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2

Equation Solving

Branislava Ćurčić-Blake

After reading this chapter you know:

- what equations are and the different types of equations,
- how to solve linear, quadratic and rational equations,
- how to solve a system of linear equations,
- what logarithmic and exponential equations are and how they can be solved,
- what inequations are and
- how to visualize equations and solve them graphically.

2.1 What Are Equations and How Are They Applied?

An *equation* is a mathematical expression; a statement that two quantities are equal. A simple example is given by

$$5 = 5$$

or

$$3 + 2 = 5$$

These are equations without *unknowns*. For statements like these to be true the values of the expressions on each side of the equal sign have to be the same. Often, equations have one *variable* that is unknown, such as

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$$3 + x = 5$$

Here, variable x is unknown, and to solve the equation, we have to find its value so that the above equation becomes true. The solution is $x=2$, because if we substitute it into the above equation, it becomes

$$3 + 2 = 5$$

which is true. Solving equations dates back several thousands of years. For example, the Babylonians (2000–1000 BC) already used equation solving to calculate the dimensions of a rectangle given its surface and the difference between its height and width.

2.1.1 Equation Solving in Daily Life

You may not be aware that we use equations every day. Often, equation solving is involved when dealing with money, e.g. when one needs to calculate percentages, differences or taxes. An example is the following:

Example 2.1

Marc wants to buy three beers (€1.50 each) and one bottle of wine (€10.00). He has €15.00. How much money does he have left after he finished shopping?

If we denote the change with x, we can write the given information in the form of an equation as follows:

$$3 \cdot 1.5 + 1 \cdot 10 + x = 15$$

Here, on the left side all expenses and change x are included. Together, they need to add up to the €15.00 that Marc has. The equation can be solved in two steps by first adding all like terms:

$$14.5 + x = 15 \quad \rightarrow \quad x = 0.5$$

Thus if Marc buys all the drinks he wants, he will have €0.50 left.

In everyday shopping we thus use equations, without even thinking about it. A similar example is provided by calculating sale prices:

Example 2.2

The bag that originally cost €70.00 is now on sale at a 25% discount. What is the sale price of the bag?

If the new price is denoted as x we can compose and solve an equation as follows:

$$x = 70 - 70 \cdot \frac{25}{100} \rightarrow x = 70(1 - 0.25) \rightarrow x = 70 \cdot 0.75 \rightarrow x = 52.5$$

Thus, the sale price of the bag is €52.50.

In this section some specific examples of equations were provided, to get you introduced to the topic. In the next section, we will generalize these examples and introduce some definitions associated with equations.

2.2 General Definitions for Equations

2.2.1 General Form of an Equation

Linear equations in one unknown, such as the examples above, can generally be written as:

$$ax = b, \tag{2.1}$$

where x is the unknown or the variable that we aim to solve the equation for and a and b are constants.

2.2.2 Types of Equations

The equations that were introduced as examples in Sect. 2.1 were quite simple and were examples of linear equations with one unknown (x). More generally, we can distinguish between *linear* and *nonlinear* equations that have one or more unknowns.

A linear equation with one unknown is an equation that can be rewritten to the form of Eq. (2.1). It only includes terms that are constant or that are the product of a constant and a single variable to its first power. Interestingly, such an equation describes a straight line. You can think of linear as straight in two dimensions or flat (as a plane) in three dimensions. Linear equations in two unknowns can be represented by:

$$ax + by + c = 0,$$

where x and y are the unknowns or the variables that we aim to solve the equation for and a, b and c are constants. To find a solution for both variables, a minimum of two *independent* equations is necessary.

Nonlinear (*polynomial*) equations are equations in which one or more terms contain a variable with a power different from one and/or in case of a nonlinear equation with more unknowns there is a term with a combination of different variables. Such a combination could e.g. be a product or a quotient of variables. Other non-polynomial nonlinear equations will be introduced in Sects. 2.6 and 2.7.

2.3 Solving Linear Equations

In Sect. 2.1.1 we already solved some equations without thinking too much about our approach. This can often be done, as long as the equations are very simple. However, if equations become a bit more complicated, it is useful to have a recipe for equation solving that always works, as long as a solution exists, of course. The general goal of solving any linear equation with one unknown x is to get it into the form

$$x = c, \tag{2.2}$$

where c is a constant. ANY linear equation can be solved, i.e. transformed into the form of Eq. (2.2) by following the general rules listed below:

1. Expand terms, when the equation is not (yet) written as a sum of linear terms.
2. Combine like terms by adding or subtracting the same values from both sides.
3. Clear out any *fractions* by multiplying every term by the *least common denominator*.
4. Divide every term by the same non-zero value to make the constant in front of the variable in the equation equal to 1.

These rules may sound rather abstract now, but in the next sections, some of these rules are explained in more detail and you can practice their application in some exercises.

2.3.1 Combining Like Terms

To solve a linear equation in one unknown one of the first steps is to combine like terms, which means that all terms with the same variable (e.g. x) are gathered. In other words, if you start with an equation

$$ax + b = c + dx$$

you will gather all terms in x on the same side to rewrite it to:

$$(a - d)x = c - b$$

Here, to transfer a term to the other side of the equality sign, we had to subtract it from the side we transfer it from. To keep the equation true, we thus also have to subtract that term from the side of the equation we transfer it to. This means that when you are transferring a term from one side of the equation to the other you are basically changing its sign. In this case, by subtracting dx from the right hand side of the equation, we also had to subtract it from the left hand side of the equation, where it thus got a negative sign. Something similar happened to the constant term b.

Example 2.3

Sjoerd ordered three bottles of wine online. Postage was €9.00 and the total costs were €45.00. How much did each bottle of wine cost?

To solve this problem we represent the price of a bottle of wine by x. Then we can compose an equation as follows:

(continued)

Example 2.3 (continued)

$$3x + 9 = 45$$

Here the like terms are 9 and 45 because they are both constants. So we combine them:

$$3x = 45 - 9 \quad \rightarrow \quad 3x = 36 \quad \rightarrow \quad x = 12$$

We find that each bottle of wine costs €12.00.

Exercise

2.1. Grandma Jo left 60 pieces of rare coins to her heirs. She had two daughters. However, the older of the two daughters, Mary, died just a few days before Grandma Jo, so her coins are to be divided among Mary's three daughters. How many rare coins will each granddaughter of Grandma Jo inherit?

2.2. Solve the following linear equations in one unknown:

a) $7x + 2 = -54$
b) $-5x - 7 = 108$
c) $3x - 9 = 33$
d) $5x + 7 = 72$
e) $4x - 6 = 6x$
f) $8x - 1 = 23 - 4x$

2.3.2 Simple Mathematical Operations with Equations

Solving an equation may be fun, if you look at it as solving a puzzle. You may swap 'pieces' around, and you may try to fit 'pieces' in different ways. Often you may have to think out of the box and find creative solutions. The easiest way is to perform simple mathematical operations such as addition, subtraction, multiplication or division, as described in rules 2 and 3 above. In case of solving a linear equation with one unknown you may want to e.g. add a constant or multiply the equation by a constant. Consider the following example:

$$\frac{5}{3}x + \frac{1}{3} = 5$$

In this case it might help to first multiply the whole equation by 3 to get rid of the fractions. To keep the equation true, you have to multiply both sides by the same number:

$$\frac{5}{3}x + \frac{1}{3} = 5 \;\Big|\; \cdot 3 \quad \rightarrow \quad 5x + 1 = 5 \cdot 3 \rightarrow \quad 5x = 15 - 1 \quad \rightarrow \quad x = \frac{14}{5}$$

Note that we also applied rule number 4. Similarly, to solve

$$100x + 50 = 200$$

you may first want to divide by 100:

$$100x + 50 = 200 \left| \cdot \frac{1}{100} \quad \rightarrow \quad x + \frac{1}{2} = 2 \quad \rightarrow \quad x = \frac{3}{2}\right.$$

In Box 2.1 a useful set of rules to solve equations is given, summarizing more formally what we just did.

Box 2.1 Useful *Arithmetic* Rules for Solving Linear Equations

If $a = b$ and $c \in \mathbb{C}$ then

$$a + c = b + c$$
$$ac = bc$$

If $a = b$, $c \in \mathbb{C}$ and $c \neq 0$ then

$$\frac{a}{c} = \frac{b}{c}$$

If F is any *function* and $a = b$ then

$$F(a) = F(b)$$

Exercise

2.3. Solve the following equations:

a) $\frac{2x}{3} = \frac{x}{3}$
b) $\frac{2x}{3} = 5x + 3$

2.4 Solving Systems of Linear Equations

We briefly mentioned, when we introduced linear equations with two unknowns, that to find a solution for both variables, a minimum of two independent equations is necessary. Such a set of linear equations is the smallest possible system of linear equations. In general, a system of linear equations is a set which contains two or more linear equations with two or more unknowns. This might sound rather abstract, but actually such a system is regularly encountered in practice.

Example 2.4

Suppose John only ate bread with peanut butter or jam during 1 month. Each jar of jam contains 30 g of sugar and 20 g of other nutrients. Each jar of peanut butter contains 15 g of sugar and 40 g of other nutrients. How many jars of jam and peanut butter did John use in this month if he consumed 150 g of sugar and 450 g of other nutrients?

(continued)

Example 2.4 (continued)

We can rewrite the problem as a system of two linear equations in two unknowns. Suppose that x is the number of jars of jam, and y is the number of jars of peanut butter, then the system of linear equations describing this problem is:

$$150 = 30x + 15y$$
$$450 = 20x + 40y$$

This system of equations has the solution $x = -\frac{5}{6}$, $y = 11\frac{2}{3}$, which you can for now verify by substituting it in the system of equations.

Here is another example, in which a system of three linear equations in three unknowns can help solve a practical problem encountered when three friends have dinner together.

Example 2.5

Mary, Jo and Sandy had dinner together and agree to go Dutch and split the bill according to what they had. They received two bills, one for drinks (total of €26.00) and one for food (€36.00) without further details. Mary had two glasses of wine, one juice and salmon. Jo had three glasses of wine and salmon. Sandy had beefsteak and two glasses of juice. They remember that juice cost €2.00 a glass, and that beefsteak was €3.00 more expensive than salmon. How much does each colleague have to pay?

We thus need to determine what wine (x), salmon (y) and beefsteak (z) cost. We know that beefsteak was €3.00 more expensive than salmon, which will give us the first equation. Further, we know that the three of them together ate two salmons and one beefsteak, giving us the second equation, and had five glasses of wine and three glasses of juice, giving us the third equation. We can thus write:

$$z = y + 3$$
$$2y + z = 36$$
$$5x + 3 \cdot 2 = 26$$

In total, these three equations are a system of three linear equations with three unknowns. The third equation is straightforward to solve as it has only one unknown x:

$$5x = 26 - 6 \quad \rightarrow \quad 5x = 20 \quad \rightarrow x = 4$$

Further, we can substitute the first equation $z=y+3$ into the second:

$$2y + (y + 3) = 36 \quad \rightarrow \quad 3y + 3 = 36 \quad \rightarrow \quad 3y = 33 \quad \rightarrow \quad y = 11$$

And if we now substitute y back into the first equation we get the price of the beefsteak

$$z = 11 + 3 = 14$$

(continued)

Example 2.5 (continued)

Now, we can calculate how much Mary, Jo and Sandy each have to pay:

$$Mary : 2x + 2 + y = 2 \cdot 4 + 2 + 11 = 21$$
$$Jo : 3y + x = 3 \cdot 4 + 11 = 23$$
$$Sandy : z + 2 \cdot 2 = 14 + 4 = 18$$

As a final check we should verify whether the total bills sum up to the same total amount. The two bills for drinks and food add up to €36.00 + €26.00 = €62.00. The bills for Mary, Jo and Sandy add up to €21.00 + €23.00 + €18.00 = €62.00. Thus, the final bills match and we solved the problem.

The approach that we followed for solving the problem in Example 2.5 is called *substitution*, which is a bit of an ad-hoc method. There are also more systematic *algebraic* methods of solving systems of linear equations and a graphical one. The methods for solving systems of linear equations that we explain in the following sections in more detail are the following:

a. Solving by substitution
b. Solving by elimination
c. Solving graphically
d. Solving by Cramer's rule

2.4.1 Solving by Substitution

This is the basic method to solve a system of any type of equations. It is not always easy to implement, but if possible gives straightforward answers. Basically, if you have a system of *n* equations with *n* unknowns, you first solve one equation to obtain the solution for one variable, then substitute that solution in the next equation to obtain the solution for the next variable etcetera. This procedure is further illustrated in the next examples for systems of linear equations.

Example 2.6

Solve the following system of two linear equations in two unknowns:

$$3x + 5 = 5y$$
$$2x - 5y = 6$$

To solve this system, you can start by e.g. expressing y in terms of x based on the first equation: $y = \frac{3}{5}x + 1$ and then substitute y into the second equation: $2x - 5\left(\frac{3}{5}x + 1\right) = 6$, to solve this equation:

$$2x - 3x - 5 = 6 \rightarrow -x = 11 \quad \rightarrow \quad x = -11$$

(continued)

Example 2.6 (continued)

The solution for x can then be substituted into the expression for y to find its solution:

$$y = \frac{3}{5}x + 1 = \frac{-33}{5} + 1 \quad \rightarrow y = \frac{-28}{5} = -5\frac{3}{5}$$

A similar procedure can be followed for a system of more than two unknowns:

Example 2.7

Solve the following system of three linear equations in three unknowns:

$$2x + y + z = 4$$
$$x - 7 - 2y = -3z$$
$$2y + 10 - 2z = 3x$$

We can solve this system by substitution e.g., by first obtaining x from the second equation:

$$x = 2y - 3z + 7$$

We can now substitute x into the first and third equations and gather like terms:
First equation: $2(2y-3z+7)+y+z=4 \rightarrow 4y-6z+14+y+z=4 \rightarrow 5y-5z=-10$
Third equation: $2y+10-2z=3(2y-3z+7) \rightarrow 2y+10-2z=6y-9z+21 \rightarrow -4y+7z=11$
Thus, now we have two equations in two unknowns (y and z):

$$5y - 5z = -10 \quad \rightarrow y = z - 2$$
$$-4y + 7z = 11$$

Finally, we substitute the expression for y in the last equation: $-4(z-2)+7z=11 \rightarrow 3z=3$. Thus $z=1$ and by substituting this in the other equations, we find that $y=-1$ and $x=2$. To verify whether this solution is correct, we can insert the values of x, y and z into the original system of equations to see if the equations become true.

$$2x + y + z = 4 \quad \rightarrow \quad 2 \cdot 2 - 1 + 1 = 4 \quad \rightarrow \quad 4 = 4 \quad \textit{True}$$
$$x - 7 - 2y = -3z \quad \rightarrow \quad 2 - 7 - 2 \cdot (-1) = -3 \cdot 1 \quad \rightarrow \quad -5 + 2 = -3 \quad \textit{True}$$
$$2y + 10 - 2z = 3x \quad \rightarrow \quad 2 \cdot (-1) + 10 - 2 \cdot 1 = 3 \cdot 2 \quad \rightarrow \quad 6 = 6 \quad \textit{True}$$

We have thus found a correct solution for the system of equations.

Exercises

2.4. Solve the following systems of linear equations ($\wedge$ means 'and'):

a) $x - 2y = 4 \wedge \frac{x}{3} - y = \frac{4}{3}$
b) $2x - 2y = 5 \wedge -4x + 8y = -16$
c) $\frac{2x}{3} - y = 2 \wedge x + \frac{y}{2} = 1$
d) $0.2x - 0.4y = 0.6 \wedge -0.8x + 1.8y = -2.4$

2.4.2 Solving by Elimination

As illustrated in the previous section, solving a system of more than two linear equations by substitution can be very lengthy and clumsy. Mistakes are almost guaranteed. A more elegant way, which is sometimes—but not always—easier to implement, is to solve a system of linear equations by elimination. In essence, one then tries to manipulate one equation (e.g. by multiplication or division by a constant) to make the coefficients of one of the variables the same in two equations. Then you can subtract or add these two equations to eliminate that particular variable. Let's make this more clear by an example:

Example 2.8

Consider the following system of two linear equations in two unknowns:

$$x + y = 10$$
$$x - y = 2$$

As you can see the coefficient of x is the same in both equations. This implies that by simply subtracting the second equation from the first we can eliminate x. We do that by subtracting the terms on the left side from each other, and the terms on the right side, separately!

$$x - x + y - (-y) = 10 - 2 \rightarrow 0 + 2y = 8 \rightarrow y = 4$$

We can now substitute y in any of the two equations. If we substitute it in the first, we find that:

$$x + 4 = 10 \quad \rightarrow \quad x = 6$$

You can now verify yourself whether we found the correct solution by substituting x and y in the second equation, similar to how this was done in Example 2.7. Note that this system can also be solved by first eliminating y by adding the two equations.

In the previous example we did not have to manipulate any of the equations. However, often this is necessary to simplify the procedure, as illustrated in the next example:

Example 2.9

Consider the following system of two linear equations in two unknowns:

$$2x + y = 4$$
$$x - 3y = -1$$

To solve this system we first multiply the second equation by -2:

$$2x + y = 4$$
$$-2x + 6y = 2$$

Then we add the two equations to find that:

$$7y = 6$$

Thus, $y = \frac{6}{7}$ and $x = -1 + 3 \cdot \frac{6}{7} = -\frac{7}{7} + \frac{18}{7} = \frac{11}{7}$. Of course, this system can also be solved by choosing other elimination strategies.

When solving a system of three linear equations in three unknowns, in the first step one should eliminate one variable by manipulating two of the three equations as illustrated in the next example:

Example 2.10

Consider the following system of three linear equations in three unknowns:

$$x + y + z = 3$$
$$2x + 4y - z = -7$$
$$-3x + 2y + z = -7$$

This system can be solved by first eliminating y, for example by first multiplying the first equation by 2 and dividing the second equation by 2. This gives the following system of equations:

$$2x + 2y + 2z = 6$$
$$x + 2y - \frac{1}{2}z = -3\frac{1}{2}$$
$$-3x + 2y + z = -7$$

In the next step, we reduce the system of three equations, to a system of two equation by eliminating y. We do this by subtracting the second equation from both the first and the third equation:

$$x + 2\frac{1}{2}z = 9\frac{1}{2}$$
$$-4x + 1\frac{1}{2}z = -3\frac{1}{2}$$

(continued)

Example 2.10 (continued)

Now we have a system of two equations with two variables, and we know how to solve that. We can do this e.g., by elimination again, by multiplying the first equation by 4 and then adding the two equations. This gives $z=3$. The values for x and y can then be found by substitution in other suitable equations to be $x=2$ and $y=-2$.

Exercise

2.5. Solve the following systems of linear equations:

a) $7x + 4y = 2 \wedge 9x - 4y = 30$
b) $2x + y = 4 \wedge x - y = -1$
c) $2x - 5y = 5 \wedge -6x + 7y = -39$
d) $x - 2y + 3z = 7 \wedge 2x + y + z = 4 \wedge -3x + 2y - 2z = -10$

2.4.3 Solving Graphically

Each linear equation in two unknowns describes a straight line in the two-dimensional plane. Notice that a straight line in the (x,y)-plane is described by $y=ax+b$, where a is a coefficient describing the steepness of the line and b indicates the intersection with the y-axis. Sometimes it can help to solve a system of two linear equations in two unknowns by plotting the associated lines. To solve the system, you just have to find the intersection of the two lines.

Example 2.11

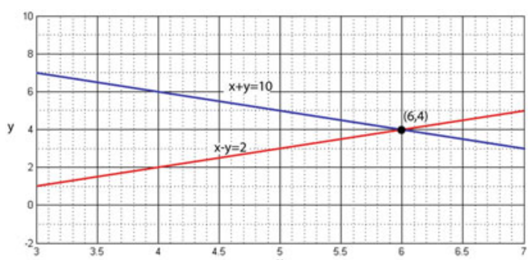
Consider the following system of two linear equations in two unknowns that was introduced in Example 2.8:

$$x + y = 10$$
$$x - y = 2$$

The associated curves are:

$$y = -x + 10$$
$$y = x - 2$$

The graph looks like this:



As the lines intersect at (6,4) the solution to the system is $x=6$, $y=4$.

Such a plot can also illustrate when a system of two linear equations in two unknowns has no solution: when the two associated lines run in parallel, there is no intersection. Of course, plotting the associated lines to find a solution to a system of equations only works when the plotting range is chosen such that the intersection is actually within the plotted part of the plane.

In principle, a graphical approach to find the solution of a system of three linear equations in three unknowns would also work, but this is not very practical. In this case, one would be looking for the intersection of three planes, which is much harder to plot and visualize.

2.4.4 Solving Using Cramer's Rule

One of the most practical ways of solving systems of linear equations is Cramer's rule. It offers a solution to a system of n equations in n unknowns by using *determinants* that are explained in detail in Sect. 5.3.1.

In its most simple form Cramer's rule can be applied to solve a system of two equations in two unknowns, according to the rule indicated in Box 2.2.

Box 2.2 Cramer's Rule for a System of 2 Linear Equations

The system of two equations with two unknowns x and y

$$ax + by = c$$
$$dx + ey = f$$

has the solution

$$x = \frac{ce - bf}{ae - bd}$$
$$y = \frac{af - cd}{ae - bd}$$

when $ae - bd \neq 0$.

Cramer's rule also applies to systems of n linear equations in n unknowns for $n > 2$, which is explained in Sect. 5.3.

2.5 Solving Quadratic Equations

Not all polynomial equations are linear. If a variable in a polynomial equation has power >1 the equation is nonlinear. Specifically, a quadratic equation in one unknown is a second-order polynomial equation of the form:

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. Quadratic equations can be used, for example, to model relationships between variables when a linear relationship is not appropriate, e.g. when modeling walking speed as a function of age. Walking speed is slow when you are very young, reaches a peak and then decreases again when you get older. Its curve would thus show an inverted U-shape, i.e. a second-order polynomial shape. Statistical programs such as SPSS provide the option to do regression analysis (see also Sect. 4.3.2 for an explanation of linear regression) and model such nonlinear relationships.

It is important to note that solutions of quadratic equations are not always *unique*—meaning that there might be more than one solution. In fact, quadratic equations can have no, one or two solutions. For example, the equation $x^2 - 4 = 0$ has two solutions: $x = 2$ and $x = -2$. More generally, a quadratic equation can be solved by following the general rules outlined in Sect. 2.3, extended with the following rules that will be explained in more detail below:

1. If needed, apply a *function* to both sides (e.g. take the square root of both sides to get rid of a square).
2. Recognize a pattern you have seen before, like the difference of squares, or square of differences.
3. Factorize in order to simplify.
4. Equate each factor to zero.

Regarding rule 1, it is useful to remember that if F is any kind of function (such as the square or square root function, or the logarithmic function), and $a = b$, then also $F(a) = F(b)$. For example, if $a = b$, then $\log a = \log b$, $c^a = c^b$ and $e^a = e^b$ (see Box 2.1).

Example 2.12

Solve the equation:

$$2^{3x} = 4$$

This type of equation often occurs in chemistry calculations, for instance when calculating drug doses (see Sect. 2.9 for an example). To solve it we will take the logarithm with base 2 of the whole equation and apply arithmetic rules for logarithms (Sect. 1.2.2). This yields

$$\log_2 2^{3x} = \log_2 4 \quad \rightarrow \quad 3x = 2 \quad \rightarrow \quad x = \frac{2}{3}$$

Example 2.13

Solve the equation:

$$\sqrt{3x + 5} = 7$$

To solve this equation both sides can be squared. This yields:

$$3x + 5 = 7^2 \quad \rightarrow \quad 3x = 49 - 5 \quad \rightarrow \quad x = \frac{44}{3}$$

We will now consider three different ways of solving quadratic equations.

2.5.1 Solving Graphically

To get a better understanding of quadratic equations, we first discuss the graphical approach to solving them. This is done by first drawing the function $f(x) = ax^2 + bx + c$ in the (x,y)-plane. Its curve is a parabola. The solution of the equation is given by the points on the curve that intersect with the x-axis, since there $f(x) = 0$. When the parabola's peak just touches the x-axis, there is one solution to the equation, when the x-axis cuts the parabola into three parts, there are two solutions and in all other cases there are none.

The parabola associated with $f(x)$ has a maximum or a minimum, depending on the sign of a (illustrated in Fig. 2.1)

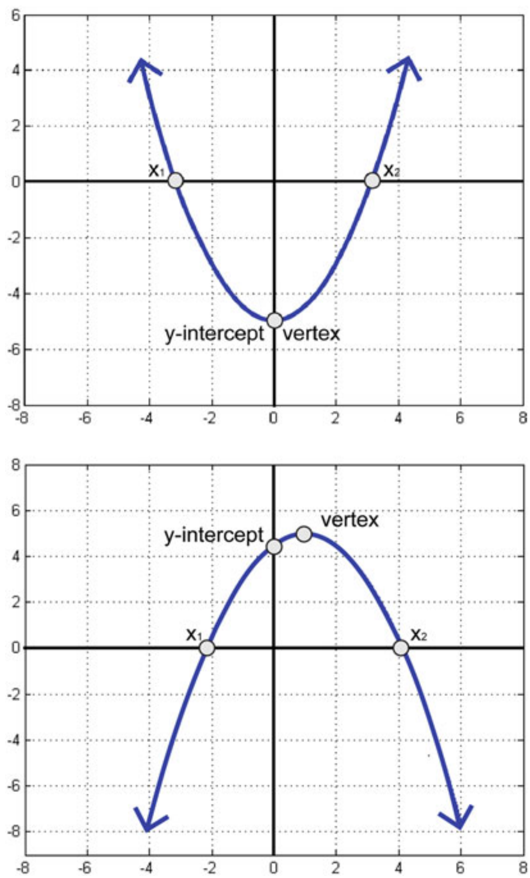


Fig. 2.1 Example of parabolas associated with quadratic equations. *Top*: The solutions indicated by x_1 and x_2 of the equation $0.5x^2 - 5 = 0$. Here $a > 0$, thus the curve is concave up. *Bottom*: The solutions indicated by x_1 and x_2 of the equation $-0.5(x-1)^2 + 5 = 0$. Here $a < 0$, thus the curve is concave down. For both parabolas, the *vertex* and the *y-intercept* are also indicated.

1. If $a > 0$, $f(x)$ is facing up (*concave* up) and the parabola has a minimum
2. If $a < 0$, $f(x)$ is facing down (concave down) and the parabola has a maximum
3. In any case, the peak of the parabola (or vertex) is at the position

$$x = \frac{-b}{2a}$$

2.5.2 Solving Using the Quadratic Equation Rule

Solving equations graphically, although intuitive, might be difficult, and not always exact, depending on whether the solution is a grid point. One method of solving quadratic equations that always works is the quadratic equation rule (Box 2.3):

Box 2.3 Quadratic Equation Rule

The solution of

$$ax^2 + bx + c = 0$$

is given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, the solution consists of two *roots*, x_1 and x_2, where the first is found for the plus sign and the second for the minus sign. Applying this rule is relatively easy, as illustrated in this example:

Example 2.14

Solve the following equation by applying the quadratic equation rule:

$$2x^2 + 3 = -5$$

Here $a=2, b=0$ *and* $c=3+5=8$. By applying the quadratic equation rule we find the two solutions

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{0 + \sqrt{0 - 4 \cdot 2 \cdot 8}}{2 \cdot 2} = \frac{\sqrt{-64}}{4} = \frac{8i}{4} = 2i$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{0 - \sqrt{0 - 4 \cdot 2 \cdot 8}}{2 \cdot 2} = \frac{\sqrt{-64}}{4} = \frac{-8i}{4} = -2i$$

As you can see there are two solutions for this equation and both are complex numbers (see Sect. 1.2.4).

Exercise

2.6. Find the roots of the following equations using the quadratic equation rule:

a) $2x^2 + 7x + 5 = 0$
b) $x^2 - 9 = 0$
c) $x^2 - 3 = 5x + 2$
d) $x(x - 5) = 3$
e) $2(y^2 - 6y) = 3$
f) $1 + \frac{6}{x} + \frac{9}{4x^2} = 0$

2.5.3 Solving by Factoring

In my opinion this is the most interesting way of solving equations as it requires bringing out one's creativity and imagination. The goal is to recognise factors (simpler forms) and rewrite the equation using only a product of these simple forms, which are subsequently each set to zero. This is not always possible, but often it is. Factors are (in the case of an *n*-th order polynomial equation) simpler polynomials than the original equation. For example, quadratic equations should be factored into two first-order polynomials.

Example 2.15

Consider the quadratic equation $x(x-1)=0$ which is already factored into two factors: the factor x and the factor $(x - 1)$. To find the solution of this equation, at least one of the factors has to be set to zero. Hence, solutions are found by equating each of the factors to zero: $x=0$ *OR* $x-1=0$. Thus, the solutions are $x_1=0$ and $x_2=1$.

To solve polynomial equations by factoring, it is important to remember the following rule and special cases regarding multiplication of such factors:

Box 2.4 Factor Multiplication Rule

General rule:

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

Special cases:

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$
$$x^2 - y^2 = (x - y)(x + y)$$

Example 2.16

Solve the following equation by factoring:

$$x^2 - 5x - 24 = 0$$

Thus, we need to find numbers a and b such that $(x-a)(x-b)=x^2-5x-24=0$. There are many different approaches to find these numbers. First, we will here use that if we add and subtract the same number we do not change the outcome:

$$x^2 - 5x - 24 = x^2 - 5x - 3x + 3x - 24$$

Now, we can rewrite the equation as

$$x^2 - 8x + 3x - 24 = 0,$$

allowing to take the factor $x - 8$ out which results in the other factor being $x + 3$:

$$x(x - 8) + 3(x - 8) = 0$$

$$(x + 3)(x - 8) = 0$$

Thus, the solutions are $x_1=-3$ and $x_2=8$.

Example 2.17

Solve the following equation by factoring:

$$x^2 - 9 = 0$$

Here you have to realize that 9 is a square of 3 ($3^2 = 9$), and to remember one of the special cases of the factor multiplication rule (Box 2.4). We can then rewrite the left-hand side of the equation, which is a difference of squares, to:

$$x^2 - 9 = (x - 3)(x + 3)$$

Thus, the equation has two solutions $x-3=0$ or $x+3=0$; thus $x_1=3$ and $x_2=-3$.

Example 2.18

Solve the following equation by factoring:

$$x^2 + 6x + 9 = 0$$

If we realize that $3^2=9$ and that $3+3=6$, we see that the above equation is the square of a sum (a special case of the factor multiplication rule again). Thus we can rewrite the left-hand side of the equation to:

(continued)

Example 2.18 (continued)

$$x^2 + 6x + 9 = (x + 3)^2$$

and the above equation to:

$$(x + 3)^2 = 0$$

Thus, the solutions are $x_{1,2} = -3$. For this equation there is only one (double) solution.

To find the factors of a quadratic equation, there is also a more systematic approach. Suppose that you are trying to solve the equation $x^2 + cx + d = 0$. Taking the general factor multiplication rule into account, we know that we are looking for two numbers a and b such that their sum is c and their product is d. Hence, assuming that the solutions of the equation are integers, we can make a table of all possible pairs of factors of d that are likely solutions and calculate their sum. The pair that has the right sum and product then provides two solutions. Let's illustrate this approach for Example 2.16:

Example 2.16 (continued)

Solve the following equation by factoring:

$$x^2 - 5x - 24 = 0$$

We first construct a table with integer factors of −24 that are likely candidates for factors and add their sums:

Factor 1	Factor 2	Sum
−2	12	10
2	−12	−10
−3	8	5
3	−8	−5
−4	6	2
4	−6	−2

Now, we immediately see that the pair of factors 3 and −8 has the right sum. Thus, the equation can be factored into $(x+3)(x-8)=0$ and its solutions are $x_1 = -3$ and $x_2 = 8$.

Exercise

2.7. Find the roots of the following equations using factoring:

a) $x^2 - 7x + 2 = -8$
b) $x^2 - 4x + 4 = x$
c) $x^2 + 2x - 8 = 0$
d) $2x^3 - 14x^2 + 20x = 0$

2.6 Rational Equations (Equations with Fractions)

We already touched upon fractions in Sect. 1.2.1. Rational equations use fractions. These are equations that have a rational expression on one or both sides in which the unknown variable is in one or more of the denominators. For example, in the following equation, the variable x is in the denominator on the left side:

$$\frac{3}{x+2} = 7 + \frac{x}{3}$$

Such equations are used for example in percentage calculations, or when calculating the speed at which a computer job will be done, as illustrated in the next example.

Example 2.19

You need to do a heavy calculation on a two core PC to obtain a fit to your data. Normally, core 1 would take 6 h to fit the data. However, core 2 is faster and would take 3 h to fit the data. How long will it take to calculate the data fit using both cores of the PC?

This problem that requires P calculations can be solved using rational equations. First, we realize that core 1 takes 6 h to do P calculations. Thus it can do $\frac{P}{6}$ calculations per hour. Second, we know that core 2 takes 3 h to do P calculations. Thus it can do $\frac{P}{3}$ calculations per hour. Thus if we say that T is the total time needed for both cores to calculate the data fit together, we find that we need to solve a rational equation:

$$\frac{P}{6} + \frac{P}{3} = \frac{P}{T}$$

Solving this equation we find that

$$\frac{9}{18} = \frac{1}{T} \quad \rightarrow \quad T = 2$$

Thus, using rational equations we find that it takes 2 h to calculate the fit using both cores of the PC. Note that this problem can also be solved in other ways that do not involve rational equations.

2.7 Transcendental Equations

These are equations where at least one side contains a so called *transcendental* (i.e. non-algebraic) function. This is a whole world of equations, and usually they can be written in the form $F(x) = 0$, where $F(x)$ can be any function. Some examples of functions, discussed in this book are logarithmic, trigonometric or exponential functions. Importantly, these equations are not always solvable, or the solutions can only be approximated *numerically*, with the help of computer algorithms. In some cases they can be expressed in algebraic form and then solved. We will only consider such cases here for two types of functions, exponential and logarithmic functions.

2.7.1 Exponential Equations

Exponential equations can be solved in two cases:

1) In case the equation consists of exponentials with different bases that are not added or subtracted, we can find a solution by applying a logarithm with any base to the equation.
2) In case exponentials of the same base k are added (e.g. 2^1, 2^2 and 2^0, that have the same base $k = 2$), we can substitute $y = k^x$ for an exponential equation in x.

Example 2.20

(Example of case 1) Solve the equation:

$$3^x = 4^{x-2} \cdot 2^x$$

In this case no exponentials are added or subtracted and we can apply a logarithm with any base to both sides of the equation. We will try the natural logarithm with base e, but it could be any base, and in fact base 2 might be more straightforward.

Remember that $\log_a x^b = b\log_a x$ and $\log_a (x \cdot y) = \log_a x + \log_a y$ (Sect. 1.2.2). If we now apply *ln* to both sides of the equation we get:

$$\ln 3^x = \ln\left(4^{x-2} \cdot 2^x\right) \quad \rightarrow \quad x \ln 3 = \ln 4^{x-2} + \ln 2^x \rightarrow$$
$$x \ln 3 = (x - 2) \ln 4 + x \ln 2$$

If we group over x we find that

$$x = \frac{2 \ln 4}{-\ln 3 + \ln 4 + \ln 2}$$

Example 2.21

(Example of case 2) Solve the equation:

$$3^{2x-1} = 9^x - 3$$

Remember that $3^2=9$, thus we can substitute $y=3^x$. Since $3^{2x}=(3^x)^2=y^2$, and $3^{2x-1} = \frac{3^{2x}}{3} = \frac{y^2}{3}$ we can rewrite the equation to

$$\frac{y^2}{3} = y^2 - 3$$

which we can then rewrite to $3y^2 - y^2 - 9 = 0$. Thus $2y^2 - 9 = 0$, and we know how to solve this by using e.g. the quadratic equation rule in Sect. 2.5.2.

Exercise

2.8. Find the roots of the following equations using logarithms or substitution:

a) $3^{6x} = 9$
b) $8 + 6^{2x+1} = 44$
c) $10e^{2x} - 30e^x + 15 = 0$

2.7.2 Logarithmic Equations

Logarithmic equations can be solved in the following two cases:

1) If the equation contains one or more logarithms of the same expression (say $P(x)$), then we can use the substitution:

$$y = \log_a P(x)$$

2) If the equation contains a linear combination of logarithms with the same base, we can use the arithmetic rules for logarithms (Sect. 1.2.2) to solve the equation.

Example 2.22

(Example of case 1). Solve the equation:

$$m\left(\ln\left(x^2+4\right)\right)^2 + n = a\sqrt{\left(\ln\left(x^2+4\right)\right)^2 + b}$$

According to the solution provided for case 1) we can rewrite

$$y = \ln\left(x^2+4\right)$$

(continued)

Example 2.22 (continued)

to get

$$my^2 + n = a\sqrt{y^2 + b}$$

If we square both sides, this results in

$$\left(my^2 + n\right)^2 = a^2y^2 + a^2b$$

which has now become a regular polynomial equation in y. After solving for y, x can be determined according to

$$x = \sqrt{e^y - 4}$$

Example 2.23

(Example of case 2). Solve the equation:

$$2\log_5(3x - 1) - \log_5(12x + 1) = 0$$

Rewriting this equation using the arithmetic rules for logarithms yields:

$$\log_5 \frac{(3x - 1)^2}{12x + 1} = \log_5 1$$

and thus

$$\frac{(3x - 1)^2}{12x + 1} = 1$$

This equation has two solutions ($x_1 = 0$ and $x_2 = 2$). However, if we now verify the solutions by substituting x_1 and x_2 in the original equation, the first logarithm has $3x_1 - 1 = -1$ as its argument, and $\log_5 -1$ is not defined, whereas the second logarithm ($\log_5(12x_1 + 1) = \log_5 1 = 0$) is well defined, so we will discard the first solution. Therefore, the solution of the equation is only $x = 2$.

Exercise

2.9. Solve the following equations using substitution and logarithmic arithmetic:

a) $\log_5 x - (\log_5 x)^2 = 0$
b) $2\log_5(3x - 1) - \log_5(12x + 1) = 0$

2.8 Inequations

2.8.1 Introducing Inequations

Inequations are used in daily life almost as frequently as equations. They are used whenever we have to consider a lower or higher limit. For example, when driving, there is a maximum speed limit. If we take over a car that is driving slowly, are we going to have to drive too fast? If you spend 200 Euros on a new printer today, what is the maximum amount you can spend on food to not go over the daily withdrawal limit of your bank? For how many family members can you buy a plane ticket, before you hit the limit of your credit card? While equations include an equality (=) sign, inequations have an inequality symbol between two mathematical expressions as given in Table 2.1.

2.8.2 Solving Linear Inequations

As you need to know how to solve equations before you start solving any inequation, we started this chapter with an extensive introduction to equation solving. Inequations are solved using the same techniques as for equations, except that one needs to take care of the direction of the inequality symbol. The main rule is that if you multiply an inequation with a negative number you need to swap the direction of the inequality symbol, i.e. greater than becomes less than and vice versa.

For example, the inequation

$$-3x > 5$$

is solved by multiplying the whole inequation by $-\frac{1}{3}$. In this case the inequality symbol needs to change direction so the inequation becomes:

$$-\frac{1}{3}(-3x) < -\frac{1}{3} \cdot 5$$

Therefore, the solution is

$$x < -\frac{5}{3}$$

Table 2.1 Inequality symbols and their meanings

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to

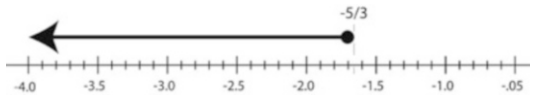


Fig. 2.2 Illustration of the set of solutions for the inequation $-3x > 5$. The *black arrow* represents all solutions. Any number smaller than $-5/3$ is a solution. The *black dot* indicates that the solution does not include the boundary value.

Table 2.2 Rules for changing the inequality symbol direction

Inequality symbol	Inequality symbol after multiplication by -1
$<$	$>$
$>$	$<$
$\leq$	$\geq$
$\geq$	$\leq$

Another important thing to remember is that the solution of an inequation is not a single value, as for linear equations, but a set of numbers that satisfies a certain criterion. I will explain that for the example above.

The solution of the above inequation is any x smaller than $-\frac{5}{3}$. This is illustrated in Fig. 2.2.

Table 2.2 summarizes what happens to inequality symbols in an inequation when both sides are multiplied by -1.

Inequations can also have three parts in the case where the solution of the inequation is limited both from above and from below. An example is given by:

$$-2 < 3x < 5$$

The solution of this inequation can be found by dividing all parts by 3. Since three is a positive number, the directions of all inequality symbols will remain the same:

$$-\frac{1}{3} \cdot 2 < x < \frac{1}{3} \cdot 5 \quad \rightarrow \quad -\frac{2}{3} < x < \frac{5}{3}$$

We can also write this as

$$x \in \left(-\frac{2}{3}, \frac{5}{3}\right)$$

Here, $\in$ stands for 'is an element of' or 'belongs to' and the ( ) brackets indicate that the boundary values are not included in the solution. Graphically this solution is illustrated in Fig. 2.3.

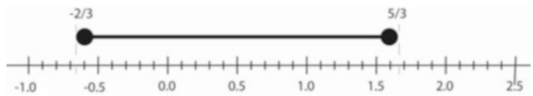


Fig. 2.3 Illustration of the set of solutions for the inequation $-2 < 3x < 5$. The *black dots* indicate that the solution does not include the boundary values.

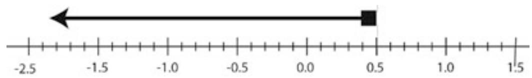


Fig. 2.4 Illustration of the set of solutions for the inequation $3 \geq 6x$. The *black square* indicates that the solution includes the boundary value.

Table 2.3 Examples of bracket use in linear inequations

Example	Same as
$x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$	$-\frac{1}{2} < x < \frac{1}{2}$
$x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$	$-\frac{1}{2} \leq x < \frac{1}{2}$
$x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$	$-\frac{1}{2} \leq x \leq \frac{1}{2}$
$x \in \left(-\infty, \frac{1}{2}\right)$	$x < \frac{1}{2}$
$x \in \left(-\infty, \frac{1}{2}\right]$	$x \leq \frac{1}{2}$

But what about inequations like $3 \geq 6x$? This inequation includes the inequality symbol $\geq$, which stands for 'greater than or equal to' (See Table 2.1). If we apply already known rules for equation solving to solve this inequation we find that:

$$\frac{3}{6} \geq x \rightarrow \frac{1}{2} \geq x \rightarrow x \leq \frac{1}{2} \rightarrow x \in \left(-\infty, \frac{1}{2}\right]$$

This means that the solutions of the equation $3 \geq 6x$ are all numbers smaller than $\frac{1}{2}$ *including* $\frac{1}{2}$. The bracket] indicates that the boundary value is included in the solution. In line with the previous examples we can then illustrate this set of solutions as in Fig. 2.4.

Some further examples of the use of brackets to denote sets of numbers are provided in Table 2.3.

Exercise

2.10. Find the solution to the following inequations:

a) $3x + 7 > 2x - 5$
b) $3 - 5x < 7x - 2$
c) $4 \leq 7x < 6$

2.8.3 Solving Quadratic Inequations

Quadratic inequations can be solved by using general knowledge about solving linear inequations while at the same time taking into account the quadratic nature of the inequation. Remember that the solution to a quadratic equation is determined by the roots of a parabola. To solve a quadratic inequation you thus need to:

1. Decide whether the parabola is concave up or concave down,
2. find the roots of that parabola and then
3. create the set of solutions.

This may sound pretty abstract but will become clear when discussing an example.

Example 2.24

Determine the solution of the inequation:

$$4x^2 - 9 < 0$$

First we realize that $4 > 0$, thus the parabola is concave up. That means that there are two solutions to the equation

$$4x^2 - 9 = 0$$

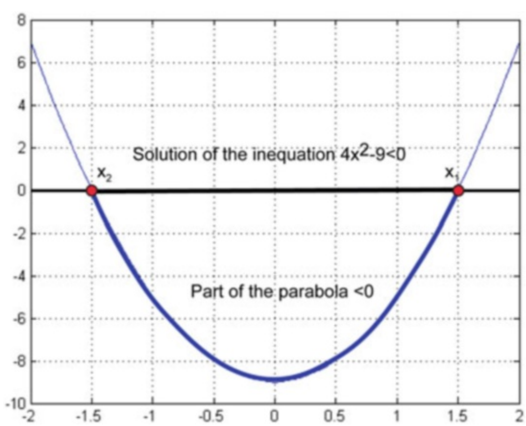
We will look at the part of the parabola that is <0, thus the part between the two solutions of the equation. Let's solve this equation by factoring (Sect. 2.5.3).

$$4x^2 - 9 = (2x - 3)(2x + 3)$$

Thus its roots are $x_1 = \frac{3}{2}$ and $x_2 = -\frac{3}{2}$. Here $x_2 < x_1$. Visually, we can now determine that the solution of the quadratic inequation is

$$x \in \left(\frac{-3}{2}, \frac{3}{2}\right)$$

as is also illustrated in the figure.



Example 2.25

Determine the solution of the inequation in Example 2.24, but then with the opposite inequality symbol:

$$4x^2 - 9 > 0$$

We can now immediately determine the solution with the help of the figure in Example 2.24:

$$x \in \left(-\infty, \frac{-3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$$

Here, ∪ denotes the *union* of two sets.

Exercise

2.11. Find the solution to the following inequations:

a) $2x^2 + 7x + 5 \geq 0$
b) $x^2 - 9 < 0$
c) $x^2 - 3 > 5x + 2$

2.9 Scientific Example

Equivalent dose for anti-psychotic medication

My favorite example of equation use in neuroscience is based on a now widely used method to calculate the *equivalent dose* for *antipsychotics* (Andreasen et al. 2010). These are medications that can be used to suppress symptoms of *psychosis* and an equivalent dose is a dose which would offer an equal effect between different antipsychotics. The term equivalent dose is also used for other types of medications, such as e.g., *analgesics*. The importance of being able to calculate equivalent doses is that patients sometimes already use other types of antipsychotics (or analgesics) and doctors want to be able to track the total strength of the medication. Also, when comparing medication dosages between patients who use different medications, this is important. Here, we discuss a specific example.

Example 2.26

A patient is known to daily use 45 mg of Mirtazipine, 10 mg of Zyprexa and 15 mg of Abilify. How much is that expressed in mg of haloperidol?

Mirtazipine is an antidepressant, which will not count towards the equivalent dose of haloperidol. The other two drugs are antipsychotics where it should be known that Abilify is a brand of aripiprazole, and Zyprexa is based on olanzapine.

From the table in Fig. 2.5 we can derive that a dose of x mg of haloperidol is equivalent to

$$y_a = 4.343x^{0.645}$$

(continued)

Example 2.26 (continued)

Medications	Formulas and Equivalents			
	Formula[a] (x = CPZ)	Chlorpromazine Equivalent (mg)	Formula[b] (x = Haloperidol)	Haloperidol Equivalent (mg)
Atypical Antipsychotics				
Aripiprazole	$y = 0.255x^{0.700d}$	6.42	$y = 4.343x^{0.645d}$	6.79
Clozapine	$y = 2.027x^{0.863f}$	108	$y = 66.58x^{0.796e}$	115.61
Olanzapine	$y = 0.086x^{0.870e}$	4.75	$y = 2.900x^{0.805e}$	5.07
Quetiapine	$y = 2.806x^{0.852f}$	142	$y = 88.16x^{0.786e}$	151.97
Risperidone	$y = 0.019x^{0.924f}$	1.32	$y = 0.790x^{0.851e}$	1.43
Ziprasidone	$y = 2.805x^{0.628d}$	50.5	$y = 35.59x^{0.578d}$	53.13
Typical Antipsychotics				
Chlorpromazine	$y = x$	100	$y = 56.98x^{0.923f}$	108.04
Fluphenazine	$y = 0.011x^{1.112d}$	1.76	$y = 0.940x^{1.028d}$	1.92
Haloperidol	$y = 0.013x^{1.082f}$	1.84	$y = x$	2
Perphenazine	$y = 0.071x^{0.994f}$	6.90	$y = 3.937x^{0.919f}$	7.44
Thioridazine	$y = 0.989x^{0.973f}$	87.3	$y = 50.51x^{0.898f}$	94.14
Thiothixene	$y = 0.057x^{0.967f}$	4.91	$y = 2.852x^{0.892f}$	5.29
Trifluoperazine	$y = 0.066x^{0.939f}$	5.09	$y = 3.001x^{0.866f}$	5.47
Fluphenazine decanoate (mg/2–3 wk)	$y = 0.163x^{0.843c}$	7.91	$y = 4.921x^{0.778c}$	8.44
Haloperidol decanoate (mg/4 wk)	$y = 0.635x^{0.872d}$	35.3	$y = 21.662x^{0.803d}$	37.8

[a] Chlorpromazine is represented by "x" in this column, such that CPZ 100 mg would yield the equivalents in the next column over to the right.
[b] Haloperidol is represented by "x" in this column, such that haloperidol 2 mg would yield the equivalents in the next column over to the right.
[c] $R^2 > .96$.
[d] $R^2 > .97$.
[e] $R^2 > .98$.
[f] $R^2 > .99$.

Fig. 2.5 Formulas for calculating dose equivalents using regression with power transformation, and chlorpromazine and haloperidol equivalents based on them. Table reprinted from Andreasen et al. (2010). Antipsychotic dose equivalents and dose-years: a standardized method for comparing exposure to different drugs. Biol. Psychiatry 67, 255–262, with permission from Elsevier.

mg of Aripiprazole, and that a dose of x mg of haloperidol is equivalent to

$$y_o = 2.900x^{0.805}$$

mg of Olanzapine.

Thus, we want to determine x from the equations above, while we know the doses of aripiprazole ($y_a = 15$) and olanzapine ($y_o = 10$), as well as the full dosage per day (the sum of the two doses). You now know how to do this:

$$x_a = \left(\frac{y_a}{4.343}\right)^{\frac{1}{0.645}}$$

$$x_o = \left(\frac{y_o}{2.9}\right)^{\frac{1}{0.805}}$$

Here, x_a is the equivalent dose of haloperidol for the daily dose of aripiprazole and x_o is the equivalent dose of haloperidol for the daily dose of olanzapine. Note that $x = x_a + x_o$ is the daily dose of antipsychotics in haloperidol terms. Thus

$$x_a = \left(\frac{15}{4.343}\right)^{\frac{1}{0.645}} = 6.8324$$

$$x_o = \left(\frac{10}{2.9}\right)^{\frac{1}{0.805}} = 4.6540$$

and the daily equivalent dose of haloperidol (in mg) is

$$x = x_a + x_o = 11.4864$$

Thus this patients receives the equivalent of 11.49 mg of haloperidol daily.

Glossary

Analgesic medication to relief pain; painkiller
Algebraic using an approach in which only mathematical symbols and arithmetic operations are used
Antipsychotic medication used to treat psychosis
Arithmetic operations between numbers, such as addition, subtraction, multiplication and division
Concave hollow inward
Determinant a scalar calculated from a matrix; can be seen as a scaling factor when calculating the inverse of a matrix (see also Sect. 5.3.1)
Elimination eliminating an unknown by expressing it in terms of other unknowns
Equation a mathematical expression that states that two quantities are equal
Equivalent dose dose which would offer an equal effect between different medications
Function a mathematical relation, like a recipe, describing how to get from an input to an output
Independent here: equations that cannot be transformed into each other by multiplication
Least common denominator the least number that is a multiple of all denominators
Linear a function or mathematical relationship that can be represented by a straight line in 2D and a plane in 3D; can be thought of as 'straight'
Nonlinear not linear
Numerically (solving) to find an approximate answer to a mathematical problem using computer algorithms
Polynomial an expression consisting of a sum of products of different variables raised to different non-negative integer powers
Psychosis a mental condition that can have many different symptoms including hallucinations
Rational equation equation that has a rational expression on one or both sides in which the unknown variable is in one or more of the denominators
Root solution of a polynomial equation
Substitution replacing a symbol or variable by another mathematical expression
Transcendental a number that is not the root of a polynomial with integer coefficients; most well-known are e and π
Union the union of two sets is the set that contains all elements in both sets
Unique here: a single solution to an equation
Unknown variable in an equation for which the equation has to be solved; an equation can have multiple unknowns
Variable alphabetic character representing a number
Vertex peak of a parabola
Y-intercept intercept of a curve with the y-axis

Symbols Used in This Chapter (in Order of Their Appearance)

$=$	equal to
$\rightarrow$	implies
$\neq$	not equal to
$\mathbb{C}$	complex numbers
log	logarithm
ln	natural logarithm (base e)
i	complex unity

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to
$x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$	$-\frac{1}{2} < x < \frac{1}{2}$
$x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$	$-\frac{1}{2} \leq x < \frac{1}{2}$
$x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$	$-\frac{1}{2} \leq x \leq \frac{1}{2}$
∞	infinity
$\cup$	unification

Overview of Equations for Easy Reference

General form of linear equation
Any linear equation with one unknown x can be written as $ax=b$ where a and b are constants.

Arithmetic rules useful for solving linear equations
If $a = b$ and $c \in \mathbb{C}$ then

$$a + c = b + c$$
$$ac = bc$$

If $a = b$, $c \in \mathbb{C}$ and $c \neq 0$ then

$$\frac{a}{c} = \frac{b}{c}$$

If F is any function and $a = b$ then $F(a) = F(b)$

$$F(a) = F(b)$$

Cramer's rule for a system of 2 linear equations
The system of two equations with two unknowns x and y

$$ax + by = c$$

$$dx + ey = f$$

has the solution

$$x = (ce - bf)/(ae - bd)$$

$$y = (af - cd)/(ae - bd)$$

when

$$ae - bd \neq 0.$$

Quadratic equation rule
The solution of

$$ax^2 + bx + c = 0$$

is given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factor multiplication rule
General rule:

$$(x + y)(x + z) = x^2 + (y + z)x + yz$$

Special cases:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$x^2 - y^2 = (x - y)(x + y)$$

Rules for changing the direction of the inequality symbol

Inequality symbol	Inequality symbol after multiplication by −1
$<$	$>$
$>$	$<$
$\leq$	$\geq$
$\geq$	$\leq$

Answers to Exercises

2.1. Suppose each of the granddaughters inherits x coins. Then we can write $3x + \frac{1}{2} \cdot 60 = 60 \rightarrow 3x = 60 - 30 \rightarrow 3x = 30 \rightarrow x = 10$. Thus each granddaughter will inherit ten rare coins.

2.2. a. $x = -8$
b. $x = -23$
c. $3x - 9 = 33 \rightarrow 3x = 33 + 9 \rightarrow x = \frac{42}{3} \rightarrow x = 14$
d. $x = \frac{65}{5} = 13$
e. $4x - 6 = 6x \rightarrow (4 - 6)x - 6 = 0 \rightarrow -2x = 6 \rightarrow x = -3$
f. $8x - 1 = 23 - 4x \rightarrow 12x = 24 \rightarrow x = 2$

2.3. a) $x = 0$
b) $x = \frac{-9}{13}$

2.4. a) $x = 4; y = 0$
b) $x = 1; y = -\frac{3}{2}$
c) $x = \frac{3}{2}; y = -1$
d) $x = 3; y = 0$

2.5. a) $x = 2; y = -3$
b) $x = 1; y = 2$
c) $x = 10; y = 3$
d) $x = 2; y = -1; z = 1$

2.6. a) $x_1 = -2\frac{1}{2}$ and $x_2 = -1$
b) $x_1 = -3$ and $x_2 = 3$
c) $x_1 = \frac{5-3\sqrt{5}}{2}$ and $x_2 = \frac{5+3\sqrt{5}}{2}$
d) $x_1 = \frac{5-\sqrt{37}}{2}$ and $x_2 = \frac{5+\sqrt{37}}{2}$
e) $x_1 = 3 - \frac{1}{2}\sqrt{42}$ and $x_2 = 3 + \frac{1}{2}\sqrt{42}$
f) $x_1 = -3 - \frac{3}{2}\sqrt{3}$ and $x_2 = -3 + \frac{3}{2}\sqrt{3}$

2.7. a) $(x-5)(x-2)=0$ thus $x_1=2$ and $x_2=5$
b) $(x-4)(x-1)=0$ thus $x_1=1$ and $x_2=4$
c) $(x+4)(x-2)=0$ thus $x_1=-4$ and $x_2=2$
d) We can rewrite this equation to $2x(x-5)(x-2)=0$ thus $x_1=0$, $x_2=2$ and $x_3=4$

2.8. a) Using that $9=3^2$, we have to solve $6x=2$ and thus $x = \frac{1}{3}$.
b) We can rewrite $6^{2x+1}=36=6^2$ so that we have to solve $2x+1=2$ and thus $x = \frac{1}{2}$.
c) Substitute $y=e^x$, so that $y^2=e^{2x}$, resulting in the quadratic equation $10y^2-30y+15=0$, which has solutions $y_1 = \frac{3}{2} + \frac{1}{2}\sqrt{3}$ and $y_2 = \frac{3}{2} - \frac{1}{2}\sqrt{3}$. Then $x_1 = \ln\left(\frac{3}{2} + \frac{1}{2}\sqrt{3}\right)$ and $x_2 = \ln\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\right)$.

2.9. a) First we substitute $y=\log_5 x$. Then the equation can be rewritten as $y-y^2=0$, which has two solutions $y_1 = 0$ and $y_2 = 1$. Hence $\log_5 x_1=0$ and $\log_5 x_2=1$ and thus $x_1=1$ and $x_2=5$
b) We can rewrite this equation using the rules for logarithms to $\log_5 \frac{(3x-1)^2}{12x+1} = \log_5 1$, so that we have to solve the equation $\frac{(3x-1)^2}{12x+1} = 1$, which has solutions $x_1=0$, $x_2=2$.

2.10. a) $x \in (-12, \infty)$
b) $x \in \left(\frac{5}{12}, \infty\right)$
c) $x \in \left[\frac{4}{7}, \frac{6}{7}\right)$

2.11. a) We first solve the equation $2x^2+7x+5=0$ which has two solutions $x_1 = \frac{-5}{2}$ and $x_2 = -1$. Because the coefficient of x^2 is larger than zero, $a > 0$, the parabola has a minimum and is concave up. We are looking for those solutions where the curve is ≥ 0. Thus

$$x \in \left(-\infty, \frac{-5}{2}\right] \cup [-1, \infty)$$

b) In a similar approach as for Exercise 2.11a we find that:

$$x \in (-3, 3)$$

c) We first rewrite the inequation to $x^2 - 5x - 5 > 0$. Its related equation has two solutions $x_{1,2} = \frac{5 \pm \sqrt{45}}{2}$. Thus

$$x \in \left(-\infty, \frac{5 - 3\sqrt{5}}{2}\right) \cup \left(\frac{5 + 3\sqrt{5}}{2}, \infty\right)$$

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