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# COMMONLY KNOWING WHETHER

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A PREPRINT

Jie Fan\*, Davide Grossi, Barteld Kooi, Xingchi Su\*, Rineke Verbrugge

14th January 2020

## ABSTRACT

This paper introduces ‘commonly knowing whether’, a non-standard version of classical common knowledge which is defined on the basis of ‘knowing whether’, instead of classical ‘knowing that’. After giving five possible definitions of this concept, we explore the logical relations among them both in the multi-agent case and in the single-agent case. We focus on one definition and treat it as a modal operator. It is found that the expressivity of this operator is incomparable with the classical common knowledge operator. Moreover, some special properties of it over binary-tree models and  $\mathcal{KD45}$ -models are investigated.

## 1 Introduction

Common knowledge, as the strongest concept among group epistemic notions, has been studied extensively in artificial intelligence [13], epistemic logic [12], epistemology [10] and philosophy of language [2]. Over Kripke semantics, common knowledge among group  $G$  that  $p$ , formulated as  $C_G p$ , is interpreted as: on every node that can be reached from the current node via the reflexive-transitive closure of relation  $R_G^2$ ,  $p$  is satisfied. As the definition suggests, common knowledge is defined on the basis of classical knowledge, ‘knowing that’, to represent propositional knowledge in a group.

There also exists a large amount of ‘wh-’style knowledge<sup>3</sup> in natural language. To describe them formally, several works on epistemic logic have been undertaken, such as the logic of ‘knowing whether’ [5], the logic of ‘knowing how’ [15] [6], and the logic of ‘knowing why’ [17]. Especially the notion of ‘knowing whether’, which means that an agent knows that  $p$  is true or knows that  $p$  is false, has been studied from distinct perspectives [7] [11] [8].

Considering that classical common knowledge is based on ‘knowing that’, the natural question arises of what the common knowledge based on ‘knowing whether’ is. There is no agreement on the definition of ‘commonly knowing whether’ yet and some possible definitions can currently only be expressed with an infinite language. In this paper, we suggest one of the definitions is the most plausible and deserves specific study.

In this paper, we give alternative definitions of ‘commonly knowing whether’ based on different intuitions and prove that they are not equivalent, following with a further study on one of them on its logical properties. In detail, Section 2 gives five different definitions of ‘commonly knowing whether’. Section 3 studies how these definitions are logically related over different frames. Section 4 discusses the expressivity of the language  $KwCw^5L$ . Some properties of  $Cw^5$  over two special classes of frames are investigated in Section 5. Due to the limitation on pages, we only show full proofs for these vital conclusions and proof sketches for some important results. As for other facts and lemmas, proof details are omitted.

## 2 Definitions of ‘Commonly Knowing Whether’ ( $Cw$ )

One way of approaching standard common knowledge is an infinite conjunction of all finite iterations of ‘everyone knowing’. Before defining ‘commonly knowing whether’, two different definitions of ‘everyone knowing whether’

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\*Jie Fan and Xingchi Su are main authors of this paper.

<sup>2</sup> $R_G$  is the union of accessible relations of the agents in  $G$ .

<sup>3</sup>‘Knowing How’ is also considered as a ‘wh-’style knowledge since it also describes a kind of non-propositional knowledge.

will be introduced. The following definitions of ‘everyone knowing whether’ and ‘commonly knowing whether’ are due to Yanjing Wang [16]. The group  $G$  concerned in this paper is finite. Since only one group is considered, we omit  $G$  in notations for group knowledge in the remainder of the paper. Infinite conjunctions are temporarily used to define some of the following notions since they have not been expressed in any finite language yet. And  $\mathcal{K}$ ,  $\mathcal{T}$ ,  $\mathcal{S5}$  refer to the classes of all Kripke frames, reflexive frames and equivalence relation frames, respectively.  $\mathcal{KD45}$  refers to the class of serial, transitive and Euclidean frames.  $G^{+4}$  refers to the set of finite sequences only consisting of agents from  $G$ , excluding the empty sequence. Let  $P$  be a fixed set of propositional variables. An infinite language is introduced to define ‘commonly knowing whether’.

**Definition 1** Given  $P$  and  $G$ , the infinite language  $KwEL^\infty$  is given as follows:

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid E\varphi \mid C\varphi \mid \bigwedge \Phi$$

where  $i \in G$ ,  $p \in P$ , and  $\Phi$  is a countably infinite set of formulas.

**Definition 2**  $Ew^1\varphi := E\varphi \vee E\neg\varphi$

Intuitively, this definition imitates the definition of ‘knowing whether’, describing ‘everyone knows whether  $\varphi$ ’ as ‘everyone knows  $\varphi$  or everyone knows not  $\varphi$ ’.

**Definition 3**  $Ew^2\varphi := \bigwedge_{i \in G} Kw_i\varphi$ , where the semantics of  $Kw_i\varphi$  is defined over Kripke pointed models  $(M, s)$  as:  $M, s \models Kw_i\varphi$  iff for any  $t_1$  and  $t_2$  with  $s \rightarrow_i t_1$  and  $s \rightarrow_i t_2$ ,  $M, t_1 \models \varphi \Leftrightarrow M, t_2 \models \varphi$ .

This definition is inspired by classical ‘everyone knows’, defining ‘everyone knows whether’ as ‘everyone in the group knows whether the proposition’.

On the basis of these definitions of ‘everyone knows whether’, five definitions of ‘commonly knows whether’ are given based on distinct interpretations.

**Definition 4**  $Cw^1\varphi := C\varphi \vee C\neg\varphi$

A group commonly knows whether  $\varphi$  is possibly interpreted as they have common knowledge that  $\varphi$  or they have common knowledge that  $\neg\varphi$ .

**Definition 5**  $Cw^2\varphi := CEw\varphi$

A group commonly knows whether  $\varphi$  plausibly means that everyone knows whether  $\varphi$  and it is a common knowledge of the group that everyone knows whether  $\varphi$ . Since there are two different definitions of ‘everyone knows whether’, we should separate  $Cw^2\varphi$  into  $Cw^{2,1}\varphi := CEw^1\varphi$  and  $Cw^{2,2}\varphi := CEw^2\varphi$ .

**Definition 6**  $Cw^3\varphi := \bigwedge_{k \geq 1} (Ew)^k\varphi$

‘Commonly knowing whether  $\varphi$ ’ can also be defined as everyone knows whether  $\varphi$  and everyone knows whether everyone knows whether  $\varphi$ , etc. We should again separate  $Cw^3\varphi$  into  $Cw^{3,1}\varphi := \bigwedge_{k \geq 1} (Ew^1)^k\varphi$  and  $Cw^{3,2}\varphi := \bigwedge_{k \geq 1} (Ew^2)^k\varphi$ . As mentioned in Section 1, this is an infinitary form corresponding to intuition.

**Definition 7**  $Cw^4\varphi := \bigwedge_{i \in G} Cw^1Kw_i\varphi$ , that is  $Cw^4\varphi := \bigwedge_{i \in G} (CKw_i\varphi \vee C\neg Kw_i\varphi)$

A group commonly knows whether  $\varphi$  possibly means for every member in the group, it is common knowledge of the group that she knows whether  $\varphi$  or it is common knowledge of the group that she does not know whether  $\varphi$ .

**Definition 8**  $Cw^5\varphi := \bigwedge_{s \in G^+} Kw_s\varphi$ . For instance: if  $s = \langle i, j, l \rangle$ , then  $Kw_s\varphi = Kw_iKw_jKw_l\varphi$ .

Listing all the possible inter-‘knowing whether’ states among every subset of the group is an alternative way to define ‘commonly knowing whether’.

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<sup>4</sup>For any set  $T$ ,  $T^+$  contains all finite sequences only consisting of elements in  $T$ .

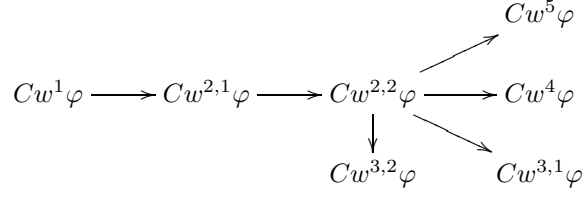


Figure 1: Logical relationships over  $\mathcal{K}$

### 3 Implication Relations among the definitions of $Cw$

The above five definitions of ‘commonly knowing whether’ do not boil down to the same thing, especially over  $\mathcal{K}$  and  $\mathcal{KD}45$ . In this section, we show how these definitions are related. The semantics presented in this paper uses Kripke models. A Kripke model is generally defined as  $M = \langle W, R_i, R_j, \dots, R_m, V \rangle$ , where  $W$  is a non-empty set of nodes,  $R_i, R_j, \dots, R_m$  are accessibility relations for the agents in group  $G$  and  $V$  is the valuation which is a function  $V : P \rightarrow \mathcal{P}(W)$ . If  $(s, t) \in R_i$ , we write  $s \rightarrow_i t$ ; if  $(s, t) \in R_i \cup R_j \cup \dots \cup R_m$ , we write  $s \rightarrow t$ ;  $\rightarrow$  is the reflexive-transitive closure of  $\rightarrow$ .

#### 3.1 Over $\mathcal{K}$

In order to clarify the relations among definitions of ‘commonly knowing whether’, we first give the logical relations between  $Ew^1$  and  $Ew^2$ .

**Fact 1** *The following statements hold:*

$$\mathcal{K} \models Ew^1\varphi \rightarrow Ew^2\varphi;$$

$$\mathcal{K} \not\models Ew^2\varphi \rightarrow Ew^1\varphi;$$

$$\mathcal{K} \models Ew^1\varphi \leftrightarrow Ew^1\neg\varphi;$$

$$\mathcal{K} \models Ew^2\varphi \leftrightarrow Ew^2\neg\varphi.$$

**Theorem 1** *Over  $\mathcal{K}$ , the logical relationships among  $Cw^1$ ,  $Cw^2$ ,  $Cw^3$ ,  $Cw^4$  and  $Cw^5$  are shown in Figure 1.<sup>5</sup>*

Every pair of these definitions are not equivalent over  $\mathcal{K}$ . Conversely, the ‘knowing that’-versions of  $Cw^2$ ,  $Cw^3$  and  $Cw^5$  exactly correspond to three approaches to common knowledge [1] which are logically equivalent.

We will show the proof sketches on an interesting case where  $Cw^2\varphi$  implies  $Cw^5$  but the converse does not hold.

$$\mathcal{K} \models Cw^2\varphi \rightarrow Cw^5\varphi \quad \text{Proof Sketch: prove with induction.}$$

Consider  $Cw^2\varphi$  in two cases:

The first case is  $Cw^2\varphi := Cw^{2,2}\varphi$ . Let  $(M, r)$  be an arbitrary pointed model, such that  $M = \langle W, R_i, R_j, \dots, R_m, V \rangle$  and  $M, r \models Cw^2\varphi$ . Then, for any node  $t$  with  $r \rightarrow t$ , we have  $M, t \models \bigwedge_{i \in G} Kw_i\varphi$ . For arbitrary  $s \in G^+$ , let  $s = \langle i_1, i_2, \dots, i_k \rangle$ , where  $\{i_n \mid 1 \leq n \leq k\} \subseteq G$ . We show a stronger result:  $M, r \models Kw_s\varphi$  and for any  $t$  such that  $r \rightarrow t$ , we have that  $M, t \models Kw_s\varphi$ .

By induction on the length  $n = |s|$  of  $s$ :

When  $n = 1$ ,  $s = \langle i_1 \rangle$ , where  $i_1$  is an arbitrary agent. By  $M, r \models Cw^2\varphi$ , we have  $M, r \models Kw_{i_1}\varphi$ . And since for any  $t$  with  $r \rightarrow t$ , there is  $M, t \models Ew^2\varphi$ . Thus there is  $M, t \models Kw_{i_1}\varphi$ .

Induction hypothesis: when  $n = k$ ,  $s = \langle i_1, i_2, \dots, i_k \rangle$ , there is  $M, r \models Kw_s\varphi$  and for any  $t$  with  $r \rightarrow t$ , there is  $M, t \models Kw_s\varphi$ .

When  $n = k + 1$ ,  $s = \langle i_1, i_2, \dots, i_k, i_{k+1} \rangle$ . By induction hypothesis, for any  $t$  such that  $r \rightarrow t$ , for any  $|s_k| = k$ , there is  $M, t \models Kw_{s_k}\varphi$ . Thus, for any  $i \in G$ , there is  $M, r \models Kw_i Kw_{s_k}\varphi$ , saying, for any  $|s_{k+1}| = k + 1$ , we have

<sup>5</sup>In the following figures, the directed arrows between two definitions refer to the strict implications. The logical relations are surely transitive.

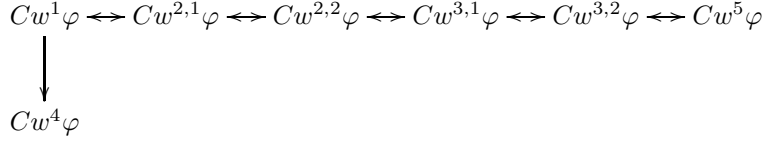


Figure 2: Logical relationships over  $\mathcal{T}$  and  $S5$  (multi-agent).

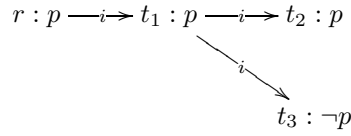
$M, r \models Kw_{s_{k+1}}\varphi$ . Now considering any  $u$  with  $r \twoheadrightarrow u$ , let  $u \rightarrow v$ . By the definition of  $\twoheadrightarrow$ ,  $r \twoheadrightarrow v$ . By induction hypothesis, we get  $M, v \models Kw_{s_k}\varphi$ . And since  $v$  is an arbitrary node such that  $u \rightarrow v$ , we have for any  $i \in G$ ,  $M, u \models Kw_i Kw_{s_k}\varphi$ . That means, for any  $|s_{k+1}| = k + 1$ , there is  $M, u \models Kw_{s_{k+1}}\varphi$ .

Therefore, we proved that, for arbitrary  $s \in G^+$  and arbitrary  $t$  with  $r \twoheadrightarrow t$ , there are  $M, r \models Kw_s\varphi$  and  $M, t \models Kw_s\varphi$ , which means  $M, r \models \bigwedge_{s \in G^+} Kw_s\varphi$ . Thus, we have  $M, r \models Cw^5\varphi$ . Above all, we obtain  $\mathcal{K} \models Cw^{2,2}\varphi \rightarrow Cw^5\varphi$ .

For the second case where  $Cw^2\varphi := Cw^{2,1}\varphi$ , since  $\mathcal{K} \models Cw^{2,1}\varphi \rightarrow Cw^{2,2}\varphi$ , we have  $\mathcal{K} \models Cw^{2,1}\varphi \rightarrow Cw^5\varphi$ . Therefore, we proved  $\mathcal{K} \models Cw^2\varphi \rightarrow Cw^5\varphi$ .

$\mathcal{K} \not\models Cw^5\varphi \rightarrow Cw^2\varphi$  Proof Sketch: find a counter-model.

Consider the following model  $M$ :



In this model, there is only one successor of  $r$ . By semantics, we have  $M, r \models Cw^5p$ . And since  $M, t_1 \not\models Ew^1p$  and  $M, t_1 \not\models Ew^2p$ , it follows that  $M, r \not\models Cw^2p$ . Thus,  $\mathcal{K} \not\models Cw^5\varphi \rightarrow Cw^2\varphi$ .

### 3.2 Over $\mathcal{KD45}$

$\mathcal{KD45}$  is considered to be the usual class of frames for doxastic (belief) logic [9, 14]. The implication relations over  $\mathcal{KD45}$  among these five definitions are the same as those over  $\mathcal{K}$ .

**Theorem 2** Over  $\mathcal{KD45}$ , the implication relations are shown as Figure 1.

### 3.3 Over $\mathcal{T}$ and $S5$

The model of knowledge is generally defined over  $S5$  [3, 14], which commits knowledge must be true via axiom  $T$ . The implication relations among five definitions over  $\mathcal{T}$  or  $S5$  are given in Theorem 3.

**Theorem 3** Over  $\mathcal{T}$  and  $S5$ , the implication relations are shown in Figure 2.

$Cw^1\varphi$ ,  $Cw^2\varphi$ ,  $Cw^3\varphi$  and  $Cw^5\varphi$  boil down to the same thing once the frame is reflexive, which should be attributed to agents' agreements on the values of  $\varphi$ . For example, if there is  $M, s \models Kw_i\varphi \wedge Kw_j\varphi$  where  $M$  is reflexive, the values of  $\varphi$  on all  $i$ -successors must agree with those on all  $j$ -successors since they share a common successor  $s$ . Comparatively, if  $M$  is not reflexive, the case of  $M, s \models K_i\varphi \wedge K_j\neg\varphi$  is possible.

### 3.4 Single-agent Case

The results above concern the multi-agent case. Considering the single-agent case, observe that  $\mathcal{K} \models Ew^1\varphi \leftrightarrow Ew^2\varphi$  which implies that  $Cw^{21}$  is equivalent to  $Cw^{22}$  and that  $Cw^{31}$  is equivalent to  $Cw^{32}$ . Moreover, since only one agent is involved, it is trivial to find that  $Cw^3$  is equivalent to  $Cw^5$ .

In terms of logical relationships, the five definitions share the same relations with the multi-agent case over  $\mathcal{K}$ ,  $\mathcal{KD45}$ ,  $\mathcal{T}$  and  $S5$ , respectively.

The following sections concentrate on  $Cw^5$  in the single-agent case.

## 4 The Case of $Cw^5$

We focus on  $Cw^5$  since the idea of it is inspired by the hierarchy of inter-knowledge of a group given in [13]. The language of ‘commonly knowing whether’ is defined in the same way as  $KwL^6$ , except introducing a new operator  $Cw^5$ . We will investigate the expressivity between  $Cw^5$  and  $C$  in Section 4.3, so the language of classical common knowledge is also given here.

**Definition 9** Given  $P$  and  $G$ , the language  $KwCw^5L$  is given as follows:

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw^5\varphi$$

where  $p \in P, i \in G$ .

**Definition 10** Given  $P$  and  $G$ , the language  $KCL$  is given as follows:

$$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid C\varphi$$

where  $p \in P, i \in G$ .

### 4.1 Some Valid or Invalid $KwCw^5L$ -formulas

In the single-agent case, we consider some validities and invalidities over  $\mathcal{K}$ :

**Fact 2**  $\mathcal{K} \models Cw^5\varphi \rightarrow Kw_iCw^5\varphi$

This valid formula illustrates that when  $Cw^5\varphi$  is true at the current node, all successors of it agree the values of  $Cw^5\varphi$ . However, the converse formula is invalid, saying  $\mathcal{K} \not\models Kw_iCw^5\varphi \rightarrow Cw^5\varphi$ .

**Fact 3**  $\mathcal{K} \models \neg Kw_i\psi \rightarrow (Kw_iCw^5\varphi \wedge Kw_i(\psi \rightarrow Cw^5\varphi) \wedge Kw_i\varphi \rightarrow Cw^5\varphi)$

This valid formula is established in the light of Almost Definability (AD) from [4], which is  $\neg Kw_i\psi \rightarrow (K_i\varphi \leftrightarrow Kw_i\varphi \wedge Kw_i(\psi \rightarrow \varphi))$ . AD shows that under the precondition  $\neg Kw_i\psi$  for some  $\psi$ , the classical knowledge operator  $K_i$  can be defined in terms of  $Kw_i$ . Since  $\mathcal{K} \models K_iCw^5\varphi \wedge Kw_i\varphi \rightarrow Cw^5\varphi$ , we can replace  $K_iCw^5\varphi$  with AD.

**Fact 4**  $\mathcal{K} \not\models Cw^5(\chi \rightarrow \varphi) \wedge Cw^5(\neg\chi \rightarrow \varphi) \rightarrow Cw^5\varphi$

Although  $Kw_i(\chi \rightarrow \varphi) \wedge Kw_i(\neg\chi \rightarrow \varphi) \rightarrow Kw_i\varphi$  is the basic axiom of  $KwL$  according to [5], its  $Cw^5$ -version is not valid.

**Fact 5**  $\mathcal{K} \not\models Cw^5\varphi \rightarrow Cw^5(\varphi \rightarrow \psi) \vee Cw^5(\neg\varphi \rightarrow \chi)$

Similarly, for another basic axiom  $Kw_i\varphi \rightarrow Kw_i(\varphi \rightarrow \psi) \vee Kw_i(\neg\varphi \rightarrow \chi)$  in  $KwL$ , the  $Cw^5$ -version is invalid.

**Fact 6**  $\mathcal{K} \not\models Cw^5\varphi \wedge Cw^5\psi \rightarrow Cw^5(\varphi \wedge \psi)$

This formula is invalid, which leads to the failure in defining  $Cw^5$  with accessibility relations.

### 4.2 $KwCw^5L$ is Bisimulation Invariant

**Definition 11** Let  $M = \langle W, R, V \rangle$  and  $M' = \langle W', R', V' \rangle$  be two Kripke models. A non-empty binary relation  $Z \subseteq W \times W'$  is called bisimulation between  $M$  and  $M'$ , written as  $M \cong M'$ , if the following conditions are satisfied:

- (i) If  $wZ'w'$ , then  $w$  and  $w'$  satisfy the same proposition letters.
- (ii) if  $wZ'w'$  and  $wRv$ , then there is a  $v' \in W'$  such that  $vZv'$  and  $w'R'v'$ .

<sup>6</sup>The language for the logic of ‘knowing whether’ defined in [5].

(iii) If  $wZ'w'$  and  $w'R'v'$ , then there exists  $v \in W$  such that  $vZv'$  and  $wRv$ .

When  $Z$  is a bisimulation linking two states  $w$  in  $M$  and  $w'$  in  $M'$ , we say that two pointed models are bisimilar and write  $Z : (M, w) \cong (M', w')$ . If a language  $L$  cannot distinguish any pair of bisimilar models,  $L$  is bisimulation invariant.

**Theorem 4**  $KwCw^5L$  is bisimulation invariant.

**Proof 1** By induction on formulas  $\varphi$  of  $KwCw^5L$ .

When  $\varphi$  is a Boolean formula, the proof is classical.

When  $\varphi = Kw_i\psi$ , we prove it in three cases. For arbitrary two bisimilar models  $(M, r)$  and  $(N, t)$ , we have:

- if  $M, r \models Kw_i\psi$  and for all  $r_n$  with  $r \rightarrow_M r_n$ ,  $M, r_n \models \psi$ . Since  $M, r \cong N, t$ , for any  $t_n$  with  $t \rightarrow_N t_n$ , there is an  $r_n$  such that  $r \rightarrow_M r_n$  and  $M, r_n \cong N, t_n$ . By induction hypothesis,  $\psi$  is bisimulation invariant. Thus  $N, t_n \models \psi$ . So  $N, t \models K_i\psi$ .
- if  $M, r \models Kw_i\psi$  and for all  $r_n$  with  $r \rightarrow_M r_n$ ,  $M, r_n \models \neg\psi$ , similar to above case.
- if  $M, r \models \neg Kw_i\psi$ , that means there are  $r_1$  with  $r \rightarrow_M r_1$  and  $r_2$  with  $r \rightarrow_M r_2$ , such that  $M, r_1 \models \psi$  and  $M, r_2 \models \neg\psi$ . Since  $M, r \cong N, t$ , there are  $t_1$  with  $t \rightarrow_N t_1$  and  $t_2$  with  $t \rightarrow_N t_2$ , such that  $M, r_1 \cong N, t_1$  and  $M, r_2 \cong N, t_2$ . By induction hypothesis,  $\psi$  is bisimulation invariant. Thus  $N, t_1 \models \psi$  and  $N, t_2 \models \neg\psi$ . Thus  $N, t \models \neg Kw_i\psi$ .

Thus,  $Kw_i\psi$  is bisimulation invariant.

When  $\varphi = Cw^5\psi$ , assume two bisimilar models  $(M, r)$  and  $(N, t)$ , such that  $M, r \models Cw^5\psi$  and  $N, t \models \neg Cw^5\psi$ . That means there exists a sequence of agents  $s$ , such that  $M, r \models Kw_s\psi$  and  $N, t \models \neg Kw_s\psi$ . Let  $s = \langle i_1, i_2, \dots, i_n \rangle$ . So  $M, r \models Kw_{i_1}\gamma$  and  $N, t \models \neg Kw_{i_1}\gamma$ , where  $\gamma = Kw_{\langle i_2, i_3, \dots, i_n \rangle}\psi$ . However, we have proved that for any formula of the form  $Kw_{i_1}\gamma$ , they are bisimulation invariant. Thus, if  $M, r \models Kw_{i_1}\gamma$ , there must be  $N, t \models Kw_{i_1}\gamma$ . Contradiction.

Therefore, we proved that  $KwCw^5L$  is bisimulation invariant.

### 4.3 $KwCw^5L$ and $KCL$

Although  $Cw^5$  is formed merely with  $Kw_i$ , which can be defined by classical operator  $K_i$ , surprisingly,  $Cw^5$  cannot be defined with  $K_i$  and  $C$ .

**Theorem 5** Over  $\mathcal{K}$ ,  $KwCw^5L$  is not weaker than  $KCL$  in expressivity.

We prove Theorem 5 by defining two classes of models, which no  $KCL$ -formula can distinguish while there exists one  $KwCw^5L$ -formula that can distinguish them.

**Definition 12** For every  $n \geq 1$ , define two sets of possible worlds  $T_n$  and  $Z_n$  with induction:

- $t_{00} \in T_n; z_0 \in Z_n$
- If  $t_i \in T_n$ , then  $t_{i0} \in T_n$  and  $t_{i1} \in T_n$ ; if  $z_i \in Z_n$ , then  $z_{i0} \in Z_n$  and  $z_{i1} \in Z_n$
- For every  $t_j \in T_n$ ,  $|j| \leq n + 2$ ; for every  $z_j \in Z_n$ ,  $|j| \leq n + 1$
- Besides  $t_i$  defined above,  $T_n$  has no more possible worlds; besides  $z_i$  defined above,  $Z_n$  has no more possible worlds.

Then define the class of models  $M = \{M_n = \langle W_n, R_n, V_n \rangle \mid n \in \mathbb{N}\}$  as follows: for every  $n \geq 1$ ,

- $W_n = \{r\} \cup T_n \cup \{t_0\}$
- $R_n = \{\langle t_i, t_{i0} \rangle, \langle t_i, t_{i1} \rangle \mid t_i \in T_n, t_{i0} \in T_n\} \cup \{\langle r, t_0 \rangle, \langle t_0, t_{00} \rangle\}$
- $V_n(p) = W_n - \{t_{0i}\}$ , where  $|i| = n + 1$ ,  $i \in \{0\}^+$ .

Define the class of models  $N = \{N_n = \langle W'_n, R'_n, V'_n \rangle \mid n \in \mathbb{N}\}$  with  $M$ : for every  $n \geq 1$ :

- $W'_n = W_n \cup Z_n$
- $R'_n = R_n \cup \{\langle z_i, z_{i0} \rangle, \langle z_i, z_{i1} \rangle \mid z_i \in Z_n, z_{i0} \in Z_n\}$
- $V'_n(p) = V_n(p) \cup Z_n - \{z_{0i}\}$ , where  $|i| = n$ ,  $i \in \{0\}^+$

The first model  $M_1$  and  $N_1$  in  $M$  and  $N$  are shown as Figure 3.

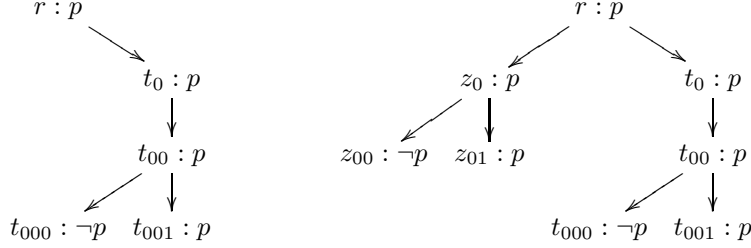


Figure 3:  $M_1$ (left) and  $N_1$ (right)

The model  $N_n$  is constructed by adding a new subtree rooted with  $z_0$  to the root  $r$  and just make  $p$  unsatisfied on the leaf node whose index only consists of 0.

We will prove that no  $KCL$ -formula can distinguish  $M$  and  $N$  with the  $CL$ -game<sup>7</sup>. If there is a winning strategy for duplicator in  $n$ -round games,  $(M_n, r)$  and  $(N_n, r)$  agree on all  $KCL$ -formulas whose modal depth is  $n$ .

**Theorem 6** For arbitrary  $n \in \mathbb{N}$ , duplicator has a winning strategy in the  $CL$ -game on  $(M_n, r)$  and  $(N_n, r)$  in  $n$  rounds.

**Proof 2** We describe Duplicator's winning strategy case by case. Starting with the initial state  $(r, r)$ , we mainly concerns the case where spoiler does a  $K$ -move. Otherwise, duplicator can move to a isomorphic sub-model such that there must be a winning strategy in following rounds. Thus, the cases below exhaust all possibilities.

- The initial state is  $(r, r)$ :
  - If spoiler does a  $K$ -move or a  $C$ -move on  $M_n$  reaching  $t_i$ , then duplicator does a  $K$ -move or a  $C$ -move on  $N_n$  to reach the corresponding  $t_i$ . Since  $(M_n, t_i) \cong (N_n, t_i)$ , there is a winning strategy after this move.
  - If spoiler does a  $K$ -move or a  $C$ -move on  $N_n$  reaching  $t_i$ , then duplicator does a  $K$ -move or a  $C$ -move on  $M_n$  to reach the corresponding  $t_i$ . Since  $(M_n, t_i) \cong (N_n, t_i)$ , there is a winning strategy after this move.
  - If spoiler does a  $K$ -move on  $N_n$  reaching  $z_0$ , duplicator moves to  $t_0$ .
  - If spoiler does a  $C$ -move on  $N_n$  reaching an arbitrary node  $z_{i(i \neq 0)}$  in  $Z_n$ , then duplicator does a  $C$ -move to reach  $t_{0i}$ . Since  $(M_n, t_{0i}) \cong (N_n, z_i)$ , there is a winning strategy after this move.
- The current state is  $(z_0, t_0)$ :
  - If spoiler does a  $K$ -move reaching  $z_{00}$  or  $z_{01}$ , then duplicator moves on  $M_n$  to reach  $t_{00}$ .
  - If spoiler does a  $K$ -move reaching  $t_{00}$  on  $M_n$ , then duplicator moves to  $z_{00}$  on  $N_n$ .
  - If spoiler does a  $C$ -move reaching  $z_{i(i \neq 0)}$ , then duplicator moves to  $t_{0i}$ . Since  $(M_n, t_{0i}) \cong (N_n, z_i)$ , there is a winning strategy after this move.
  - If spoiler does a  $C$ -move reaching  $t_{00}$  on  $M_n$ , then duplicator moves to  $z_{00}$  on  $N_n$ .
  - If spoiler does a  $C$ -move reaching  $t_{0i(i \neq 0)}$ <sup>8</sup>, then duplicator moves to  $z_i$  on  $N_n$ . Since  $(M_n, t_{0i}) \cong (N_n, z_i)$ , there is a winning strategy after this move.
- The current state is  $(z_i, t_i)$  and  $i \neq 0$ : this means before the game gets to this state, both players have only done  $K$ -moves. In the current state, there have been at most  $(n - 1)$  rounds. Thus,  $i \leq (n - 1)$  and players can do next round as follows:

<sup>7</sup>The definition of the  $CL$ -game is given in Appendix.

<sup>8</sup>The notation  $t_{0i}$  is correct since the index for every node in  $T_n$  begins with 0.



- If spoiler does a  $K$ -move reaching  $z_{i0}$  or  $z_{i1}$ , then duplicator does a  $K$ -move to reach  $t_{i0}$  where  $M_n, t_{i0} \models p$  since  $|i0| = |i1| = (i + 1) \leq n$  and there are  $N_n, z_{i0} \models p$  and  $N_n, z_{i1} \models p$ .
- If spoiler does a  $K$ -move or a  $C$ -move reaching  $t_{i0}$  or  $t_{i1}$ , then duplicator does a  $K$ -move or a  $C$ -move to reach  $z_{i0}$  where  $M_n, z_{i0} \models p$  since  $|i0| = |i1| = (i + 1) \leq n$  and there are  $N_n, t_{i0} \models p$  and  $N_n, t_{i1} \models p$ .
- If spoiler does a  $C$ -move reaching  $z_{j(|j|>|i|)}$ , then duplicator does a  $C$ -move to reach  $t_{0j}$ . Since  $(M_n, t_{0j}) \cong (N_n, z_j)$ , there is a winning strategy after this move.
- If spoiler does a  $C$ -move reaching  $t_{0i(i \neq 0)}$ , duplicator moves to  $z_i$  on  $N_n$ . Since  $(M_n, t_{0i}) \cong (N_n, z_i)$ , there is a winning strategy after this move.

Therefore, for arbitrary  $n \in \mathbb{N}$ , there is a winning strategy for duplicator in the  $n$ -round  $CL$ -game over  $(M_n, r)$  and  $(N_n, r)$ .

The above proof shows the nonexistence of some  $KCL$ -formula which can distinguish  $M$  and  $N$  since for any  $KCL$ -formula, its modal depth is a natural number  $n$  which results in that it cannot distinguish  $M_n$  and  $N_n$ . Comparatively, there is a  $KwCw^5L$ -formula,  $Kw_iCw^5p$  such that  $M_n, r \models Kw_iCw^5p$  and  $N_n, r \models \neg Kw_iCw^5p$  for every  $n \in \mathbb{N}$ . Moreover, consider the following two pointed models,  $M$  and  $M'$ :

$$r : p \xrightarrow{i} t : p \qquad r' : p \xrightarrow{i} t' : \neg p$$

Figure 4:  $M$ (left) and  $M'$ (right)

We can find a  $KCL$ -formula to distinguish them, where  $M, r \models Kp$  and  $M', r' \models \neg Kp$ . But there is no any  $KwCw^5L$ -formula can distinguish  $M$  and  $M'$ . It follows that  $KCL$  is not weakly expressive than  $KwCw^5L$ .

Therefore, together with Theorem 5, the following theorem is obtained:

**Theorem 7** *Language  $KCL$  and  $KwCw^5L$  are incomparable with respect to expressivity.*

## 5 $KwCw^5L$ over Special Frames

Because of the invalidity of the formula  $(Cw^5\varphi \wedge Cw^5\psi) \rightarrow Cw^5(\varphi \wedge \psi)$ , the operator  $Cw^5$  is not normal, in the sense that it cannot be defined with accessibility relations. However, an interesting observation over binary-tree models can be proved.

**Definition 13**  *$(M, r)$  is a binary-tree model with root  $r$  if  $(M, r)$  is a tree model with root  $r$  and for any node  $t$  in  $M$ ,  $t$  has precisely two successors.*

### 5.1 $KwCw^5L$ over Binary Trees

**Theorem 8** *Consider the single-agent case. If  $M, r \models Cw^5\varphi$  where  $(M, r)$  is a binary tree with root  $r$ , then on every layer of  $(M, r)$ , the number of the nodes where  $\varphi$  is satisfied is even.*

In order to prove Theorem 8, we need to prove a stronger theorem:

**Theorem 9** *For an arbitrary formula  $\varphi$ , if  $M$  is a binary-tree model, then  $M, v_m \models Kw_i^n\varphi$  ( $1 \leq n$ ) iff the number of the  $\varphi$ -satisfied nodes on the  $(|m| + n)$ th layer that  $v_m$  can reach via relation  $\rightarrow$  is even.*

**Proof sketch** By induction on  $n$ :

- When  $n = 1$ , it is trivial.
- (Induction Hypothesis) When  $n = k$ , it holds.
- When  $n = k + 1$ , suppose  $M, v_m \models Kw_i^{k+1}\varphi$ , which is equivalent to  $M, v_m \models Kw_i^k Kw_i\varphi$ . Then we can use induction hypothesis. Suppose  $M, v_m \not\models Kw_i^k\varphi$ , which means on one successor,  $Kw_i^k\varphi$  is satisfied, and on other one successor,  $Kw_i^k\varphi$  is not satisfied. Then we can use induction hypothesis.

Proof details are shown in Appendix.

**Remark 1** *Theorem 8 can be extended into a more general conclusion considering the multi-agent case: on any  $G$ -binary-tree model<sup>9</sup>  $(M, r)$  where  $r$  is the root,  $M, r \models Cw^5\varphi$  iff for any sequence of agents  $s$  in  $G$ , on every layer of the subtree (of  $(M, r)$ ) generated with  $s^{10}$ , the number of the  $\varphi$ -satisfied nodes is even.*

## 5.2 $KwCw^5L$ over $\mathcal{KD45}$

By conclusions drawn in Section 3.2,  $Cw^5\varphi$  is not equivalent to  $Cw^1\varphi$  over  $\mathcal{KD45}$ , where  $Cw^5$  still preserves its non-triviality.

**Single-agent Case** When the group concerned is a singleton, there is nestings reducing for operator  $Kw_i$  over  $\mathcal{KD45}$ . The proof idea of Theorem 10 is similar to the classical case proved in [12] where the nestings of operator  $K$  can also be reduced over  $S5$ .

**Theorem 10** *Over  $\mathcal{KD45}$ , every  $KwL_{(1)}$ -formula<sup>11</sup> is equivalent to some formula without nestings of the modal operator  $Kw_i$ .*

**Corollary 1** *In the single-agent case,  $\mathcal{KD45} \models Cw^5\varphi \leftrightarrow Kw_i\varphi$*

**Multi-agent Case** However, when more than one agent are involved, we cannot reduce nestings. Fortunately, there are still some interesting findings.

**Theorem 11** *For any sequence  $s = \langle i_1, i_2, \dots, i_n \rangle$  ( $n \geq 2$ ) of agents, if there exists  $m$  such that  $1 \leq m < n$  and  $i_m = i_{m+1}$ , then  $\mathcal{KD45} \models Kw_s\varphi$  for any formula  $\varphi$ .*

The proof of Theorem 11 is given in Appendix.

**Corollary 2** *Over  $\mathcal{KD45}$ ,  $Cw^5\varphi = \bigwedge_{s \in G^+} Kw_s\varphi$  where  $s = \langle i_1, i_2, \dots, i_n \rangle$  is a sequence of agents and there is no  $m$  ( $1 \leq m < n$ ) such that  $i_m = i_{m+1}$ .*

## 6 Conclusions

In this paper, we provide some initial results on definitions of ‘commonly knowing whether’, their relationships and properties, with special focus on  $Cw^5$ . We also show that over  $\mathcal{K}$ , the languages  $KCL$  and  $KwCw^5L$  are incomparable with respect to expressivity.

There is much more expected work to be done in terms of ‘commonly knowing whether’. For instance, giving a Kripke semantics of  $Cw^5$ , axiomatizing  $KwCw^5L$ , proving the completeness of that axiomatization with respect to models of  $KwCw^5L$ , studying other definitions of ‘commonly knowing whether’ ( $Cw^2$  is also a good choice), etc.

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<sup>9</sup>A  $G$ -binary-tree model is a tree model where every node exactly has two  $R_i$ -successors for every  $i \in G$ .

<sup>10</sup>A subtree (of some tree model  $(M, r)$ ) generated with a sequence of agents  $s$  is a subtree rooted with  $r$  which only consists of all  $s$ -paths starting with  $r$  in  $(M, r)$ .

<sup>11</sup> $KwL_{(1)}$  is the language of  $KwL$  with only one agent.

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## A Proof of Theorem 9

**Proof 3** Given a binary tree  $(M, v_0)$ , where  $v_0$  is the root, we firstly define the index of  $M$  as follows: if there are nodes  $v_m, t, r$  in  $M$  and  $v_m \rightarrow_i t, v_m \rightarrow_i r$ , then define the index of  $t$  as  $v_{m0}$  and the index of  $r$  as  $v_{m1}$ .

Let  $v_m$  be an arbitrary node in  $M$ . Do induction on  $n$ :

- When  $n = 1$ ,
  - Assume  $M, v_m \models Kw_i\varphi$ . Since  $M$  is a binary tree, there must be two nodes,  $v_{m0}$  and  $v_{m1}$  such that  $v_m \rightarrow_i v_{m0}$  and  $v_m \rightarrow_i v_{m1}$ . Since  $M, v_m \models Kw_i\varphi$ , we have  $(M, v_{m0} \models \varphi \text{ and } M, v_{m1} \models \varphi)$  or  $(M, v_{m0} \models \neg\varphi \text{ or } M, v_{m1} \models \neg\varphi)$ . Thus, on the  $(|m| + 1)$ th layer, the number of nodes where  $\varphi$  is satisfied is 2 or 0, both of which are even.
  - Assume the number of the nodes on the  $(|m| + 1)$ th layer that  $v_m$  can reach is even. That means there are only two possible cases:  $(M, v_{m0} \models \varphi \text{ and } M, v_{m1} \models \varphi)$  or  $(M, v_{m0} \models \neg\varphi \text{ or } M, v_{m1} \models \neg\varphi)$ . Thus, we have  $M, v_m \models Kw_i\varphi$ .
- Induction hypothesis: when  $n = k$ ,  $M, v_m \models Kw_i^k\varphi (1 \leq n) \Leftrightarrow$  the number of the  $\varphi$ -satisfied nodes on the  $(|m| + k)$ th layer that  $v_m$  can reach via relation  $\rightarrow$  is even.
- When  $n = k + 1$ ,
  - Assume  $M, v_m \models Kw_i^{k+1}\varphi$ , which is equivalent to  $M, v_m \models Kw_i^k Kw_i\varphi$ . Let  $T$  be the set of nodes exactly consisting of all  $Kw_i\varphi$ -satisfied nodes on the  $(|m| + k)$ th layer that  $v_m$  can reach via relation  $\rightarrow$ . By induction hypothesis,  $|T|$  is even. let  $|T| = 2a$ . Thus, among all the successors of  $T$ , the number of  $\varphi$ -satisfied nodes is  $2x + 0y$ , where  $x + y = 2a$ .  $2x + 0y$  is surely an even number. Let  $S$  be a set of nodes only consisting of  $\neg Kw_i\varphi$ -satisfied nodes on the  $(|m| + k)$ th layer that  $v_m$  can reach via relation  $\rightarrow$ . Since  $M$  is a binary tree, let  $|S| = 2b$ . For every node in  $S$  has only one  $\varphi$ -satisfied successor, among all the successors of  $S$ , the number of  $\varphi$ -satisfied nodes is  $2b$ . Thus, the number of  $\varphi$ -satisfied nodes on the  $(|m| + k + 1)$ th layer is  $2x + 2b = 2(x + b)$  which must be even.
  - Assume  $M, v_m \not\models Kw_i^{k+1}\varphi$ , which means  $M, v_{m0} \models Kw_i^k\varphi$  and  $M, v_{m1} \models \neg Kw_i^k\varphi$ . By induction hypothesis, the number of the  $\varphi$ -satisfied nodes on the  $(|m| + k + 1)$ th layer that  $v_{m0}$  can reach via relation  $\rightarrow$  is even. And the number of the  $\varphi$ -satisfied nodes on the  $(|m| + k + 1)$ th layer that  $v_{m1}$  can reach via relation  $\rightarrow$  is odd. That means that the  $\varphi$ -satisfied nodes on the  $(|m| + k + 1)$ th layer that  $v_m$  can reach via relation  $\rightarrow$  is an even number plus an odd number, which equals to an odd number.

## B The definition of $CL$ -game

**The Definition of  $CL$ -Game** A  $CL$ -game is a game with two players, duplicator and spoiler, playing on a Kripke-model. Given two Kripke models  $M = \langle W, R, V \rangle$  and  $M' = \langle W', R', V' \rangle$ , from an arbitrary node  $w$  in  $W$  and an arbitrary node  $w'$  in  $W'$ , play games in  $n$  rounds between duplicator and spoiler as following rules:

- When  $n = 0$ , if the sets of satisfied formulas on node  $w$  and  $w'$  are the same, then duplicator wins; otherwise, spoiler wins.
- When  $n \neq 0$ ,
  - $K$ -move: If spoiler starting from node  $w$  does  $K$ -move to node  $x$  which can be reached by  $R$ , then duplicator starting from  $w'$  does  $K$ -move to a node  $y$  in  $W'$  with the same set of satisfied propositional variables as  $x$ . If spoiler starts from  $w'$ , then duplicator starts from  $w$  with similar way to move.
  - $C$ -move: If spoiler starting from node  $w$  does  $C$ -move to node  $x$  which can be reached by  $\rightarrow$ , then duplicator starting from  $w'$  does  $C$ -move to a node  $y$  in  $W'$  with same set of satisfied propositional variables to  $x$ . If spoiler starts from  $w'$ , then duplicator starts from  $w$  with similar way to move.

In the game, for arbitrary  $x \in W$  and  $y \in W'$ , we say  $(x, y)$  or  $(y, x)$  is a state of  $CL$ -game.

## C Proof of Theorem 11

**A Corollary** For arbitrary formula  $\varphi$ ,  $Kw_i Kw_i\varphi$  is valid over any  $\mathcal{KD}45$ , saying  $\mathcal{KD}45 \models Kw_i Kw_i\varphi$ .

This is a corollary of the results in [5].

**The Proof of Theorem 11** Let  $s = \langle i_1, i_2, \dots, i_n \rangle$  where there is an  $m \in N$  such that  $1 \leq m < n$  and  $i_m = i_{m+1}$ . By the corollary above,  $\mathcal{KD}45 \models Kw_{\langle i_m, i_{m+1}, \dots, i_n \rangle} \varphi$ , which means  $\mathcal{KD}45 \models Kw_{\langle i_m, i_{m+1}, \dots, i_n \rangle} \varphi \leftrightarrow \top$  for an arbitrary formula  $\varphi$ . Since  $\mathcal{KD}45 \models Kw_{\langle i_1, i_2, \dots, i_{m-1} \rangle} \top$ , we have  $\mathcal{KD}45 \models Kw_{\langle i_1, i_2, \dots, i_n \rangle} \varphi$ .