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*Published in:*
Oxford Bulletin of Economics and Statistics

*DOI:*
10.1111/obes.12602

OXFORD BULLETIN OF ECONOMICS AND STATISTICS, 86, 5 (2024) 0305-9049
doi: 10.1111/obes.12602

# Multivariate Trend-Cycle-Seasonal Decompositions with Correlated Innovations*

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## Abstract

Multivariate analysis can help to focus on important phenomena, including trend and cyclical movements, but any economic information in seasonality is typically ignored. The present paper aims to more fully exploit time series information through a multivariate unobserved component model for quarterly data that exhibits seasonality together with cross-variable component correlations. We show that economic restrictions, including common trends, common cycles and common seasonals can aid identification. The approach is illustrated using Italian GDP and consumption data.

## I. Introduction

Over many decades, economists have sought to understand the drivers underlying the time series evolution of series of interest. The notion that a series can be decomposed into components that are not directly observed is, therefore, deeply embedded in empirical economic analyses. In particular, unobserved components (UC) models enable researchers to study trends, cycles and seasonality as separate phenomena of interest. Although many researchers employ seasonally adjusted data, there is now a large body of evidence that seasonal movements are related to trend and/or cycle movements and that the use of

JEL Classification numbers: C32, E21, E32.

*This paper is dedicated to Mardi H. Dungey (1966–2019), our friend, co-author and mentor. A preliminary version was presented at the Mardi Dungey Memorial Research Conference, Washington DC, 21 February 2020, organized by Renee Fry-McKibbin and Tara Sinclair. We are also grateful to Paul Bekker, Gianluca Cubadda, James Morley, Simon van Norden, Tom Wansbeek, and especially Tara Sinclair for insightful comments on earlier versions of this paper. Comments of participants at various seminars are also gratefully acknowledged. Finally, thanks to the editor and the referees for constructive feedback that have led to substantial improvements in this version of the paper.

conventional seasonally adjusted data can distort economic decision-making; see, among others, Osborn (1988), Barsky and Miron (1989), Cecchetti and Kashyap (1996), Krane and Wascher (1999), Matas-Mir and Osborn (2004), Koopman and Lee (2009), Stock (2013), Wright (2013) and Abeln and Jacobs (2023). To account for these considerations, our approach is to analyze seasonal data using a UC model that allows for correlations across trend, cycle and seasonal components.

In particular, the analysis here builds on Hindrayanto, Jacobs, Osborn and Tian (2019, henceforth HJOT), who (for quarterly data) study a univariate seasonal UC specification that allows the innovations to be correlated across the three components. Although attractive in principle, HJOT show that the general model employing a conventional AR(2) cycle is under-identified, with one correlation restriction required for estimation. The present paper examines whether multivariate data can overcome this identification issue by exploiting cross-equation relationships in a correlated UC setting with trend, cycle and seasonal components.

Although conventional UC models assume that innovations in individual components are uncorrelated, a substantial literature has questioned this in the context of trend-cycle models; important contributions include Clark (1989), Morley, Nelson and Zivot (2003, henceforth MNZ), Morley (2007), Sinclair (2009) and Dungey *et al.* (2013, 2015).[1] Although univariate UC models are often employed, multivariate analysis can shed important light on underlying economic phenomena, such as common trends and/or common cycles (e.g. Harvey and Koopman, 1997; Morley, 2007; Fleischman and Roberts, 2011). Trenkler and Weber (2016, hereafter TW), provide a discussion of identification for the multivariate trend-cycle model with correlated innovations. However, to our knowledge, our analysis is the first to employ a correlated multivariate UC model with trend, cyclical and seasonal components.

Focusing on a quarterly bivariate specification, we show that the baseline unrestricted correlated seasonal UC model with a second-order cyclical component fails the rank condition for identification. Although this implies that the addition of more series does not in itself aid identification, we further show that when such series have common components then identification can be achieved by imposing the corresponding cross-equation restrictions. In particular, the use of common trend or common seasonality restrictions yields an identified model. However, the specification of a common cycle across series, with common AR coefficients, is not sufficient for identification and requires one additional restriction. The implications of some extensions of the baseline model in terms of the seasonal and cyclical components are also considered.

The paper proceeds as follows. Section II discusses multivariate correlated seasonal UC models, focusing on identification issues. Section III then illustrates our common component approach using Italian GDP and consumption data previously analysed by Cubadda (1999). The final section concludes.

[1] Harvey and Koopman (1992) point out that UC models with uncorrelated components can produce correlated trends and cycles. Hence they do not include correlated components in their models or their preferred state-space modeling programme suite SSF-Pack. However outcomes for UC models specified with uncorrelated components can differ from those with correlated components.

## II. Multivariate UC models

This section describes the model and discusses its identification, including economically plausible restrictions that may apply. A baseline specification is discussed in the first four subsections, with some extensions considered in the last subsection.

### Baseline seasonal UC model

Many macroeconomic variables exhibit trend, cycle and seasonal characteristics. Hence, for an observed $k \times 1$ vector $\boldsymbol{Y}_t$, consider a baseline multivariate UC model that explicitly recognizes these characteristics through the measurement equation

$$\boldsymbol{Y}_t = \boldsymbol{T}_t + \boldsymbol{C}_t + \boldsymbol{S}_t, \tag{1}$$

in which the trend, cycle and seasonal components ($\boldsymbol{T}_t$, $\boldsymbol{C}_t$ and $\boldsymbol{S}_t$, respectively) are $k \times 1$ vectors.

Following MNZ and many others, we assume the trend for each variable can be represented as a random walk with drift, so that

$$\boldsymbol{T}_t = \boldsymbol{T}_{t-1} + \boldsymbol{\beta} + \boldsymbol{\eta}_t, \tag{2}$$

where $\boldsymbol{\eta}_t = (\eta_{1t}, \ldots, \eta_{kt})'$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)'$ and the $k \times k$ covariance matrix $E[\boldsymbol{\eta}_t\boldsymbol{\eta}_t'] = \boldsymbol{\Sigma}_{\eta\eta}$ is not *a priori* assumed to be diagonal. The multivariate cyclical component of (1) is represented by the AR processes

$$\boldsymbol{\Phi}(L)\boldsymbol{C}_t = \boldsymbol{\varepsilon}_t, \tag{3}$$

where $\boldsymbol{\Phi}(L)$ is a $k \times k$ matrix in the lag operator $L$, with $\boldsymbol{\Phi}(L) = \boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \ldots - \boldsymbol{\Phi}_p L^p$ ($\boldsymbol{I}_k$ being a $k \times k$ identity matrix) having all roots strictly outside the unit circle, and with disturbance $\boldsymbol{\varepsilon}_t$, $E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'] = \boldsymbol{\Sigma}_{\varepsilon\varepsilon}$. As usual in economic applications of multivariate UC models (such as Morley, 2007, Sinclair, 2009 or Ma and Wohar, 2013) and to aid identification (see subsection II below), $\boldsymbol{\Phi}(L)$ is assumed diagonal with the cycle in each variable having the same order $p$. Empirical analyses typically employ $p = 2$, since this can adequately capture short-term movements in economic data while also allowing the parameters of the correlated UC trend-cycle model to be identified; see MNZ and TW for the univariate and multivariate cases, respectively.

As in HJOT and many other papers, seasonality is modeled using the so-called dummy variable form

$$\Psi(L)\boldsymbol{S}_t = \boldsymbol{\omega}_t, \tag{4}$$

where $\Psi(L)$ is the scalar annual summation polynomial over a year ($\Psi(L) = 1 + L + L^2 + L^3$ for quarterly data) and the disturbance vector $\boldsymbol{\omega}_t$ has covariance matrix $E[\boldsymbol{\omega}_t\boldsymbol{\omega}_t'] = \boldsymbol{\Sigma}_{\omega\omega}$. This specification views seasonality as a long term phenomenon, with $\Psi(L)$ implying that the series contain the full set of unit roots at seasonal (namely the annual and biannual) frequencies; see, for example, Ghysels and Osborn (2001, chapter 3). When combined with (2) and (3), (4) then allocates all short term stochastic fluctuations to the cyclical component, with the trend and seasonal components capturing long term non-seasonal and seasonal movements, respectively.

It may be noted that the nature of seasonality as represented in this baseline model is somewhat restrictive, with no deterministic seasonal component included and unit roots assumed to be present at all relevant seasonal frequencies. These and other relevant issues are discussed at the end of this section.

To facilitate later discussion, stack the UC model disturbances of (2)–(4) to form the $3k \times 1$ vector $\boldsymbol{U}_t$ as

$$\boldsymbol{U}_t = [\boldsymbol{\eta}_t', \boldsymbol{\varepsilon}_t', \boldsymbol{\omega}_t']', \tag{5}$$

and define the $3k \times 3k$ covariance matrix

$$E[\boldsymbol{U}_t\boldsymbol{U}_t'] \equiv \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{\eta\eta} & \boldsymbol{\Sigma}_{\eta\varepsilon} & \boldsymbol{\Sigma}_{\eta\omega} \\ \boldsymbol{\Sigma}_{\eta\varepsilon}' & \boldsymbol{\Sigma}_{\varepsilon\varepsilon} & \boldsymbol{\Sigma}_{\varepsilon\omega} \\ \boldsymbol{\Sigma}_{\eta\omega}' & \boldsymbol{\Sigma}_{\varepsilon\omega}' & \boldsymbol{\Sigma}_{\omega\omega} \end{bmatrix}, \tag{6}$$

where in an obvious notation,

$$E[\boldsymbol{\eta}_t\boldsymbol{\varepsilon}_t'] = \boldsymbol{\Sigma}_{\eta\varepsilon}, E[\boldsymbol{\eta}_t\boldsymbol{\omega}_t'] = \boldsymbol{\Sigma}_{\eta\omega}, E[\boldsymbol{\varepsilon}_t\boldsymbol{\omega}_t'] = \boldsymbol{\Sigma}_{\varepsilon\omega}. \tag{7}$$

It is important to note that our correlated UC model does not make the assumption of the conventional multivariate UC model that all cross-covariances in (7) are zero. Although the disturbances are possibly cross-correlated at $t$, they are assumed uncorrelated over time, so that

$$E[\boldsymbol{U}_{t_1}\boldsymbol{U}_{t_2}'] = \boldsymbol{0}, \qquad t_1 \neq t_2.$$

It is also assumed that all diagonal elements of $\boldsymbol{\Sigma}$ are strictly positive, so that stochastic trend, cycle and seasonal components exist for all variables in the system.

### Reduced form

As a preliminary to identification, we consider the reduced form and autocovariances of the multivariate seasonal UC model for quarterly data.[2] It is straightforward to see that the system (1)–(4) implies the reduced form

$$\boldsymbol{\Phi}(L)\Delta_4\boldsymbol{Y}_t = \boldsymbol{\Phi}(1)\Psi(1)\boldsymbol{\beta} + \boldsymbol{\Phi}(L)\Psi(L)\boldsymbol{\eta}_t + \Delta_4\boldsymbol{\varepsilon}_t + \boldsymbol{\Phi}(L)\Delta_1\boldsymbol{\omega}_t, \tag{8}$$

where $\Delta_4 = 1 - L^4$ is the annual difference and $\Delta_1$ is the usual first difference. In this general model, each element of $\boldsymbol{Y}_t$ is seasonally integrated (see, e.g. Ghysels and Osborn 2001, chapter 3), due to the presence of a zero-frequency unit root in its trend component (2) and the full set of unit roots at seasonal frequencies through the nonstationary seasonal process of (4). Hence annual differencing is required to reduce each univariate process in $\boldsymbol{Y}_t$ to stationarity, but this does not rule out cointegration across the components of $\boldsymbol{Y}_t$.

[2]The expressions in this section can be easily generalized to monthly data.

Since $\boldsymbol{\Phi}(L)$ is of order $p$ and the disturbance process $\boldsymbol{Z}_t \equiv \boldsymbol{\Phi}(L)\Psi(L)\boldsymbol{\eta}_t + \Delta_4\boldsymbol{\varepsilon}_t + \boldsymbol{\Phi}(L)\Delta_1\boldsymbol{\omega}_t$ is the sum of moving averages, the reduced form (8) is a VARMA($p, q$) process, with $q = p + 3$ for $p > 0$. Note that the AR matrix polynomial $\boldsymbol{\Phi}(L)$ carries over to the reduced form, but the reduced form vector MA parameters are complicated functions of the parameters in $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Sigma}$.

Ensuring the uniqueness of the reduced form (8) is important for subsequent analysis. Specifically, it is necessary to ensure that no other finite order VARMA representation exists which has the same autocovariance properties as those of (8). However, achieving uniqueness is difficult when $\boldsymbol{\Phi}(L)$ is non-diagonal; see discussions in Dufour and Pelletier (2022) for general VARMA models and TW (section III.5) in the context of the correlated trend-cycle UC model. Therefore, and in line with previous economic applications, our assumptions of section II assume diagonal $\boldsymbol{\Phi}(L)$. Ruling out the AR and MA polynominals in each equation $i = 1, \ldots, k$ having any factor in common,[3] (8) is then unique and its parameters are identified.

An immediate consequence is that the reduced form (8) yields the AR parameters of $\boldsymbol{\Phi}(L)$. Also, noting that $\Psi(L)$ is the (known) annual summation operator, the UC drift vector $\boldsymbol{\beta}$ can be obtained from the reduced form intercept vector.

To focus on the disturbances, write the disturbance process $\boldsymbol{Z}_t$ of (8) as

$$\begin{aligned}\boldsymbol{Z}_t &= \boldsymbol{A}(L)\boldsymbol{\eta}_t + (1 - L^4)\boldsymbol{\varepsilon}_t + \boldsymbol{B}(L)\boldsymbol{\omega}_t \\ &= \boldsymbol{H}(L)\boldsymbol{U}_t,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\boldsymbol{A}(L) &= (1 + L + L^2 + L^3)\boldsymbol{\Phi}(L) = \boldsymbol{I}_k + \boldsymbol{A}_1 L + \cdots + \boldsymbol{A}_{p+3}L^{p+3}, \\ \boldsymbol{B}(L) &= (1 - L)\boldsymbol{\Phi}(L) = \boldsymbol{I}_k + \boldsymbol{B}_1 L + \cdots + \boldsymbol{B}_{p+1}L^{p+1},\end{aligned} \tag{10}$$

while $\boldsymbol{U}_t$ is defined in (5) and $\boldsymbol{H}(L)$ is the $k \times 3k$ matrix

$$\begin{aligned}\boldsymbol{H}(L) &\equiv \begin{bmatrix}\boldsymbol{A}(L) & (1 - L^4)\boldsymbol{I}_k & \boldsymbol{B}(L)\end{bmatrix} \\ &= \boldsymbol{H}_0 + \boldsymbol{H}_1 L + \boldsymbol{H}_2 L^2 + \cdots + \boldsymbol{H}_q L^q,\end{aligned} \tag{11}$$

where $q = \max(p + 3, 4)$. For the specific case of $p = 2$ (see the discussion of the next subsection), $q = 5$ and it can be seen that

$$\begin{aligned}\boldsymbol{H}_0 &= \begin{bmatrix}\boldsymbol{I}_k & \boldsymbol{I}_k & \boldsymbol{I}_k\end{bmatrix}, \\ \boldsymbol{H}_1 &= \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{0} & \boldsymbol{B}_1\end{bmatrix} = \begin{bmatrix}(\boldsymbol{I}_k - \boldsymbol{\Phi}_1) & \boldsymbol{0} & -(\boldsymbol{I}_k + \boldsymbol{\Phi}_1)\end{bmatrix}, \\ \boldsymbol{H}_2 &= \begin{bmatrix}\boldsymbol{A}_2 & \boldsymbol{0} & \boldsymbol{B}_2\end{bmatrix} = \begin{bmatrix}(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2) & \boldsymbol{0} & (\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2)\end{bmatrix}, \\ \boldsymbol{H}_3 &= \begin{bmatrix}\boldsymbol{A}_3 & \boldsymbol{0} & \boldsymbol{B}_3\end{bmatrix} = \begin{bmatrix}(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2) & \boldsymbol{0} & \boldsymbol{\Phi}_2\end{bmatrix},\end{aligned}$$

[3]The polynomial $\phi_i(L)$ in the $i$th equation will cancel in (8) when the corresponding cycle disturbance has zero variance. However, this would imply the absence of any stochastic cycle component in the $i$th variable.

$$\boldsymbol{H}_4 = \begin{bmatrix} \boldsymbol{A}_4 & -\boldsymbol{I}_k & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} -(\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2) & -\boldsymbol{I}_k & \boldsymbol{0} \end{bmatrix},$$
$$\boldsymbol{H}_5 = \begin{bmatrix} \boldsymbol{A}_5 & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\Phi}_2 & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Therefore, the primary issue for identification of the correlated UC model concerns whether the elements of the covariance matrix $\boldsymbol{\Sigma}$ of (6) can be deduced from the properties of the reduced form. With $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\beta}$ treated as given, the key properties are the non-zero autocovariance matrices of $\boldsymbol{Z}_t$, namely

$$\boldsymbol{\Gamma}_\ell = \sum_{i=0}^{q-\ell} \boldsymbol{H}_{i+\ell} \boldsymbol{\Sigma} \boldsymbol{H}_i' \qquad \ell = 0, 1, \ldots, q. \tag{12}$$

Using (6) and (11), (12) can be used to express the autocovariances of $\boldsymbol{Z}_t$ as functions of the elements of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Phi}(L)$.

### Covariance matrix identification

Identification proceeds by considering the relationship between the autocovariances of (12) and the covariance matrix $\boldsymbol{\Sigma}$ of the underlying UC model. MNZ show that $p \geq 2$ is sufficient for the identification of the univariate correlated trend-cycle model, while TW generalize this result to the multivariate context. The addition of seasonality complicates identification, with HJOT showing not only that univariate models of the form (1)–(6) for quarterly data with $k = 1$ are under-identified for $p \leq 1$, but also that one additional disturbance covariance restriction is required for identification when $p = 2$.

#### *Order condition*

Following the line of analysis used by the above authors, the previous subsection has already noted that $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\beta}$ are identified from the multivariate ARMA reduced form. Further, from (10) and (11), $\boldsymbol{H}_i$ $(i = 0, \ldots, q)$ involve only the parameters of $\boldsymbol{\Phi}(L)$. Therefore, the autocovariances of $\boldsymbol{Z}_t$ defined by (12) can be used to provide information about the $3k(3k+1)/2$ distinct elements of $\boldsymbol{\Sigma}$, effectively treating the other parameters as given. The order condition for identification of the baseline model then requires $\boldsymbol{\Gamma}_\ell$ of (12) for $\ell = 0, 1, \ldots, q$ to contain at least $3k(3k+1)/2$ distinct elements.

The $q+1$ non-null autocovariance matrices of (12) have $qk^2 + k(k+1)/2$ distinct elements, of which $k(k+1)/2$ are contributed by the contemporaneous covariance matrix $\boldsymbol{\Gamma}_0$. As discussed above, the VMA order $q$ is a consequence of both the data frequency and cycle order $p$. For quarterly data and $p \leq 1$, $q = 4$ and hence the number of distinct autocovariance elements in $\boldsymbol{\Gamma}_\ell$ for $\ell = 0, \ldots, q$, namely $(9k^2 + k)/2$, is less than the number of distinct elements of $\boldsymbol{\Sigma}$, $(9k^2 + 3k)/2$. Consequently, as for the univariate case, the parameters of the quarterly unrestricted correlated multivariate UC model with seasonality are not identified when $p \leq 1$. We therefore concentrate on the case $p = 2$, which is of interest for empirical as well as theoretical reasons.

With $p = 2$, $\boldsymbol{\Gamma}_\ell$ of (12) for $\ell = 0, \ldots, 5$ contain $(11k^2 + k)/2$ distinct elements. It is easily seen that $(11k^2 + k)/2 = (9k^2 + 3k)/2$ for $k = 1$ (the case discussed by HJOT) and $(11k^2 + k)/2 > (9k^2 + 3k)/2$ for $k > 1$. Therefore, the order condition for identification is then satisfied for the correlated seasonal UC model. However, the rank condition also needs to be satisfied and HJOT show that this fails in the univariate case.

*Rank condition*

Using a similar notation to TW, define $\boldsymbol{\gamma}_0^* = \text{vech}(\boldsymbol{\Gamma}_0)$, where the vech operator columnwise stacks the elements of $\boldsymbol{\Gamma}_0$ on and below the main diagonal into the $k(k+1)/2$ vector $\boldsymbol{\gamma}_0^*$, starting with the first column of $\boldsymbol{\Gamma}_0$ and with the elements of each subsequent column placed below the immediately preceding one. Also define the $k^2$ vectors $\boldsymbol{\gamma}_i = \text{vec}(\boldsymbol{\Gamma}_i)$, $i = 1, \ldots, 5$, where the conventional vec operator stacks all elements in the columns of the relevant matrix below each other. The vector $\boldsymbol{\gamma}^* = [\boldsymbol{\gamma}_0^{*\prime}, \boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2', \boldsymbol{\gamma}_3', \boldsymbol{\gamma}_4', \boldsymbol{\gamma}_5']'$ then contains the $(11k^2 + k)/2$ distinct autocovariance elements for $\boldsymbol{Z}_t$ at lags $\ell = 0, \ldots, 5$. Similarly, define the vector $\boldsymbol{\sigma}^* = \text{vech}(\boldsymbol{\Sigma})$ containing the $(9k^2 + 3k)/2$ distinct elements of the component covariance matrix $\boldsymbol{\Sigma}$ and also the $(11k^2 + k)/2 \times (9k^2 + 3k)/2$ matrix $\boldsymbol{D}$ whose elements depend only on $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ to write the relationships of (12) as the system of equations

$$\boldsymbol{\gamma}^* = \boldsymbol{D}\boldsymbol{\sigma}^*, \tag{13}$$

in which the elements of $\boldsymbol{\sigma}^*$ are unknown. Identification requires $\boldsymbol{D}$ to have rank $(9k^2 + 3k)/2$.

*Bivariate case*

In order to progress, we examine the rank of $\boldsymbol{D}$ for a bivariate model. With $k = 2$, the dimension of $\boldsymbol{D}$ is $23 \times 21$ and satisfaction of the rank condition requires this matrix to have a full rank of 21. Recalling that $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are diagonal and denoting the non-zero AR(2) coefficients for the first and second variables of the system as $\phi_{11}, \phi_{12}$ and $\phi_{21}, \phi_{22}$, respectively, Appendix A shows explicit expressions for the elements of $\boldsymbol{D}$. Employing these expressions, symbolic analysis in MATLAB[4] shows that this matrix has rank 19, indicating linear dependencies exist between the 21 columns of $\boldsymbol{D}$. This analytical result is based on the assumption that each symbol (that is, each of $\phi_{11}$, $\phi_{12}$, $\phi_{21}$ and $\phi_{22}$) represents a non-zero and distinct value, and hence it assumes that the AR coefficients are not common across the two variables. In the special case where the two cyclical components have identical AR dynamics, namely $\phi_{11} = \phi_{21}$ and $\phi_{12} = \phi_{22}$, this analysis shows that the rank of $\boldsymbol{D}$ reduces to 18.

The rank of $\boldsymbol{D}$ therefore implies that at least two covariance restrictions are required for identification in the baseline bivariate seasonal UC model, with three required when the AR dynamics are common across the two variables. These results generalise those of HJOT for the univariate case, where a single restriction is required for identification.

The implication of the baseline correlated trend-cycle-seasonal UC model with $p = 2$ not being identified in the bivariate case is that the addition of equations does not in itself solve the identification problem that arises in the single equation case. Fortunately,

[4]The Symbolic Math Toolbox in MATLAB allows the manipulation of symbolic expressions and can be employed to obtain the rank of a matrix containing symbols and numerical values. The webpage https://au.mathworks.com/help/symbolic/rank.html provides the information of the use of a MATLAB function to calculate the exact rank of a matrix.

however, multivariate analysis is frequently undertaken in order to investigate phenomena that are common across variables and, as considered in the next subsection, common component restrictions can be used to achieve identification.

Before considering common components, it may be noted that the conventional multivariate UC model, used by Harvey (1989) among many others, imposes zero correlations across different components. In other words, the covariance matrix $\boldsymbol{\Sigma}$ of (6) is assumed to be block diagonal, with $6k^2$ zero restrictions thereby imposed on its $9k^2$ elements. The discussion above implies that this uncorrelated multivariate UC specification is consequently substantially over-identified.

**Common component restrictions**

Some previous studies employing (non-seasonal) UC models, including Morley (2007), Ma and Wohar (2013), Clark (1989), Fleischman and Roberts (2011) and McElroy (2017), specify common components across variables based on economic rationale and to improve efficiency of estimation. Where justified on economic grounds, we propose that such restrictions can be used to aid identification of the correlated seasonal UC model.

For $\boldsymbol{T}_t = (\tau_{1t}, \tau_{2t}, \ldots, \tau_{kt})'$, $\boldsymbol{C}_t = (c_{1t}, c_{2t}, \ldots, c_{kt})'$ and $\boldsymbol{S}_t = (s_{1t}, s_{2t}, \ldots, s_{kt})'$, possible common component restrictions include:

1. Common trends, which imposes in (2)

$$\tau_{it} = d_i \tau_{1t} = d_i \tau_{1,t-1} + d_i \beta_1 + d_i \eta_{1t}, i = 2, \ldots, k,$$

   so that both the deterministic and stochastic trend components of the *ith* element of $T_t$ are the same scalar multiple $d_i$ of $\tau_{1t}$.
2. Common cycles, for which in (3)

$$c_{it} = b_i c_{1t}, i = 2, \ldots, k, \tag{14}$$

   for $b_i > 0$, implying that $\boldsymbol{\Phi}(L) = \phi(L)\boldsymbol{I}_k$ where $\phi(L)$ is scalar and

$$\boldsymbol{\varepsilon}_{it} = b_i \boldsymbol{\varepsilon}_{1t} i = 2, \ldots, k. \tag{15}$$

3. Common seasonals, with $s_{it} = a_i s_{1t}$, for $i = 2, \ldots, k$ so that

$$\omega_{it} = a_i \omega_{1t}, i = 2, \ldots, k. \tag{16}$$

4. Perfectly correlated trend innovations, which is a less restrictive case of common trends that places no restriction on the drift parameters, but imposes

$$\eta_{it} = d_i \eta_{1t}, i = 2, \ldots, k. \tag{17}$$

5. Perfectly correlated cycle innovations, in which no cross-equation restrictions are placed on the AR parameters, but the cycle shocks satisfy (15).

Note that perfectly correlated trend innovations, and hence also common trends, imply cointegration between the $k$ series which are linked through a single stochastic trend. Similarly, common seasonals implies the existence of seasonal cointegration across the variables.[5] A concept of common cycles is used in the analysis of non-stationary multivariate time series, with Cubadda (1999) focusing on the seasonal case. However, the unit root framework used by Cubadda (1999) defines common cycles from an unrestricted VMA representation of the seasonally differenced series, whereas the corresponding representation in section II is derived from the component models and hence does not have an unrestricted form.[6] Although the setup of the decomposition differs between Cubadda (1999) and our study, for the bivariate case common cycles in both imply the same dynamics apply for the cycle components across variables together with perfectly correlated cycle shocks.

The list of possible restrictions is not exhaustive; for example more than one common stochastic trend may be appropriate for $k > 2$, cointegration may apply at only some seasonal frequencies or zero restrictions could be appropriate for some elements of $\boldsymbol{\Sigma}$. However, the list above is useful for illustration and gives rise to specifications employed in the application of the next section.

Focusing again on the bivariate model ($k = 2$), Table 1 sets out the restrictions implied by the common trend, common cycle and common seasonal specifications. In addition, the table includes the model where the only non-zero correlations are within components, namely the conventional uncorrelated seasonal UC model.

TABLE 1

*Restrictions and the resulting ranks for bivariate seasonal UC models*

| | Uncorrelated | Common components | | |
|---|---|---|---|---|
| | Components | Trend | Cycle | Seasonal |
| Within component covariances | | $\sigma^2_{\eta_2} = d_2^2\sigma^2_{\eta_1}$<br>$\sigma_{\eta_1\eta_2} = d_2\sigma^2_{\eta_1}$ | $\sigma^2_{\varepsilon_2} = b_2^2\sigma^2_{\varepsilon_1}$<br>$\sigma_{\varepsilon_1\varepsilon_2} = b_2\sigma^2_{\varepsilon_1}$ | $\sigma^2_{\omega_2} = a_2^2\sigma^2_{\omega_1}$<br>$\sigma_{\omega_1\omega_2} = a_2\sigma^2_{\omega_1}$ |
| Trend-cycle covariances | $\sigma_{\eta_2\varepsilon_1} = \sigma_{\eta_1\varepsilon_1} = 0$<br>$\sigma_{\eta_2\varepsilon_2} = \sigma_{\eta_1\varepsilon_2} = 0$ | $\sigma_{\eta_2\varepsilon_1} = d_2\sigma_{\eta_1\varepsilon_1}$<br>$\sigma_{\eta_2\varepsilon_2} = d_2\sigma_{\eta_1\varepsilon_2}$ | $\sigma_{\eta_1\varepsilon_2} = b_2\sigma_{\eta_1\varepsilon_1}$<br>$\sigma_{\eta_2\varepsilon_2} = b_2\sigma_{\eta_2\varepsilon_1}$ | |
| Trend-seasonal covariances | $\sigma_{\eta_2\omega_1} = \sigma_{\eta_1\omega_1} = 0$<br>$\sigma_{\eta_2\omega_2} = \sigma_{\eta_1\omega_2} = 0$ | $\sigma_{\eta_2\omega_1} = d_2\sigma_{\eta_1\omega_1}$<br>$\sigma_{\eta_2\omega_2} = d_2\sigma_{\eta_1\omega_2}$ | | $\sigma_{\eta_1\omega_2} = a_2\sigma_{\eta_1\omega_1}$<br>$\sigma_{\eta_2\omega_2} = a_2\sigma_{\eta_2\omega_1}$ |
| Cycle-seasonal covariances | $\sigma_{\varepsilon_2\omega_1} = \sigma_{\varepsilon_1\omega_1} = 0$<br>$\sigma_{\varepsilon_2\omega_2} = \sigma_{\varepsilon_1\omega_2} = 0$ | | $\sigma_{\varepsilon_2\omega_1} = b_2\sigma_{\varepsilon_1\omega_1}$<br>$\sigma_{\varepsilon_2\omega_2} = b_2\sigma_{\varepsilon_1\omega_2}$ | $\sigma_{\varepsilon_1\omega_2} = a_2\sigma_{\varepsilon_1\omega_1}$<br>$\sigma_{\varepsilon_2\omega_2} = a_2\sigma_{\varepsilon_2\omega_1}$ |
| Other restrictions | | $\frac{\beta_2}{\beta_1} = d_2$ | $\phi_{21} = \phi_{11}$<br>$\phi_{22} = \phi_{12}$ | |
| Rank($[\boldsymbol{D}', \tilde{\boldsymbol{D}}']'$) | 21 | 21 | 20 | 21 |

[5]Ghysels and Osborn (2001, section 3.6), provide an introductory discussion of seasonal cointegration, while a more detailed technical analysis can be found in Johansen and Schaumburg (1999). In general, seasonal cointegration allows different cointegration vectors to apply at each seasonal frequency, whereas the form of (16) requires the same cointegrating relationship to apply at the seasonal (annual and biannual) frequencies.

[6]For the univariate nonseasonal case, MNZ show that trend-cycle decomposition from an ARMA model in the first differenced series is equivalent to that from the correlated UC model. However, to our knowledge, similar analyses have not been undertaken for multivariate or seasonal models.

As discussed above, a bivariate correlated seasonal UC model requires at least two restrictions in order to identify all elements of $\boldsymbol{\Sigma}$. To examine whether the common component models satisfy the rank condition, the $m$ restrictions of Table 1are expressed in the form $\tilde{\boldsymbol{D}}\boldsymbol{\sigma}^* = \boldsymbol{0}$ where $\boldsymbol{0}$ is an $m \times 1$ vector of zeros and $\tilde{\boldsymbol{D}}$ is the $m \times 21$ coefficient matrix implied by the restrictions.[7] The additional equations are then combined with (13) and the rank of $[\boldsymbol{D}', \tilde{\boldsymbol{D}}']'$ is investigated for each specification, with symbolic analysis in MATLAB again used for this purpose. As discussed above, the rank condition for identification requires $[\boldsymbol{D}', \tilde{\boldsymbol{D}}']'$ to be of rank 21. As shown in Table 1, the common trend and common seasonality specifications have full column rank and are identified. However, the common cycle requires one further restriction for identification.

For simplicity, Table 1 includes only the specifications that are examined in the illustration of the next section. However, it can be noted that, when analyzed in the same way, the bivariate perfectly correlated trend and perfectly correlated cycle innovations models both satisfy the rank condition for identification. This is unsurprising for the perfectly correlated trend shock model, as the drift parameters are identified from the reduced form and they do not enter the expressions for the autocovariances of $\boldsymbol{Z}_t$ in (12). On the other hand, while the AR coefficients of the cycle component are also identified from the reduced form, their values enter the autocovariance matrices of (12), as noted above. In principle, therefore, the more highly parameterized perfectly correlated cycle innovations model is identified provided that the two series have distinct AR coefficients, whereas the AR restrictions of the common cycle model lead to a failure of identification.

## Extensions and discussion

The specification used for the baseline correlated UC model above follows conventional practice in the UC model literature. However, the specification may be considered restrictive in some respects, so that this subsection discusses some possible extensions to the model.[8]

### *Modelling seasonality*

From the perspective of modelling seasonality, it may be surprising that deterministic seasonality is not explicitly considered in the baseline model above or, indeed, typically in seasonal UC analyses. Deterministic seasonality, which allows the mean of a process to vary across the seasons of the year (see, e.g. Ghysels and Osborn, 2001, chapter 2), can be viewed as a special case of (4) when the covariance matrix satisfies $\boldsymbol{\Sigma}_{\omega\omega} = \boldsymbol{0}$.

[7] Table 1 introduces unknown scale parameters $d_2$, $b_2$ and $a_2$ for the common trend, common cycle and common seasonal specifications, respectively. One of the restrictions is reserved to identify the scale parameter. More specifically, the common trend model specifies that $d_2$ is the ratio of two (identifiable) drift terms and thus the remaining six restrictions can be used to aid identification of the component covariance parameters with $d_2$ considered known. For the common cycle and common seasonality models, the restrictions $\sigma^2_{\varepsilon_2} = b_2^2\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\omega_2} = a_2^2\sigma^2_{\omega_1}$ are retained for solving for $b_2$ and $a_2$, respectively.

[8] The constructive comments of the referees of an earlier version of this paper have led to the extensions considered in this section.

However, this would rule out the presence of seasonal unit roots in all variables and would effectively mean that the issues relating to identification are those of the correlated UC trend-cycle model.

Of more interest is to generalize the baseline model to allow both deterministic and non-stationary stochastic seasonality, which can be achieved by replacing the measurement equation of (1) by

$$\boldsymbol{Y}_{q,t} = \boldsymbol{M}_{q,t} + \boldsymbol{T}_t + \boldsymbol{C}_t + \boldsymbol{S}_t, \tag{18}$$

where the first subscript of $\boldsymbol{Y}_{q,t}$ indicates the period (in our case the quarter $q$) of the year in which observation $t$ falls, $\boldsymbol{M}_{q,t} = E[\boldsymbol{Y}_{q,t}]$ is the vector of expected values corresponding to that observation and each of the remaining components has zero mean. Specifically, $\boldsymbol{M}_{q,t}$ allows trends that vary over the quarters of the year and hence incorporates drift effects. Therefore, the trend component of (2) is replaced by the zero-mean stochastic trend vector

$$\boldsymbol{T}_t = \boldsymbol{T}_{t-1} + \boldsymbol{\eta}_t. \tag{19}$$

The expressions for $\boldsymbol{C}_t$ and $\boldsymbol{S}_t$ in (3) and (4), respectively, remain unchanged, as does the covariance matrix (6).

Following the same line of argument as in section II, the reduced form for the more general model is

$$\boldsymbol{\Phi}(L)\Delta_4\boldsymbol{Y}_t = \boldsymbol{\Phi}(1)\Psi(1)\Delta_4\boldsymbol{M}_{q,t} + \boldsymbol{\Phi}(L)\Psi(L)\boldsymbol{\eta}_t + \Delta_4\boldsymbol{\varepsilon}_t + \boldsymbol{\Phi}(L)\Delta_1\boldsymbol{\omega}_t. \tag{20}$$

Although conventional seasonal dummy effects in $\boldsymbol{M}_{q,t}$ are annihilated by the seasonal differencing operator, any seasonally-varying drift terms can be identified from the reduced form intercept vector, as above for the baseline model. More importantly, the discussions above relating to the identification of the elements of the disturbance covariance matrix $\boldsymbol{\Sigma}$ of (6) continues to apply when the model explicitly includes seasonal dummy variables and/or seasonal trends.

As noted above, the seasonal component (4) of the baseline model assumes the presence of unit roots at all seasonal frequencies. For quarterly data, where $\Psi(L) = 1 + L + L^2 + L^3 = (1 + L)(1 + L^2)$, these unit roots are $-1$ and the complex pair $\pm i$ arising from the factors $(1 + L)$ and $(1 + L^2)$, respectively. Since all seasonal unit roots are treated together, the only common component restriction on the seasonals available for consideration is the common seasonals specification discussed in section II. However, the seasonal cointegration implied by common seasonals requires the same cointegrating relationship to apply at all seasonal frequencies, whereas the literature relating to seasonal cointegration typically allows any cointegrating relationships to differ across these frequencies (see, e.g. Ghysels and Osborn, 2001, section 3.6 or Johansen and Schaumburg, 1999).

Rather than specifying the evolution of seasonality through the use of the annual summation operator $\Psi(L)$, the UC framework can be used to separately consider the evolution at each seasonal frequency, as noted by, for example, Durbin and Koopman (2012, subsection 3.2.2) in the context of the uncorrelated seasonal UC model. For quarterly data, the seasonal component of (4) can be replaced by the two relationships[9]

$$
\begin{aligned}
(1+L^2)\boldsymbol{S}_{1,t} &= \boldsymbol{\omega}_{1,t} \\
(1+L)\boldsymbol{S}_{2,t} &= \boldsymbol{\omega}_{2,t},
\end{aligned}
\tag{21}
$$

where the $k \times 1$ vectors $\boldsymbol{S}_{1,t}$ and $\boldsymbol{S}_{2,t}$ relate to the unit roots at the annual and semi-annual frequencies, respectively, while $\boldsymbol{\omega}_{1,t}$ and $\boldsymbol{\omega}_{2,t}$ are temporally uncorrelated disturbance vectors. The total seasonal effect is then given by $\boldsymbol{S}_t = \boldsymbol{S}_{1,t} + \boldsymbol{S}_{2,t}$. It is, however, important to appreciate that $\boldsymbol{S}_t$ defined in this way satisfies

$$
\Psi(L)\boldsymbol{S}_t = (1+L)\boldsymbol{\omega}_{1,t} + (1+L^2)\boldsymbol{\omega}_{2,t},
\tag{22}
$$

so that $\Psi(L)\boldsymbol{S}_t$ is an MA(2) process, in contrast to the vector white noise process followed by $\Psi(L)\boldsymbol{S}_t$ in the baseline specification of (4).

Although separately considering the unit roots at each seasonal frequency adds flexibility to the seasonal component, it comes at the cost of additional parameters to be estimated. The focus of his paper is to consider how common component restrictions can be used to identify the parameters of the correlated UC model with the simpler seasonal form of (4). Therefore, the potential use of (21) is left as an issue for further research.

*Seasonal and cross-frequency cointegration*

The literature concerned with seasonal cointegration, including Johansen and Schaumburg (1999), Cubadda (1999) and many others, predominately considers possible cointegration separately at each seasonal frequency. While noting that such cointegration could be useful for identification by providing some cross-equation restrictions, it also increases the complexity of the model by requiring separate consideration of the unit roots at each seasonal frequency and hence is beyond the scope of this paper.

It was noted above that restrictions other than common components could be imposed for identification. Indeed, HJOT use perfect (positive or negative) disturbance correlations for identification of the univariate seasonal correlated UC model. Where appropriate, restrictions of this type can also be employed in the multivariate case, not only within variables but also across variables. For example, perfect correlation of the trend disturbance $\eta_{1t}$ of $y_{1t}$ with the seasonal disturbance $\omega_{2t}$ of $y_{2t}$ would give rise to cointegration between the trend component of $y_{1t}$ and the seasonal component $y_{2t}$. Such cross-frequency cointegration has been analyzed recently in a VAR framework by del Barrio, Cubadda, and Osborn (2022), for which the UC approach provides a complementary perspective.

[9]For univariate data, Durbin and Koopman (2012) use a two-equation trigonometric representation for the component at the annual frequency $\lambda_1 = \pi/2$ rather than the form used in the first equation of (21). Using their notation except for a superscript $K$ on their mutually and temporally uncorrelated disturbances series, here denoted $\omega_{1t}^K$, $\omega_{1t}^{*K}$, the two trigonometric relationships together imply $(1+L^2)\gamma_{1t} = (\omega_{1t}^K + \omega_{1,t-1}^{*K})$. With $S_{1,t}$ replacing $\gamma_{1t}$ and noting that the right-hand side of the last equation is a serially uncorrelated disturbance, this is the scalar version of the equation in (21). For ease of interpretation, we prefer to use this simple expression.

This brief discussion indicates that, in addition to possible zero restrictions on individual covariances and common component restrictions as discussed above cross-frequency cointegration could be useful for identification in the baseline seasonal correlated UC model with $p = 2$.

*Higher autoregressive orders ($p > 2$)*
Section II investigates the order and rank conditions relating to the autocovariances of $\boldsymbol{Z}_t$ defined by (12) for quarterly data with $p = 2$ and $k = 2$. As also discussed there, for $p > 0$ lags and $k$ variables, the order condition requires $\boldsymbol{\Gamma}_\ell$ of (12) for $\ell = 0, 1, \ldots, p + 3$ to contain at least $3k(3k + 1)/2$ distinct elements. The $p + 4$ non-null autocovariances have $(p + 3)k^2 + k(k + 1)/2 = (2p + 7)^2 + k/2$ distinct elements. Hence the order condition for identification is always satisfied for $k \geq 2$ and $p \geq 2$.

Following the same approach as in section II, namely using symbolic analysis in MATLAB, we further investigated the rank condition for the bivariate case ($k = 2$) when $p = 3$. Matrix $\boldsymbol{D}$ now becomes $27 \times 21$ and this matrix is found to have full column rank of 21. This implies that an unrestricted correlated seasonal UC model can be identified when a sufficiently high AR order applies. However, it is an open question as to how useful this result is in practice, since all variables in the system would need to have the required AR lag order[10] of $p > 2$ and it may be difficult to satisfy this requirement empirically.

## III. Illustration: Italian GDP and consumption

This section returns to the baseline correlated seasonal UC model with $p = 2$. Our purpose here is not to examine all possible specifications, which would include individual zero covariance restrictions and potentially separate treatment of unit roots at each seasonal frequency, but rather to illustrate how common components can be used to achieve identification for the baseline model with a conventional AR(2) cycle.

The empirical illustration builds on the analysis of Cubadda (1999), who investigates testing issues associated with cointegration and common cycles in seasonal data and applies his methods to Italian GDP and consumption data. In particular, the tests applied by Cubadda (1999) do not reject the presence of unit roots at the zero or seasonal frequencies, so that both series can be treated as being seasonally integrated, as required by the UC model of section II. Further, using his VAR framework, Cubadda (1999) finds evidence to support both (zero frequency) cointegration and common cycles in this data, so that our use of the UC model provides an interesting comparison.

### Data and estimation

Cubadda (1999) provides data[11] for seasonally unadjusted quarterly real GDP and real consumption of non-durable goods and services for Italy from 1970Q1 to 1996Q1.

[10]Using the symbolic approach, the $27 \times 21$ matrix $\boldsymbol{D}$ is found to have rank of 20 when one variable in a bivariate system has an AR order of 3 and the other has order 2.
[11]Data are available at http://econ.queensu.ca/jae/.

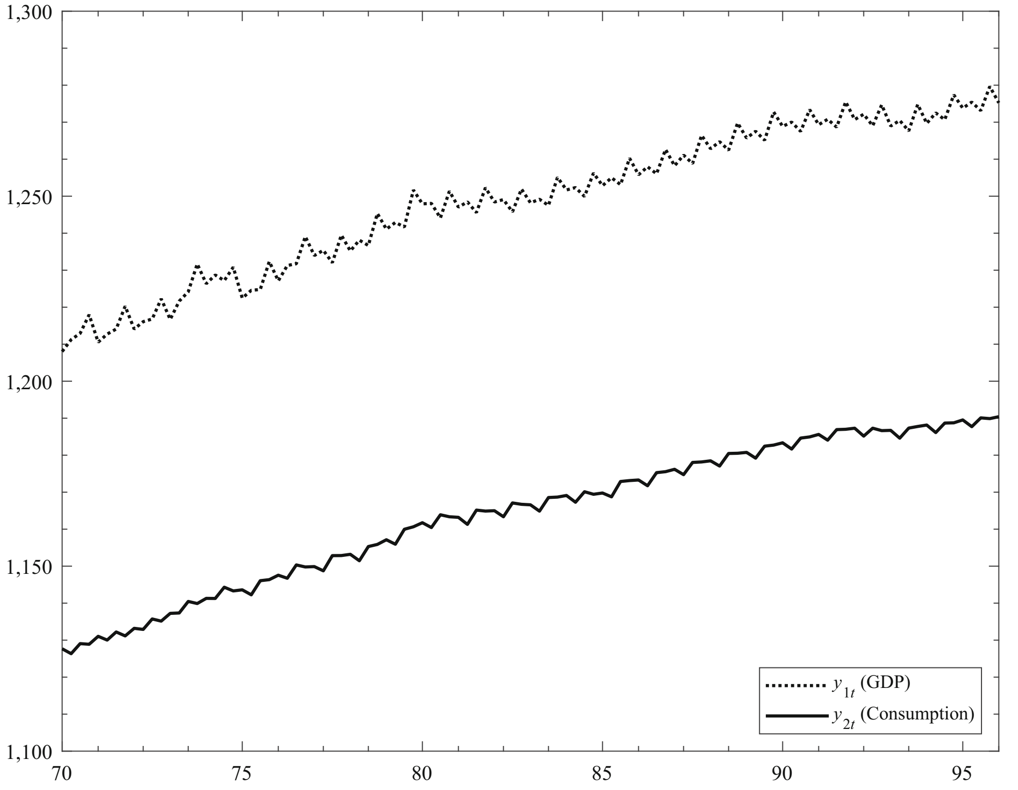


Figure 1. GDP and consumption in Italy 1970:Q1 to 1996:Q1, expressed as 100 times the natural logarithm

Figure 1 depicts these values in natural logarithms, scaled by multiplication by 100. It is clear that both GDP and consumption exhibit upward trends of a similar steepness. Seasonality is also evident in both series, although the seasonal fluctuations appear to be more prominent in GDP than in consumption.

For estimation purposes, the correlated seasonal UC model described in section II is written in linear Gaussian state-space form with appropriate covariance restrictions imposed. Assuming normality, maximum likelihood and the Kalman filter are implemented to estimate the unknown parameters, and hence the unobserved trend, cycle and seasonal components. The estimation results for all UC models are obtained using MATLAB, version R2022b, with the Econometrics Toolbox$^{\text{TM}}$ state-space functionality for building the UC models in state-space forms. The elements in the covariance matrix $\boldsymbol{\Sigma}$ are computed via nonlinear transformation of the parameters from the state-space forms, and the delta method is used for computing the standard errors of the estimated variances and correlations for component shocks.

## Estimation results

Table 2 presents estimation results for four bivariate seasonal UC models for GDP and consumption in Italy. The first is the standard uncorrelated component model, which allows non-zero disturbance correlations across variables only within each component, so that all cross-component correlations are assumed to be zero. The next three models, the common trend, the common cycle with perfectly correlated trend innovations, and the common seasonal models, allow cross-component correlations to be non-zero but impose restrictions as discussed in section II and specified in Table 1, with additional restrictions as implied by (zero frequency) cointegration imposed on the common cycle specification. The test results of Cubadda (1999) support such cointegration, which implies the same restrictions as for the common trend model in Table 1 except that the drift terms $\beta_1$ and $\beta_2$ are not restricted.

Appendix B provides a Monte Carlo comparison of these four models. Each of the four specifications in turn is taken as the true data generating process with the estimated values of Table 2 used as the true parameters, with all four model specifications then estimated. The results are reassuring, in that conventional information criteria (either AIC or BIC) select the true model in a clear majority of replications.

Figures 2,3 and 4 compare estimated trends, cycles and seasonals, respectively, for the four models applied to the Italian GDP and consumption data. Results for the perfectly correlated trend and cycle innovations models, which do not restrict the drift term or the AR coefficients (as appropriate), are not included in order to conserve space, but are available upon request.[12]

The common trend component model is preferred over the other specifications in Table 2 according to both BIC and AIC. It may be noted that the uncorrelated components model is, according to information criteria, the worst of those considered. Hence the results provide strong support for allowing non-zero cross-component correlations. Although the uncorrelated UC model supports cointegration, with an estimated trend innovation correlation of effectively one, the correlation of the cyclical innovations is estimated close to $-1$, in contrast to the insignificant estimate in the preferred common trend components specification.

Estimation results for the preferred common trend model indicate a slightly larger variation in trend innovations than in cyclical innovations for both GDP and consumption. For both variables the variation in seasonal innovations is smaller again, with this especially true for consumption, in line with the relatively small seasonal variation seen for this series in Figure 1. Imposing the common cycle restriction results in the cyclical innovation having larger variation than the trend innovation for GDP, whereas the estimated trend innovation for consumption remains more volatile than the cyclical innovation. In terms of the estimated components, the imposition of a common cycle has most effect (perhaps not surprisingly) in Figure 3, with consumption then showing markedly more cyclical variation for this model than in the others shown.

On the other hand, and again compared with the common trend model, imposing common seasonals increases the estimated volatilities of the trend innovations for GDP and the seasonal innovations for consumption. The effects of the restriction can be seen

[12]In fact, no satisfactory estimates could be obtained for the perfectly correlated trend innovations (i.e. zero frequency cointegration) model, due to local maxima in the empirical likelihood function. Although the perfectly correlated cycle innovations model improves on the common cycle specification in terms of information criteria values, it is inferior to the common trends specification in this sense. Also, the two-sided $P$-value for the estimate of $\phi_{22}$ in this latter model is only around the 10% level, possibly questioning the AR(2) assumption.

TABLE 2

*Estimation results for bivariate UC models with uncorrelated and common components for real GDP and consumption in Italy*

| Parameter | Uncorrelated | Common trend | Common cycle with $\rho_{\eta_1\eta_2} = 1$ | Common seasonality |
|---|---|---|---|---|
| GDP | | | | |
| $\sigma_{\eta_1}$ | 0.448 (0.060) | 0.512 (0.067) | 0.457 (0.115) | 0.951 (0.093) |
| $\sigma_{\varepsilon_1}$ | 0.449 (0.060) | 0.417 (0.072) | 0.825 (0.215) | 0.391 (0.052) |
| $\sigma_{\omega_1}$ | 0.199 (0.027) | 0.226 (0.051) | 0.226 (0.038) | 0.219 (0.043) |
| $\rho_{\eta_1\varepsilon_1}$ | 0 (-) | 0.426 (0.191) | −0.621 (0.216) | −0.872 (0.063) |
| $\rho_{\eta_1\omega_1}$ | 0 (-) | −0.687 (0.419) | 0.557 (0.390) | −0.219 (0.106) |
| $\rho_{\varepsilon_1\omega_1}$ | 0 (-) | 0.012 (0.346) | −0.816 (0.228) | −0.287 (0.156) |
| Consumption | | | | |
| $\sigma_{\eta_2}$ | 0.429 (0.037) | $d_2 \times \sigma_{\eta_1}$ | 0.494 (0.092) | 0.323 (0.021) |
| $\sigma_{\varepsilon_2}$ | 0.054 (0.020) | 0.393 (0.077) | $b_2 \times \sigma_{\varepsilon_1}$ | 0.225 (0.023) |
| $\sigma_{\omega_2}$ | 0.079 (0.015) | 0.090 (0.021) | 0.143 (0.034) | $a_2 \times \sigma_{\omega_1}$ |
| $\rho_{\eta_2\varepsilon_2}$ | 0 (-) | $\rho_{\eta_1\varepsilon_2}$ | $\rho_{\eta_2\varepsilon_1}$ | 0.874 (0.034) |
| $\rho_{\eta_2\omega_2}$ | 0 (-) | $\rho_{\eta_1\omega_2}$ | −0.893 (0.088) | $\rho_{\eta_2\omega_1}$ |
| $\rho_{\varepsilon_2\omega_2}$ | 0 (-) | 0.140 (0.493) | $\rho_{\varepsilon_1\omega_2}$ | $\rho_{\varepsilon_2\omega_1}$ |
| Cross-series | | | | |
| $\rho_{\eta_1\eta_2}$ | 1.000 (0.000) | 1 (-) | 1 (-) | 0.610 (0.068) |
| $\rho_{\eta_1\varepsilon_2}$ | 0 (-) | −0.922 (0.037) | $\rho_{\eta_1\varepsilon_1}$ | 0.150 (0.114) |
| $\rho_{\eta_1\omega_2}$ | 0 (-) | −0.509 (0.450) | $\rho_{\eta_2\omega_2}$ | $\rho_{\eta_1\omega_1}$ |
| $\rho_{\eta_2\varepsilon_1}$ | 0 (-) | $\rho_{\eta_1\varepsilon_1}$ | $\rho_{\eta_1\varepsilon_1}$ | −0.144 (0.134) |
| $\rho_{\eta_2\omega_1}$ | 0 (-) | $\rho_{\eta_1\omega_1}$ | $\rho_{\eta_1\omega_1}$ | −0.906 (0.029) |
| $\rho_{\varepsilon_1\varepsilon_2}$ | −0.999 (0.001) | −0.052 (0.178) | 1 (-) | 0.354 (0.148) |
| $\rho_{\varepsilon_1\omega_2}$ | 0 (-) | −0.954 (0.129) | 0.245 (0.198) | $\rho_{\varepsilon_1\omega_1}$ |
| $\rho_{\varepsilon_2\omega_1}$ | 0 (-) | 0.810 (0.410) | $\rho_{\varepsilon_1\omega_1}$ | −0.997 (0.001) |
| $\rho_{\omega_1\omega_2}$ | −0.343 (0.304) | −0.094 (0.251) | −0.382 (0.253) | 1 (-) |
| Others | | | | |
| $\beta_1$ | 0.622 (0.045) | 0.651 (0.057) | 0.592 (0.047) | 0.624 (0.102) |
| $\beta_2$ | 0.605 (0.043) | $d_2 \times \beta_1$ | 0.603 (0.050) | 0.603 (0.041) |
| $a_2$ | | | | 1.027 (0.172) |
| $b_2$ | | | 0.295 (0.112) | |
| $d_2$ | | 0.932 (0.000) | | |
| $\phi_{11}$ | 1.535 (0.097) | 1.534 (0.106) | 1.393 (0.127) | 0.013 (0.031) |
| $\phi_{12}$ | −0.735 (0.095) | −0.701 (0.097) | −0.535 (0.146) | −0.934 (0.030) |
| $\phi_{21}$ | 1.075 (0.046) | 0.560 (0.174) | | −0.001 (0.001) |
| $\phi_{22}$ | −0.885 (0.057) | −0.393 (0.189) | | 0.998 (0.000) |
| Log Lik. | −236.726 | −196.721 | −219.812 | −208.818 |
| AIC | 503.452 | 435.442 | 471.623 | 461.635 |
| BIC | 543.261 | 491.175 | 514.087 | 520.022 |

*Notes*: The sample period is 1970Q1 to 1996Q1. SEs are shown in parentheses.

particularly in Figure 3, with estimated cyclical variation for both series appearing to be highly seasonal in nature.[13] A further implausible consequence is that the estimated trend for consumption is typically below the actual values in Figure 2.

[13] A reviewer has observed that this is due to estimates of the AR(2) parameters, where the estimated AR polynomial for GDP is close to $(1 + L^2)$ which has complex roots $\pm i$, and that for consumption is close to $(1 - L^2) = (1 - L)(1 + L)$, both of which imply non-stationary seasonal patterns, together with a zero frequency unit root in the latter case. These are inappropriate for cycle components and indicate that the common seasonals model is misspecified.

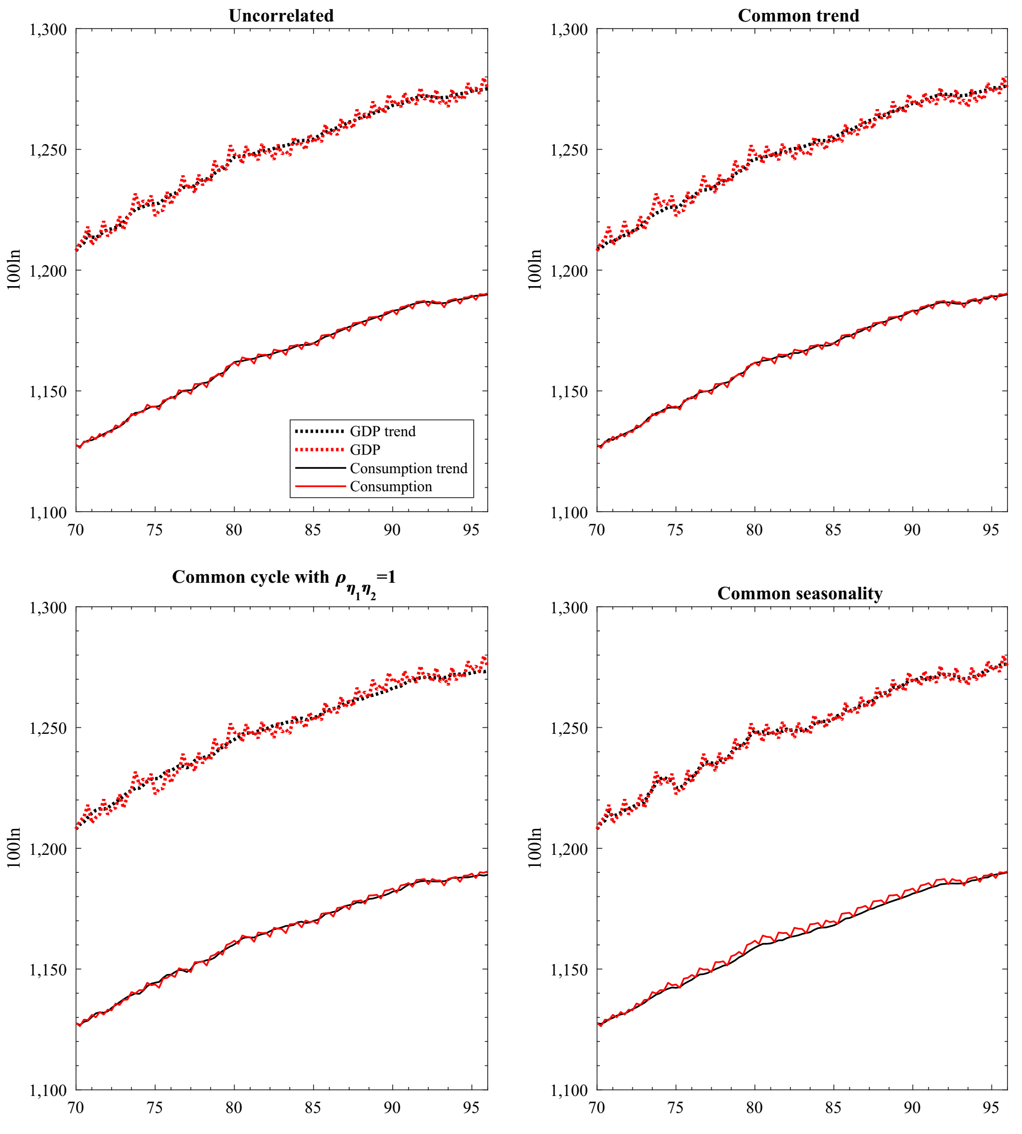


Figure 2. Estimated trends in Italian GDP and Consumption from 1970Q1 to 1996Q1 based on the following four restricted bivariate seasonal UC models: a model that imposes zero correlations between cross-component innovations, a common trend model, a common cycle model with an additional restriction of perfectly correlated trend innovations, and a common seasonality model. The Black lines are estimated components and the red lines are the values of GDP and Consumption in natural logarithm times 100. The dotted lines are for GDP and the solid lines are for consumption [Colour figure can be viewed at wileyonlinelibrary.com]

Overall, the estimated correlations for cross-component innovations in the common trend model are mainly strong or modest. For instance, the common trend innovation is strongly negatively correlated with the cycle innovations for consumption ($\widehat{\rho}_{\eta_1\varepsilon_2} = -0.922$) and moderately positively correlated with cycle innovation for GDP ($\widehat{\rho}_{\eta_1\varepsilon_1} = 0.426$). While the cyclical innovations across the two series and also the

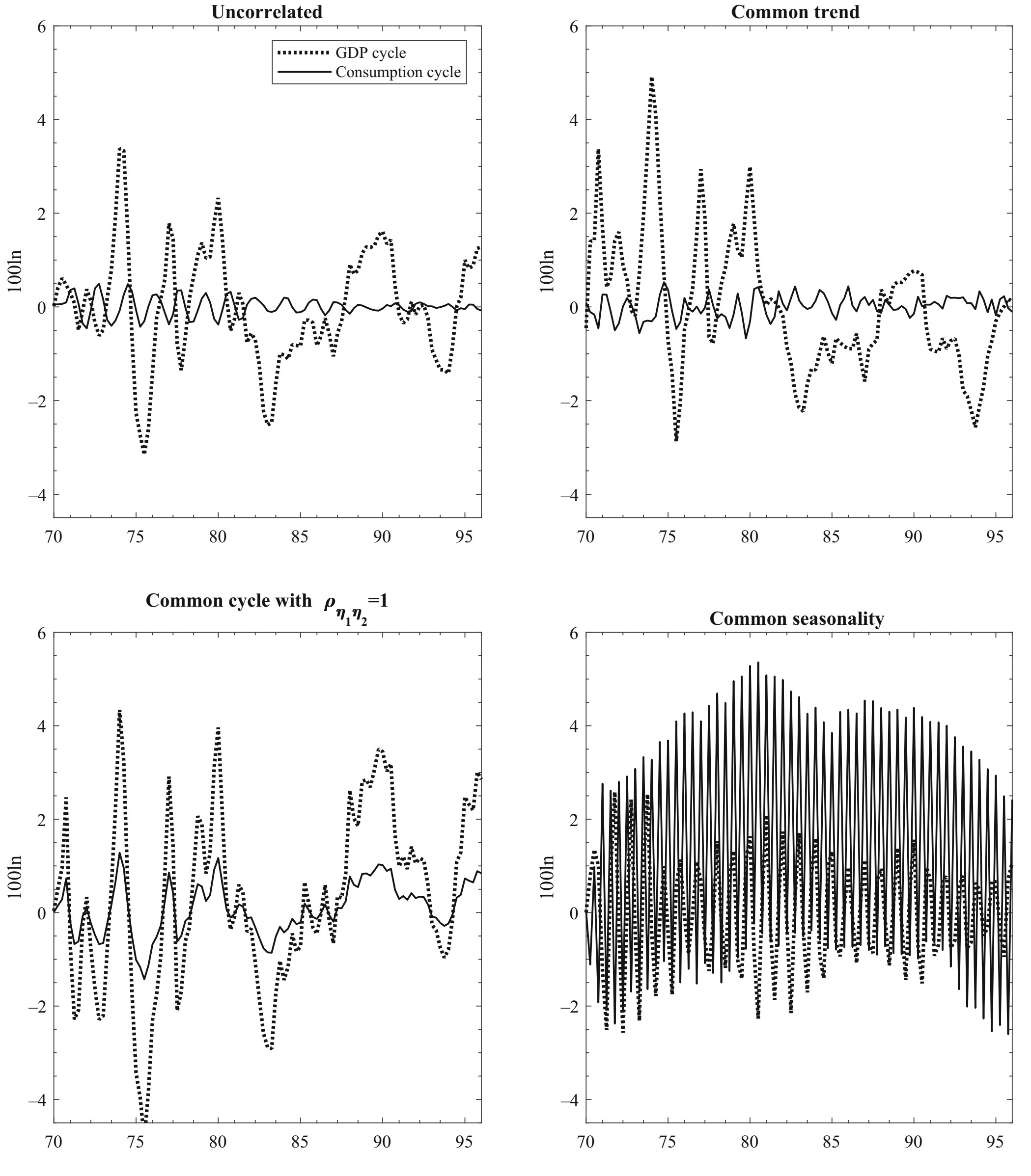


Figure 3. Estimated cycles in Italian GDP and Consumption from 1970Q1 to 1996Q1 based on the following four restricted bivariate seasonal UC models: a model that imposes zero correlations between cross-component innovations, a common trend model, a common cycle model with an additional restriction of perfectly correlated trend innovations, and a common seasonality model. The dotted lines are for GDP and the solid lines are for consumption

two seasonal innovations are effectively uncorrelated, strong estimated correlations apply between cyclical and seasonal innovations across components ($\widehat{\rho}_{\varepsilon_1\omega_2} = -0.954$, $\widehat{\rho}_{\varepsilon_2\omega_1} = 0.810$).

As just discussed, a number of cross-correlations in the common trend model are relatively close to plus or minus one between the innovations of the (stationary)

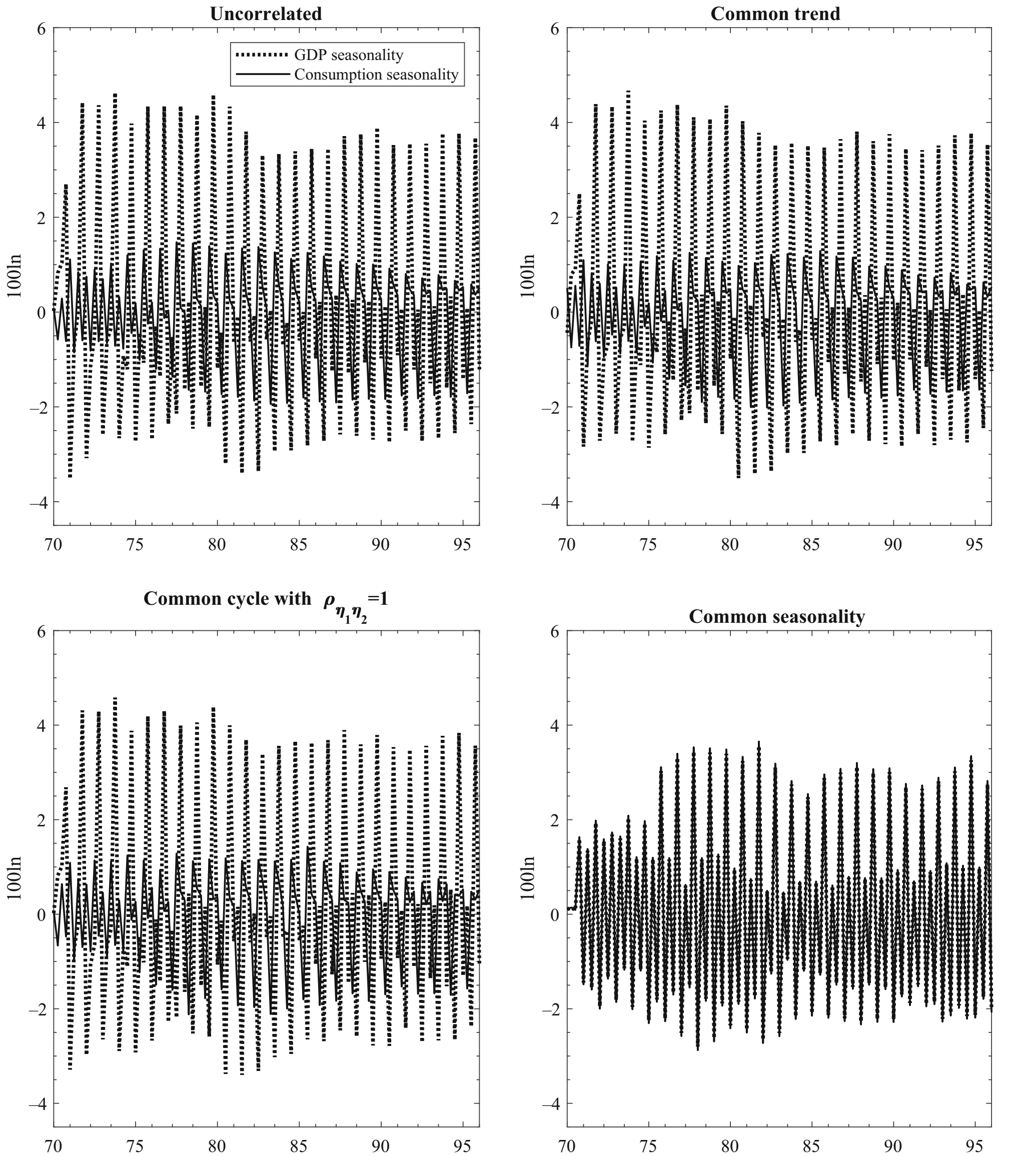


Figure 4. Estimated seasonals in Italian GDP and Consumption from 1970Q1 to 1996Q1 based on the following four restricted bivariate seasonal UC models: a model that imposes zero correlations between cross-component innovations, a common trend model, a common cycle model with an additional restriction of perfectly correlated trend innovations, a common seasonality model, and a model with perfectly correlated trend innovations and a model with perfectly correlated cycle innovations. The dotted lines are for GDP and the solid lines are for consumption

cycle components and the innovations of the (non-stationary) trend or seasonal components. However, the common trend and seasonal innovations are moderately negatively correlated for both GDP and consumption ($\widehat{\rho}_{\eta_1\omega_1} = -0.687, \widehat{\rho}_{\eta_1\omega_2} = -0.509$). Therefore, cross-frequency cointegration (see section II) does not appear to apply in this specification.

The apparently good performance of the common trend model for this data is in line with the tests of Cubadda (1999) which support cointegration between the series, while the relatively poor performance of the common seasonals model is compatible with Cubadda (1999) rejecting cointegration at the seasonal frequencies. On the other hand, the Cubadda (1999) test supports a common cycle, which appears at odds with the (effectively) zero estimated correlation between cycle innovations for the common trends model in Table 2. However, as noted in section II, the specification of the components in the UC approach leads to the particular reduced form in (8), whereas Cubadda (1999) begins from an unrestricted form. Further, our empirical common cycle model imposes zero frequency cointegration, which is not imposed by Cubadda (1999) when testing for common cycles. Consequently, our implied results regarding a common cycle may differ from those of Cubadda (1999).

### Impulse responses and further discussion

Impulse responses are relatively infrequently used in UC analyses, presumably because the conventional uncorrelated specification rules out cross-component effects. However, with cross-component correlations allowed, they can provide insights that are useful for the interpretation of empirical results.[14] Figure 5 provides generalized impulse response functions (GIRFs, see Pesaran and Shin, 1998) for the common trend specification of Table 2, showing the responses of GDP and consumption to each of the five disturbances in the system (namely, $\eta_1$, $\varepsilon_1$, $\varepsilon_2$, $\omega_1$ and $\omega_2$).

Because GIRFs take account of correlations across disturbances without making any causality assumptions, a cycle shock applied to an element of $\boldsymbol{\varepsilon}_t$ in (3) will have permanent effects when this disturbance is correlated with trend and/or seasonal disturbances. With substantial cross-correlations estimated across components for Italian GDP and consumption, a feature of Figure 5 is the pervasive effects of seasonality: in particular, GDP shows marked seasonality in its responses to both a common trend and a consumption cycle shock. Interestingly, although a (positive) GDP cycle shock has positive long-run effects on both GDP and consumption, a (positive) consumption cycle shock has negative long-run effects on both variables.

Based on the above analysis of Italian GDP and consumption, a few final comments are in order. Although the purpose of this section is to illustrate the application of common component restrictions for the correlated seasonal UC model, rather than to develop a full modelling framework, nevertheless the illustration contains useful pointers for empirical practice. Specifically, pretests for zero frequency and seasonal unit roots should be conducted to ensure that the series satisfy the implicit assumptions made when specifying the trend and seasonal components. Cointegration tests then provide further information about whether common trends and/or common seasonals restrictions may be appropriate. It is also worth emphasizing that capturing the types of cross-component effects shown in Figure 5 and discussed in the previous paragraph requires use of a multivariate correlated UC specification that takes account of seasonality.

[14]We thank a reviewer who raised the question of the use of impulse responses in this context.

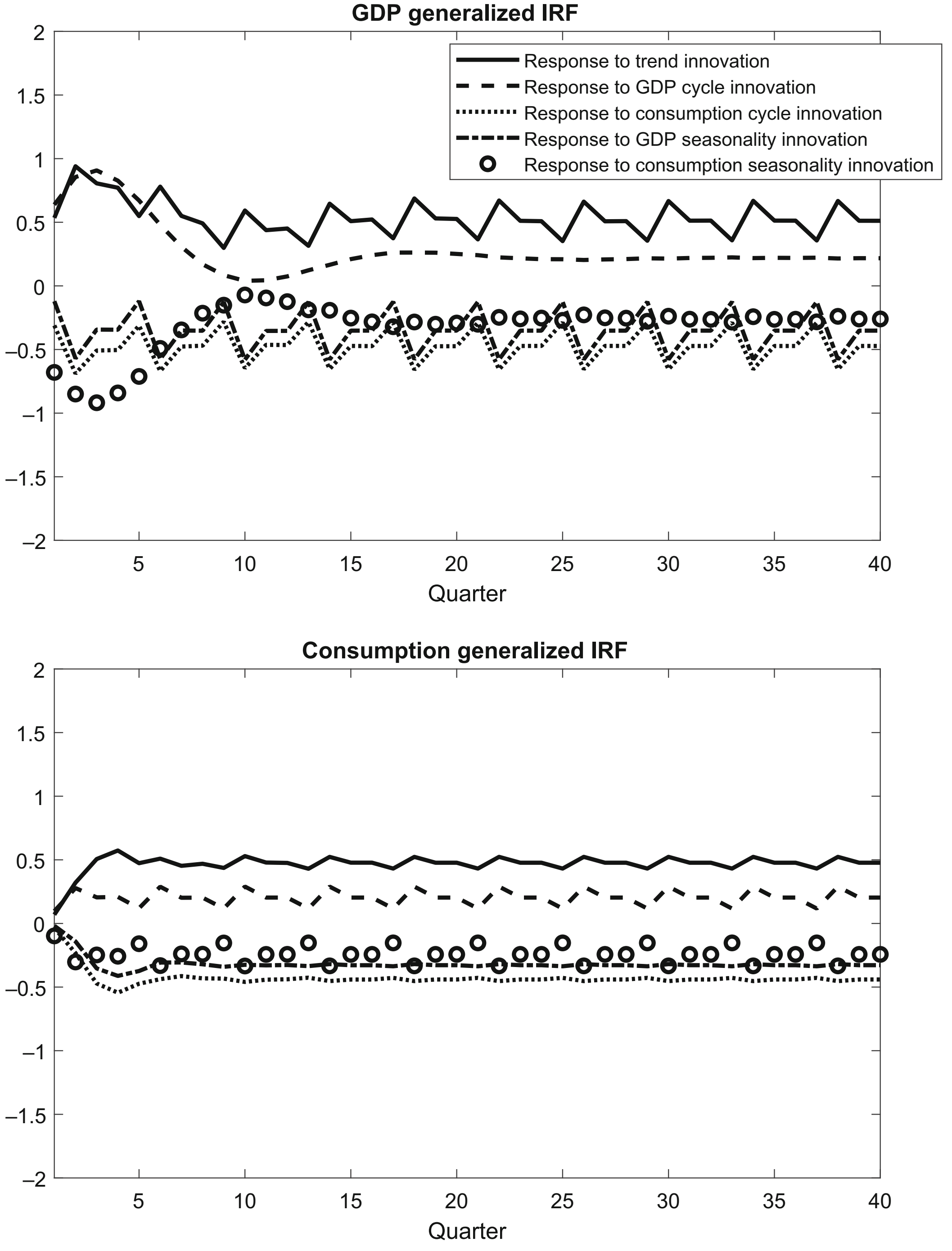


Figure 5. Generalized impulse response functions of Italian GDP and consumption to one standard deviation shock to each of the five innovations in the bivariate common trend model

## IV. Conclusions

Multivariate analysis of economic time series can throw important light on underlying economic phenomena, including trend, cyclical and seasonal movements. In order to analyze such movements when they are potentially correlated, a correlated multivariate unobserved components model is required. Although previously considered in a univariate context, to the best of our knowledge the present paper is the first to study identification conditions for a multivariate trend-cycle-seasonal model with correlated shocks. A key result that carries over from the case of univariate correlated seasonal UC models studied by Hindrayanto *et al.* (2019) is that restrictions are required for identification when a conventional AR(2) specification is used for the cycle. Therefore, the addition of variables does not, in itself, solve the identification problem that arises in the case of a univariate uncorrelated seasonal UC model.

Although restrictions are required to deliver identification in the multivariate case, we believe that forms of cross-equation restrictions that we propose are intuitive and can allow the approach to be applied in a variety of real-world situations. Indeed, a strong advantage of a multivariate rather than a univariate approach is that cross-equation restrictions can be applied. Our focus is on common trend, common cycle and common seasonals restrictions for a bivariate specification. Although the common trend and common seasonals restrictions provide sufficient information for identification, a further restriction is required with common cycles when the same AR coefficients apply across the variables.

The primary focus of our analysis is a conventional model with cycles of AR(2) form and seasonality specified through an annual sum process, but the implications of some generalizations to these are also discussed.

We illustrate our common component approach using Italian GDP and consumption data previously analyzed by Cubadda (1999). A range of specifications is considered, including four restricted bivariate seasonal UC models: a model that imposes zero correlations between cross-component innovations, a common trend model, a common cycle model with an additional restriction of perfectly correlated trend innovations (cointegration), and a common seasonals model. From these, the common trends specification is preferred. The illustration not only shows that the correlated seasonal UC model with a conventional AR(2) cycle can be estimated when plausible restrictions are imposed, but an impulse response analysis emphasizes the pervasive effects of seasonality and the importance of cross-component correlations.

## Appendix A: Matrix *D*

For the bivariate case, the matrix $\boldsymbol{D}$ of (13) is $\boldsymbol{D} \equiv [\boldsymbol{D}_1\ \boldsymbol{D}_2\ \boldsymbol{D}_3\ \boldsymbol{D}_4\ \boldsymbol{D}_5]$, where

$$
\boldsymbol{D}_1 = \begin{bmatrix}
4*(\phi_{11}+\phi_{12}-1)^2+2*(\phi_{11}+\phi_{12})-2*\phi_{12}*(\phi_{11}-1) & 0 \\
0 & 0 \\
0 & 1+2*(\phi_{21}+\phi_{22}-1)^2+(\phi_{21}+\phi_{22})^2+\phi_{22}^2+(\phi_{21}-1)^2 \\
(\phi_{11}+\phi_{12}-1)*(3*\phi_{11}+2*\phi_{12}-2)-(\phi_{11}-1)+\phi_{12}*(\phi_{11}+\phi_{12}) & 0 \\
0 & 0 \\
0 & 0 \\
0 & (\phi_{21}+\phi_{22}-1)*(3*\phi_{21}+3*\phi_{22}-2)-\phi_{21}+\phi_{22}+1 \\
2*(\phi_{11}+\phi_{12}-1)^2 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 2*(\phi_{21}+\phi_{22}-1)^2 \\
(\phi_{11}-1)*(\phi_{11}+\phi_{12})+(\phi_{12}-1)*(\phi_{11}+\phi_{12}-1) & 0 \\
0 & 0 \\
0 & 0 \\
0 & (\phi_{21}-1)*(\phi_{21}+\phi_{22})+(\phi_{22}-1)*(\phi_{21}+\phi_{22}-1) \\
\phi_{11}*\phi_{12}-\phi_{11}-2*\phi_{12} & 0 \\
0 & 0 \\
0 & 0 \\
0 & \phi_{21}*\phi_{22}-\phi_{21}-2*\phi_{22} \\
-\phi_{12} & 0 \\
0 & 0 \\
0 & 0 \\
0 & -\phi_{22}
\end{bmatrix}
$$

$$
\boldsymbol{D}_2 = \begin{bmatrix}
0 & 2 & 0 \\
1 + 2*(\phi_{11}+\phi_{12}-1)*(\phi_{21}+\phi_{22}-1) + (\phi_{11}+\phi_{12})*(\phi_{21}+\phi_{22}) + (\phi_{11}-1)*(\phi_{21}-1) + \phi_{12}*\phi_{22} & 0 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0 \\
(\phi_{21}+\phi_{22}-1)*(2*\phi_{11}+3*\phi_{12}-1) + (\phi_{21}-1)*(\phi_{11}+\phi_{12}-1) - (\phi_{11}-1) + \phi_{12} & 0 & 0 \\
(\phi_{11}+\phi_{12}-1)*(2*\phi_{21}+3*\phi_{22}-1) + (\phi_{11}-1)*(\phi_{21}+\phi_{22}-1) - (\phi_{21}-1) + \phi_{22} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
(\phi_{11}+\phi_{12}-1)*(\phi_{21}-2) + (\phi_{21}+\phi_{22}-1)*(\phi_{11}+2*\phi_{12}) & 0 & 0 \\
(\phi_{11}+\phi_{12}-1)*(\phi_{21}+2*\phi_{22}) + (\phi_{21}+\phi_{22}-1)*(\phi_{11}-2) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
(\phi_{21}-1)*(\phi_{11}+\phi_{12}) - (\phi_{11}+\phi_{12}-1) + \phi_{12}*(\phi_{21}+\phi_{22}-1) & 0 & 0 \\
(\phi_{11}-1)*(\phi_{21}+\phi_{22}) - (\phi_{21}+\phi_{22}-1) + \phi_{22}*(\phi_{11}+\phi_{12}-1) & 0 & 0 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
\phi_{12}*\phi_{21} - \phi_{11} - 2*\phi_{12} & 0 & 0 \\
\phi_{22}*\phi_{11} - \phi_{21} - 2*\phi_{22} & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
-\phi_{12} & 0 & 0 \\
-\phi_{22} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{D}_3 = \begin{bmatrix}
2*(1+\phi_{11}^2+\phi_{12}^2+\phi_{11}-\phi_{11}*\phi_{12}) & 0 & 0 \\
0 & 0 & 1+\phi_{12}*\phi_{22}+(\phi_{11}+1)*(\phi_{21}+1)+(\phi_{21}-\phi_{22})*(\phi_{11}-\phi_{12}) \\
0 & 1+\phi_{22}^2+(\phi_{21}+1)^2+(\phi_{21}-\phi_{22})^2 & 0 \\
(\phi_{11}-\phi_{12})*(\phi_{12}-\phi_{11}-1)-(\phi_{11}+1) & 0 & 0 \\
0 & 0 & \phi_{12}*(\phi_{21}-\phi_{22})-(\phi_{21}+1)*(\phi_{11}-\phi_{12})-(\phi_{11}+1) \\
0 & 0 & \phi_{22}*(\phi_{11}-\phi_{12})-(\phi_{11}+1)*(\phi_{21}-\phi_{22})-(\phi_{21}+1) \\
0 & (\phi_{21}-\phi_{22})*(\phi_{22}-\phi_{21}-1)-(\phi_{21}+1) & 0 \\
\phi_{11}-\phi_{11}*\phi_{12}-2*\phi_{12} & 0 & 0 \\
0 & 0 & \phi_{11}-\phi_{12}*\phi_{21}-2*\phi_{12} \\
0 & 0 & \phi_{21}-\phi_{11}*\phi_{22}-2*\phi_{22} \\
0 & \phi_{21}-\phi_{21}*\phi_{22}-2*\phi_{22} & 0 \\
\phi_{12} & 0 & 0 \\
0 & 0 & \phi_{12} \\
0 & 0 & \phi_{22} \\
0 & \phi_{22} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{D}_4 = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+\phi_{11}+\phi_{12} \\
0 & 2+2*(\phi_{21}+\phi_{22}) & 2*\phi_{21}*(1-\phi_{22}) & 2 & 0 \\
-(\phi_{11}+\phi_{12}+1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \phi_{12}-\phi_{11}+1 \\
0 & 0 & 0 & 0 & \phi_{11}+\phi_{12}-1 \\
0 & 2*\phi_{22} & -\phi_{21}^2-\phi_{22}^2-1 & -(\phi_{21}+\phi_{22}+1) & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\phi_{11}+\phi_{12}-1) \\
0 & 0 & 0 & 0 & \phi_{11}+\phi_{12}-1 \\
0 & 0 & 0 & 0 & 0 \\
\phi_{11}+\phi_{12}+1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\phi_{11}+\phi_{12}-1) \\
0 & 0 & 0 & 0 & \phi_{11}-1 \\
0 & -\phi_{22} & \phi_{21}^2+\phi_{22}^2+\phi_{22}+1 & \phi_{21}+\phi_{22}+1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\phi_{11}+\phi_{12}) \\
0 & 0 & 0 & 0 & -1 \\
0 & -(\phi_{21}+\phi_{22}+1) & \phi_{21}*\phi_{22}-\phi_{21} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\phi_{12} \\
0 & 0 & 0 & 0 & 0 \\
0 & -\phi_{22} & -\phi_{22} & 0 & 0
\end{bmatrix}
$$

$$
\boldsymbol{D}_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\phi_{11} - \phi_{21}\phi_{12} & 1 + \phi_{21} + \phi_{22} & 1 & \phi_{21} - \phi_{11}\phi_{22} & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\phi_{12} - \phi_{22} & \phi_{21} + \phi_{22} - 1 & -\phi_{22} & -\phi_{11} * \phi_{21} - \phi_{12}\phi_{22} - 1 & -(\phi_{11} + 1) \\
-\phi_{11} * \phi_{21} - \phi_{12} * \phi_{22} - 1 & -\phi_{21} + \phi_{22} + 1 & -(\phi_{21} + 1) & \phi_{22} - \phi_{12} & -\phi_{12} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\phi_{21} + \phi_{11} * \phi_{22} & \phi_{21} + \phi_{22} - 1 & -(\phi_{21} - \phi_{22}) & \phi_{11} - \phi_{12} * \phi_{21} & \phi_{11} - \phi_{12} \\
\phi_{21} - \phi_{11} * \phi_{22} & -(\phi_{21} + \phi_{22} - 1) & \phi_{21} - \phi_{22} & \phi_{12} * \phi_{21} - \phi_{11} & \phi_{12} - \phi_{11} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\phi_{21} * (\phi_{11} - \phi_{12}) + \phi_{12} * (\phi_{21} + \phi_{22}) + 1 & \phi_{21} - 1 & \phi_{21} + 1 & \phi_{12} & \phi_{12} \\
\phi_{22} & -(\phi_{21} + \phi_{22} - 1) & \phi_{22} & \phi_{11} * \phi_{21} + \phi_{12} * \phi_{22} + 1 & \phi_{11} + 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\phi_{12} * \phi_{21} - \phi_{11} & -1 & -1 & 0 & 0 \\
0 & -(\phi_{21} + \phi_{22}) & 0 & \phi_{11}\phi_{22} - \phi_{21} & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\phi_{12} & 0 & 0 & 0 & 0 \\
0 & -\phi_{22} & 0 & -\phi_{22} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

## Appendix B: Simulations

Four simulation experiments are undertaken to compare the performances of alternative restricted seasonal UC models. The four data generating processes (DGPs) for bivariate series are a model that imposes zero correlations between cross-component innovations, a common trend model, a common cycle model with perfectly correlated trend innovations and a common seasonality model. The model specifications are described in section II, and the parameter values are taken from Table 2 for each respective model. All four specifications are estimated using the data generated from each DGP.

The experiments use a sample size of 100, equivalent to 25 years of quarterly data, and are repeated 1000 times. In Table B1, the percentage frequencies of selecting a restricted bivariate seasonal UC model using AIC and BIC for each DGP are reported. The high values along the table's diagonal suggest that both information criteria are generally able to choose the correct bivariate seasonal UC model.

In particular, when the DGP is the common cycle model with perfectly correlated trend innovations, both AIC and BIC prefer this model to the other restricted seasonal UC models in 99.8% of the 1,000 repeated simulations. When the common trend specification is the true DGP, both criteria select the true model in more than 80% of the cases. In the scenario where the DGP is an UC model with zero correlations among cross-component innovations, the DGP sets the correlation between two trend innovations ($\rho_{\eta_1\eta_2}$) and the correlation between two cycle innovations ($\rho_{\varepsilon_1\varepsilon_2}$) to 1 and $-1$, respectively, reflecting the estimates for the uncorrelated UC model in Table 2. Nevertheless, Table B1 shows that the common cycle model with perfectly correlated trend innovations (which imposes the true cointegration restriction $\rho_{\rho_{\eta_1\eta_2}} = 1$) is favoured by AIC only 37.2% of the time and by BIC only 21.4% of the time.

TABLE B1

*Simulations: percentage frequencies of a true restricted bivariate seasonal UC model being selected by information criteria*

| DGP | Alternative restricted seasonal UC models | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Uncorrelated | | Common trend | | Common cycle with $\rho_{\eta_1\eta_2} = 1$ | | Common seasonality | |
| | AIC | BIC | AIC | BIC | AIC | BIC | AIC | BIC |
| Uncorrelated | 60.9 | 78.6 | 0.1 | 0 | 37.2 | 21.4 | 1.8 | 0 |
| Common Trend | 1.5 | 2.6 | 90.5 | 82.1 | 7.3 | 15.1 | 0.7 | 0.2 |
| Common Cycle with $\rho_{\eta_1\eta_2} = 1$ | 0.1 | 0.2 | 0.1 | 0 | 99.8 | 99.8 | 0 | 0 |
| Common Seasonality | 0.4 | 7.8 | 2.5 | 1.8 | 0.5 | 24.5 | 96.6 | 65.9 |

*Notes*: The simulations consider four data generating processes for bivariate series: the UC model with zero correlations imposed between cross-component innovations, a common trend model, a common cycle model with perfectly correlated trend innovations, and a common seasonality model. The parameter values for each DPG are taken from the empirical estimation results for each model in Table 2. For each DGP, the four restricted seasonal UC models are estimated. The table reports the percentage frequencies for which each model is selected by the Akaike Information Criterion and the Bayesian Information Criterion. The sample size is 100 and the number of replications is 1,000.

*Final Manuscript Received: May 2023*

## References

Abeln, B. and Jacobs, J. P. A. M. (2023). *Seasonal Adjustment without Revisions: A Real-Time Data Appoach*, SpringerBrief, Springer, Berlin.

Barsky, R. B. and Miron, J. A. (1989). 'The seasonal cycle and the business cycle', *Journal of Political Economy*, Vol. 97, pp. 503–534.

Cecchetti, S. G. and Kashyap, A. K. (1996). 'International cycles', *European Economic Review*, Vol. 40, pp. 331–360.

Clark, P. K. (1989). 'Trend reversion in real output and unemployment', *Journal of Econometrics*, Vol. 40, pp. 15–32.

Cubadda, G. (1999). 'Common cycles in seasonal non-stationary time series', *Journal of Applied Econometrics*, Vol. 14, pp. 273–291.

del Barrio, T., Cubadda, G. and Osborn, D. R. (2022). 'On cointegration for processes integrated at different frequencies', *Journal of Time Series Analysis*, Vol. 43, pp. 412–435.

Dufour, J.-M. and Pelletier, D. (2022). 'Practical methods for modelling weak VARMA processes: Identification, estimation and specification with a macroeconomic application', *Journal of Business and Economic Statistics*, Vol. 40, pp. 1140–1152.

Dungey, M., Jacobs, J. P. A. M., Tian, J. and van Norden, S. (2013). 'On the correspondence between data revision and trend-cycle decomposition', *Applied Economics Letters*, Vol. 20, pp. 312–315.

Dungey, M., Jacobs, J. P. A. M., Tian, J. and van Norden, S. (2015). 'Trend in cycle or cycle in trend? New structural identifications for unobserved components models of U.S. real GDP', *Macroeconomic Dynamics*, Vol. 19, pp. 776–790.

Durbin, J. and Koopman, S. J. (2012). *Time Series Analysis by State Space Methods* 2nd ed., Oxford University Press, Oxford.

Fleischman, C. A. and Roberts, J. M. (2011). *From Many Series, One Cycle: Improved Estimates of the Business Cycle from a Multivariate Unobserved Components Model*. Finance and Economics Discussion Series No. 2011-46, Federal Reserve Board, Washington D.C.

Ghysels, E. and Osborn, D. R. (2001). *The Econometric Analysis of Seasonal Time Series*, Cambridge University Press, Cambridge, UK.

Harvey, A. C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press, Cambridge, UK.

Harvey, A. C. and Koopman, S. J. (1992). 'Diagnostic checking of unobserved-components time series models', *Journal of Business & Economic Statistics*, Vol. 10, pp. 377–389.

Harvey, A. C. and Koopman, S. J. (1997). 'Multivariate structural time series models', in Heij C., Schumacher J. M., Hanzon B., and Praagman C. (eds), *Systematic Dynamics in Economic and Financial Models*, John Wiley & Sons Ltd, Chichester,UK, pp. 269–298.

Hindrayanto, I., Jacobs, J. P. A. M., Osborn, D. R. and Tian, J. (2019). 'Trend-cycle-seasonal interactions: identification and estimation', *Macroeconomic Dynamics*, Vol. 23, pp. 3163–3188.

Johansen, S. and Schaumburg, E. (1999). 'likelihood analysis of seasonal cointegration', *Journal of Econometrics*, Vol. 88, pp. 301–339.

Koopman, S. J. and Lee, K. M. (2009). 'Seasonality with trend and cycle interactions in unobserved components models', *Journal of the Royal Statistical Society Series C*, Vol. 58, pp. 427–448.

Krane, S. and Wascher, W. (1999). 'The cyclical sensitivity of seasonality in U.S. employment', *Journal of Monetary Economics*, Vol. 44, pp. 523–553.

Ma, J. and Wohar, M. E. (2013). 'An unobserved components model that yields business and medium-run cycles', *Journal of Money, Credit and Banking*, Vol. 45, pp. 1351–1373.

Matas-Mir, A. and Osborn, D. R. (2004). 'Does seasonality change over the business cycle? An investigation using monthly industrial production series', *European Economic Review*, Vol. 48, pp. 1309–1332.

McElroy, T. (2017). 'Multivariate seasonal adjustment, economic identities and seasonal taxonomy', *Journal of Business and Economic Statistics*, Vol. 35, pp. 611–625.

Morley, J. C. (2007). 'The slow adjustment of aggregate consumption to permanent income', *Journal of Money, Credit and Banking*, Vol. 39, pp. 615–638.
Morley, J. C., Nelson, C. R. and Zivot, E. (2003). 'Why are the Beveridge-Nelson and unobserved-components decompositions of GDP so different?', *The Review of Economics and Statistics*, Vol. 85, pp. 235–243.
Osborn, D. R. (1988). 'Seasonality and habit persistence in a life cycle model of consumption', *Journal of Applied Econometrics*, Vol. 3, pp. 255–266.
Pesaran, M. H. and Shin, Y. (1998). 'Generalized impulse response analysis in linear multivariate models', *Economics Letters*, Vol. 58, pp. 17–29.
Sinclair, T. M. (2009). 'The relationships between permanent and transitory movements in U.S. output and the unemployment rate', *Journal of Money, Credit and Banking*, Vol. 41, pp. 529–542.
Stock, J. H. (2013). 'Comments on Unseasonal Seasonals?', *Brookings Papers on Economic Activity*, Vol. Fall, pp. 111–119.
Trenkler, C. and Weber, E. (2016). 'On the identification of multivariate correlated unobserved components models', *Economics Letters*, Vol. 138, pp. 15–18.
Wright, J. H. (2013). 'Unseasonal seasonals? (Including comments and discussion)', *Brookings Papers on Economic Activity*, Vol. Fall, pp. 65–126.