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# Scalar quasinormal modes of Kerr-AdS$_5$

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An analytic expression for the scalar quasinormal modes of generic, spinning Kerr-AdS$_5$ black holes was previously proposed by the authors [J. High Energy Phys. 08 (2017) 094], in terms of transcendental equations involving the Painlevé VI (PVI) $\tau$ function. In this work, we carry out a numerical investigation of the modes for generic rotation parameters, comparing implementations of expansions for the PVI $\tau$ function in terms of both conformal blocks (Nekrasov functions) and Fredholm determinants. We compare the results with standard numerical methods for the subcase of Schwarzschild black holes. We then derive asymptotic formulas for the angular eigenvalues and the quasinormal modes in the small black hole limit for generic scalar mass and discuss, both numerically and analytically, the appearance of superradiant modes.

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## I. INTRODUCTION

The quasinormal fluctuations of black holes play an important role in general relativity. Improving the precision of the quantitative knowledge of the decay rates is required to advance our understanding of gravitation, from the interpretation of gravitational wave data to the study of the linear stability of a given solution to Einstein equations.

A completely different motivation to analyze quasinormal oscillation of black holes arises from the gauge-gravity correspondence. In the context of Maldacena's conjecture, black hole solutions in asymptotic anti–de Sitter (AdS) spacetimes describe thermal states of the corresponding conformal field theory (CFT) with the Hawking temperature, and the perturbed black holes describe the near-equilibrium states. Namely, the perturbation—parametrized by a scalar field in our case of study—induces a small deviation from the equilibrium, so that the (scalar) quasinormal mode spectrum of the black hole is dual to poles in the retarded Green's function on the conformal side. Thus, one can compute the relaxation times in the dual theory by equating them to the imaginary part of the eigenfrequencies [1]. There have been many studies of quasinormal modes of various types of perturbations on several background solutions in AdS spacetime, and we refer to Ref. [2] for further discussions.

We turn our attention to a specific background, the Kerr-AdS$_5$ black hole [3]. The motivation to put on a firmer basis the linear perturbation problem of the Kerr-AdS$_5$ system is threefold. First, the calculation of scattering coefficients and quasinormal modes depends on the connection relations of different solutions to Fuchsian ordinary differential equations—the so-called connection problem, for which we present the exact solution in terms of transcendental equations. Second, by the AdS/CFT duality, perturbations of the Kerr-AdS$_5$ black hole serve as a tool to study the associated CFT thermal state [4,5] with a sufficiently general set of Lorentz charges (mass and angular momenta). Small Schwarzschild-AdS$_5$ black holes, with a horizon radius smaller than the AdS scale, are known to be thermodynamically unstable; it would be thus interesting to have some grasp on the generic rotating case. Finally, numerical and analytic studies hint at the existence of unstable (superradiant) massless scalar modes [6–8], which should also be well described by the isomonodromy method.

The Painlevé VI (PVI) $\tau$ function was introduced in this context by Refs. [9,10]—see also Ref. [11]—as an approach to study rotating black holes in four dimensions and a positive cosmological constant. The method has deep ties to integrable systems and the Riemann-Hilbert problem in complex analysis, relating scattering coefficients to monodromies of a flat holomorphic connection of a certain matricial differential

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system associated to the Heun equation—the isomonodromic deformations. For the Heun equation related to the Kerr–de Sitter and Kerr–anti–de Sitter black holes, the solution for the scattering problem has been given in terms of transcendental equations involving the PVI $\tau$ function.

In turn, the PVI $\tau$ function has been interpreted as a chiral $c = 1$ conformal block of Virasoro primaries, through the Alday-Gaiotto-Tachikawa conjecture [12]. In the latter work, the authors have given asymptotic expansions for the PVI $\tau$ function in terms of Nekrasov functions, expanding early work by Jimbo *et al.* [13]. More recently, the authors of Refs. [14,15] have reformulated the PVI $\tau$ function in terms of the determinant of a certain class of Fredholm operators. We will see that this formulation has computational advantages over the Nekrasov sum expansion and will allow us to numerically solve the transcendental equations posed by the quasinormal modes with high accuracy.

The paper is organized as follows. In Sec. II, we review the five-dimensional Kerr-AdS metric and write the linear scalar perturbation equation of motion in terms of the radial and the angular Heun differential equation. In Sec. II B, we review the isomonodromy method. First, the solutions of each Heun equation are linked to a differential matricial differential equation, which in turn can be seen as a flat holomorphic connection. Then, we identify gauge transformations of each connection as a Hamiltonian system which is directly linked to the Painlevé VI $\tau$ function. Finally, we recast the conditions to obtain our original differential equations and their quantization conditions in terms of the PVI $\tau$ function.

In Sec. III, we give approximate expressions for the monodromy parameters in terms of the isomonodromy time $t_0$. Applying these results to the angular equation, we obtain an approximate expression for the separation constant for slow rotation or near equally rotating black holes. We then set out to calculate numerically the quasinormal modes for the Schwarzschild-AdS$_5$ and compare with the established Frobenius methods and quadratic eigenvalue problem (QEP).

In Sec. IV, we turn to the general-rotation Kerr-AdS$_5$ black holes. We study numerically the quasinormal modes for increasing outer horizon radii, again comparing with the Frobenius method. We then use the analytical results for the monodromy parameters for the radial equation to give an asymptotic formula for the quasinormal modes in the subcase where the field does not carry any azimuthal angular momenta $m_1 = m_2 = 0$ (and therefore the angular eigenvalue quantum number $\ell$ even). We close by discussing the existence of superradiant modes for $\ell$ odd.

We conclude in Sec. V. In Appendix A, we describe the Nekrasov expansion and the Fredholm determinant formulation of the PVI $\tau$ function, reviewing work done in Ref. [14]. In Appendix B, we give an explicit parametrization of the monodromy matrices given the monodromy parameters.

## II. SCALAR FIELDS IN KERR-AdS$_5$

Let us review the five-dimensional Kerr-AdS$_5$ black hole metric as presented in Ref. [3]:

$$
\begin{aligned}
ds^2 = {} & -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a_1\sin^2\theta}{1-a_1^2}d\phi - \frac{a_2\cos^2\theta}{1-a_2^2}d\psi\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a_1 dt - \frac{(r^2+a_1^2)}{1-a_1^2}d\phi\right)^2 \\
& + \frac{1+r^2}{r^2\rho^2}\left(a_1a_2 dt - \frac{a_2(r^2+a_1^2)\sin^2\theta}{1-a_1^2}d\phi - \frac{a_1(r^2+a_2^2)\cos^2\theta}{1-a_2^2}d\psi\right)^2 + \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(a_2 dt - \frac{(r^2+a_2^2)}{1-a_2^2}d\psi\right)^2 \\
& + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2,
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
\Delta_r &= \frac{1}{r^2}(r^2+a_1^2)(r^2+a_2^2)(1+r^2) - 2M = \frac{1}{r^2}(r^2-r_0^2)(r^2-r_-^2)(r^2-r_+^2), \\
\Delta_\theta &= 1 - a_1^2\cos^2\theta - a_2^2\sin^2\theta, \qquad \rho^2 = r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta,
\end{aligned}
\tag{2}
$$

with $M$, $a_1$, and $a_2$ real parameters, related to the Arnowitt-Deser-Misner mass and angular momenta by [16–18]

$$
\mathcal{M} = \frac{\pi M(2\Xi_1 + 2\Xi_2 - \Xi_1\Xi_2)}{4\Xi_1^2\Xi_2^2}, \quad \mathcal{J}_\phi = \frac{\pi M a_1}{2\Xi_1^2\Xi_2}, \quad \mathcal{J}_\psi = \frac{\pi M a_2}{2\Xi_1\Xi_2^2},
$$
$$
\Xi_1 = 1 - a_1^2, \quad \Xi_2 = 1 - a_2^2. \tag{3}
$$

When $M > 0$, $a_1^2, a_2^2 < 1$, all these quantities are physically acceptable, and we have that $r_-$ and $r_+$, the real roots of $\Delta_r$, correspond to the inner and outer horizons, respectively, of the black hole [16], whereas $r_0$ is purely imaginary:

$$
-r_0^2 = 1 + a_1^2 + a_2^2 + r_-^2 + r_+^2. \tag{4}
$$

For the purposes of this article, we will see the radial variable, or rather $r^2$, as a generic complex number. It will be interesting for us to treat all three roots of $\Delta_r$, $r_+^2$, $r_-^2$, and $r_0^2$, as Killing horizons. Actually, in the complexified version of the metric (1), in all three hypersurfaces defined by $r = r_0, r_-, r_+$ we have that each of the Killing fields

$$\xi_k = \frac{\partial}{\partial t} + \Omega_1(r_k)\frac{\partial}{\partial \phi} + \Omega_2(r_k)\frac{\partial}{\partial \psi}, \quad k = 0, -, +, \tag{5}$$

becomes null. The temperature and angular velocities for each horizon are given, respectively, by

$$\Omega_{k,1} = \frac{a_1(1-a_1^2)}{r_k^2 + a_1^2}, \quad \Omega_{k,2} = \frac{a_2(1-a_2^2)}{r_k^2 + a_2^2},$$
$$T_k = \frac{r_k^2 \Delta_r'(r_k)}{4\pi(r_k^2+a_1^2)(r_k^2+a_2^2)} = \frac{r_k}{2\pi}\frac{(r_k^2-r_i^2)(r_k^2-r_j^2)}{(r_k^2+a_1^2)(r_k^2+a_2^2)}, \quad i,j \neq k. \tag{6}$$

Within the physically sensible range of parameters, $T_+$ is positive, $T_-$ is negative, and $T_0$ is purely imaginary.

### A. Kerr–anti–de Sitter scalar wave equation

The Klein-Gordon equation for a scalar of mass $\mu$ in the background (1) is separable by the factorization $\Phi = \Pi(r)\Theta(\theta)e^{-i\omega t + i m_1 \phi + i m_2 \psi}$. To wit, $\omega$ is the frequency of the mode, and $m_1, m_2 \in \mathbb{Z}$ are the azimuthal components of the mode's angular momentum. The angular equation is given by [7]

$$\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\Delta_\theta\frac{d\Theta(\theta)}{d\theta}\right) - \left[\omega^2 + \frac{(1-a_1^2)m_1^2}{\sin^2\theta} + \frac{(1-a_2^2)m_2^2}{\cos^2\theta} - \frac{(1-a_1^2)(1-a_2^2)}{\Delta_\theta}(\omega + m_1 a_1 + m_2 a_2)^2 + \mu^2(a_1^2\cos^2\theta + a_2^2\sin^2\theta)\right]\Theta(\theta) = -C_j\Theta(\theta), \tag{7}$$

where $C_j$ is the separation constant and $j$ an integer index which will be defined later. By two consecutive transformations $\chi = \sin^2\theta$ and $u = \chi/(\chi - \chi_0)$, with $\chi_0 = (1-a_1^2)/(a_2^2 - a_1^2)$,[1] we can take the four singular points of Eq. (7) to be located at

$$u = 0, \quad u = 1, \quad u = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \quad u = \infty, \tag{8}$$

and the indicial exponents[2] are

$$\alpha_0^\pm = \pm\frac{m_1}{2}, \qquad \alpha_1^\pm = \frac{1}{2}\left(2 \pm \sqrt{4+\mu^2}\right),$$
$$\alpha_{u_0}^\pm = \pm\frac{m_2}{2}, \qquad \alpha_\infty^\pm = \pm\frac{1}{2}(\omega + a_1 m_1 + a_2 m_2). \tag{9}$$

The exponents have a sign symmetry, except for $\alpha_1^\pm$, which corresponds $\Delta/2$ and $(4-\Delta)/2$, where $\Delta$ is the conformal dimension of the CFT primary field associated to the $AdS_5$ scalar. We define the single monodromy parameters $\varsigma_i$ through $\alpha_i^\pm = \frac{1}{2}(\alpha_i \pm \varsigma_i)$. Writing them explicitly,

$$\varsigma_0 = m_1, \qquad \varsigma_1 = 2 - \Delta, \qquad \varsigma_{u_0} = m_2,$$
$$\varsigma_\infty \equiv \varsigma = \omega + a_1 m_1 + a_2 m_2. \tag{10}$$

We note an obvious sign symmetry $\varsigma_i \to -\varsigma_i$, so we will take the positive sign as standard.

Coming back to Eq. (7), by introducing the following transformation:

$$\Theta(u) = u^{m_1/2}(u-1)^{\Delta/2}(u-u_0)^{m_2/2}S(u), \tag{11}$$

we bring the angular equation to the canonical Heun form

$$\frac{d^2S}{du^2} + \left(\frac{1+m_1}{u} + \frac{1+\sqrt{4+\mu^2}}{u-1} + \frac{1+m_2}{u-u_0}\right)\frac{dS}{du} + \left(\frac{q_1 q_2}{u(u-1)} - \frac{u_0(u_0-1)Q_0}{u(u-1)(u-u_0)}\right)S = 0 \tag{12}$$

with $q_1$, $q_2$, and the accessory parameter $Q_0$ given, respectively, by

$$q_1 = \frac{1}{2}(m_1 + m_2 + \Delta - \varsigma), \quad q_2 = \frac{1}{2}(m_1 + m_2 + \Delta + \varsigma), \tag{13}$$

$$4u_0(u_0-1)Q_0 = -\frac{\omega^2 + a_1^2\mu^2 - C_j}{a_2^2 - 1} - u_0[(m_2 + \Delta - 1)^2 - m_2^2 - 1] - (u_0 - 1)[(m_1 + m_2 + 1)^2 - \varsigma^2 - 1]. \tag{14}$$

We note that Eq. (12) has the same AdS spheroidal harmonic form as the problem in four dimensions, the eigenvalues reducing to those ones when $m_1 = m_2$, $\ell \to \ell/2$, $a_1 = 0$, and $a_2 = i\alpha$ [11]. Also, according to Eq. (7) we have that $u_0$ in Eq. (12) is close to zero for $a_2 \simeq a_1$, the equal rotation limit.

[1] The second change of variables is justified in terms of the asymptotic expansion for the $\tau$ function close to 0.

[2] The asymptotic behavior of the function near the singular points $\Theta(u) \simeq (u - u_i)^{\alpha_i^\pm}$ or $\Theta(u) \simeq u^{-\alpha_\infty^\pm}$ for the point at infinity.

The radial equation is given by

$$\frac{1}{r\Pi(r)}\frac{d}{dr}\left(r\Delta_r\frac{d\Pi(r)}{dr}\right)-\left[C_j+\mu^2r^2+\frac{1}{r^2}(a_1a_2\omega-a_2(1-a_1^2)m_1-a_1(1-a_2^2)m_2)^2\right]+\frac{(r^2+a_1^2)^2(r^2+a_2^2)^2}{r^4\Delta_r}\left(\omega-\frac{m_1a_1(1-a_1^2)}{r^2+a_1^2}-\frac{m_2a_2(1-a_2^2)}{r^2+a_2^2}\right)^2=0, \quad (15)$$

which again has four regular singular points, located at the roots of $r^2\Delta_r(r^2)$ and infinity. The indicial exponents $\beta_i^{\pm}$ are defined analogously to the angular case. Schematically, they are given by

$$\beta_k=\pm\frac{1}{2}\theta_k, \quad k=+,-,0 \text{ and } \beta_\infty=\frac{1}{2}(2\pm\theta_\infty), \quad (16)$$

where $\theta_k$, $k=+,-,0,\infty$ are the single monodromy parameters, given in terms of the physical parameters of the problem by

$$\theta_k=\frac{i}{2\pi}\left(\frac{\omega-m_1\Omega_{k,1}-m_2\Omega_{k,2}}{T_k}\right), \quad \theta_\infty=2-\Delta, \quad (17)$$

where $k=0,+,-$. To bring this equation to the canonical Heun form which we can use, we perform the change of variables[3]:

$$z=\frac{r^2-r_-^2}{r^2-r_0^2}, \quad \Pi(z)=z^{-\theta_-/2}(z-z_0)^{-\theta_+/2}(z-1)^{\Delta/2}R(z), \quad (18)$$

where

$$z_0=\frac{r_+^2-r_-^2}{r_+^2-r_0^2}. \quad (19)$$

The equation for $R(z)$ is

$$\frac{d^2R}{dz^2}+\left[\frac{1-\theta_-}{z}+\frac{-1+\Delta}{z-1}+\frac{1-\theta_+}{z-z_0}\right]\frac{dR}{dz}+\left(\frac{\kappa_1\kappa_2}{z(z-1)}-\frac{z_0(z_0-1)K_0}{z(z-1)(z-z_0)}\right)R(z)=0, \quad (20)$$

where

$$\kappa_1=\frac{1}{2}(\theta_-+\theta_+-\Delta-\theta_0), \quad \kappa_2=\frac{1}{2}(\theta_-+\theta_+-\Delta+\theta_0), \quad (21)$$

$$4z_0(z_0-1)K_0=-\frac{C_j+\mu^2r_-^2-\omega^2}{r_+^2-r_0^2}-(z_0-1)[(\theta_-+\theta_+-1)^2-\theta_0^2-1]-z_0[2(\theta_+-1)(1-\Delta)+(2-\Delta)^2-2]. \quad (22)$$

Both Eqs. (12) and (20) can be solved by usual Frobenius methods in terms of the Heun series near each of the singular points. We are, however, interested in solutions for Eq. (12) which satisfy

$$S(u)=\begin{cases}1+\mathcal{O}(u), & u\to 0,\\ 1+\mathcal{O}(u-1), & u\to 1,\end{cases} \quad (23)$$

which will set a quantization condition for the separation constant $C_j$. For the radial equation with $\mu^2>0$, the conditions that $\Pi(z)$ corresponds to a purely ingoing wave at the outer horizon $z=z_0$ and normalizable at the boundary $z=1$ read as follows[4]:

$$R(z)=\begin{cases}1+\mathcal{O}(z-z_0), & z\to z_0,\\ 1+\mathcal{O}(z-1), & z\to 1,\end{cases} \quad (24)$$

where $R(z)$ is a regular function at the boundaries. This condition will enforce the quantization of the (not necessarily real) frequencies $\omega$, which will correspond to the (quasi)normal modes.

### B. Radial and angular $\tau$ functions

The functions described in this section will be the main ingredient to compute the separation constant $C_j$ and the quasinormal modes, which will be the focus of the next section. A more extensive discussion of the strategy can be found in Ref. [19]. Let us begin by rewriting the Heun equation in the standard form as a first-order differential equation. Consider the system given by

[3]Note that, with this choice of variables, we have that at infinity the radial solution will behave as $\Pi(z)\sim z^{\pm\theta_0/2}$.

[4]The computation of the accessory parameters and the boundary conditions of the radial equation are slightly different with respect to those shown in Ref. [19]. We have chosen a more suitable Möbius transformation for the asymptotic expansion of the PVI $\tau$ function in the limit $z_0\to 0$.

$$\frac{d\Phi}{dz} = A(z)\Phi, \qquad \Phi(z) = \begin{pmatrix} y^{(1)}(z) & y^{(2)}(z) \\ w^{(1)}(z) & w^{(2)}(z) \end{pmatrix},$$
$$A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}, \tag{25}$$

where $\Phi(z)$ is a matrix of fundamental solutions and the coefficients $A_i$, $i = 0, t, 1$, are $2\times 2$ matrices that do not depend on $z$. Using Eqs. (25), we can derive a second-order ordinary differential equation (ODE) for one of the two linearly independent solutions $y^{(1,2)}(z)$ given by

$$y'' - (\mathrm{Tr}A + (\log A_{12})')y' + (\det A - A_{11}' + A_{11}(\log A_{12})')y = 0, \tag{26}$$

which, by the partial fraction expansion of $A(z)$, will have singular points at $z = 0, t, 1, \infty$ and at the zeros and poles of $A_{12}(z)$. Let us investigate the latter. By a change of basis of solutions, we can assume that the matrix $A(z)$ becomes diagonal at infinity and, thus,

$$A_\infty = -(A_0 + A_1 + A_t), \qquad A_\infty = \begin{pmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{pmatrix}. \tag{27}$$

This leads to the assumption that $A_{12}$ vanishes like $\mathcal{O}(z^{-2})$ as $z \to \infty$. By the partial fraction form of $A(z)$, we then have

$$A_{12}(z) = \frac{k(z-\lambda)}{z(z-1)(z-t)}, \qquad k, \lambda \in \mathbb{C}, \tag{28}$$

where $k$ and $\lambda$ do not depend on $z$ but can be expressed explicitly in terms of the entries of $A_i$, as can be seen in Ref. [20]. For our purposes, it suffices to check that $z = \lambda$ is a zero of $A_{12}(z)$ and necessarily of the order of 1. Therefore, $z = \lambda$ is an extra singular point of Eq. (26), which does not correspond to the poles of $A(z)$. A direct calculation shows that this singular point has indicial exponents 0 and 2, with no logarithmic tails, and hence corresponds to an apparent singularity, with trivial monodromy. The resulting equation for (26) is, in general, not quite the Heun equation but has five singularities:

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_t}{z-t} + \frac{1-\theta_1}{z-1} - \frac{1}{z-\lambda}\right)y' + \left(\frac{\kappa_+(\kappa_-+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-t)(z-1)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}\right)y = 0, \tag{29}$$

where $\theta_i = \mathrm{Tr}A_i$ and we set by gauge transformation $\det A_i = 0$ for $i = 0, t, 1$. The accessory parameters are $\mu = A_{11}(z = \lambda)$ and $K$, which is defined below. We will refer to this equation as the deformed Heun equation.

The absence of logarithmic behavior at $z = \lambda$ results in the following algebraic relation between $K$, $\mu$, and $\lambda$:

$$K(\mu, \lambda, t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{\theta_0}{\lambda} + \frac{\theta_1}{\lambda-1} + \frac{\theta_t - 1}{\lambda - t}\right)\mu + \frac{\kappa_+(1+\kappa_-)}{\lambda(\lambda-1)}\right]. \tag{30}$$

Now, since we are interested in properties of the solutions of Eq. (26), and therefore of Eq. (25), which depend solely on the monodromy data—phases and change of bases picked as one continues the solutions around the singular points—we are free to change the parameters of the equations as long as they do not change the monodromy data. The isomonodromy deformations parametrized by a change of $t$ view $A(z)$ as the "$z$ component" of a flat holomorphic connection $\mathcal{A}$. The "$t$ component" can be guessed immediately:

$$\mathcal{A}_z = A(z), \qquad \mathcal{A}_t = -\frac{A_t}{z-t}, \tag{31}$$

and the flatness condition gives us the Schlesinger equations:

$$\frac{\partial A_0}{\partial t} = -\frac{1}{t}[A_0, A_t], \qquad \frac{\partial A_1}{\partial t} = -\frac{1}{t-1}[A_1, A_t],$$
$$\frac{\partial A_t}{\partial t} = \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t]. \tag{32}$$

When integrated, these equations will define a family of flat holomorphic connections $\mathcal{A}(z, t)$ with the same monodromy data, parametrized by a possibly complex parameter $t$. The set of corresponding $A(z, t)$ will be called the isomonodromic family. It has been known since the pioneering work of the Kyoto school in the 1980s—see Ref. [21] for a mathematical review and Ref. [10] for the specific case we consider here—that the flow defined by these equations is Hamiltonian, conveniently defined by the $\tau$ function

$$\frac{d}{dt}\log\tau(t; \{\vec{\theta}, \vec{\sigma}\}) = \frac{1}{t}\mathrm{Tr}(A_0 A_t) + \frac{1}{t-1}\mathrm{Tr}(A_1 A_t). \tag{33}$$

In terms of $\mu$, $\lambda$, the Schlesinger flow can be cast as

$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}, \tag{34}$$

and the ensuing second-order differential equation for $\lambda$ is known as the PVI transcendent. The relation between the $\tau$ function and the Hamiltonian can be obtained by direct algebra:

$$\frac{d}{dt}\log\tau(t;\{\vec{\theta},\vec{\sigma}\})$$
$$=K(\mu,\lambda,t)+\frac{\theta_0\theta_t}{t}+\frac{\theta_1\theta_t}{t-1}-\frac{\lambda(\lambda-1)}{t(t-1)}\mu-\frac{\lambda-t}{t(t-1)}\kappa_+. \quad (35)$$

Expansions for the PVI $\tau$ function near $t=0$, 1, and $\infty$ were given in Refs. [12,22] and Appendix A. For $t$ sufficiently close to zero, we have

$$\tau(t)=Ct^{\frac{1}{4}(\sigma^2-\theta_0^2-\theta_t^2)}(1-t)^{\frac{1}{2}\theta_1\theta_t}\times\left(1+\left(\frac{\theta_1\theta_t}{2}+\frac{(\theta_0^2-\theta_t^2-\sigma^2)(\theta_\infty^2-\theta_1^2-\sigma^2)}{8\sigma^2}\right)t-\frac{(\theta_0^2-(\theta_t-\sigma)^2)(\theta_\infty^2-(\theta_1-\sigma)^2)}{16\sigma^2(1+\sigma)^2}\kappa t^{1+\sigma}-\frac{(\theta_0^2-(\theta_t+\sigma)^2)(\theta_\infty^2-(\theta_1+\sigma)^2)}{16\sigma^2(1-\sigma)^2}\kappa^{-1}t^{1-\sigma}+\cdots\right). \quad (36)$$

The parameters in these expansions are related to the monodromy data $\{\vec{\theta},\vec{\sigma}\}=\{\theta_0,\theta_t,\theta_1,\theta_\infty;\sigma_{0t},\sigma_{1t}\}$, where $\theta_i=\mathrm{Tr}A_i$ are as above and $\sigma_{ij}$ are the composite monodromy parameters

$$2\cos\pi\sigma_{ij}=\mathrm{Tr}M_iM_j, \quad (37)$$

where $M_i$ ($M_j$) is the matrix that implements the analytic continuation around the singular point $z_i$ ($z_j$). Given the monodromy data, the $\sigma$ parameter is related to $\sigma_{0t}$ by the addition of an even integer $\sigma_{0t}=\sigma+2p$ so that the coefficients above will give the largest term in the series. We will defer the procedure to calculate $p$ until Sec. IV. The parameter $\kappa$ is given in terms of the monodromy data by Eq. (A12).

The usefulness of the PVI $\tau$ function for the solution of the scattering and quasinormal modes for the scalar AdS perturbations is based on the relation between the scattering coefficients and the monodromy data [9,11]. For the quasinormal modes, the relationship was shown in Ref. [19]. Succinctly, it states that conditions like Eqs. (23) and (24) require the relative connection matrix between the Frobenius solutions constructed at the singular points to be upper or lower triangular. In turn, this means that, in the basis where one monodromy matrix is diagonal, the other will be upper or lower triangular. A direct calculation shows that

$$\cos\pi\sigma_{ij}=\cos\pi(\theta_i+\theta_j). \quad (38)$$

As derived in Ref. [19], the converse is also true: If the composite monodromy is given by Eq. (38), then the monodromy matrices $M_i$ and $M_j$ are both either lower or upper triangular. We note that this formulation views the problem of finding eigenvalues for the angular equation similar in spirit to finding the quasinormal frequencies for the radial equation.

For the problem under consideration, the expressions for the composite monodromies condition (38) in terms of the quantities in each ODE (12) and (20) are, respectively,

$$\sigma_{0u_0}(m_1,m_2,\varsigma,\Delta,u_0,C_j)=m_1+m_2+2j,\quad j\in\mathbb{Z}, \quad (39)$$

$$\sigma_{1z_0}(\theta_k,\Delta,z_0,\omega_n,C_j)=\theta_++\Delta+2n-2,\quad n\in\mathbb{Z}. \quad (40)$$

These conditions on the $\tau$ function for the radial and angular system can be obtained by first placing conditions on the matricial system (25) such that the equation for the first line of $\Phi(z)$ (26) recovers the differential equation we are considering—Eq. (12) for the angular case and Eq. (20) for the radial case. We need, from the generic form of the equation satisfied by the first line (29), that the canonical variables $\lambda(t_0)=t_0$, $\mu(t_0)$, and $K(t_0)$ are to be chosen so that Eq. (30) has a well-defined limit as $\lambda(t_0)\to t_0$. These conditions, expressed in terms of the $\tau$ function (33), are

$$\left.\frac{d}{dt}\log\tau(t;\{\vec{\theta},\vec{\sigma}\}^-)\right|_{t=t_0}=\frac{(\theta_t-1)\theta_1}{2(t_0-1)}+\frac{(\theta_t-1)\theta_0}{2t_0}+K_0,$$
$$\left.\frac{d}{dt}\left[t(t-1)\frac{d}{dt}\log\tau(t;\{\vec{\theta},\vec{\sigma}\}^-)\right]\right|_{t=t_0}=\frac{\theta_t-1}{2}(\theta_t-\theta_\infty-\theta_0-\theta_1-2), \quad (41)$$

where $K_0$ is the accessory parameter of the corresponding Heun equation (radial or angular) and the parameters of the $\tau$ function are given by

$$\{\vec{\theta},\vec{\sigma}\}^-=\{\theta_0,\theta_t-1,\theta_1,\theta_\infty+1;\sigma_{0t}-1,\sigma_{1t}-1\}. \quad (42)$$

These conditions can be understood as an initial value problem of the dynamical system defined by Eq. (34). Given the expansion of the $\tau$ function (36), these conditions provide an analytic solution to the system and can be inverted to find the composite monodromy parameters $\sigma_{0t}$ and $\sigma_{1t}$. We plan to apply these conditions to both the radial equation (20) and the angular equation (12) and view Eq. (41) as the set of (exact) transcendental equations which can be solved numerically.

The solution for the quasinormal modes means finding for $\omega$, given the rest of the parameters of the differential equations (20) and (12), by solving the set of four transcendental equations, the pair in the conditions on the $\tau$ functions (41) for each condition in the angular and radial equations (39) and (40). The parameters for each pair are given explicitly by

| | $t_0$ | $\theta_0$ | $\theta_t$ | $\theta_1$ | $\theta_\infty$ |
|---|---|---|---|---|---|
| $\tau_{\rm Rad}(t)$ | $z_0$ | $\theta_-$ | $\theta_+$ | $2-\Delta$ | $\theta_0$ |
| $\tau_{\rm Ang}(t)$ | $u_0$ | $-m_1$ | $-m_2$ | $2-\Delta$ | $\varsigma$ |

It should be noted that the conditions (41) give an analytic solution for the quasinormal frequencies. The set of transcendental (and implicit) equations is probably the best that can be done: Save for a few special cases—see Ref. [22]—the solution for the dynamical system (34) cannot be given in terms of elementary functions. On the other hand, the true usefulness of the result (41) relies on the control we have over the calculation of the PVI $\tau$ function.

In previous work [19], we considered the interpretation of the expansion (36) in terms of conformal blocks, which in turn allow us to interpret the $\tau$ function as the generating function for the accessory parameters of classical solutions of the Liouville differential equation—an important problem in the constructive theory of conformal maps [23]. On the other hand, expressions like the first equation in (41) could be interpreted in the gauge-gravity correspondence as an equilibrium condition on the angular and radial "systems," if one could interpret the radial (20) and angular (12) equations as Ward identities for different sectors in the purported boundary CFT—see Ref. [24] for comments on that direction in the simpler case of Bañados-Teitelboim-Zanelli black holes. The second condition in Eq. (41) is related to an associated $\tau$ function, with shifted monodromy arguments

$$\tau(t;\{\vec{\theta},\vec{\sigma}\}) \equiv \tau(t;\{\theta_0,\theta_t,\theta_1,\theta_\infty\},\{\sigma_{0t},\sigma_{1t}\}) \quad (43)$$

via the so-called "Toda equation"—see Proposition 4.2 in Ref. [25], or Ref. [23], for a sketch of proof. With help from the Toda equation, the second condition in Eq. (41) can be more succinctly phrased as

$$\tau(t_0;\{\vec{\theta},\vec{\sigma}\}) = 0, \quad (44)$$

for which we will give an interpretation in terms of the Fredholm determinant in Appendix A. In would be interesting to further that line and explore the holographic aspects of the structure outlined by the analytic solution, but we will leave that for future work.

The expression for the $\tau$ function in terms of conformal blocks (36), called the Nekrasov expansion, is suitable for the small black hole limit which we will treat algebraically in this article. From the numerical analysis perspective, however, it suffers from the combinatorial nature of its coefficients—see Appendix A, which takes exponential computational time $\mathcal{O}(e^{\alpha N})$ to achieve $\mathcal{O}(t^N)$ precision. Because of this, we have used for the numerical analysis an alternative formulation of the PVI $\tau$ function through Fredholm determinants, introduced in Refs. [14,15], also outlined in Appendix A. This formulation achieves $\mathcal{O}(t^N)$ precision for the $\tau$ function in polynomial time $\mathcal{O}(N^\alpha)$.

## III. PAINLEVÉ VI $\tau$ FUNCTION FOR KERR-AdS$_5$ BLACK HOLE

For $u_0$ or $z_0$ sufficiently close to a critical value of the PVI $\tau$ function ($t=0,1,\infty$), both the Nekrasov expansion and the Fredholm determinant will converge fast. It makes sense then to begin exploring solutions with this property. If $u_0$ is close to 0, this corresponds to the almost equally rotating $a_1 \simeq a_2$ or to the slowly rotating $a_1$, $a_2 \simeq 0$ cases. For $z_0$ close to 0, we are considering the near-extremal limit $r_+ \simeq r_-$ or small $r_+, r_- \simeq 0$ black holes.

The procedure of solving Eq. (41) can be summarized by first using the second equation to find the parameter $s$ in the Nekrasov expansion (A2) and then substituting this back in the first equation in order to find the monodromy parameter $\sigma$—see Refs. [26,27]. In our application, there are some remarks on the procedure. The first observation is that the $\tau$ function is quasiperiodic with respect to shifts of $\sigma_{0t}$ by even integers $\sigma_{0t} \to \sigma_{0t} + 2p$:

$$\tau(t;\{\vec{\theta}\},\{\sigma_{0t}+2p,\sigma_{1t}\}) = s^{-p}\tau(t;\{\vec{\theta}\},\{\sigma_{0t},\sigma_{1t}\}), \quad p\in\mathbb{Z}. \quad (45)$$

This means that, upon inverting Eqs. (39) and (40), we will obtain, rather than the $\sigma_{0t}$ associated to the system, a parameter, which we will call $\sigma$, related to $\sigma_{0t}$ by the shift $\sigma_{0t} = \sigma + 2p$. Let us digress over the consequences of this periodicity by analyzing the structure of the expansion (A2). Schematically,

$$\tau(t_0) = t_0^{\frac{1}{4}(\sigma^2-\theta_0^2-\theta_t^2)} \sum_{m\in\mathbb{Z}} P(\sigma+2m;t_0) s^m t_0^{m^2+m\sigma}, \quad (46)$$

where $P(\sigma+2m;t_0)$ is analytic in $t_0$, and to find the zero of $\tau(t_0)$ as per Eq. (44) is useful to define $X = st_0^\sigma$, making the expansion analytic in $t_0$ and meromorphic in $X$. We can now solve Eq. (44) and thus define $X(\sigma,t_0)$ in terms of $\sigma$ as a series in $t_0$. Let us classify these solutions by their leading term:

$$X_p(\sigma;t_0) \equiv s_p t_0^\sigma = t_0^{2p+1}(x_0 + x_1 t_0 + x_2 t_0^2 + \cdots). \quad (47)$$

Depending on the sign of Re$\sigma$, the leading term will depend on $t_0$ or $t_0^{-1}$. We will suppose Re$\sigma > 0$ for the discussion. The "fundamental" solution $X_0$ is written as [see Eq. (A12)]

$$X_0(\sigma;t_0)=\frac{\Gamma^2(1+\sigma)}{\Gamma^2(1-\sigma)}\frac{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\sigma))}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\sigma))}\frac{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty-\sigma))\Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty-\sigma))}{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty+\sigma))\Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty+\sigma))}Y(\sigma;t_0) \quad (48)$$

with

$$Y(\sigma;t_0)=\left[\frac{((\theta_t+\sigma)^2-\theta_0^2)((\theta_1+\sigma)^2-\theta_\infty^2)}{16\sigma^2(\sigma-1)^2}t_0\right]\times\left(1-(\sigma-1)\frac{(\theta_0^2-\theta_t^2)(\theta_1^2-\theta_\infty^2)+\sigma^2(\sigma-2)^2}{2\sigma^2(\sigma-2)^2}t_0+\mathcal{O}(t_0^2)\right). \quad (49)$$

Solutions with $\mathrm{Re}\sigma<0$ can be obtained by sending $\sigma$ to $-\sigma$ and inverting the term in square brackets in the expression for $Y$. Solutions with a higher value for $p$ will also be of interest. These will have the leading term of the order of $t_0^{2p+1}$ and can be obtained from the quasiperiodicity property (45), which translates to a shifting property for $X_p$. From the generic structure (46) above, we have

$$\sum_{m\in\mathbb{Z}}P(\sigma+2m;t_0)X^m t_0^{m^2}=\tilde{X}^{-p}t_0^{-p^2}\sum_{m\in\mathbb{Z}}P(\tilde{\sigma}+2m;t_0)\tilde{X}^m t_0^{m^2}, \quad (50)$$

where

$$\tilde{\sigma}=\sigma-2p,\qquad X=\tilde{X}t_0^{2p}. \quad (51)$$

By this property, assuming $\mathrm{Re}\sigma>0$, we have that a solution $X_p(\sigma;t_0)$ for Eq. (44) with a leading term of higher order in $t_0$ can be obtained from a fundamental solution of leading order $t_0$ with shifted $\sigma$:

$$X_p(\sigma;t_0)=t_0^{2p}X_0(\sigma-2p;t_0). \quad (52)$$

This allows us to construct a class of solutions for the conditions (41) which are generic enough for our purposes. From $X_p(\sigma;t_0)$ or $Y(\sigma;t_0)$ we can define the parameter $\kappa$ entering the expansion (36):

$$\kappa(t_0;\{\vec{\theta},\vec{\sigma}\})=Y(\sigma;t_0)t_0^{-\sigma} \quad (53)$$

and the family of parameters $s_p$:

$$s_p=X_p(\sigma;t_0)t_0^{-\sigma}=X_0(\sigma-2p;t_0)t_0^{-\sigma+2p}, \quad (54)$$

with $X_0$ given in terms of $Y$ as above. The knowledge of both parameters $s_p$ and $\sigma$ is sufficient to determine the monodromy data by Eq. (A5).

We can now proceed to compute the accessory parameter $K_0$ in terms of the monodromy parameter $\sigma$ by substituting $\kappa$ found through Eq. (53) back to the first equation in Eq. (41). We note that this equation has for argument the shifted monodromy parameters $\{\vec{\theta},\vec{\sigma}\}^-$ defined by Eq. (42). This shift leaves the $s$ parameter invariant $s(\{\vec{\theta},\vec{\sigma}\}^-)=s(\{\vec{\theta},\vec{\sigma}\})$, but, because of the string of gamma functions in Eq. (A12), the $\kappa$ parameter entering the asymptotic formula (36) will change as

$$\kappa(\{\vec{\theta},\vec{\sigma}\}^-)=-\frac{16\sigma^2(\sigma-1)^2}{((\theta_t+\sigma)^2-\theta_0^2)((\theta_\infty-\sigma+1)^2-(\theta_1+1)^2)}\kappa(\{\vec{\theta},\vec{\sigma}\}). \quad (55)$$

Using the fundamental solution for $Y(\sigma,t_0)$ (49) and (53), we find the first terms of the expansion of the accessory parameter

$$\begin{aligned}4t_0K_0={}&(\sigma-1)^2-(\theta_t+\theta_0-1)^2+2(\theta_1-1)(\theta_t-1)t_0+\frac{((\sigma-1)^2-1-\theta_0^2+\theta_t^2)((\sigma-1)^2-1-\theta_\infty^2+\theta_1^2)}{2\sigma(\sigma-2)}t_0\\&+2(\theta_1-1)(\theta_t-1)t_0^2+\frac{(\theta_0^2-\theta_t^2)^2(\theta_1^2-\theta_\infty^2)^2}{64}\left(\frac{1}{\sigma^3}-\frac{1}{(\sigma-2)^3}\right)t_0^2\\&-\frac{((\theta_0^2-\theta_t^2)(\theta_1^2-\theta_\infty^2)+8)^2-2(\theta_0^2+\theta_t^2)(\theta_1^2-\theta_\infty^2)^2-2(\theta_0^2-\theta_t^2)^2(\theta_1^2+\theta_\infty^2)-64}{32\sigma(\sigma-2)}t_0^2\\&+\frac{((\theta_0-1)^2-\theta_t^2)((\theta_0+1)^2-\theta_t^2)((\theta_1-1)^2-\theta_\infty^2)((\theta_1+1)^2-\theta_\infty^2)}{32(\sigma+1)(\sigma-3)}t_0^2\\&-\frac{1}{32}(5+14\theta_0^2-18\theta_t^2-18\theta_1^2+14\theta_\infty^2)t_0^2+\frac{13}{32}\sigma(\sigma-2)t_0^2+\mathcal{O}(t_0^3)\end{aligned} \quad (56)$$

for $\mathrm{Re}\sigma > 0$. The corresponding expression for $\mathrm{Re}\sigma < 0$ can be obtained by sending $\sigma \to -\sigma$. The higher-order corrections can be consistently computed from the series derived in Ref. [27]. Note that, since any solution for $X$ in the series (52) will yield the same value for $s$ in Eq. (A2), and hence the same value for $K_0$, the difference between $\sigma$ and $\sigma_{0t}$ is tied to which terms of the expansion are dominant and depends on the particular value for $s$ and $t_0$. The generic structure of the conformal block expansion, of which $K_0$ is the semiclassical limit, was discussed at some length in the classical CFT literature [28,29]. The relevant facts for our following discussion, given the generic expansion

$$4t_0K_0 = k_0 + k_1t_0 + k_2t_0^2 + \cdots + k_nt_0^n + \cdots, \tag{57}$$

are as follows: $k_n$ is a rational function of the monodromy parameters, the numerator is a polynomial in the "external" parameters $\theta_i$ and $\sigma$, and the denominator is a polynomial of $\sigma$ alone. Secondly, $k_n$ is invariant under the reflection $\sigma \leftrightarrow 2-\sigma$. Thirdly, $k_n$ has simple poles at $\sigma = 3, 4, \ldots, n+1$ and $\sigma = -1, -2, \ldots, -n+1$ and poles of the order of $2n-1$ at $\sigma = 0, 2$ and is analytic at $\sigma = 1$. Fourthly, the leading order term of $k_n$ near $\sigma \simeq 2$ is (for $n \geq 1$)

$$k_n = -4\mathbf{C}_{n-1}\frac{(\theta_0^2-\theta_t^2)^n(\theta_1^2-\theta_\infty^2)^n}{16^n(\sigma-2)^{2n-1}} + \cdots, \quad \mathbf{C}_n = \frac{1}{n+1}\binom{2n}{n}, \tag{58}$$

where $\mathbf{C}_n$ is the $n$th Catalan number. A similar structure exists for the fundamental solution $X_0(\sigma; t_0)$ or, rather, $Y(\sigma; t_0)$:

$$Y(\sigma; t_0) = \chi_1 t_0 + \chi_2 t_0^2 + \cdots \tag{59}$$

with leading order for each $\chi_n$ given by (for $n \geq 3$)

$$\chi_n = -\mathbf{C}_{n-2}\frac{((\theta_t+\sigma)^2-\theta_0^2)((\theta_1+\sigma)^2-\theta_\infty^2)}{16\sigma^2(1-\sigma)^2} \times \frac{(\theta_1^2-\theta_\infty^2)^{n-1}(\theta_0^2-\theta_t^2)^{n-1}}{4^{n-1}\sigma^{2(n-1)}(\sigma-2)^{2(n-1)}} + \cdots, \tag{60}$$

where the implicit terms are of the order of $\mathcal{O}((\sigma-2)^{-2n+3})$ or higher.

### A. The angular eigenvalues

The separation constant can be calculated from the $\tau$ function expansion by imposing the quantization condition (39). For equal rotation parameters $a_1 = a_2$, the Heun equation reduces to a hypergeometric, and an analytic expression in terms of finite combinations of elementary functions can be obtained [7]. We can recover the result with the PVI $\tau$ function by taking the limit $u_0 \to 0$. The leading term of Eq. (56) gives the exact result

$$C_j = (1-a_1^2)[(m_1+m_2+2j)(m_1+m_2+2j-2) - 2\omega a_1(m_1+m_2) - a_1^2(m_1+m_2)^2] + a_1^2\omega^2 + a_1^2\Delta(\Delta-4), \tag{61}$$

which recovers the literature if we set the integer labeling the angular mode as

$$\ell = -(m_1+m_2+2j). \tag{62}$$

We note that (some of) the SO(4) selection rules are encoded in the requirement that $j$ is an integer [30].

For generic angular parameters, the monodromy data of the angular equation (12) is composed of the single monodromy parameters (10) $\{\varsigma_0, \varsigma_{u_0}, \varsigma_1, \varsigma_\infty\}$ and the composite monodromy parameters $\{\varsigma_{0u_0}, \varsigma_{1u_0}\}$. Using the formula (56), the separation constant (61) can be written up to third order in $u_0$ (remember that $\varsigma = \omega + a_1m_1 + a_2m_2$):

$$\begin{aligned} C_\ell = {} & \omega^2 + \ell(\ell+2) - \varsigma^2 - \frac{a_1^2+a_2^2}{2}(\ell(\ell+2) - \varsigma^2 - \Delta(\Delta-4)) - \frac{(a_1^2-a_2^2)(m_1^2-m_2^2)}{2\ell(\ell+2)}(\ell(\ell+2) - \varsigma^2 + (\Delta-2)^2) \\ & - \frac{(a_1^2-a_2^2)^2}{1-a_2^2}\Bigg[\frac{(\ell(\ell+2)+m_2^2-m_1^2)(\ell(\ell+2)+(\Delta-2)^2-\varsigma^2)}{2\ell(\ell+2)} - \frac{13}{32}\ell(\ell+2) + \frac{1}{32}(5+14(m_1^2+\varsigma^2) - 18(m_2^2+(\Delta-2)^2)) \\ & - \frac{((m_1+1)^2-m_2^2)((1-m_1)^2-m_2^2)((\Delta-1)^2-\varsigma^2)((\Delta-3)^2-\varsigma^2)}{32(\ell-1)(\ell+3)} \\ & + \frac{((m_1^2-m_2^2)((\Delta-2)^2-\varsigma^2)+8)^2 - 64 - 2(m_1^2+m_2^2)((\Delta-2)^2-\varsigma^2)^2}{32\ell(\ell+2)} \\ & - \frac{2(m_1^2-m_2^2)^2((\Delta-2)^2+\varsigma^2)}{32\ell(\ell+2)} - \frac{(m_1^2-m_2^2)^2((\Delta-2)^2-\varsigma^2)^2}{64}\left(\frac{1}{(\ell+2)^3} - \frac{1}{\ell^3}\right)\Bigg] + \mathcal{O}\left(\left(\frac{a_1^2-a_2^2}{1-a_2^2}\right)^3\right). \end{aligned} \tag{63}$$

TABLE I. The massless scalar field $s$-wave $\ell = 0$ and fundamental $n = 0$ quasinormal mode $\omega_{0,0}$ in a Schwarzschild-AdS$_5$ background for some values of $r_+$. The results were obtained using the Fredholm determinant expansion for the $\tau$ function with $N = 16$.

| $r_+$ | $z_0$ | $\omega_{0,0}$ |
|---|---|---|
| 0.005 | $2.49988 \times 10^{-5}$ | $3.9998498731325748 - 1.5044808171834238 \times 10^{-6}i$ |
| 0.01 | $9.99800 \times 10^{-5}$ | $3.9993983005189682 - 1.2123793015872442 \times 10^{-5}i$ |
| 0.05 | $2.48756 \times 10^{-3}$ | $3.9844293869590734 - 1.7525974895168137 \times 10^{-3}i$ |
| 0.1 | $9.80392 \times 10^{-3}$ | $3.9355764849860639 - 1.7970664179740506 \times 10^{-2}i$ |
| 0.2 | $3.70370 \times 10^{-2}$ | $3.7906778316981978 - 0.1667439940917780i$ |
| 0.4 | 0.121212 | $3.7173879743704008 - 0.7462495474087164i$ |
| 0.6 | 0.209302 | $3.8914015767067012 - 1.3656095289384492i$ |

TABLE II. The same quasinormal mode frequency $\omega_{0,0}$ computed using numerical matching from Frobenius solutions (with 15 terms) and the quadratic eigenvalue problem (with 120-point lattice).

| $r_+$ | Frobenius | QEP |
|---|---|---|
| 0.005 | $3.9998498731325743 - 1.5044808171845522 \times 10^{-6}i$ | $3.9998483860043481 - 2.8895543908757586 \times 10^{-5}i$ |
| 0.01 | $3.9993983005189876 - 1.2123793015712405 \times 10^{-5}i$ | $3.9993981402971502 - 2.3439366987252536 \times 10^{-5}i$ |
| 0.05 | $3.9844293869590911 - 1.7525974895155961 \times 10^{-3}i$ | $3.9844293921364538 - 1.7526437924554161 \times 10^{-3}i$ |
| 0.1 | $3.9355764849860673 - 1.7970664179739766 \times 10^{-2}i$ | $3.9355763694852816 - 1.7970671629389028 \times 10^{-2}i$ |
| 0.2 | $3.7906778316982394 - 0.1667439940917505i$ | $3.7906771832980760 - 0.1667441392742093i$ |
| 0.4 | $3.7173879743704317 - 0.7462495474087220i$ | $3.7173988607936563 - 0.7462476412816416i$ |
| 0.6 | $3.8914015767126869 - 1.3656095289361863i$ | $3.8913340701538795 - 1.3656086881322822i$ |

This expression reduces to the ones found in Ref. [7] when $a_1 \simeq a_2$. It also agrees with the expression in Ref. [31] for $\Delta = 4$, at least to the order given.

With an expression for the separation constant, we can address the computation of the quasinormal modes using the two initial conditions for the radial PVI $\tau$ function at $t_0 = z_0$. We will next explore this and compare with numerical results obtained from well-established methods in numerical relativity.

### B. The quasinormal modes for Schwarzschild

In the limit $a_i \to 0$, one recovers the Schwarzschild-AdS metric, and accordingly the radial differential equation coming from the Klein-Gordon equation for massless scalar fields (15) can be reduced to the standard form of the Heun equation. The exponents $\theta_k$ are given by

$$\theta_+ = \frac{i}{2\pi}\frac{\omega}{T}, \quad \theta_- = 0, \quad \theta_0 = \frac{1}{2\pi}\frac{\omega}{T}\frac{\sqrt{1+r_+^2}}{r_+}, \quad \theta_\infty = 2-\Delta, \tag{64}$$

where $2\pi T = 2\pi T_+ = (1 + 2r_+^2)/r_+$ is the temperature of the black hole, given by Eq. (6) by setting $a_1 = a_2 = r_- = 0$. The mass of the black hole is given by $\mathcal{M} = \frac{1}{2} r_+^2 (1 + r_+^2)$. We note that the system of coordinates is different from Ref. [32], and the singular point at $r = r_+$ is mapped by Eq. (18) to $z_0 = r_+^2/(1 + 2r_+^2)$.

Likewise, the angular equation (7) reduces to a standard hypergeometric form. The angular eigenvalues can be seen to be the usual SO(4) Casimir: $C_\ell = \ell(\ell + 2)$. In terms of $\omega$, $\Delta$, and $r_+$, the accessory parameter $K_0$ in Eq. (22) is

$$K_0 = -\frac{\omega^2}{4(1 + r_+^2)} + \frac{1 + 2r_+^2}{1 + r_+^2}\left[\frac{\ell(\ell + 2)}{4r_+^2} + \frac{\Delta(\Delta - 2)}{4}\right] + \frac{i\omega}{2r_+}\frac{1 + r_+^2(2 - \Delta)}{1 + r_+^2}. \tag{65}$$

This, along with the quantization condition for the radial monodromies (40), provides through Eq. (41) an implicit solution for the quasinormal modes $\omega_n$ along with the composite monodromy $\sigma_{0t}$, as we will tackle in Sec. IV B.

In order to test the method, we present in Tables I and II the numerical solution $\omega_{n,\ell}$ for the first quasinormal mode $n = 0, \ell = 0$ $s$-wave case and compare with known methods, the pseudospectral method with a Chebyshev-Gauss-Lobatto grid to solve the associated QEP and the usual numerical matching method based on the Frobenius expansion of the solution near the horizon and spatial infinity.[5] The Frobenius method implements the smoothness on the first derivative at the matching point of the two series solutions constructed with 15 terms, at the horizon and the boundary [33]. On the other hand, the pseudospectral method relies on a grid with 120 points between 0 and 1. For a more comprehensive reading, we recommend Refs. [34,35]. The results for $\omega_{0,0}$ are reported in Tables I and II.

The Schwarzschild-AdS case has been considered before [1,8,32,36,37] and should be thought of as a test of the new

[5] It should be noted that the Frobenius method is, in spirit, similar to the old combinatorial approach for the PVI $\tau$ function given by Jimbo [13].

TABLE III. Fundamental modes for Kerr-AdS$_5$, $\ell = m_1 = m_2 = 0$, $a_1 = 0.002$, $a_2 = 0.00199$, and the mass of the scalar field is $7.96 \times 10^{-8}$.

| $r_+$ | $z_0$ | $\tau$ function | Frobenius |
|---|---|---|---|
| 0.00200 | $4.0 \times 10^{-8}$ | $3.9999043938966996 - 3.9179009496192059 \times 10^{-7} i$ | $3.9999043938967028 - 3.9179009496196828 \times 10^{-7} i$ |
| 0.02185 | 0.000476717 | $3.9970574292783057 - 1.3247529381539807 \times 10^{-4} i$ | $3.9970574292783089 - 1.3247529381539848 \times 10^{-4} i$ |
| 0.06154 | 0.003758101 | $3.9760894388470440 - 3.4698629309308322 \times 10^{-3} i$ | $3.9760894388470473 - 3.4698629309308430 \times 10^{-3} i$ |
| 0.10123 | 0.010040760 | $3.9339314599984108 - 1.8761575868127569 \times 10^{-2} i$ | $3.9339314599984140 - 1.8761575868127629 \times 10^{-2} i$ |
| 0.14092 | 0.019098605 | $3.8762906043241960 - 5.7537333581688194 \times 10^{-2} i$ | $3.8762906043241993 - 5.7537333581688376 \times 10^{-2} i$ |
| 0.18061 | 0.030620669 | $3.8166724096683002 - 1.2480348073108545 \times 10^{-1} i$ | $3.8166724096683035 - 1.2480348073108582 \times 10^{-1} i$ |
| 0.22030 | 0.044236431 | $3.7668353453284391 - 2.1574723769724682 \times 10^{-1} i$ | $3.7668353453284420 - 2.1574723769724741 \times 10^{-1} i$ |
| 0.29968 | 0.076131349 | $3.7116288122171590 - 4.3786490332401062 \times 10^{-1} i$ | $3.7116288122171622 - 4.3786490332401161 \times 10^{-1} i$ |
| 0.37906 | 0.111610120 | $3.7104224042819611 - 6.8107859662243775 \times 10^{-1} i$ | $3.7104224042819692 - 6.8107859662244147 \times 10^{-1} i$ |
| 0.49813 | 0.165833126 | $3.7816024214536172 - 1.0519267755676109 i$ | $3.7816024214748239 - 1.0519267755684242 i$ |
| 0.61720 | 0.216209245 | $3.9134030353146323 - 1.4181181443831172 i$ | $3.9134030400737264 - 1.4181181441373386 i$ |
| 0.73627 | 0.260096962 | $4.0879586460765776 - 1.7776344225896197 i$ | $4.0879588168442726 - 1.7776344550831753 i$ |

method. Even without an optimized code,[6] the Fredholm determinant evaluation of the PVI $\tau$ function provides a faster way of computing the normal modes than both the numerical matching and the QEP method. Convergence is significantly faster when compared to the other methods for small $z_0 \sim 10^{-5}$ and can provide at least 14 significant digits for the fundamental frequencies.

## IV. MONODROMY PARAMETERS FOR KERR-AdS

The fast convergence and high accuracy of the $\tau$ function calculation is suitable for the study of small black holes. Turning our attention to Kerr-AdS$_5$, we consider spinning black holes of different angular momenta and radii. In view of holographic applications, we make use of an extra parameter given by the mass of the scalar field scattered by the black hole. Numerical results are presented in Table III.[7]

One can use the initial condition for the first derivative and Eq. (44) to determine an asymptotic formula for the composite monodromy parameters $\sigma$ and $s$ as functions of the frequency. In the spirit of establishing the occurrence of instabilities, it is worth looking at the small black hole limit. To better parametrize this limit, let us define

$$a_1^2 = \epsilon_1 r_+^2, \qquad a_2^2 = \epsilon_2 r_+^2, \tag{66}$$

with the understanding that $r_+^2$ is a small number. The three parameters $r_+^2$, $\epsilon_1$, and $\epsilon_2$ are sufficient to express the other roots of $\Delta_r$ as follows:

$$r_-^2 = \frac{1+(1+\epsilon_1+\epsilon_2)r_+^2}{2}\left(\sqrt{1+\frac{4\epsilon_1\epsilon_2 r_+^2}{(1+(1+\epsilon_1+\epsilon_2)r_+^2)^2}}-1\right), \tag{67}$$

$$-r_0^2 = \frac{1+(1+\epsilon_1+\epsilon_2)r_+^2}{2} \times\left(\sqrt{1+\frac{4\epsilon_1\epsilon_2 r_+^2}{(1+(1+\epsilon_1+\epsilon_2)r_+^2)^2}}+1\right). \tag{68}$$

Since we want $r_-^2 \leq r_+^2$, the $\epsilon_i$ will satisfy

$$\epsilon_1\epsilon_2 \leq 1 + (2+\epsilon_1+\epsilon_2)r_+^2 \simeq 1, \tag{69}$$

and we remind the reader that $\epsilon_{1,2}$ are also constrained by the extremality condition $a_i < 1$ the space of allowed $\varepsilon_{1,2}$ is illustrated in Fig. 1.

We will focus on the case $m_1 = m_2 = 0$ (and therefore $\ell$ even) in order to keep the expressions reasonably short. It will be convenient to leave $z_0$ implicit at times:

$$z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2} = \frac{1+(3+\epsilon_1+\epsilon_2)r_+^2 - \sqrt{(1+(1+\epsilon_1+\epsilon_2)r_+^2)^2+4\epsilon_1\epsilon_2 r_+^2}}{1+(3+\epsilon_1+\epsilon_2)r_+^2 + \sqrt{(1+(1+\epsilon_1+\epsilon_2)r_+^2)^2+4\epsilon_1\epsilon_2 r_+^2}}, \tag{70}$$

which asymptotes as $z_0 = (1-\epsilon_1\epsilon_2)r_+^2 + \mathcal{O}(r_+^4)$. The expansions of the single monodromy parameters are, up to terms of the order of $\mathcal{O}(r_+^3)$,

$$\theta_0 = \omega\left(1 - \frac{3}{2}(1+\epsilon_1)(1+\epsilon_2)r_+^2 + \cdots\right), \tag{71}$$

$$\theta_+ = i\omega\frac{(1+\epsilon_1)(1+\epsilon_2)}{1-\epsilon_1\epsilon_2} r_+ + \cdots, \tag{72}$$

[6] Using PYTHON's standard libraries for arbitrary precision floats. The PYTHON code for both the Nekrasov expansion and the Fredholm determinant can be provided upon request.

[7] In the table values, we have neglected some precision in the results for the sake of clarity, but we can provide more accurate values upon request.

$$\theta_- = -i\omega \frac{(1+\epsilon_1)(1+\epsilon_2)}{1-\epsilon_1\epsilon_2}\sqrt{\epsilon_1\epsilon_2}r_+ + \cdots. \quad (73)$$

The single monodromy parameters can be seen to have the structure

$$\theta_- = -i\phi_- r_+, \qquad \theta_+ = i\phi_+ r_+, \quad (74)$$

where $\phi_\pm$ are real and positive for real and positive $\omega$. We also observe that $\theta_0$ is parametrically close to the frequency $\omega$, and the correction is negative for positive $r_+$.

We now proceed to solve for the composite monodromy parameter $\sigma_\ell \equiv \sigma_{0z_0}(\ell)$ using the series expansion (56). For even $\ell \geq 2$, the first correction is

$$\begin{aligned}\sigma_\ell &\equiv \ell + 2 - \nu_\ell r_+^2 \\ &= \ell + 2 - \frac{(1+\epsilon_1)(1+\epsilon_2)}{4(\ell+1)}(3\omega^2 + 3\ell(\ell+2) \\ &\quad - \Delta(\Delta-4))r_+^2 + \mathcal{O}(r_+^4), \quad \ell \geq 2, \end{aligned} \quad (75)$$

and, due to the pole structure of Eq. (57), a naive series inversion will yield the expansion for $\sigma$ up to the order of $r_+^{2\ell}$. The case $\ell = 0$ is then special and will be dealt with shortly. One can see from Eq. (54) that, for $p = 0$, the monodromy parameter $s$ will behave asymptotically as $z_0^{-\sigma}$, diverging for small $z_0$. Changing the value of $p$ will change this behavior. Changing the value of $p$ means shifting the argument $\sigma$ that enters the definition of $X_0(\sigma, t_0)$ in Eq. (52) and therefore of $Y(\sigma, t_0)$ in Eq. (48). Let us call $Y_{\ell,2p}$ the expression in Eq. (49) for generic $p$ and $\sigma \simeq 2 + \ell$. The expression for $p = 0$ is given by

$$\begin{aligned} Y_{\ell,0} &\equiv Y(\sigma_\ell; z_0) \\ &= -(1-\epsilon_1\epsilon_2)\frac{\omega^2 - (\Delta - \ell - 4)^2}{16(\ell+1)^2} \\ &\quad \times \left(1 + \frac{2i}{\ell+2}\phi_+ r_+\right) r_+^2 + \cdots, \quad \ell \geq 2. \end{aligned} \quad (76)$$

We point out that this value is actually independent of $p$, except when $2p = \ell$, as we will see below. We anticipate, from Eq. (53), that $Y_{\ell,p}$ for $2p < \ell$ will yield a larger value for $s_\ell$ for smaller $r_+$. We also remark that $s_\ell$ will have a nonanalytic expansion in $r_+$, due to the term $z_0^{-\sigma_\ell}$. Finally, from the expansion we conclude that $Y_{\ell,p}$ has an imaginary part of subleading order.

### A. $\ell = 0$

The "$s$-wave" case $\ell = 0$ is singular, since the leading behavior of $\sigma - 2$ is of the order of $r_+^2$. The expansion (57) does not converge, in general, due to the denominator structure of the coefficients $\kappa_n$. For the small $r_+$ black hole application, however, we are really dealing with a scaling limit where

$$\theta_- = \varphi_- \sqrt{z_0}, \quad \theta_+ = \varphi_+ \sqrt{z_0}, \quad \text{and} \quad \sigma = 2 - \upsilon z_0 \quad (77)$$

have finite limits for $\varphi_\pm$ and $\upsilon$ as $z_0 \to 0$. Because of the poles of increasing order in $\sigma$ in Eq. (57), in the $\ell = 0$ case one has to resum the whole series in order to compute $\upsilon$.

Thankfully, the task is amenable due to the fact that, in the scaling limit, the term of the order of $z_0$ in each of the factors $k_n t_0^n$ in the expansion (57) comes from the leading order pole (58):

$$k_n z_0^n = -4\mathbf{C}_{n-1}\frac{(\varphi_-^2 - \varphi_+^2)^n(\theta_1^2 - \theta_\infty^2)^n}{16^n \upsilon^{2n-1}} z_0 + \mathcal{O}(z_0^2). \quad (78)$$

The series can be resummed using the generating function for the Catalan numbers

$$1 + x + 2x^2 + 5x^3 + \cdots = \sum_{n=0}^{\infty} \mathbf{C}_n x^n = \frac{1 - \sqrt{1-4x}}{2x}, \quad (79)$$

and the result for $\upsilon$ readily written

$$\begin{aligned} &4z_0 K_0(\ell = 0) + (\theta_+ + \theta_- - 1)^2 + 2(\theta_1 - 1)(\theta_+ - 1)\frac{z_0}{z_0 - 1} \\ &\quad = 1 + \frac{1}{2}(\theta_1^2 - \theta_\infty^2)z_0 - 2\upsilon z_0 \sqrt{1 + \frac{(\varphi_+^2 - \varphi_-^2)(\theta_1^2 - \theta_\infty^2)}{4\upsilon^2}} \\ &\quad + \mathcal{O}(z_0^2). \end{aligned} \quad (80)$$

A similar procedure allows us to compute the parameter $Y(\upsilon) \equiv Y(2 - \upsilon z_0; z_0)$ up to the order of $z_0^{3/2}$:

$$\begin{aligned} Y(\upsilon) &= -z_0(1 + \varphi_+\sqrt{z_0})\frac{\theta_\infty^2 - (\theta_1 + 2)^2}{64} \\ &\quad \times \left(1 + \sqrt{1 + \frac{(\varphi_+^2 - \varphi_-^2)(\theta_1^2 - \theta_\infty^2)}{4\upsilon^2}}\right)^2 + \cdots. \end{aligned} \quad (81)$$

For the application to the $\ell = 0$ case of the scalar field, we will use the notation (74) and again use $\sigma_0 = 2 - \nu_0 r_+^2$. In terms of the black hole parameters, $\nu_0$ has a surprisingly simple form:

$$\nu_0 = \frac{1}{4}(1+\epsilon_1)(1+\epsilon_2)\sqrt{(3\omega^2 - \Delta(\Delta-4))^2 - 4\omega^2(\omega^2 - (\Delta-2)^2)} + \mathcal{O}(r_+^2), \quad (82)$$

and

$$Y_{0,0} \equiv Y(\sigma_0; z_0) = -(1-\epsilon_1\epsilon_2) r_+^2 (1+i\phi_+ r_+) \frac{\omega^2-(\Delta-4)^2}{64} \left(1+\frac{3\omega^2-\Delta(\Delta-4)}{\sqrt{(3\omega^2-\Delta(\Delta-4))^2-4\omega^2(\omega^2-(\Delta-2)^2)}}\right)^2 + \cdots. \tag{83}$$

Finally, let us define the shifted $Y_{\ell,\ell}$ for $2p=\ell$. Since the shifted argument $\sigma-2p$ is close to 2, we need the same scaling limit as above in Eq. (81). The result is

$$Y_{\ell,\ell} \equiv Y(\sigma_\ell-\ell; z_0) = -(1-\epsilon_1\epsilon_2) r_+^2 (1+i\phi_+ r_+) \frac{\omega^2-(\Delta-4)^2}{64} \left(1+\sqrt{1+\frac{4(\ell+1)^2\omega^2(\omega^2-(\Delta-2)^2)}{(3\omega^2+3\ell(\ell+2)-\Delta(\Delta-4))^2}}\right)^2 + \cdots, \tag{84}$$

where $\nu_\ell$ is taken from Eq. (75).

To sum up, we exhibit the overall structure for small $r_+$:

$$\sigma_\ell = \ell + 2 - \nu_\ell r_+^2 + \cdots, \tag{85}$$

$$Y_{\ell,\ell} = -(1-\epsilon_1\epsilon_2)\vartheta_\ell (1+i\phi_+ r_+) r_+^2 + \cdots, \tag{86}$$

where $\nu_\ell$ and $\vartheta_\ell$ have nonzero limits as $r_+ \to 0$, have corrections of the order of $r_+^2$, and, most importantly, are positive for $\omega$ real and greater than $\Delta-4$.

### B. The quasinormal modes

Implementation of the quantization condition (40) can be done with the formula (B7). This yields a transcendental equation for $\omega$ whose solutions will give all complex frequencies for the radial quantization condition. These include negative real-part frequencies, as well as non-normalizable modes. Since we are interested in positive real-part frequencies, we will consider a small correction to the vacuum $AdS_5$ result [38,39]

$$\omega_{n,\ell} = \Delta + 2n + \ell + \eta_{n,\ell} r_+^2, \tag{87}$$

under the hypothesis that $\eta_{n,\ell}$ has a finite limit as $r_+ \to 0$. One notes by Eq. (71) that $\theta_0$ and $\omega$ are perturbatively close, so $\eta_{n,\ell}$ can be calculated perturbatively from the expansion of $\theta_0$. We will assume that $\Delta$ is *not* an integer.

The parametrization (87) allows us to expand Eq. (54) as a function of $r_+$. The procedure is straightforward: We use $Y_{\ell,0}$ from Eq. (85), as it gives the right asymptotic behavior, to compute $X_0$ using Eq. (48) and then the $s$ parameter (54). To second order in $r_+$, we have

$$s_{n,\ell} = -\frac{16\Gamma(n+\ell/2+1)\Gamma(\Delta-2+n+\ell/2)}{\Gamma(n+\ell/2+3)\Gamma(\Delta+n+\ell/2)} \frac{\nu_\ell^2}{(1-\epsilon_1\epsilon_2)^2(\phi_+^2-\phi_-^2)} \left(1-i\phi_+ r_+ + 2i\frac{\phi_+\nu_\ell r_+}{\phi_+^2-\phi_-^2}\right) Y_{\ell,\ell} r_+^{-2+2\nu_\ell r_+^2}, \tag{88}$$

and the leading behavior for the parameter $s_{n,\ell}$ given $Y_{\ell,\ell}$ in Eq. (85) is

$$s_{n,\ell} = \Sigma_{n,\ell}\left(1+\frac{2i\nu_{n,\ell} r_+}{(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n+\ell)}\right) r_+^{2\nu_{n,\ell} r_+^2} + \cdots, \tag{89}$$

where we defined $\nu_{n,\ell}$ as the correction for $\sigma$ as in Eq. (85) calculated at the vacuum frequency $\nu_\ell(\omega=\Delta+2n+\ell)$. Finally,

$$\Sigma_{n,\ell} = \frac{16\Gamma(n+\ell/2+1)\Gamma(\Delta-2+n+\ell/2)}{\Gamma(n+\ell/2+3)\Gamma(\Delta+n+\ell/2)} \frac{\nu_{n,\ell}^2 \vartheta_{n,\ell}}{(1+\epsilon_1)^2(1+\epsilon_2)^2(\Delta+2n+\ell)^2}, \tag{90}$$

again, with $\vartheta_{n,\ell} = \vartheta(\omega=\Delta+2n+\ell)$. We also note that $\Sigma_{n,\ell}$ is real and positive under the same conditions as Eq. (85). Moreover, the choice of $p$ implicit in $Y_{\ell,\ell}$ guarantees that $s_{n,\ell}$ has a finite limit as $r_+ \to 0$, although its dependence on $r_+$ is nonanalytic.

Equation (B7) can now be used, setting $\cos\pi\sigma_{1t} = \cos\pi(\theta_1+\theta_t)$ for the radial parameters, to find a perturbative equation for $\eta_{n,\ell}$. We expand each of the terms in Eq. (B7) using Eq. (74) as well as

$$\theta_0 = \omega_0 - \beta r_+^2, \qquad \omega_0 = \Delta+2n+\ell, \quad \text{and} \quad \sigma = 2+\ell-\nu_\ell r_+^2. \tag{91}$$

Now, the following two relations hold:

$$\sin^2\pi\sigma\cos\pi(\theta_1+\theta_t) - \cos\pi\theta_0\cos\pi\theta_\infty - \cos\pi\theta_t\cos\pi\theta_1 + \cos\pi\sigma(\cos\pi\theta_0\cos\pi\theta_1 + \cos\pi\theta_t\cos\pi\theta_\infty)$$
$$= \frac{\pi^3}{2}\sin(\pi\Delta)(\phi_+^2-\phi_-^2)\left(\beta + \frac{2i\nu_\ell^2 r_+}{(1+\epsilon_1)(1+\epsilon_2)\omega_0}\right)r_+^4 + \cdots, \quad (92)$$

$$-\frac{1}{2}(\cos\pi\theta_\infty - \cos\pi(\theta_1\pm\sigma))(\cos\pi\theta_0 - \cos\pi(\theta_t\pm\sigma)) = \frac{\pi^3}{2}\sin(\pi\Delta)(\phi_+^2-\phi_-^2)\left(\frac{\beta\pm\nu}{2}\right)\left(1\pm\frac{2i\nu_\ell r_+}{(1+\epsilon_1)(1+\epsilon_2)\omega_0}\right)r_+^4 + \cdots. \quad (93)$$

We can now proceed to calculate the first correction to the eigenfrequencies (87). By using the approximations (92) and (93) above, we find the correction to $\theta_0$ for each of the modes $n,\ell$:

$$\beta_{n,\ell} = \nu_{n,\ell}\frac{\Sigma_{n,\ell}+1}{\Sigma_{n,\ell}-1} + 4i\frac{\nu_{n,\ell}^2}{(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n+\ell)}\frac{\Sigma_{n,\ell}}{(\Sigma_{n,\ell}-1)^2}r_+ + \mathcal{O}(r_+^2\log r_+). \quad (94)$$

Finally, after some laborious calculations, we find

$$\eta_{n,\ell} = -\frac{(1+\epsilon_1)(1+\epsilon_2)}{2}\left[\frac{Z_{n,\ell}}{2(\ell+1)} - 3(\Delta+2n+\ell)\right] - \frac{i}{4}(2n+\ell+1)(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n+\ell)(2\Delta+2n+\ell-2)r_+$$
$$+ \mathcal{O}(r_+^2\log r_+), \quad \ell\geq 2, \quad (95)$$

with

$$Z_{n,\ell}^2 = (3(\Delta+2n+\ell)^2 + 3\ell(\ell+2) - \Delta(\Delta-4))^2 + 4(\ell+1)^2(\Delta+2n+\ell)^2(2n+\ell+1)(2\Delta+2n+\ell-2). \quad (96)$$

For $\ell=0$, the form of the correction is slightly different. Repeating the calculation, we see that $\eta_{n,\ell=0}$ has the simpler form

$$\eta_{n,0} = -\frac{(1+\epsilon_1)(1+\epsilon_2)}{4}(3(\Delta+2n-1)^2 - (\Delta-2)^2 + 1) - i(n+1)(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n)(\Delta+n-1)r_+$$
$$+ \mathcal{O}(r_+^2\log r_+). \quad (97)$$

We note that both the real and imaginary parts of the corrections $\eta_{n,\ell}$ are negative, the real part of the order of $r_+^2$ as expected, and the imaginary part of the order of $r_+^3$. We stress that we are taking $m_1=m_2=0$ an illustration of the fundamental mode $\omega_0$ as a function of $r_+$ is depicted in Fig. 2.

In the midst of the calculation, we see that the imaginary part of $\eta_{n,0}$ has the same sign as the imaginary part of $\theta_+$, which in turn is essentially the entropy intake of the black hole as it absorbs a quantum of frequency $\omega$ and angular momenta $m_1$ and $m_2$:

$$\theta_+ = \frac{i}{2\pi}\delta S = \frac{i}{2\pi}\frac{\omega - m_1\Omega_{+,1} - m_2\Omega_{+,2}}{T_+}, \quad (98)$$

giving the same sort of window for unstable mode parameters $m_1$ and $m_2$ as in superradiance, so a closer look at higher values for $m_{1,2}$ is perhaps in order for future work. A full consideration of linear perturbations of the five-dimensional Kerr-AdS black hole, involving higher spin [40,41], can be done within the same theoretical framework presented here and will be left for the future. We close by observing that the expressions (95) and (97) above seem to represent a distinct limit than the results in Ref. [7]—which are, however, restricted to $\Delta=4$—and therefore not allowing for a direct comparison.

### C. Some words about the $\ell$ odd case

Let us illustrate the parameters for the subcase $m_1=\ell$, $m_2=0$. The single monodromy parameters admit the expansion

$$\theta_0 = \omega + \sqrt{\epsilon_1}\ell r_+ - \frac{3}{2}(1+\epsilon_1)(1+\epsilon_2)\omega r_+^2 + \cdots, \quad (99)$$

$$\theta_+ = -i\ell\frac{\sqrt{\epsilon_1}(1+\epsilon_2)}{1-\epsilon_1\epsilon_2} + i\omega\frac{(1+\epsilon_1)(1+\epsilon_2)}{1-\epsilon_1\epsilon_2}r_+ + \cdots, \quad (100)$$

$$\theta_- = i\ell\frac{\sqrt{\epsilon_2}(1+\epsilon_1)}{1-\epsilon_1\epsilon_2} - i\omega\frac{(1+\epsilon_1)(1+\epsilon_2)}{1-\epsilon_1\epsilon_2}\sqrt{\epsilon_1\epsilon_2}r_+ + \cdots, \quad (101)$$

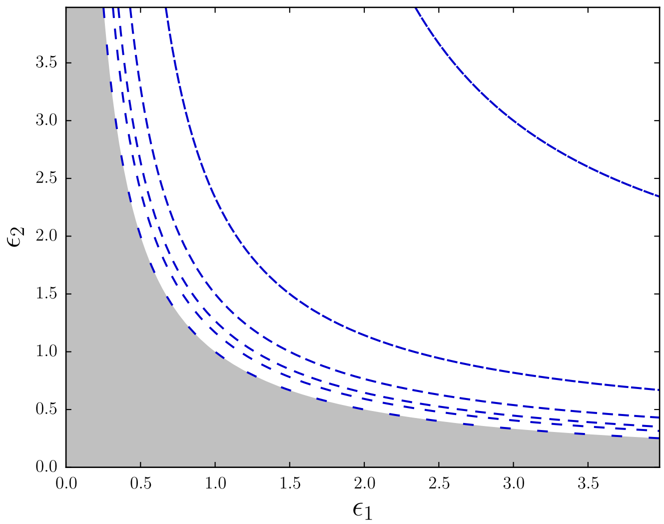


FIG. 1. The space of parameters defined by $\epsilon_1\epsilon_2 < 1$ corresponds to the gray area. The dashed lines represent the extremal black holes where $r_+ = r_-$, for $r_+ = 0.002, 0.2, 0.25, 0.333, 0.5, 1$ with increasing dash density. The curve $r_+ = 1$, the one closest to the right upper corner, is drawn for comparison.

with all of them finite and nonzero as $r_+ \to 0$. As usual, $\theta_\pm$ are purely imaginary, whereas $\theta_0$ is real for real $\omega$. These properties hold for any value of $m_1$ and $m_2$.

For $\ell \geq 1$ and odd, the composite monodromy parameters are found much in the same way as the case $\ell \geq 2$ considered above, by inverting Eq. (56). In the following, we set $\omega_0 = \Delta + 2n + \ell$ as the limit of the frequency as $r_+ \to 0$. We have for the composite monodromy parameter

$$\sigma_\ell = 2 + \ell - \nu_\ell r_+^2 + \mathcal{O}(r_+^4), \tag{102}$$

with $\nu_\ell$, defined as in Eq. (89), now for $\ell > 1$:

$$\nu_\ell = (1+\epsilon_1)(1+\epsilon_2) \times \frac{3\omega_0^2 + 3\ell(\ell+2) - \Delta(\Delta-4)}{4(\ell+1)}, \quad \ell \geq 3. \tag{103}$$

For $\ell = 1$, finding $\nu_1$ from condition (41) requires going to higher order in $z_0$, due to the pole at $\sigma = 3$ in the expansion (56):

$$\nu_1 = \frac{(1+\epsilon_1)(1+\epsilon_2)}{32}(3\omega_0^2 + 9 - \Delta(\Delta-4))\left(2 + \frac{1}{3}\sqrt{34 - 8\frac{2\Delta^4 - 16\Delta^3 + (50 - 3\omega_0^2)\Delta^2 + 12(\omega_0^2 - 6)\Delta - 36\omega_0^2}{(3\omega_0^2 + 9 - \Delta(\Delta-4))^2}}\right). \tag{104}$$

For the following discussion, we take from this calculation that the $\nu_\ell$'s are real and greater than 1 for $\Delta > 1$, which we will assume to hold for any $m_1$ and $m_2$. Apart from these properties, the particular form for $\nu_\ell$ will be left implicit. Given $\nu_\ell$, we can use the same procedure as in the even $\ell$ case to compute the $s$ parameter. Again, in order to have a finite $r_+ \to 0$ limit, we take $p = (\ell+1)/2$. After some calculations, we have

$$s_{n,\ell} = 1 + 2\nu_\ell r_+^2 \log r_+ + \Xi_{n,\ell}\nu_\ell r_+^2 + \mathcal{O}(r_+^4(\log r_+)^2) \tag{105}$$

with

$$\Xi_{n,\ell} = 2\gamma + \log(1-\epsilon_1\epsilon_2) + \Psi\left(\frac{1+\theta_+ + \theta_-}{2}\right) + \Psi\left(\frac{1+\theta_+ - \theta_-}{2}\right) + \Psi\left(\frac{3+2n+\ell}{2}\right) + \Psi\left(\frac{3-2\Delta-2n-\ell}{2}\right), \tag{106}$$

where $\Psi(z)$ is the digamma function and $\gamma = -\Psi(1)$ the Euler-Mascheroni constant. In the definition above, we have already set $\theta_0 = \Delta + 2n + \ell - \beta_{n,\ell} r_+^2$, but as we can see from Eq. (105), now we need $s_{n,\ell}$ to second order in the expansion parameter $r_+$. We again assume that $\Delta$ is, in general, *not* an integer, since this is irrelevant for the determination of the imaginary part of the frequency. However, having $\Delta$ integer will change the behavior of the real part of the correction to the eigenfrequency with respect to $r_+$.

We note that $s_{n,\ell}$ is nonanalytic, and therefore the expansion for $\beta_{n,\ell}$ will include terms like $\log r_+$. We expand Eq. (B7) with $\sigma_{1t} = 2 - \Delta + \theta_+$ (up to an even integer) to fourth order and find as a first approximation to the correction to the frequency

$$\eta_{n,\ell} = \cdots + \nu_{n,\ell}\left(\frac{\nu_{n,\ell}+1}{\nu_{n,\ell}-1} + \Xi_{n,\ell}\right)\frac{r_+^2}{\log r_+} + \cdots, \tag{107}$$

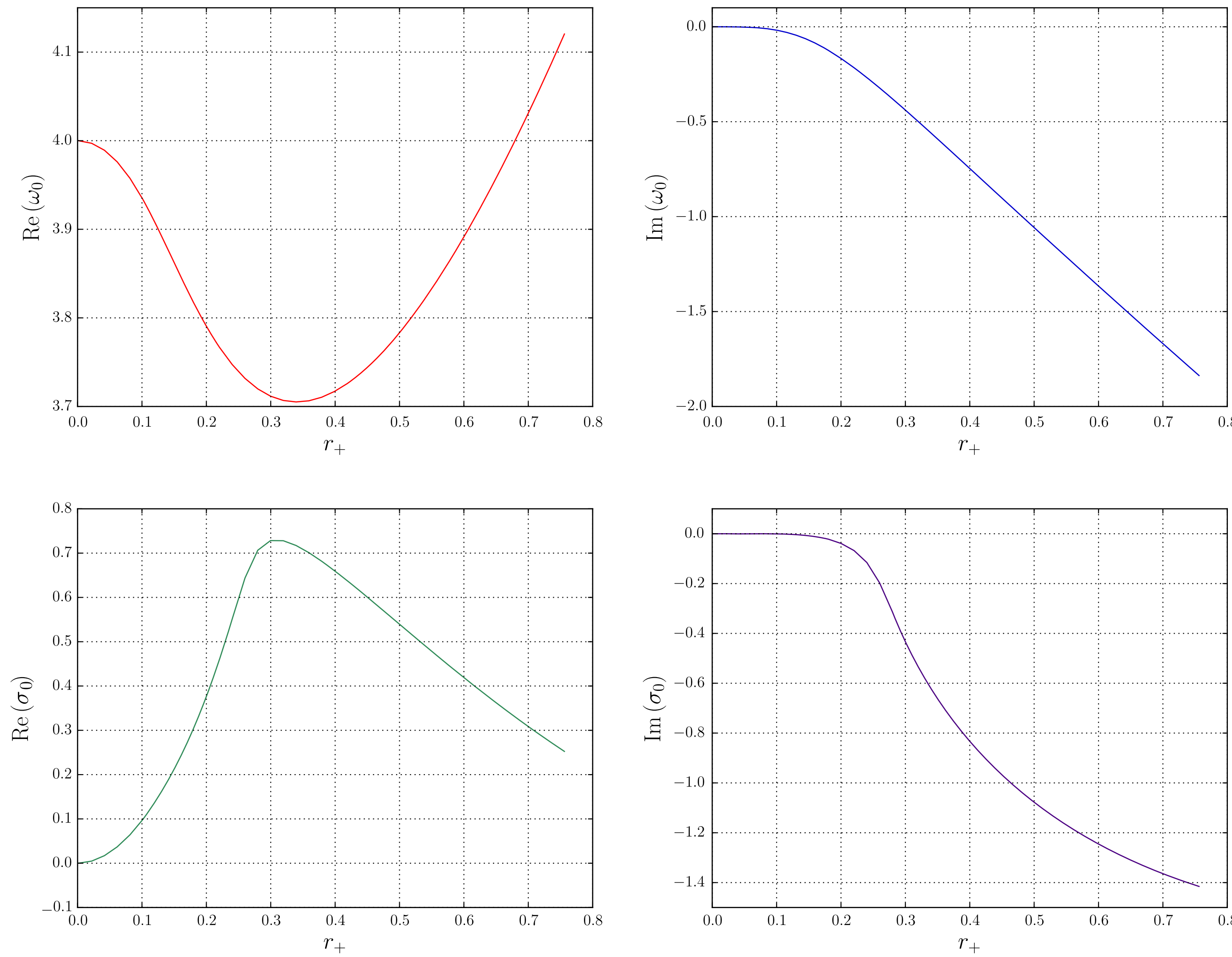


FIG. 2. In the first row, the dependence of the real and imaginary parts of the first quasinormal mode frequency $\omega_0$ for small Kerr-AdS$_5$ black holes ($a_1 = 0.002$, $a_2 = 0.00199$, $\mu = 7.96 \times 10^{-8}$). In the second, the dependence of the composite parameter $2 - \sigma$.

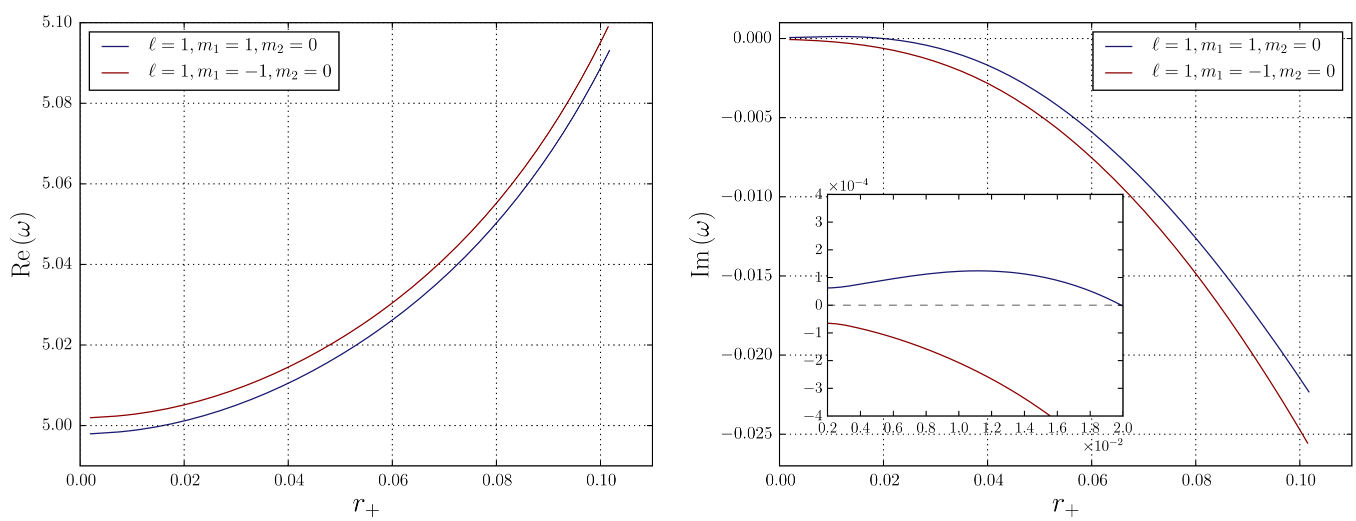


FIG. 3. The dependence of the real (left) and imaginary part (right) of the first quasinormal mode frequency $\omega_0$ at $\ell = 1$ and $m_1 = \pm 1$ for small Kerr-AdS$_5$ black holes ($a_1 = 0.002$, $a_2 = 0.00199$, $\mu = 7.96 \times 10^{-8}$).

where the terms left out are real, stemming from the relation between $\theta_0$ and $\omega$.

From Eq. (107), any possible imaginary part for the eigenfrequency will then come from the imaginary part of $\Xi_{n,\ell}$. The latter can be calculated by using the reflection property of the digamma function

$$\mathrm{Im}\Xi_{n,\ell} = -\frac{i\pi}{2}\left(\tan\frac{\pi}{2}(\theta_+ + \theta_-) + \tan\frac{\pi}{2}(\theta_+ - \theta_-)\right), \tag{108}$$

or, in terms of $m_1$ and $m_2$,

$$\mathrm{Im}\Xi_{n,\ell} = \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\frac{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}}{1+\sqrt{\epsilon_1\epsilon_2}}(m_1 - m_2)\right) + \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\frac{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}{1-\sqrt{\epsilon_1\epsilon_2}}(m_1 + m_2)\right). \tag{109}$$

We then see that the imaginary part of $\Xi_{n,\ell}$ can have any sign, a strong indication that the $\ell$ odd modes are unstable. Numerical support for this is included in Fig. 3, in which we use an arbitrary-precision PYTHON code (capped at 50 decimal places) to show a slightly positive imaginary part for the resonant frequency at $r \lesssim 0.02$. We point out that, indeed, instabilities in asymptotically anti–de Sitter spaces are expected from general grounds [42], and odd $\ell$ instabilities for the massless case ($\Delta = 4$) were found in Ref. [7].

## V. DISCUSSION

In this paper, we used the isomonodromy method to derive asymptotic expressions for the separation constant for the angular equation (angular eigenvalue) in (63) as well as the frequencies for the scalar quasinormal modes in a five-dimensional Kerr-AdS background in the limit of small black holes; see, in particular, Eqs. (95) and (97). The numerical analysis carried out for the Schwarzschild-AdS and Kerr-AdS cases showed that the $\tau$ function approach has advantages when compared to standard methods, in terms of faster processing times. For $\ell$ even, the correction to the vacuum AdS frequencies is negative with a negative imaginary part for $\Delta > 1$, the scalar unitarity bound, showing no instability in the range studied. For $\ell$ odd, there are strong indications for instability due to the general structure of the corrections in Eq. (109). In particular, for $\ell = 1$, the numerical results shown in Fig. 3 exhibit an unstable mode for $r_+ \leq 0.02$ and nearly equal rotational parameters. We plan to address the phase space of instabilities and holographic consequences in future work.

The method in this paper relies on the construction of the $\tau$ function of the PVI transcendent proposed in the literature following the Alday-Gaiotto-Tachikawa conjecture. The conditions in Eq. (41) translate the accessory parameters in the ODEs governing the propagation of the field—themselves depending on the physical parameters—into monodromy parameters, and the quantization condition (39) allows us to derive the angular separation constant (63). In turn, the quantization condition for the radial equation (40), through series solutions for the composite monodromy parameters $s$ and $\sigma$, allows us to solve for the eigenfrequencies $\omega_{n,\ell}$, even in the generic complex case.

The interpretation of the ODEs involved as the level-2 null vector condition of the semiclassical Liouville field theory allows us to conclude that all descendants are relevant for the calculation of the monodromy parameters, even though, for angular momentum parameter $\ell \geq 2$, one can consider just the conformal primary (first channel) for the parameter $\sigma_{0t}$.

The scaling limit resulting from this analysis gives the monodromy parameter $\sigma$ in Eq. (75). For the parameter $\sigma_{1t}$, the requisite of a smooth $r_+ \to 0$ limit forces us to consider the asymptotics of the whole series (56), thus involving all descendants. This means that a naive matching of the solution obtained from the near horizon approximation to the asymptotic solution near infinity is not a suitable tool for dealing with small black holes. For the composite monodromy parameter $\sigma_{1t}$, more suitably parametrized by $s$ in Eq. (B7), the requirement of a finite $r_+ \to 0$ limit allows us to select the solutions (89) for $\ell$ even and (105) for $\ell$ odd. Although finite in the small black hole limit, the $s$ parameter has a nonanalytic expansion in terms of $r_+^m(\log r_+)^n$.

For the $s$-wave $\ell = 0$ calculations, we had to consider a scaling limit in Eq. (56) where the Liouville momenta associated to $\theta_+$ and $\theta_-$ go to zero as $r_+$, at the same time as $z_0$ and $\sigma - 2$ scales as $r_+^2$. The formulas (80) and (81) are reminiscent of the light-light-heavy-heavy limit of Witten diagrams for conformal blocks [43]. It would be interesting to understand the CFT meaning of this limit.

The Toda equation, which allows us to interpret the second condition (41) on the Painlevé $\tau$ function, also merits further study. As for the first condition, we note that it provides the accessory parameters for both the angular and radial equations—$Q_0$ in Eq. (12) and $K_0$ in Eq. (20), respectively—as the derivative of the logarithm of the $\tau$ function for each system. On the other hand, these accessory parameters are both related to the separation constant of the Klein-Gordon equation, as can be verified through Eqs. (14) and (22). Including these terms in the definition of a $\tau$ function for the angular and radial systems, we can represent the fact that the separation constant is the same for Eqs. (12) and (20) as the condition

$$\frac{d}{du_0}\log\tau_{\mathrm{angular}} = \frac{d}{dz_0}\log\tau_{\mathrm{radial}}, \tag{110}$$

which in turn can be interpreted as a thermodynamical equilibrium condition. Given the usual interpretation of the $\tau$ function as the generating functional of a quantum theory, the elucidation of this structure can shed light on the spacetime approach to conformal blocks. The present work gives, in our opinion, convincing evidence that the PVI $\tau$ function is the best tool—both numerically and analytically—to study connection problems for Fuchsian equations, in particular, scattering and resonance problems for a wide class of black holes.

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## APPENDIX A: NEKRASOV EXPANSION AND FREDHOLM DETERMINANT FOR PAINLEVÉ VI

In what follows, we will assume the "sufficient generality condition"

$$\sigma_{0t}\notin\mathbb{Z},\qquad \sigma_{0t}\pm\theta_0\pm\theta_t\notin\mathbb{Z},\qquad \sigma_{0t}\pm\theta_1\pm\theta_\infty\notin\mathbb{Z}. \tag{A1}$$

The Nekrasov expansion of the PVI $\tau$ function is given as a double expansion [22,23]

$$\tau(t)=\sum_{n\in\mathbb{Z}}\mathcal{N}^{\theta_1}_{\theta_\infty,\sigma_{0t}+2n}\mathcal{N}^{\theta_t}_{\sigma_{0t}+2n,\theta_0}s^n t^{\frac{1}{4}((\sigma_{0t}+2n)^2-\theta_0^2-\theta_t^2)}(1-t)^{\frac{1}{2}\theta_1\theta_t}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}(\vec{\theta},\sigma_{0t}+2n)t^{|\lambda|+|\mu|}, \tag{A2}$$

where

$$\mathcal{N}^{\theta_3}_{\theta_2,\theta_1}=\frac{\prod_{\epsilon=\pm}G(1+\frac{1}{2}(\theta_3+\epsilon(\theta_2+\theta_1)))G(1+\frac{1}{2}(\theta_3+\epsilon(\theta_2-\theta_1)))}{G(1-\theta_1)G(1+\theta_2)G(1-\theta_3)} \tag{A3}$$

with $G(z)$ the Barnes function, defined by the solution of the functional equation $G(z+1)=\Gamma(z)G(z)$, with $G(1)=1$ and $\Gamma(z)$ the Euler gamma.[8] The other parameters in Eq. (A2) are the coefficients of the $c=1$ Virasoro conformal block

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}(\vec{\theta},\sigma)=&\prod_{(i,j)\in\lambda}\frac{((\theta_t+\sigma+2(i-j))^2-\theta_0^2)((\theta_1+\sigma+2(i-j))^2-\theta_\infty^2)}{16h_\lambda^2(i,j)(\lambda'_j-i+\mu_i-j+1+\sigma)^2}\\ &\times\prod_{(i,j)\in\mu}\frac{((\theta_t-\sigma+2(i-j))^2-\theta_0^2)((\theta_1-\sigma+2(i-j))^2-\theta_\infty^2)}{16h_\lambda^2(i,j)(\mu'_j-i+\lambda_i-j+1-\sigma)^2},\end{aligned} \tag{A4}$$

where $\mathbb{Y}$ denotes the space of Young diagrams and $\lambda$ and $\mu$ are two of its elements, with the number of boxes $|\lambda|$ and $|\mu|$. For each box situated at $(i,j)$ in $\lambda$, $\lambda_i$ are the number of boxes at row $i$ of $\lambda$, $\lambda'_j$, the number of boxes at column $j$ of $\lambda$; $h(i,j)=\lambda_i+\lambda'_j-i-j+1$ is the hook length of the box at $(i,j)$. Finally, the parameter $s$ is given in terms of monodromy data by

$$s=\frac{(w_{1t}-2p_{1t}-p_{0t}p_{01})-(w_{01}-2p_{01}-p_{0t}p_{1t})\exp(\pi i\sigma_{0t})}{(2\cos\pi(\theta_t-\sigma_{0t})-p_0)(2\cos\pi(\theta_1-\sigma_{0t})-p_\infty)}, \tag{A5}$$

where

$$\begin{aligned}&p_i=2\cos\pi\theta_i,\qquad p_{ij}=2\cos\pi\sigma_{ij},\\ &w_{0t}=p_0p_t+p_1p_\infty,\qquad w_{1t}=p_1p_t+p_0p_\infty,\\ &w_{01}=p_0p_1+p_tp_\infty.\end{aligned} \tag{A6}$$

The Fredholm determinant representation for the PVI $\tau$ function uses the usual Riemann-Hilbert problem formulation in terms of Plemelj (projection) operators and jump matrices. The idea is to introduce projection operators which act on the space of (a pair of) functions on the complex plane to give analytic functions with prescribed monodromy (Cauchy-Riemann operators). Details can be found in Ref. [14]. One should point out that the two expansions agree as functions of $t$ up to a multiplicative constant:

$$\tau(t)=\mathrm{const}\cdot t^{\frac{1}{4}(\sigma^2-\theta_0^2-\theta_t^2)}(1-t)^{-\frac{1}{2}\theta_t\theta_1}\det(\mathbb{1}-AD), \tag{A7}$$

where the Plemelj operators $A$ and $D$ act on the space of pairs of square-integrable functions defined on $\mathcal{C}$, a circle on the complex plane with radius $R<1$:

$$\begin{aligned}&(Ag)(z)=\oint_{\mathcal{C}}\frac{dz'}{2\pi i}A(z,z')g(z'),\\ &(Dg)(z)=\oint_{\mathcal{C}}\frac{dz'}{2\pi i}D(z,z')g(z'),\qquad g(z')=\begin{pmatrix}f_+(z)\\ f_-(z)\end{pmatrix},\end{aligned} \tag{A8}$$

[8] Since $\tau$ is defined up to a multiplicative constant, this functional relation is the only property of the Barnes function necessary for obtaining the expansion.

with kernels given, for $|t| < R$, explicitly by

$$A(z,z') = \frac{\Psi(\theta_1,\theta_\infty,\sigma;z)\Psi^{-1}(\theta_1,\theta_\infty,\sigma;z') - \mathbb{1}}{z-z'},$$
$$D(z,z') = \Phi(t)\frac{\mathbb{1} - \Psi(\theta_t,\theta_0,-\sigma;t/z)\Psi^{-1}(\theta_t,\theta_0,-\sigma;t/z')}{z-z'} \times \Phi^{-1}(t). \tag{A9}$$

The parametrix $\Psi(z)$ and the "gluing" matrix $\Phi(t)$ are, respectively,

$$\Psi(\alpha_1,\alpha_2,\alpha_3;z) = \begin{pmatrix} \phi(\alpha_1,\alpha_2,\alpha_3;z) & \chi(\alpha_1,\alpha_2,\alpha_3;z) \\ \chi(\alpha_1,\alpha_2,-\alpha_3;z) & \phi(\alpha_1,\alpha_2,-\alpha_3,z) \end{pmatrix},$$
$$\Phi(t) = \begin{pmatrix} t^{-\sigma/2}\kappa^{-1/2} & 0 \\ 0 & t^{\sigma/2}\kappa^{1/2} \end{pmatrix}, \tag{A10}$$

with $\phi$ and $\chi$ given in terms of Gauss' hypergeometric function:

$$\phi(\alpha_1,\alpha_2,\alpha_3;z) = {}_2F_1\left(\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3), \frac{1}{2}(\alpha_1-\alpha_2+\alpha_3); \alpha_3; z\right),$$
$$\chi(\alpha_1,\alpha_2,\alpha_3;z) = \frac{\alpha_2^2-(\alpha_1+\alpha_3)^2}{4\alpha_3(1+\alpha_3)} z\,{}_2F_1\left(1+\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3), 1+\frac{1}{2}(\alpha_1-\alpha_2+\alpha_3); 2+\alpha_3; z\right). \tag{A11}$$

Finally, $\kappa$ is a known function of the monodromy parameters:

$$\kappa = s\frac{\Gamma^2(1-\sigma)}{\Gamma^2(1+\sigma)}\frac{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0+\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0+\sigma))}{\Gamma(1+\frac{1}{2}(\theta_t+\theta_0-\sigma))\Gamma(1+\frac{1}{2}(\theta_t-\theta_0-\sigma))}\frac{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty+\sigma))\Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty+\sigma))}{\Gamma(1+\frac{1}{2}(\theta_1+\theta_\infty-\sigma))\Gamma(1+\frac{1}{2}(\theta_1-\theta_\infty-\sigma))}. \tag{A12}$$

Meaningful limits for integer $\sigma$ violating Eq. (A1) can be obtained by canceling the factors in the denominator of $s$ with poles of the Barnes function from the structure constants $\mathcal{N}^{\sigma+2n}_{\theta_t,\theta_0}$.

For the numerical implementation, we write the matrix elements of $A$ and $D$ in the Fourier basis $z^n$, truncated up to the order of $N$. Again, the structure of the matrix elements $A_{mn}$ and $D_{mn}$ can be found in Ref. [14]. This truncation gives $\tau$ up to terms $\mathcal{O}(t^N)$ and, unlike the Nekrasov expansion, can be computed in polynomial time. The formulation does, in principle, allow for the calculation for arbitrary values of $t$, by evaluating the integrals in Eq. (A8) as Riemann sums using quadratures [44], so there are good perspectives for using the method outlined here for more generic configurations.

## APPENDIX B: EXPLICIT MONODROMY CALCULATIONS

Given $\sigma_{0t}$ and $s$ satisfying Eq. (A1), we can construct an explicit representation for the monodromy matrices—up to conjugation—as follows.

The monodromy matrices are

$$M_0 = \frac{i}{\sin\pi\sigma_{0t}}\begin{pmatrix} \cos\pi\theta_t - \cos\pi\theta_0 e^{i\pi\sigma_{0t}} & 2s_i\sin\frac{\pi}{2}(\sigma_{0t}+\theta_0-\theta_t)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_0-\theta_t) \\ -2s_i^{-1}\sin\frac{\pi}{2}(\sigma_{0t}+\theta_0+\theta_t)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_0+\theta_t) & -\cos\pi\theta_t + \cos\pi\theta_0 e^{-i\pi\sigma_{0t}} \end{pmatrix}, \tag{B1}$$

$$M_t = \frac{i}{\sin\pi\sigma_{0t}}\begin{pmatrix} \cos\pi\theta_0 - \cos\pi\theta_t e^{i\pi\sigma_{0t}} & -2s_i e^{i\pi\sigma_{0t}}\sin\frac{\pi}{2}(\sigma_{0t}+\theta_0-\theta_t)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_0-\theta_t) \\ 2s_i^{-1}e^{-i\pi\sigma_{0t}}\sin\frac{\pi}{2}(\sigma_{0t}+\theta_0+\theta_t)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_0+\theta_t) & -\cos\pi\theta_0 + \cos\pi\theta_t e^{-i\pi\sigma_{0t}} \end{pmatrix}, \tag{B2}$$

$$M_1 = \frac{i}{\sin\pi\sigma_{0t}}\begin{pmatrix} -\cos\pi\theta_\infty + \cos\pi\theta_1 e^{-i\pi\sigma_{0t}} & 2s_e e^{i\pi\sigma_{0t}}\sin\frac{\pi}{2}(\sigma_{0t}+\theta_1+\theta_\infty)\sin\frac{\pi}{2}(\sigma_{0t}+\theta_1-\theta_\infty) \\ -2s_e^{-1}e^{-i\pi\sigma_{0t}}\sin\frac{\pi}{2}(\sigma_{0t}-\theta_1+\theta_\infty)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_1-\theta_\infty) & \cos\pi\theta_\infty - \cos\pi\theta_1 e^{i\pi\sigma_{0t}} \end{pmatrix}, \tag{B3}$$

$$M_\infty = \frac{i}{\sin\pi\sigma_{0t}}\begin{pmatrix} -\cos\pi\theta_1+\cos\pi\theta_\infty e^{-i\pi\sigma_{0t}} & -2s_e\sin\frac{\pi}{2}(\sigma_{0t}+\theta_1+\theta_\infty)\sin\frac{\pi}{2}(\sigma_{0t}+\theta_1-\theta_\infty) \\ 2s_e^{-1}\sin\frac{\pi}{2}(\sigma_{0t}-\theta_1+\theta_\infty)\sin\frac{\pi}{2}(\sigma_{0t}-\theta_0-\theta_\infty) & \cos\pi\theta_1-\cos\pi\theta_\infty e^{i\pi\sigma_{0t}} \end{pmatrix}. \tag{B4}$$

The matrices satisfy

$$M_t M_0 = \begin{pmatrix} e^{i\pi\sigma_{0t}} & 0 \\ 0 & e^{-i\pi\sigma_{0t}} \end{pmatrix}, \qquad M_\infty M_1 = \begin{pmatrix} e^{-i\pi\sigma_{0t}} & 0 \\ 0 & e^{i\pi\sigma_{0t}} \end{pmatrix}. \tag{B5}$$

The parameters $s_i$ and $s_e$ are related to the parameter $s$ defined in Eq. (A5) through

$$s = \frac{s_i}{s_e}. \tag{B6}$$

A direct calculation shows that

$$\begin{aligned}\sin^2\pi\sigma_{0t}\cos\pi\sigma_{1t} &= \cos\pi\theta_0\cos\pi\theta_\infty+\cos\pi\theta_t\cos\pi\theta_1-\cos\pi\sigma_{0t}(\cos\pi\theta_0\cos\pi\theta_1+\cos\pi\theta_t\cos\pi\theta_\infty)\\ &\quad -\frac{1}{2}(\cos\pi\theta_\infty-\cos\pi(\theta_1-\sigma_{0t}))(\cos\pi\theta_0-\cos\pi(\theta_t-\sigma_{0t}))s\\ &\quad -\frac{1}{2}(\cos\pi\theta_\infty-\cos\pi(\theta_1+\sigma_{0t}))(\cos\pi\theta_0-\cos\pi(\theta_t+\sigma_{0t}))s^{-1}.\end{aligned} \tag{B7}$$

We close by noting that, for the special case of interest where $\sigma_{1t}=\theta_1+\theta_t+2n$, $n\in\mathbb{Z}$, the expressions above are still valid.

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