0 - x_0 \in \mathbb{R}^n_+$.

*Proof:* (Finsler $\Rightarrow$ Incremental) As in the proof of Theorem 3.1, we show that $V_I$ in (7) is an incremental Lyapunov function when $V_D$ satisfies the conditions of item i) in Proposition 4.1.

It follows from (13) that:

$$\underline{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1\sum_{i=1}^{n}\frac{d\gamma_i(s)}{ds}ds$$
$$\leq V_I(x,x') \leq \overline{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1\sum_{i=1}^{n}\frac{d\gamma_i(s)}{ds}ds$$

where $\gamma_i : [0,1] \to \mathbb{R}$ denotes the $i$th element of the vector-valued function $\gamma$. Note that for any $\gamma \in \Gamma(x,x')$ and $i = 1,\ldots,n$

$$\int_0^1 \frac{d\gamma_i(s)}{ds}ds = x'_i - x_i \tag{16}$$

such that (15) holds. Moreover, $d\gamma_i(s)/ds \geq 0$ and $s \in [0,1]$ implies $x'_i - x_i \geq 0$. The rest is the same as the proof of Theorem 3.1.

(Incremental $\Rightarrow$ Finsler) Denoting the incremental Lyapunov function by $V_I$ as in (15), we show that

$$V_D(x,\delta x) := \inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\frac{V_I(\gamma(0),\gamma(s))}{s} \tag{17}$$

is a Finsler–Lyapunov function satisfying the properties of item i) in Proposition 4.1. Here, $\Gamma'$ is defined as in the proof of Theorem 3.1.

In particular, it follows from (15) that:

$$\underline{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\sum_{i=1}^{n}\frac{\gamma_i(s)-\gamma_i(0)}{s}$$
$$\leq V_D(x,\delta x) \leq \overline{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\sum_{i=1}^{n}\frac{\gamma_i(s)-\gamma_i(0)}{s}.$$

Note that for any $\gamma \in \Gamma'(x,\delta x)$ and $i = 1,\ldots,n$

$$\lim_{s\to 0+}\frac{\gamma_i(s)-\gamma_i(0)}{s} = \delta x_i \tag{18}$$

leading to (13). Moreover, $\gamma_i(s) - \gamma_i(0) \geq 0$ and $s \in [0,1]$ implies $\delta x_i \geq 0$. The rest is the same as the proof of Theorem 3.1. ∎

The max-separable counterpart is stated next.

*Corollary 4.3:* Item ii) of Proposition 4.1 holds for a piecewise continuous function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if and only if there exist a piecewise continuous function $V_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\underline{c}, \overline{c}$, and $\lambda > 0$ such that

$$\underline{c}\max_{i=1,\ldots,n}\{x'_{0,i} - x_{0,i}\}$$
$$\leq V_I(x_0,x'_0) \leq \overline{c}\max_{i=1,\ldots,n}\{x'_{0,i} - x_{0,i}\} \tag{19}$$

and (4b) for all $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $x'_0 - x_0 \in \mathbb{R}^n_+$.

*Proof:* (Finsler $\Rightarrow$ Incremental) Given the Finsler–Lyapunov function $V_D$ as in item ii) of Proposition 4.1, we show that $V_I$ defined as in (7) is an incremental Lyapunov function.

It follows from (14) that:

$$\underline{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1\max_{i=1,\ldots,n}\left\{\frac{d\gamma_i(s)}{ds}\right\}ds$$
$$\leq V_I(x,x') \leq \overline{c}\inf_{\gamma\in\Gamma(x,x')}\int_0^1\max_{i=1,\ldots,n}\left\{\frac{d\gamma_i(s)}{ds}\right\}ds.$$

Note that $\max_{i=1,\ldots,n}\{d\gamma_i(s)/ds\}$ is a piecewise continuous function of $s \in [0,1]$. That is, the integral exists for any $\gamma \in \Gamma(x,x')$. Next, it follows from (16) that:

$$\int_0^1\max_{i=1,\ldots,n}\left\{\frac{d\gamma_i(s)}{ds}\right\}ds \geq \max_{i=1,\ldots,n}\left\{\int_0^1\frac{d\gamma_i(s)}{ds}ds\right\}$$
$$= \max_{i=1,\ldots,n}\{x'_i - x_i\}.$$

Also, $\gamma(s) = x + s(x' - x)$ gives the following inequality:

$$\inf_{\gamma\in\Gamma(x,x')}\int_0^1\max_{i=1,\ldots,n}\left\{\frac{d\gamma_i(s)}{ds}\right\}ds \leq \max_{i=1,\ldots,n}\{x'_i - x_i\}.$$

Thus, we have (19) from (16), where again $d\gamma_i(s)/ds \geq 0$ and $s \in [0,1]$ implies $x'_i - x_i \geq 0$. The rest is the same as the proof of Theorem 3.1.

(Incremental $\Rightarrow$ Finsler) We show that (17) is a Finsler–Lyapunov function. It follows from (19) that:

$$\underline{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\max_{i=1,\ldots,n}\left\{\frac{\gamma_i(s)-\gamma_i(0)}{s}\right\}$$
$$\leq V_D(x,\delta x)$$
$$\leq \overline{c}\inf_{\gamma\in\Gamma'(x,\delta x)}\limsup_{s\to 0+}\max_{i=1,\ldots,n}\left\{\frac{\gamma_i(s)-\gamma_i(0)}{s}\right\}.$$

Since $\gamma$ is continuous, there exist $\bar{s} > 0$ and $\ell \in \{1,\ldots,n\}$ such that

$$\max_{i=1,\ldots,n}\left\{\frac{\gamma_i(s)-\gamma_i(0)}{s}\right\} = \frac{\gamma_\ell(s)-\gamma_\ell(0)}{s} \quad \forall s \in [0,\bar{s}).$$

This yields as follows:

$$\lim_{s\to 0+}\max_{i=1,\ldots,n}\left\{\frac{\gamma_i(s)-\gamma_i(0)}{s}\right\}$$
$$= \lim_{s\to 0+}\frac{\gamma_\ell(s)-\gamma_\ell(0)}{s} = \delta x_\ell = \max_{i=1,\ldots,n}\{\delta x_i\}$$

and, as a consequence, the limit superior has the same value, which does not depend on the choice of a path $\gamma \in \Gamma'(x,\delta x)$. Thus, we have (14) from (18), where again $\gamma_i(s) - \gamma_i(0) \geq 0$ and $s \in [0,1]$ implies $\delta x_i \geq 0$. The remainder follows the proof of Theorem 3.1. ∎

In a special case, where $V_D(x,\delta x) = v^\top \delta x$ with $v \in \mathbb{R}^n_+$, [30, Corollary 4] shows the existence of an incremental Lyapunov function as follows:

$$V_I(x,x') = \sum_{i=1}^{n} v_i|x'_i - x_i|.$$

A similar construction of

$$V_I(x,x') = \max_{i=1,\ldots,n}\frac{|x'_i - x_i|}{w_i}$$

is found in [30, Corollary 5]. However, the converse is not shown in contrast to Corollaries 4.2 and 4.3. Combining Proposition 4.1 and Corollary 4.2 (respectively, Corollary 4.3) gives a new incremental Lyapunov function without taking the absolute values. This means that we only have to verify such conditions for $(x,x') \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $x' - x \in \mathbb{R}^n_+$ instead of the whole $(x,x') \in \mathbb{R}^n \times \mathbb{R}^n$, which is another difference from [30, Corollaries 4 and 5] and a new finding of this article for cooperative systems.

## V. Passivity

In this section, we build on the ideas in the proof of Theorem 3.1 to show the equivalence between incremental passivity and differential passivity under a mild technical assumption.

To this end, consider the open nonlinear system (i.e., the nonlinear system with inputs and outputs)

$$\begin{cases} \dot{x} = f(x,u), & x(0) = x_0 \\ y = h(x,u) \end{cases} \tag{20}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are assumed to be of class $C^2$. Let $\phi(t, x_0, u)$ denote the solution to the system (20) at time $t \geq 0$ starting from $x(0) = x_0 \in \mathbb{R}^n$ under the input $u : \mathbb{R} \to \mathbb{R}^m$, i.e., $x(t) = \phi(t, x_0, u)$.

Throughout this section, we assume that the solution exists for all $t \geq 0$.

*Standing Assumption 5.1:* For any $x_0 \in \mathbb{R}^n$ and any bounded and continuous $u : \mathbb{R} \to \mathbb{R}^m$, the solution $\phi(t, x_0, u)$ exists for all $t \geq 0$.

As for incremental stability, *incremental passivity* is defined as a property of the auxiliary system as follows:

$$\begin{cases} \dot{x} = f(x,u), & x(0) = x_0 \\ \dot{x}' = f(x',u'), & x'(0) = x'_0 \\ y = h(x,u) \\ y' = h(x',u') \end{cases} \tag{21}$$

given as follows [17, Def. 4.7.1].

*Definition 5.2:* An open system (20) is said to be *incrementally passive* if there exists a nonnegative piecewise continuous function $V_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &V_I(x(t), x'(t)) - V_I(x_0, x'_0) \\ &\quad \leq \int_0^t (u'(\tau) - u(\tau))^\top (y'(\tau) - y(\tau)) d\tau \end{aligned} \tag{22}$$

for all $t \geq 0$, all $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$, and any bounded and continuous $(u, u') : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m$. ◁

To introduce *differential* passivity, we define the variational system of (20) along a trajectory $(x(t), u(t))$ as

$$\begin{cases} \dot{\delta x} = \dfrac{\partial f(x(t),u(t))}{\partial x}\delta x + \dfrac{\partial f(x(t),u(t))}{\partial u}\delta u \\ \delta y = \dfrac{\partial h(x(t),u(t))}{\partial x}\delta x + \dfrac{\partial h(x(t),u(t))}{\partial u}\delta u. \end{cases} \tag{23}$$

Now, differential passivity is defined as follows [22, Def. 4.1].

*Definition 5.3:* An open system (20) is said to be *differentially passive* if there exists a nonnegative piecewise continuous function $V_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$V_D(x(t), \delta x(t)) - V_D(x_0, \delta x_0) \leq \int_0^t \delta u^\top(\tau) \delta y(\tau) d\tau \tag{24}$$

for all $t \geq 0$, all $(x_0, \delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and any bounded and continuous $(u, \delta u) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m$. ◁

Using a similar reasoning as for Lyapunov functions in Theorem 3.1, it can be shown that incremental passivity and differential passivity are equivalent properties under the mild assumption that $V_D(x, \delta x)$ in (10) exists, i.e., takes a value in $\mathbb{R}_+$ for any $(x, \delta x)$. This is guaranteed in case $V_I$ satisfies (4a) as shown in Theorem 3.1; see also Remark 5.5 below for another existence condition.

*Theorem 5.4:* A system (20) is incrementally passive if it is differentially passive. Conversely, if (20) is incrementally passive with storage function $V_I$, then it is differentially passive provided that $V_D$ in (10) exists at each $(x, \delta x) \in \mathbb{R}^n \times \mathbb{R}^n$.

*Proof:* (Differential ⇒ Incremental) Aiming to show that the properties of Definition 5.2 hold, consider a pair of initial conditions $(x_0, x'_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and input functions $(u, u') : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m$. Based on those, let $\gamma \in \Gamma(x_0, x'_0)$, with $\Gamma(x_0, x'_0)$ being the collection of $C^1$ paths such that $\gamma(0) = x_0$ and $\gamma(1) = x'_0$ as before.

Similarly, we use a line segment to connect the pair of inputs $(u, u')$ and define $\nu(s) = u + s(u' - u)$. We emphasize that $\nu(s)$ is itself a function of time, i.e., $\nu(s)(t) = u(t) + s(u'(t) - u(t))$, but we omit the argument $t$ to simplify notation. Then, a direct calculation shows as follows:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial s} &= \frac{\partial}{\partial s}\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial t} \\ &= \frac{\partial f\big(\phi(t,\gamma(s),\nu(s)),\nu(s)\big)}{\partial s} \\ &= \frac{\partial f\big(\phi(t,\gamma(s),\nu(s)),\nu(s)\big)}{\partial \phi}\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial s} \\ &\quad + \frac{\partial f\big(\phi(t,\gamma(s),\nu(s)),\nu(s)\big)}{\partial \nu}\frac{\partial \nu(s)}{\partial s} \end{aligned}$$

indicating that

$$\big(\delta x(t), \delta u(t)\big) = \left(\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial s}, \frac{\partial \nu(s)}{\partial s}\right) \tag{25}$$

is a solution to the state equation of the variational system (23). In fact, we have

$$\frac{\partial \nu(s)}{\partial s} = u' - u.$$

by definition of $\nu$. A similar result can be obtained for the output equation of the variational system by observing that

$$\begin{aligned} &\frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial s} \\ &\quad = \frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial \phi}\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial s} \\ &\quad + \frac{\partial h(\phi(t,\gamma(s),\nu(s)),\nu(s))}{\partial \nu}\frac{\partial \nu(s)}{\partial s}. \end{aligned}$$

Thus, summarizing, for any $s \in [0, 1]$, the solutions

$$\begin{aligned} &\big(x(t), u(t), y(t)\big) \\ &\quad = \big(\phi(t,\gamma(s),\nu(s)), \nu(s), h\big(\phi(\tau,\gamma(s),\nu(s)),\nu(s)\big)\big) \end{aligned}$$

and

$$\begin{aligned} &\big(\delta x(t), \delta u(t), \delta y(t)\big) \\ &\quad = \left(\frac{\partial \phi(t,\gamma(s),\nu(s))}{\partial s}, u' - u, \frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial s}\right) \end{aligned} \tag{26}$$

satisfy (20) and (23), respectively.

Now, denoting the storage function for differential passivity by $V_D$ (as in Definition 5.3), we introduce $V_I$ as in (7) as a candidate storage function for incremental passivity. It is easy to see that nonnegativity of $V_D$ implies that $V_I$ is nonnegative, and thus we proceed to show that (22) holds. To this end, note that $\phi(t, \gamma(s), \nu(s)) \in \Gamma(x(t), x'(t))$ regardless of the choice of $\gamma \in \Gamma(x_0, x'_0)$, such that (7) yields as follows:

$$V_I(x(t), x'(t)) = \inf_{\bar{\gamma} \in \Gamma(x(t), x'(t))} \int_0^1 V_D\left(\bar{\gamma}(s), \frac{d\bar{\gamma}(s)}{ds}\right) ds$$

$$\leq \int_0^1 V_D\left(\phi(t,\gamma(s),\nu(s)), \frac{\partial\phi(t,\gamma(s),\nu(s))}{\partial s}\right) ds.$$

Moreover, from (24) and (26), we have as follows:

$$V_I(x(t),x'(t)) - \int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds$$
$$\leq \int_0^1 V_D\left(\phi(t,\gamma(s),\nu(s)), \frac{\partial\phi(t,\gamma(s),\nu(s))}{\partial s}\right) ds$$
$$- \int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{\partial s}\right) ds$$
$$\leq \int_0^1 \int_0^t (u'(\tau)-u(\tau))^\top \frac{\partial h\big(\phi(\tau,\gamma(s),\nu(s)),\nu(s)\big)}{\partial s} d\tau ds.$$

Since the integrand is continuous on the bounded set $[0,t]\times[0,1]$, the integral exists. Therefore, it follows that:

$$\int_0^1 \int_0^t (u'(\tau)-u(\tau))^\top \frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial s} d\tau ds$$
$$= \int_0^t (u'(\tau)-u(\tau))^\top \int_0^1 \frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial s} ds d\tau$$
$$= \int_0^t (u'(\tau)-u(\tau))^\top (y'(\tau)-y(\tau)) d\tau$$

where the latter equality follows from the fundamental theorem of calculus. In summary, we have as follows:

$$V_I(x(t),x'(t)) - \int_0^1 V_D\left(\gamma(s), \frac{d\gamma(s)}{ds}\right) ds$$
$$\leq \int_0^t (u'(\tau)-u(\tau))^\top (y'(\tau)-y(\tau)) d\tau.$$

Since this holds for any $\gamma \in \Gamma(x_0, x_0')$, we can use the definition of $V_I$ in (7) to obtain (22).

(Incremental $\Rightarrow$ Differential) Let $V_I$ denote the storage function for incremental passivity as in Definition 5.2. We show that $V_D$ in (10) satisfies (24). To this end, let $(x_0,\delta x_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and pick any bounded and continuous $(u,\delta u): \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m$. Next, take $\gamma \in \Gamma'(x_0,\delta x_0)$, with $\Gamma'(x_0,\delta x_0)$ the collection of class $C^1$ paths $\gamma:[0,1]\to\mathbb{R}^n$ such that $\gamma(0)=x_0$ and $d\gamma(s)/ds|_{s=0} = \delta x_0$ as before. Finally, denote $\nu(s) = u + s\delta u$ and note that $\nu(s)$ is itself a function of time.

From (26), we conclude that

$$\delta x(t) = \left.\frac{\partial\phi(t,\gamma(s),\nu(s))}{\partial s}\right|_{s=0}$$

is the solution to the variational system (23) starting from $\delta x_0$ under the input $\delta u$ with the variational system considered along the trajectory

$$(x(t),u(t)) = \big(\phi(t,\gamma(0),\nu(0)),\nu(0)\big) = \big(\phi(t,x_0,u(t)),u(t)\big).$$

Here, the final equality follows from the definitions of $\gamma$ and $\nu$. Similar to before, we note that this implies that, for any $t\geq 0$

$$\phi(t,\gamma,\nu) \in \Gamma'(x(t),\delta x(t)).$$

As a result, using the definition of $V_D$ in (10) gives as follows:

$$V_D(x(t),\delta x(t))$$
$$= \inf_{\bar\gamma\in\Gamma'(x(t),\delta x(t))} \limsup_{s\to 0^+} \frac{V_I(\bar\gamma(0),\bar\gamma(s))}{s^2}$$
$$\leq \limsup_{s\to 0^+} \frac{V_I\big(\phi(t,\gamma(0),\nu(0)),\phi(t,\gamma(s),\nu(s))\big)}{s^2}.$$

Moreover, we have[1] as follows:

$$V_D(x(t),\delta x(t)) - \limsup_{s\to 0^+} \frac{V_I(\gamma(0),\gamma(s))}{s^2}$$
$$\leq \limsup_{s\to 0^+} \frac{V_I(\phi(t,\gamma(0),\nu(0)),\phi(t,\gamma(s),\nu(s)))}{s^2}$$
$$- \limsup_{s\to 0^+} \frac{V_I(\gamma(0),\gamma(s))}{s^2}$$
$$\leq \limsup_{s\to 0^+} \frac{V_I(\phi(t,\gamma(0),\nu(0)),\phi(t,\gamma(s),\nu(s)))-V_I(\gamma(0),\gamma(s))}{s^2}$$

after which incremental passivity as in (22) leads to

$$V_D(x(t),\delta x(t)) - \limsup_{s\to 0^+} \frac{V_I(\gamma(0),\gamma(s))}{s^2}$$
$$\leq \limsup_{s\to 0^+} \int_0^t S(\tau,s) d\tau. \tag{27}$$

Here, $S$ is given as follows:

$$S(\tau,s)$$
$$= \left(\frac{\nu(s)-\nu(0)}{s}\right)^\top$$
$$\times \left(\frac{h\big(\phi(\tau,\gamma(s),\nu(s)),\nu(s)\big) - h\big(\phi(\tau,\gamma(0),\nu(0)),\nu(0)\big)}{s}\right).$$

By the definition of $\nu$ as $\nu(s) = u + s\delta u$, it follows that $(\nu(s) - \nu(0))/s = \delta u$, after which the limit for $s\to 0^+$ gives the following:

$$\lim_{s\to 0^+} S(\tau,s) = \delta u^\top(\tau) \left.\frac{\partial h(\phi(\tau,\gamma(s),\nu(s)),\nu(s))}{\partial s}\right|_{s=0}$$
$$= \delta u^\top(\tau)\delta y(\tau).$$

Moreover, for any bounded $t>0$ and sufficiently small $\bar s > 0$, $\int_0^t S(\tau,s)d\tau$ is bounded for any $s\in(0,\bar s)$. Therefore, taking the limit and integral is exchangeable for $S(\tau,s)$ in (27). Namely, it follows that

$$\lim_{s\to 0^+} \int_0^t S(\tau,s)d\tau = \int_0^t \delta u^\top(\tau)\delta y(\tau) d\tau$$

which implies that the limit superior of $\int_0^t S(\tau,s)d\tau$ is equivalent to the limit. In summary, (27) can be rewritten as follows:

$$V_D(x(t),\delta x(t)) - \limsup_{s\to 0^+} \frac{V_I(\gamma(0),\gamma(s))}{s^2} \leq \int_0^t \delta u^\top(\tau)\delta y(\tau) d\tau.$$

Since this holds for any $\gamma\in\Gamma'(x_0,\delta x_0)$, we have (24) from the definition of $V_D$ in (10). ■

*Remark 5.5:* The existence assumption for $V_D$ constructed by (10) is minor. For instance, this holds if $V_I$ is of class $C^2$ and satisfies $V_I(x,x) = 0$ for all $x\in\mathbb{R}^n$. Then, $V_I$ can be written as follows:

$$V_I(x,x') = \frac{\partial V_I(x,x)}{\partial x'}(x'-x)$$
$$+ \frac{1}{2}(x'-x)^\top \frac{\partial^2 V_I(x,x)}{\partial x'^2}(x'-x) + R(x,x')$$

[1]Here, we use that $\limsup(a) - \limsup(b) = \limsup(a) + \liminf(-b) \leq \limsup(a-b)$.

where $\lim_{s\to 0} R(x, x + s\delta x)/s^2 = 0$ for any $(x, \delta x) \in \mathbb{R}^n \times \mathbb{R}^n$. From $V_I(x, x') \geq 0$, we in fact obtain $\partial V_I(x, x)/\partial x' = 0$. Therefore, $V_D$ in (10) exists. ◁

*Remark 5.6:* Similar to the case of exponential stability, one can prove a discrete-time counterpart of Theorem 5.4 when the notions of incremental and differential passivity are suitably translated to discrete time. ◁

In [24, Thm. 2.17] and [23, Thm. 6], it has been shown that differential passivity implies incremental passivity. However, both papers consider a restrictive definition of differential passivity in which the storage function has the (quadratic) form $V_D(x, \delta x) = \delta x^\top M(x) \delta x$ with symmetric and positive definite $M(x)$. In particular, [24, Thm. 2.17] has considered the case where $M$ is constant (see also Remark 3.2 for this case).

In this article, we do not impose a quadratic structure or positive definiteness of $V_D(x, \delta x)$ with respect to $\delta x$. In other words, $V_D(x, \delta x)$ does not necessarily have a Finsler structure, and a geodesic is not required to exist in contrast to [24, Thm. 2.17] and [23, Thm. 6]. Moreover, these papers do not show that incremental passivity implies differential passivity. Our construction (10) of the differential storage function from an incremental storage function is conjectured by [17, Note 20 for Ch. 4] without taking the infimum with respect to the collection of paths. Theorem 5.4 proves this conjecture by taking the infimum.

## VI. Conclusion

In this article, we have shown that incremental and differential Lyapunov approaches for studying incremental exponential stability are equivalent in the sense that an incremental Lyapunov function can be constructed from a Finsler–Lyapunov function, and vice versa. The proposed constructions are applicable also for discrete-time systems and separable Lyapunov functions for analysis of cooperative systems. Moreover, we have applied the proposed technique to show the equivalence between incremental and differential passivity under a mild technical assumption. In [38], foundations of contraction theory are constructed in a coordinate-free setting. Tailoring our results to the coordinate-free setting will be the subject of future work.

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