Chapter 3

Dual-encoding for motion unwrapping in harmonic MRE

The experimental data of this work was provided by Helge Hertum and Ingrid Sack, Elastography Group, Charité University Hospital, Berlin, Germany. These results are the basis for an article currently in preparation in collaboration with Charité.

3.1 Introduction

In this chapter we briefly present an extension of the work shown in Chapter 2 in the case of magnetic resonance elastography, introduced in Chapter 1, section 1.3.3. Unwrapping in MRE has been treated in the literature, as in MRI for velocity encoding, by smoothing with respect to neighbor pixels or even the neighbor timesteps [Ito82; GP98; Fly97; Sac+09; SZ03; Bar+15; JSD15], but since our technique will not follow the same idea, we will not compare them. However, smoothing techniques can be used after applying the technique presented in this chapter.

3.2 Theory

3.2.1 Harmonic displacement encoding (Henc)

Similar to the VENC idea, we define the harmonic displacement encoding, henc, by

\[ henc = \frac{\pi}{\xi(\omega, T)} \]

Then (1.14) can be re-written as follows:

\[ \hat{v}_n(x) = \frac{\pi}{henc} u_n(x) \]
From this equation we note that $|h_{\text{enc}}|$ is the maximum possible displacement which is not aliased, because $|\vartheta_n(x)| < \pi$, which could be a problematic constraint in the application of phase contrast for recovering the displacement.

In the rest of this chapter, we emphasize the dependence of $\vartheta_n$ on $h_{\text{enc}}$ explicitly by writing

$$\vartheta_n(x; u_n, h_{\text{enc}}) = \frac{\pi}{h_{\text{enc}}} u_n(x) \quad (3.1)$$

### 3.2.2 Cost Functionals

For each time $\tau_n$ with $n = 0, 1, \ldots, N - 1$, we will now reformulate the phase-contrast problem like a least-squares estimator, as we did for the velocity encoding problem. We will denote the true displacement $\hat{u}_n$, and the phase measured for that displacement for a given $h_{\text{enc}}$ as $\hat{\vartheta}_n(x; \hat{u}_n, h_{\text{enc}})$.

For a fixed $x$, we consider the functionals

$$J_n(x; u, h_{\text{enc}}) = \frac{1}{2} \left| e^{i \hat{\vartheta}_n(x; \hat{u}_n, h_{\text{enc}})} - e^{i \vartheta_n(x; u, h_{\text{enc}})} \right|^2 \quad (3.2)$$

which is, after some calculations, equal to

$$J_n(x; u, h_{\text{enc}}) = 1 - \cos (\hat{\vartheta}_n(x; \hat{u}_n, h_{\text{enc}}) - \vartheta_n(x; u, h_{\text{enc}})) \quad (3.3)$$

Observe that $\hat{\vartheta}_n(x; \hat{u}_n, h_{\text{enc}}) = \hat{\varphi}_n(x; \hat{u}_n, h_{\text{enc}}) - \hat{\varphi}_0(x)$, so we need to measure $\hat{\varphi}_n(x; \hat{u}_n, h_{\text{enc}})$ and $\hat{\varphi}_0(x)$ by:

- $\hat{\varphi}_n(x; \hat{u}_n, h_{\text{enc}})$ is acquired by considering the equations described in the previous section, applying the gradient corresponding to the respective $h_{\text{enc}}$.
- $\hat{\varphi}_0(x)$ is acquired once in the experiment. It is obtained when we apply a null gradient.

In addition, observe that equation (3.3) asserts that, the minimum of $J_n$ is reached when $\hat{\varphi}_n = \varphi_n + 2\pi \ell$, $\ell \in \mathbb{Z}$. In terms of the displacement $u$, the periodicity of $J_n(x; \cdot, h_{\text{enc}})$ is $2h_{\text{enc}}$. Therefore, as in standard phase-contrast MRI, aliasing arises if $|h_{\text{enc}}|$ is less or equal to the true displacement.

### 3.2.3 Dual encoding strategy

To overcome aliasing, we apply the following dual encoding strategy:

- We define the sum of functionals $J_n$ with $h_{\text{enc}}$s $H_1$ and $H_2$:

$$J_{\Sigma, n}(x; u, H_1, H_2) = J_n(x; u, H_1) + J_n(x; u, H_2) \quad (3.4)$$

Note that for each $n$ we perform three measurements: the corresponding to $H_1, H_2$ and the null gradient.
3.3. METHODS

For each $n$, we can estimate $u_n(x)$ by the unwrapped displacement $u^*_n(x; H_1, H_2)$ by solving the problem

$$u^*_n(x; H_1, H_2) = \arg\min_{u \in [-u_{max}, u_{max}]} J_{\Sigma,n}(x; u, H_1, H_2)$$

where $u_{max} = \text{lcm}(2h\text{enc}_1, h\text{enc}_2)/2$, as in Section 2.2.6.

If we need to obtain a displacement in a steady-state and $\tau_n = \frac{2\pi n}{N}$, we apply the following discrete Fourier transform in time of $\{u^*_n\}_{n=0}^{N-1}$ in order to obtain $u_c$ by equation (1.15)

The advantage of the dual henc strategy is that the minimum $v_n(x; H_1, H_2)$ of $J_{\Sigma,n}(x; \cdot, H_1, H_2)$ is reached uniquely in an interval of width $2\text{lcm}\{H_1, H_2\}$, where $\text{lcm}$ is the (rational) least common multiple, and, moreover, $u^*_n(x; H_1, H_2)$ is a minimum for $J_n(x; \cdot, H_1)$ and $J_n(x; \cdot, H_2)$. Therefore, if we take a good pair $(H_1, H_2)$, such that $2\text{lcm}\{H_1, H_2\}$ is maximized, we can obtain an estimation for $v_n$ which is unique in a wide interval, even when both $(H_1, H_2)$ are smaller than the true displacement $\hat{u}_n$. Aliasing will only happen when $\text{lcm}\{H_1, H_2\} \leq |\hat{u}_n|$

3.3 Methods

We consider a phantom consisting of a plastic box of approximately 10x10x10 centimeters filled with an heparin gel to emulate soft tissue.

The scan parameters are:

- $TR = 2000$ ms, $TE = 95$ ms.
- $N = 8$ timesteps $\tau_n$ to sample one wave period of time $T = 20$ ms.
- Mechanical and MEG frequency are the same: $f_{\text{mech}} = f_{\text{grad}} = 50$ Hz, and then $\omega = 2\pi f_{\text{mech}} = 314.159$ rad/s.
- The gradient has the form

$$G(t) = \begin{cases} A & \text{if } t \in [0, T/2] \\ -A & \text{if } t \in [-T/2, 0] \\ 0 & \text{otherwise} \end{cases}$$

with MEG amplitudes $A = \{2, 8, 9, 12, 16, 18\}$ [mT/m].

According to section 1.3.3, the encoding efficiency is

$$\xi(\omega, T) = -\gamma \int_{-T/2}^{T/2} G(t) \sin(\omega t) dt = -\frac{4\gamma A}{\omega}$$
where we used the fact that $T = \frac{2\pi}{\omega}$. Hence we see that in practice, for fixed $\omega$, the encoding efficiency is controlled by the parameter $A$. Note that since aliasing occurs if $|\text{henc}(A)| < |u|$, aliasing occurs in this case if

$$|\text{henc}(A)| = \left| \frac{\pi}{\xi(\omega, T)} \right| = \frac{\pi\omega}{4\gamma A} < |u_{\text{true}}| \quad (3.7)$$

In the following table we show $\text{henc}$ as a function of the amplitude for the amplitudes used in our experiment:

<table>
<thead>
<tr>
<th>$A \times 10^{-3} mT/m$</th>
<th>2</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>16</th>
<th>18</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{henc}(A)</td>
<td>\times 10^{-4} m$</td>
<td>4.612</td>
<td>1.153</td>
<td>1.025</td>
<td>0.769</td>
<td>0.576</td>
</tr>
</tbody>
</table>

Table 3.1: Correspondence between amplitude ($A$) and the $\text{henc}$, which can be seen as the critical observed displacement.

The following table shows the critical displacement for the dual encoding technique, following Table 2.1:

<table>
<thead>
<tr>
<th>$(A_1, A_2)$</th>
<th>(9, 12)</th>
<th>(8, 12)</th>
<th>(12, 16)</th>
<th>(12, 18)</th>
<th>(16, 24)</th>
<th>(18, 24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\text{henc}_{\text{eff}}</td>
<td>\times 10^{-4} m$</td>
<td>3.074</td>
<td>2.306</td>
<td>2.306</td>
<td>1.537</td>
</tr>
</tbody>
</table>

Table 3.2: Correspondence between the pair $(A_1, A_2)$ of amplitudes and the effective $\text{henc}$, that is, the critical displacement for the dual-$\text{henc}$ technique.

### 3.4 Results

#### 3.4.1 Results for a fixed time

We show the results for the phantom experiments for $n = 3, 5, 7$ in Figures 3.1, 3.2 and 3.3, respectively. The peak displacement is different at each time, which can be seen in the single-$\text{henc}$ figure corresponding to $A = 2$. We can distinguish the shape of the phantom by separating it from the noisy part near the boundaries of each image. We first show the single-$\text{henc}$ phase contrast MRI and the dual-$\text{henc}$ technique, where we observe the aliasing in pixels corresponding to displacements according to Tables 3.1 and 3.2, respectively. That is,

- For single-$\text{henc}$, we observe aliasing from $A = 8$ onwards for $n = 3, 5$ and $A = 12$ for $n = 7$. It is also clear how the noise in the image decreases when increasing the encoding gradient.


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- For dual-henc, we observe aliasing only for the pair (12, 18) onwards for
  \( n = 3, 5 \) and we do not observe aliasing in any image for \( n = 7 \), because
  the critical value shown in Table 3.2 is not reached.

In addition, by observing the images for the different times, we can see the
propagation of the displacement.

![Images of displacement](image1.png)

Figure 3.1: Phantom data: single- (PC) and dual-HENC, for \( n = 3 \)

3.4.2 Results for the discrete Fourier transform in time

We perform a discrete Fourier transform in time and show the results for the
phantom experiments in Figure 3.4. The peak displacement can be seen in the
single-henc figure corresponding to \( A = 2 \), and we see that it is bigger than the
peak of each time. The transition to aliased images is not inferred directly from
Tables 3.1 and 3.2, but we can notice that if one of the recovered displacements
has aliasing for any time, then the Fourier transform in time has aliasing, that
is:

- For single-henc, we observe aliasing from \( A = 8 \) onwards, because aliasing
  is present for that amplitude and some \( n \), specifically, at least for \( n = 3, 5 \).

- For dual-henc, we observe aliasing in \( (A_1, A_2) = (12, 18) \) since aliasing is
  present for that amplitude pair and at least \( n = 3, 5 \).
(a) PC $A = 2$
(b) PC $A = 8$
(c) PC $A = 9$
(d) PC $A = 12$
(e) ODV 9, 12
(f) ODV 8, 12
(g) ODV 12, 16
(h) ODV 12, 18

Figure 3.2: Phantom data: single- (PC) and dual-HENC, for the time corresponding to $n = 5$.

(a) PC $A = 2$
(b) PC $A = 8$
(c) PC $A = 9$
(d) PC $A = 12$
(e) ODV 9, 12
(f) ODV 8, 12
(g) ODV 12, 16
(h) ODV 12, 18

Figure 3.3: Phantom data: single- (PC) and dual-HENC, for the time corresponding $n = 7$. 
3.5 Discussions and conclusions

We see that the theory is confirmed at least in a phantom experiment, showing properly the predictions given. At this moment, we haven’t performed any experiment with volunteers. The dual-henc MRE technique presented has a potential advantage: we can reconstruct very small displacements, which usually correspond to points distant to the mechanical source, with less noise and at the same time we can reconstruct larger displacements, corresponding to points close to the mechanical source, without aliasing.

Here we aim to keep henc constant. Equation (3.7) has as a consequence that, if the frequency is halved, the amplitude must be doubled. This does not cause serious problems in actual implementations.

However, if the frequency is halved, i.e., if the period is doubled, the corresponding repetition time and echo times become much larger than those used for velocity recovering in MRI. This could cause practical problems that are not solved by the dual-henc technique, since they are related with the relaxation times showed in equation (1.1). Therefore it is still a challenge to explore deeper in tissues by reducing the frequency.

Figure 3.4: Phantom data: single- (PC) and dual-HENC, for Re($u_c$) obtained from the discrete Fourier transform in time.