Chapter 1

Introduction

1.1 Motivation

In this thesis we study some inverse problems in MRI and in elasticity imaging. A problem is called inverse if its formulation involves the solution of other problem, called direct. In science, the direct problem comes from fundamental laws, such as energy conservation; the inverse problem consists in the determination of the parameters of the equations governing the direct problem, from the knowledge of the solution entirely or a part of it. In other words, by solving an inverse problem, parameter values that cannot be obtained directly can be found.

Magnetic resonance imaging (MRI) is a noninvasive biomedical imaging technique. Its development comes from the study of the nuclear magnetic resonance by Felix Bloch and Edward Purcell [Blo46] and later the development of the tomographic technique by Paul Lauterbur and Peter Mansfield [Lau+73; Man82]. MRI has potential advantages over other techniques: lack of ionizing radiation, free choice of imaging planes, high resolution, capability for tissue characterization, etc. The characterization of tissues can be anatomical and qualitative or even quantitative giving a map of a specific parameter which can be obtained from the MRI technique, such as: proton density, magnetic relaxation times, water diffusivity, velocity and displacement. Each of the parameters mentioned above has the capability of characterizing a biological tissue, in order to detect any disease, see for example [JBP+05; CJ05; MBV06; GME12].

In this thesis we are interested in the recovery of the following parameters from magnetization measurements in MRI: the velocity of blood flowing in vessels and the displacements of tissues under a harmonic regime. Both velocity and displacement imaging is achieved by the so-called Phase-Contrast MRI (PC-MRI), that is, the derivation of the quantity of interest is performed by the substraction (modulo $2\pi$) of two measured phases of the complex transverse
magnetization in MRI.

Imaging the velocity of blood flow finds its main application in the research into cardiovascular diseases, since heart failure is a major cause of death in Western societies and will become even more prevalent with increasing life expectancy [MBV06]. Blood velocity measurements can also lead to the recovery of further mechanical information like relative pressures [Yan+96; Ber+18; NB19; Nol+19] and wall shear stress [Sot+12; Sot+16].

Imaging the harmonic displacements in tissues finds its motivation in elastography, that is, two-step techniques consisting in measuring the displacements of a tissue and then recovering some elastic parameter of it. Examples of elastography techniques are ultrasound and magnetic resonance elastography (MRE). Several diseases involve changes in the mechanical properties of tissue and normal function of tissue, like skeletal muscle, heart, lungs, gut, etc. The changes in mechanical properties are not best seen precisely in the displacement map, but the best contrast is seen in the spatial distribution of any elastic parameter [HSB17]. These elastic parameters can be then reconstructed from the knowledge of the displacement map, using the elasticity equations.

Mathematically, elasticity imaging techniques have given rise to many problems which are challenging in many branches of mathematics. Our focus will be on partial differential equations, studying nonlinear models for elasticity and the measurements. In the literature, inverse problems in elasticity are most studied for linear models, however those models do not describe accurately some phenomena, but nonlinear elasticity models do it.

1.2 Research objectives

The topics presented in this thesis are:

1. The proposal of a new method for the recovery of the velocity of blood from MRI measurements. The proposed method is more robust, meaning less affected by noise and able to measure a wider range of velocities than the usual methods. This method is motivated, proposed and tested in Chapter 2. In addition, in Chapter 3 the natural extension to MRE is discussed.

2. Analysis of nonlinear inverse problems in elasticity, in two cases: assuming the measurements or the elasticity model being nonlinear. This analysis motivates techniques for the reconstruction of an elasticity parameter. Examples will be presented. This topic is developed in Chapter 4.

In the following two sections we will provide the reader the fundamentals of MRI for the comprehension of velocity and displacement recovery, and also fundamentals of hybrid inverse problems and elasticity equations.
1.3 Magnetic Resonance Imaging

1.3.1 Generalities of MRI

The nuclear magnetic resonance phenomenon is modeled by the Bloch equation of the magnetization $M = [M_1, M_2, M_3]^\top$ of protons in position $X(t)$ at time $t$ under the influence of a magnetic field $B$ [Abr61; Sli13]:

$$\frac{dM}{dt} = \gamma M \times B + \frac{1}{T_1} (M_0 - M_3)e_3 - \frac{1}{T_2} M_\perp$$  (1.1)

where $T_1$ and $T_2$ are relaxation times, $M_0$ is the magnetization at equilibrium, $M_\perp = M_1e_1 + M_2e_2$ and $(e_1, e_2, e_3)$ are the basis vectors of the frame of reference.

The Bloch equation represents the conservation of angular momentum of the spin of the protons, including quantum effects.

By components, this is:

$$\frac{dM_1}{dt} = \gamma (B_3M_2 - B_2M_3) - \frac{M_1}{T_2},$$  (1.2)

$$\frac{dM_2}{dt} = \gamma (-B_3M_1 - B_1M_3) - \frac{M_2}{T_2},$$  (1.3)

$$\frac{dM_3}{dt} = \gamma (B_2M_1 - B_1M_2) - \frac{M_3 - M_0}{T_1}.$$  (1.4)

The main steps in MRI are defined by different stages of the “controlled” magnetic field $B$:

- Initially, $B = B_0e_3$, where $B_0$ has a much larger magnitude compared with all the other fields described below. The field $B_0e_3$ is constant in time and space and it is present in all the experiment. Under this equilibrium condition, $M_1 = M_2 = 0$.

- The application of a radiofrequency pulse (rf pulse) causes the resonance in spins, that pulse is generated by adding a time-harmonic field $B_1(t)e_1$ for a very short period of time. This creates the transversal magnetization to tip towards the transversal plane, i.e. $M_1 \neq 0, M_2 \neq 0$.

- When the rf pulse is released, the problem is studied only using equations (1.2)-(1.3), and a non-zero initial condition, because the measurements that the scanner gives us are $M_{xy} = M_1 + iM_2$.

- In order to recover the desired physical quantity, we modify the component $B_3$. The description for velocity and displacement recovery is given in the remaining of this section.
For our purposes, we will also consider the relaxation time $T_2$ large enough for neglecting the term $\frac{M_1}{T_2}$. Therefore, from equations (1.2)-(1.3), we see that $M_{xy}$ satisfies the following equation:

$$\frac{dM_{xy}}{dt} = -i\gamma B_3 M_{xy}. \tag{1.5}$$

Hence

$$M_{xy}(X(t), t) = M_{xy}(t_0) \exp \left( -i\gamma \int_{t_0}^t B_3(X(t'), t')dt' \right).$$

We can fix $M_{xy}(t_0) \in \mathbb{R}$, and then we can write

$$M_{xy}(t) = M_{xy}(t_0)e^{i\varphi(X(t), t)}$$

where the phase

$$\varphi(X(t), t) = -\gamma \int_{t_0}^t B_3(X(t'), t')dt' \tag{1.6}$$

is the key quantity in this work.

### 1.3.2 Velocity Encoding

We now assume that the spins are moving “locally” following the equation

$$X(t) = x + tu \tag{1.7}$$

where $x, u$ are time-constant vectors. In particular, $x$ corresponds to the original position of the spins at $t = 0$, and $X(t)$ to the position of the spins at time $t$. In addition, let us consider the component $B_3$ with the form

$$B_3(X(t), t) = B_0 + X(t) \cdot G(t)$$

where $G(t)$ is a constant-in-space vector. Then, equation (1.6) becomes, at the time of measurement $T_E$, and passing the dependence from $X(T_E)$ to $x$ and $u$ due (1.7):

$$\varphi(X(T_E), T_E) = \varphi(x, u, T_E) = -\left( \gamma B_0 T_E + \gamma x \cdot m_0(G) + \gamma u \cdot m_1(G) \right)$$

where

$$m_0 = \int_0^{T_E} G(t')dt', \quad m_1 = \int_0^{T_E} t'G(t')dt'$$

are known as the zeroth and first moments, respectively [KJH18]. It is usual in MRI practice to consider a waveform $G(t)$ such that $m_0(G) = 0$ in order
to remove it from the velocity reconstruction problem. Then the model of the
phase \( \varphi = \varphi(x, u, T_E) \) reads:

\[
\varphi = \varphi_0 + \gamma u m_1(G), \quad \varphi_0 := \gamma B_0 T_E, \tag{1.8}
\]

with \( v \) the component of the velocity in direction of \( G \), and \( G \) the magnitude of the vector \( G \).

Since \( \varphi_0 \) is not fully known due to inhomogeneities in the magnetic field \( B_0 \), we cannot obtain \( v \) directly from equation (1.8). So, for each \( x \) we consider two measurements \( \varphi^G_0 \) and \( \varphi^G_1 \) satisfying (1.8) with waveforms \( G_0 \) and \( G_1 \) respectively, so that we can obtain \( u \) via the formula

\[
u = \frac{\varphi^{G_0} - \varphi^{G_1}}{\gamma (m_1(G_0) - m_1(G_1))}. \tag{1.9}
\]

In practice, the user controls indirectly the strength of the magnetic fields by setting the velocity encoding parameter, VENC, defined by:

\[
\text{VENC}(G_0, G_1) = \frac{\pi}{\gamma (m_1(G_0) - m_1(G_1))},
\]

which has the dimension of velocity. According to equation (1.9), the velocity can be determined by:

\[
u = \frac{\varphi^{G_0} - \varphi^{G_1}}{\pi \text{VENC}(G_0, G_1)}. \tag{1.10}
\]

This is the standard phase contrast MRI technique for velocity recovery (PC-MRI). Note that \( u = u(x) \).

### 1.3.3 Magnetic Resonance Elastography (MRE)

Instead of assuming that the motion is modelled by (1.7), we adopt the following model [Man+01; HSB17]:

\[
\mathbf{X}(x, t) = x + \mathbf{U}(x, t) \tag{1.11}
\]

where the displacement \( \mathbf{U}(x, t) \) is given by

\[
\mathbf{U}(x, t) = \text{Re}\left( \mathbf{u}_c(x) e^{i\omega t} \right). \tag{1.12}
\]

This motion is caused by some periodic external force, where \( \mathbf{u}_c(x) \in \mathbb{C}^d \) describes a complex steady-state behavior.

The waveform \( \mathbf{G}(t) \), called motion encoding gradient (MEG) in this context, is zeroth moment nulling. Typically, the MEG \( \mathbf{G}(t) \) is set either as:
• symmetric, that is $G(t) = G(-t)$, or
• antisymmetric, that is $G(t) = -G(-t)$,

where we set $t = 0$ at the middle of the time of the application of the MEG, that is the MEG is assumed to be applied for $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$, where $T = \frac{2\pi}{\omega}$.

Then the equation (1.6) for the phase of the spins moving under this model and under this gradient field is given by

$$\varphi(x) = \varphi_0 + \gamma \int_{-T/2}^{T/2} G(t) \cdot X(x, t) dt$$

where $G$ is the magnitude of the vector $G$ and $X$ is the component of $X$ in direction of $G$. We will denote by $u_c$ the component of $u_c$ in the direction of $G$. The phase $\varphi_0$ is static and it can be measured by a null MEG, so we are interested only in the dynamic phase:

$$\vartheta(x) = \gamma \int_{-T/2}^{T/2} G(t) \cdot X(x, t) dt$$

We have:

$$\vartheta(x) = \gamma \int_{-T/2}^{T/2} G(t) \left( x + \text{Re}(u_c(x)e^{i\omega t}) \right) dt$$

$$= \gamma \int_{-T/2}^{T/2} G(t) dt + \gamma \text{Re}(u_c(x) \int_{-T/2}^{T/2} G(t) e^{i\omega t} dt)$$

$$= \gamma \text{Re}(u_c(x) \int_{-T/2}^{T/2} G(t) e^{i\omega t} dt)$$

$$= \gamma \left( \text{Re}(u_c) \int_{-T/2}^{T/2} G(t) \cos(\omega t) dt - \text{Im}(u_c) \int_{-T/2}^{T/2} G(t) \sin(\omega t) dt \right)$$

where we use the fact that $G(t)$ is zeroth moment nulling, so the term $\gamma x \int_{-T/2}^{T/2} G(t) dt$ in the second line is null. Therefore we obtain:

$$\vartheta(x) = \begin{cases} 
\text{Re}(u_c(x)) & \text{for } G \text{ symmetric, } \xi(\omega, T) = \gamma \int_{-T/2}^{T/2} G(t) \cos(\omega t) dt \\
\text{Im}(u_c(x)) & \text{for } G \text{ antisymmetric, } \xi(\omega, T) = -\gamma \int_{-T/2}^{T/2} G(t) \sin(\omega t) dt 
\end{cases}$$

where $\xi(\omega, T)$ is known as the encoding efficiency. This is a key controlled parameter in MRE.
Several measurements and the discrete Fourier transform in time

To identify displacements at different states of the wave given by (1.12), we consider

\[ U_c(x, \tau) = u_c(x) e^{i \omega \tau} \]

where \( \tau \) is the time corresponding to the displacement observed by the MRI experiment, which corresponds to the complex displacement at the time of the rf pulse, here \( u_c \) corresponds to the complex displacement of maximum amplitude. Then

\[ U(x, \tau, t) = \Re \left( U_c(x, \tau) e^{i \omega t} \right) \]

is the displacement occurring at time \( t \) after the rf pulse. Following the analysis done before, we can obtain by PC-MRI either \( \Re( u_c(x) e^{i \omega \tau} ) \) or \( \Im( u_c(x) e^{i \omega \tau} ) \) in the direction of the MEG \( G(t) \).

For reconstructing any elastic parameter, it would be better to have a displacement field suitable in a time independent equation (see next section). For achieving that, let \( n = 0, \ldots, N - 1 \), and consider the set of \( N \) displacements as follows:

- Considering the harmonic regime as above, let \( u_n(x) \) be the PC-MRI measurement of the displacement of the spins having its position in \( x \) at a certain time \( \tau_n \). That is, we assume the model:

\[
 u_n(x) = \begin{cases} 
 \Re( u_c(x) e^{i \omega \tau_n} ) & \text{if } G \text{ is symmetric}, \\
 \Im( u_c(x) e^{i \omega \tau_n} ) & \text{if } G \text{ is antisymmetric}, 
\end{cases} \tag{1.13}
\]

where each \( u_n \) is obtained from the measurement \( \vartheta_n \):

\[
 u_n = \frac{\vartheta_n}{\xi(\omega, T)}. \tag{1.14}
\]

- We apply the following discrete Fourier transform for \( \{u_n\}_{n=0}^{N-1} \):

\[
 u_{FT} = \sum_{n=0}^{N-1} u_n e^{-i \frac{2\pi n}{N}}
\]

and we choose \( \tau_n = \frac{T n}{N} \).

- If \( G(t) \) is symmetric, then we have:

\[
 u_n(x) = \Re \left( u_c(x) e^{i \frac{2\pi n}{N}} \right)
\]
and then
\[ u^{s}_{FT} = \sum_{n=0}^{N-1} \text{Re}\left(u_{c}(x)e^{i\frac{2\pi n}{N}}\right)e^{-i\frac{2\pi n}{N}} \]
\[ = \sum_{n=0}^{N-1} \left[ \text{Re}(u_{c})\cos\left(\frac{2\pi n}{N}\right) - \text{Im}(u_{c})\sin\left(\frac{2\pi n}{N}\right) \right] \left[ \cos\left(\frac{2\pi n}{N}\right) - i \sin\left(\frac{2\pi n}{N}\right) \right] \]
\[ = \sum_{n=0}^{N-1} \left[ \text{Re}(u_{c})\cos^{2}\left(\frac{2\pi n}{N}\right) - \text{Im}(u_{c})\sin\left(\frac{2\pi n}{N}\right)\cos\left(\frac{2\pi n}{N}\right) \right] \]
\[ + i\left[ - \text{Re}(u_{c})\cos\left(\frac{2\pi n}{N}\right)\sin\left(\frac{2\pi n}{N}\right) + \text{Im}(u_{c})\sin^{2}\left(\frac{2\pi n}{N}\right) \right] \]
\[ = \frac{N}{2} u_{c} \]

since
\[ \sum_{n=0}^{N-1} \cos^{2}\left(\frac{2\pi n}{N}\right) = \sum_{n=0}^{N-1} \sin^{2}\left(\frac{2\pi n}{N}\right) = \frac{N}{2} \]

and
\[ \sum_{n=0}^{N-1} \cos\left(\frac{2\pi n}{N}\right)\sin\left(\frac{2\pi n}{N}\right) = 0. \]

- Similarly, we obtain for the case \( G \) antisymmetric:
\[ u^{a}_{FT} = i\frac{N}{2} u_{a} \]

Therefore,
\[ u_{c} = \begin{cases} \frac{2}{N} u^{s}_{FT} & \text{if } G \text{ is symmetric,} \\ \frac{2i}{N} u^{a}_{FT} & \text{if } G \text{ is antisymmetric.} \end{cases} \quad (1.15) \]

Now that we have obtained \( u_{c} \) is we can recover completely the model (1.12) in order to characterize the viscoelastic properties of a tissue [HSB17], and also we can use algorithms for recovering the shear modulus from a time independent elasticity equation, see for example [AWZ15].
1.4 Hybrid Inverse Problems in Elasticity

1.4.1 Hybrid Inverse Problems

Hybrid inverse problems are inverse problems that describe coupled-physics phenomena in order to reconstruct a parameter of interest. The main idea of hybrid inverse problems is given in two steps:

- First, a high-resolution inverse problem is solved to provide internal information involving solutions and parameters of a differential equation.
- Next, the obtained internal information, also called internal functional, is used to reconstruct with high-contrast the parameter of interest of the inverse problem.

In some settings, a single modality gives a reconstruction with either high contrast or high resolution, but not both at the same time. Under convenient conditions, the physical coupling defining a hybrid inverse problem result in a reconstruction with a high contrast and resolution.

Hybrid inverse problems are useful in, for example, medical and geophysical imaging. Several examples of physical couplings defining hybrid inverse problems are described in [Bal13], for example: the photo-acoustic effect, the ultrasound modulation effect, transient elastography, current density imaging.

The mathematical analysis of many hybrid inverse problems falls within the context of linear and nonlinear equations or systems of equations. In particular, in this thesis, the study of the second step is given by the theory of redundant systems of elliptic partial differential equations, coupling system

\[
\begin{cases}
  \mathcal{L} v = f & \text{in } \Omega, \\
  \mathcal{B} v = g & \text{on } \partial \Omega,
\end{cases}
\]

with the functionals related with the measurements

\[\mathcal{M} v = H \text{ in } \Omega,\]

where \( v = (\gamma, u) \), \( \gamma \) is the set of parameters to reconstruct, \( u \) is the solution of the direct problem, \( \mathcal{L}, \mathcal{B}, \mathcal{M} \) are operators acting on \( v \), \( \Omega \) is an open bounded subset of \( \mathbb{R}^d \) with smooth boundary.

A main tool in our study of hybrid inverse problems is the symbol of a differential operator, since it will provide important properties of equations. We use the definition of symbol given in [Esk11] and we present it here.

**Definition 1.1.** If \( A(x, D) \) (with \( D = -i \frac{\partial}{\partial x} \)) is a differential operator having the form

\[ A(x, D) u = \sum_{|k|=0}^{m} a_k(x) \left( -i \frac{\partial}{\partial x} \right)^k u, \]

we use the definition of symbol given in [Esk11] and we present it here.
then the symbol of \( A(x, D) \) is the polynomial \( A(x, \xi) \) defined by

\[
A(x, \xi) = \sum_{|k|=0}^{m} a_k(x)\xi^k
\]

and its principal symbol is the polynomial \( A_0(x, \xi) \) defined by

\[
A_0(x, \xi) = \sum_{|k|=m} a_k(x)\xi^k.
\]

Note that the definition is related with the following well-known Fourier transform property for partial derivatives

\[
\mathcal{F}[D^k u](\xi) = \xi^k \mathcal{F}[u](\xi)
\]

where we can see that the symbol of the differential operator \( D^k \) is \( \xi^k \).

### 1.4.2 Linear Elasticity Equations

We consider elastic deformations that are given by [Amm+15]:

\[
\rho U_{tt} - \nabla \cdot S[U] = F
\]

(1.16)

where \( U(x, t) \) is the displacement field, \( F(x, t) \) is the excitation force density, and \( S \) is the Cauchy stress tensor which in case of linear elasticity is given by

\[
S[U] = 2\mu \epsilon[U] + \lambda (\text{tr}(\epsilon[U]))I_d
\]

where

\[
\epsilon[U] = \frac{1}{2}(\nabla U + \nabla U^\top) = \nabla^S U
\]

is the infinitesimal strain tensor.

As in section 1.3.3, if the force is periodic in time with frequency \( \omega \), the resulting displacements will be periodic in time as well. After applying a Fourier transform in time in equation (1.16), we obtain:

\[
\omega^2 u + 2\nabla \cdot \mu \nabla^S u + \nabla(\lambda \nabla \cdot u) = 0
\]

(1.17)

where \( u = u(x, \omega) \) is the Fourier transform of \( U(x, t) \) in the variable \( t \), evaluated in the frequency \( \omega \).
1.5 Thesis Overview

The main contributions of this thesis are new methods for the reconstruction of the following parameters of interest: the velocity of blood in vessels via magnetic resonance imaging (MRI), the displacements of tissues under a harmonic regime via MRI, and the shear modulus from different models of hybrid imaging.

**Inverse problems in MRI.**

Imaging velocity fields by phase-contrast MRI has limitations, some of them are:

- Since the phase is a number in the interval \([-\pi, \pi]\), a wrong choice of the VENC parameter causes aliasing, that is, there are pixels corresponding to jumps with respect to the true velocity. This is because the PC-MRI velocity satisfies \(|u_{pc}| \leq |\text{VENC}|\), so the true velocity has to be less than \(|\text{VENC}|\) to avoid aliasing.

- The usual way to reduce the above limitation is to increase the VENC, but this adds noise to the image, since VENCxVNR (velocity-to-noise ratio).

In Chapter 2 a method (Optimal Dual-VENC, ODV) for tackling these limitations is proposed, in order to augment the range of velocities measured while keeping a low noise level. The approach uses dual-VENC measurements combined with appropriate cost function optimization having the form:

\[
J(u) = \frac{1}{2} \sum_{j=1}^{2} |e^{i\hat{\vartheta}^{G_{j}}_{} - e^{i\hat{\vartheta}^{G_{j}}_{\text{(u)}}}}|^2
\]

where for \(j = 1, 2\), \(\hat{\vartheta}^{G_{j}}_{\text{(u)}} = \hat{\vartheta}^{G_{j}} - \hat{\vartheta}^{G_{0}}_{\text{}}\) is the “measured” velocity dependent phase and \(\hat{\vartheta}^{G_{j}}_{\text{}}\) its model depending on the velocity \(u\). The VENCs involved are defined from the pairs \((G_{0}, G_{1})\) and \((G_{0}, G_{2})\), respectively. Experiments are performed with a phantom and with volunteers in order to verify the theoretical results.

**Main Result:** The global minimum of \(J\) in a “reasonable domain” (explained in Section 2.2.6) gives a new recovered velocity \(u_{ODV}\) which is not aliased even if the VENC’s involved are less than the true velocity, keeping a low noise.

On the other hand, the limitations described above are also present in imaging harmonic displacements. The adaptation of the method presented in Chapter 2 is developed to deal with harmonic displacements by MRI, and issues related with MRE are discussed. Experiments with a phantom are performed in order to confirm the theoretical results. This is presented in Chapter 3.
Inverse problems in elasticity.

Most of the inverse problems in elasticity have been mathematically studied in the frame of linear elasticity with linear measurements. In Chapter 4 the inverse problem of the recovery of the shear modulus $\mu$ is studied from the following models where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, and $u_j$ is the displacement field corresponding to the $j$-th measurement, where $j = 1, \ldots, J$ and $J$ is the number of measurements:

- Linear elasticity in harmonic regime with frequency $\omega$, with elastic energy density measurements: that is, the model reads

$$\begin{cases}
\omega^2 u_j + 2 \nabla \cdot \mu \nabla S u_j = -\nabla p_j & \text{in } \Omega, \\
\nabla \cdot u_j = 0 & \text{in } \Omega, \\
u_j = g_j & \text{on } \partial \Omega,
\end{cases}$$

and is coupled to measurements given by:

$$\frac{\mu}{2} |\nabla S u_j|^2 = H_j \quad \text{in } \Omega.$$

This problem is studied in dimension $d = 2$ assuming that the pressures $p_j$ are known. In addition, a forcing term is added in the model equation, which is described by a differential operator of order at most 1.

- Saint-Venant model with internal measurements of the displacements, that is, the model becomes:

$$\begin{cases}
(L_{\mu,\lambda} + N_{\mu,\lambda}) u_j + \omega^2 u_j = 0 & \text{in } \Omega, \\
u_j = g & \text{on } \partial \Omega,
\end{cases}$$

where

$$\begin{align*}
L_{\mu,\lambda} u_j &= 2 \nabla \cdot \mu \nabla S u_j + \nabla(\lambda \nabla \cdot u_j), \\
N_{\mu,\lambda} u_j &= 2c_\tau \nabla \cdot (\mu \nabla u_j^T \nabla u_j) + \nabla(\lambda |\nabla u_j|^2)
\end{align*}$$

and $c_\tau$ is a constant in $x$. The measurements are given by:

$$u_j = H_j \quad \text{in } \Omega.$$

This problem is studied in dimension $d = 2, 3$. In addition, we study the case of linear elasticity with the same model of measurements, which is useful for the analysis of Saint-Venant model.

In each case, the ellipticity of the respective linearized system is studied, applying the curl operator in the models coupled to internal displacements measurements. Then, the resulting systems augmented with the respective measurements can be written by:

$$\begin{cases}
\mathcal{L} v = f & \text{in } \Omega, \\
\mathcal{B} v = g & \text{on } \partial \Omega,
\end{cases}$$
where $\mathcal{L}, \mathcal{B}$ are differential operators and $v = (\delta \mu, \{\delta u_j\}_{j=1}^J)$. Necessary definitions for the development of the chapter are given in Sections 4.2 and 4.3.

**Main Results:** The following results are presented for each model:

- **Ellipticity** for the operator $\mathcal{L}(x, D)$, with $J = 2$ sets of measurements, that is, the principal symbol $\mathcal{P}(x, \xi)$ is a full rank matrix $\forall x \in \Omega$ and $\forall \xi \in \mathbb{R}^d \setminus \{0\}$. The ellipticity holds for certain conditions over the infinitesimal strain tensors $\nabla_S u_j$, where $j = 1, 2$. This is presented for the different models in Theorem 4.3, Corollary 4.1, Section 4.6.1 and Section 4.7.1.

- **Lopatinskii condition** for the operator $\mathcal{A} = (\mathcal{L}, \mathcal{B})$, and then the existence of a parametrix, which gives a useful estimate in appropriate Sobolev spaces, showing injectivity of $\mathcal{A}$ except for a finite dimensional kernel. This is presented for the different models in Lemma 4.5, Theorem 4.4, Corollary 4.1, Section 4.6.2, and Section 4.7.2.

- **A stability estimate** which allows to show that the kernel mentioned above is trivial. As a consequence, it is possible to define the inverse of $\mathcal{A}$. In the case of elastic energy density, it is necessary a lower bound for the frequency $\omega$. This is presented for the different models in Theorem 4.5, Lemma 4.2, Section 4.6.3 and Section 4.7.2.

- **The existence of $\mathcal{A}^{-1}$** allows to define an iterative algorithm for the recovery of $\mu$, which is presented and its convergence is shown. See Sections 4.4.5, 4.5.3 and 4.7.3.