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Proving routines in a lecturer's mathematical discourse for proof teaching: a case of analysis lecturing

Dimitri Lipper[a], Thomais Karavi [a*] and Angeliki Mali [b]

[a]Centre for learning and teaching, University of Groningen, Groningen, the Netherlands; [b]Department of Mathematics & Applied Mathematics, University of Crete, Heraklion, Greece

ABSTRACT

This study focuses on the characterisation of proof teaching in university lectures on mathematical analysis using the commognitive framework. The case is set around a lecturer with experience in teaching this course and a good reputation among university students. Six lectures of the course were analysed using discourse and thematic analysis. The analysis resulted in a range of characteristics of the mathematical discourse for proof teaching of a lecturer. Then, by exploring the proving routines within the characteristics, we produced a typology of teaching characteristics in the lecturer's discourse; that typology gave us insights into a pattern of proving specific to our case study. The exploration of the proving routines in the lecturer's discourse offered an in-depth understanding of teaching proof beyond the macro-level descriptions of lectures in the extant literature. We conclude the study with implications for research and education.

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Lecturing; commognition; proving routines; mathematical analysis; university mathematics

1. Introduction

In university mathematics lectures, proof teaching and learning are particularly important as 'proof of new theorems may be considered the summit of mathematical practice' (Tall et al., 2012, p. 14). Hanna (2020) emphasised that around the world, mathematics curricula highlight the central position of proof, and mathematics teachers appear as students' facilitators 'both to reflect proof's central position in mathematics and to reap its many educational benefits.' (p. 562). Contrary to university mathematics lectures, proofs are not central to high school mathematics in some countries (Tall et al., 2012). In the Netherlands, for example, the teaching of proofs in high schools is limited to the area of trigonometry, and proofs are not a central element of the Dutch curriculum. A discontinuity, then, appears between high school and university-level mathematics (Selden & Selden, 2003), and first-year university mathematics students sometimes struggle with the more formal and rigorous mathematics taught at the university level. In this study, we investigate the teaching of a lecturer in first-year, proof-oriented lectures with the aim of providing an in-depth understanding of proof teaching at this educational level.

CONTACT Thomais Karavi t.karavi@tue.nl
*Present address: Eindhoven School of Education, Eindhoven University of Technology, Eindhoven, the Netherlands

From a macro-level, lectures have been characterised as *traditional* (e.g. Petropoulou et al., 2020; Weber, 2004), *chalk and talk* (e.g. Artemeva & Fox, 2011; Viirman, 2015), or *transmissionist* (e.g. Pampaka et al., 2012; Pritchard, 2010) with a pre-set structure that follows the Definition-Theorem-Proof (DTP) paradigm (Weber, 2004). When presenting proofs, the lecturers often write on the blackboard while discussing and commenting on the mathematical content (e.g. Melhuish et al., 2022; Weber, 2004). Sometimes, they offer a meta-commentary and reflection on the mathematical content after its presentation (e.g. Artemeva & Fox, 2011; Melhuish et al., 2022).

More specifically, Weber (2004) identified three teaching styles that appeared chronologically in a course and showcased that DTP is not a single paradigm but embeds teaching styles. Briefly, the *logico-structural* style brought the use of definitions for the production of formal proofs, the *procedural* style highlighted the structure of the proof and intended to develop a 'mathematical toolbox' for use in similar proofs, and, last, the *semantic* style related to the use of diagrams and graphical representations for emphasising the thinking behind the proof. In particular, Weber (2004) found that the lecturer's semantic style offered an informal-intuitive explanation for the formal proof. Likewise, styles for the teaching of proof in lectures emerged from the study of Hemmi (2010), where the focus was on the lecturers' pedagogical perspectives on teaching. Both studies shed light on characteristics of how to work with and communicate proof in teaching.

Several studies focused on the characteristics of teaching in proof-oriented lectures in relation to lecturers' pedagogical perceptions for facilitating students' learning. For example, Lew et al. (2016) connected lecturing with students by investigating the reasons why students did not identify the same critical points of the proof as those their lecturer identified. One possible explanation was that students paid attention only to what was written on the blackboard. Another explanation was that the students focused on the algebraic manipulations rather than the lecturer's colloquialisms because the key points were delivered verbally to the students. Weber (2012) identified lecturers' intention to provide proofs primarily for explanation purposes. However, the lecturers had limited pedagogical approaches to facilitating students' problem-solving, reading, and understanding of proof. Similarly, focusing on constructing and revising proofs for pedagogical purposes, Lai and Weber (2014) argued that the lecturers emphasised certain aspects of proof because of the learning goals they set for students. However, it is not clear how the characteristics of proof teaching appear in the proving processes in the lectures informing lecturers' communication of the mathematical proofs with the students.

The studies discussed described lecturing, offering a macro-level characterisation of the teaching of proof in the lectures. In the last decade, a growing number of studies has used discursive approaches to analyse lecturing, offering valuable insights into lecturers' teaching in advanced courses and traditional settings. Studies from Viirman (2014, 2021) and Pinto (2019), using the commognitive framework, focused on identifying the fine aspects of lecturing that largely appeared tacitly in the lecturers' discourse. This study aims to characterise proof teaching at the university level using the commognitive framework (Sfard, 2008).

2. Theoretical considerations

For this study, we employ the commognitive framework (Sfard, 2008) to explore proof-teaching in lectures. We aim to gain an in-depth understanding of the emerging teaching characteristics. In the following paragraphs, we provide the theoretical considerations that outline this study.

2.1. Mathematical discourse

Mathematical discourse is distinguishable by the community's word use, visual mediators, narratives, and routines. Word use is the characteristic keywords that appear within the mathematical discourse (e.g. function, limit, and set). Visual mediators are visual objects that are used for communicational purposes (e.g. graphs and symbols). Narratives are sequences of utterances that describe objects, relations among objects, or relations of processes with objects (e.g. definitions, theorems, and proofs). Finally, routines are discursive patterns that appear repeatedly and are characteristic of the mathematical discourse (e.g. proving).

Narratives can be rejected or endorsed by the participants of the discourse. The process through which participants of the discourse become convinced that a narrative can be endorsed is called substantiation. A key form of substantiation in mathematics is proof production, 'a sequence of endorsed narratives, each of which is deductively inferred from previous ones and the last of which is the narrative that is being endorsed' (Sfard, 2008, p. 232). Moreover, Sfard (2008) distinguishes between object-level and meta-level rules of discourse. 'Object-level rules are narratives about regularities in the behaviour of objects of the discourse' (p. 201), meaning that they relate to the properties of the objects of the discourse. Metarules underlie patterns in the discourse and substantiate object-level narratives.

In this study, we are concerned with the mathematical discourse for proof teaching of a lecturer. The mathematical discourse for proof teaching relates to the lecturer's mathematical discourse actually communicated with students in the lectures. The communicational congener in this discourse is the talk about proving. In accordance with previous research literature (e.g. Viirman, 2015, 2021), in this study, we characterise the lecturer's proving routines and reserve the focus on metarules for a future study. We provide a theoretical account of proving routines in the next section.

2.2. Routines in the mathematical discourse

As Lavie et al. (2019) put forward, 'repetition is the gist of learning' (p. 153). The researchers define routines as task-procedure pairs that include the performance of a specific, given task by a person-performer, participant in the discourse, along with the procedure to execute the task's performance. For example, for a task that involves adding two numbers, the procedure reflects the execution of the task by the specific participant in the given situation. In some cases, the task-procedure pairs are indistinguishable. The routines occur when a participant in a mathematical discourse recognises a situation as familiar to one of previous participation. In these instances, the participant may choose to repeat a previously performed procedure, thinking of the past and current situations as similar through the identification of precedent events. Thus, the participant needs to recognise *when* to perform a routine and *how* (Sfard & Lavie, 2005). Hence, each routine has three parts: the initiation and closure, which are related to the *when* of the routine, and the procedure that shows the *how* of the routine (Sfard, 2008). Nachlieli and Tabach (2019) provide further insight into those three parts. During the initiation and closure, the conditions of emergence or closure and the performer of the routine come up. The procedure includes the processes performed for the implementation of the routine by the performer of the routine.

Routines are developed by individualising the routines that are common within the community of participation in an independent and agent way. In this process, Sfard and Lavie (2005) introduced two types of discursive routines, the exploration and ritual routines; lately, Lavie et al. (2019) elaborated on them. The ritual routines are developed for the sake of social approval and reward by mimicking the performance of a more experienced participant in the discourse and focusing on the process. Ritual routines appeared to be a necessary stepping stone for introducing a person to a new discourse. In distinction, implementing an exploration routine shifts the attention from the process to the outcome. For example, for the task that involves adding two numbers, the person who performs a ritual routine asks themselves, 'How do I proceed?' whereas the one who performs an exploration routine asks themselves, 'What is it that I want to get?' (Lavie et al., 2019, p. 166). One of the teaching goals is to support students' development of exploration routines.

Through the process of de-ritualisation, a routine is shifting from ritual to exploration. Lavie et al. (2019) identified six changes in students' mathematical discourse when moving from ritual to exploration routines: the student performs a task in more than one way (flexibility), each step of a procedure feeds into the next step (bondedness), the student can isolate the performance and apply it in new situations (applicability), the student makes more decisions on their own during the learning process (agentivity), the student increases the level of abstraction of a mathematical object (objectification), and the student establishes the criteria to assess the outcome of their performance based on their judgement (substantiability). Such changes from ritual to exploration in students' discourse may indicate a shift in students' routines from the performance to the outcome.

2.3. Routines for teaching

When discussing the routines in lecturers' discourse, Viirman (2014) analysed the mathematical routines in the mathematical discourse of seven lecturers teaching first-year mathematics courses (calculus, basic algebra, and linear algebra) in Sweden. The focus was on the teaching of mathematical functions. Two discursive routines were identified: the construction routines, which were, for example, 'routines for constructing definitions and for constructing examples satisfying given definitions' (p. 518), and substantiation routines, which included, for example, routines for definition verification, proof, and claim contradiction (p. 522). The proof routines appeared surprisingly common in the substantiation routines, although the observed courses were not proof-oriented. In conclusion, important differences were found in the discursive routines of different lecturers. In the follow-up study, exploring the same data set as in 2014, Viirman (2015) focused on the lecturers' teaching practices in their pedagogical discourse. In this case, three categories of didactical routines were identified: explanation, motivation, and question-posing routines. Explanation routines related to 'known mathematical facts, summary and repetition, different representations, everyday language, and concretisation and metaphor' (p. 1165). Motivation routines contained 'reference to utility, the nature of mathematics, humour and result focus' (p. 1165). Finally, the question-posing routines included 'control questions, asking for facts, enquiries and rhetorical questions' (p. 1165). Pedagogical discourse and mathematical discourse were similar regarding word use and visual mediators but differed in their routines.

Viirman's initial studies on lecturers' routines gave a categorisation of emerging routines and found that the lecturers' routines differ depending on the lecturer. This latter observation raised the discussion of where these differences can be identified in the lecturers' teaching. Viirman (2021) studied the metarules that underline lecturers' routines in teaching and explored two aspects of meta-level discourse: introducing new mathematical objects and finding out what counted as a valid endorsement of a (mathematical) narrative. An example of a meta-level rule with respect to the introduction of new mathematical objects was 'definitions first': 'to be able to talk about a new mathematical object, we first need to define it' (p. 13). An example of a meta-level rule concerning valid endorsements of narratives was 'algebraic realizations are the preferred means of endorsement of mathematical narratives' (p. 16).

Exploring further the routines for teaching and aiming at the characterisation of teaching that provides ritual opportunities to the students, Nachlieli and Tabach (2019) developed a conceptual framework, offering insights into the nature of routines for teaching. Ritual teaching has often been criticised in favour of explorative teaching, given that exploration routines are the ones aimed to be developed in students' learning. However, by exploring the nature of the routines, the researchers argued that ritual teaching, where the ritual routines for teaching emerge, is essential for the students' learning because it serves as a basis for their explorations and helps students with their initial steps when entering a new discourse. Focusing also on teachers' given opportunities for students' development of ritual or explorative routines, Christiansen et al. (2023) explored teaching in three different lessons. The researchers offered a characterisation of routines for teaching that seemed to encourage students' development of exploration routines at the beginning of the lessons but shifted to the encouragement of ritual routines towards the end of them. In doing so, they used two changes from ritual to exploration, as described by Lavie et al. (2019) (i.e. agentivity, and substantiability) to characterise the type of opportunities offered by the teachers. They described in that way hybrid opportunities to learn (i.e. recreated-exploration-requiring and guided-exploration-enabling) in the interaction of the teachers and the students, adding an extra layer to the characterisation of the routines for teaching.

An adaptation of the six changes from ritual to exploration identified by Lavie et al. (2019) was also discussed by Österling (2022), who developed a framework for studying de-ritualising in the teaching of mathematics lectures. This study did not focus on proof teaching specifically and looked at high-school mathematics. At the university level, Karavi et al. (2022) proposed an adaptation of the six changes by Lavie et al. (2019) for studying teaching in proof-oriented mathematics lectures in terms of the lecturer's support for the changes, learners of mathematics may undergo, when moving from ritual to exploration.

In this paper, the adaptations of Karavi et al. (2022) are a guide to identifying characteristics in the lecturers' mathematical discourse for proof teaching, which may support students' de-ritualisation when engaging with mathematical proofs. We thus focus on identifying the lecturer's support for the changes learners of mathematics *may* undergo when moving from ritual to exploration. For example, the lecturer's support for applicability was formulated as 'discussing the application of a proving process in other situations' and bondedness as ‘making connections between the different steps of the proof’ for the students (Karavi et al., 2022, p. 8).

The case of this study is in the Netherlands. We study the teaching of proof to a large cohort of students in university lectures of mathematical analysis, where the lecturer’s interaction with students is limited. We aim to analyse the lecturer’s mathematical discourse for proof teaching by looking at one lecturer’s proof teaching characteristics and by connecting those to the lecturer’s proving routines. The goal is to explore the proving routines and get an in-depth understanding of proof teaching at the university level. The research questions are the following:

What are the characteristics of a lecturer’s proof teaching in first-year analysis lectures at a Dutch university?

How are these characteristics related to the proving routines of the lecturer’s mathematical discourse for proof teaching?

3. Methods

3.1. Context

To characterise proof teaching in university mathematics lectures, a first-year, proof-oriented analysis course was selected for a single case study at a university in the Netherlands. The course was compulsory for all students registered for mathematics bachelor degrees, and knowledge of calculus was a prerequisite for attending the course (i.e. familiarity with concepts like limits and sequences, basic computational skills, etc.). According to the course description, the goal was ‘to provide the fundamental mathematical underpinning of many concepts and techniques from previous calculus courses [...], the course on analysis treats the underlying mathematical fundaments and their full proofs’. Thus, the course mostly focused on proofs of elementary mathematical theorems and not so much on their applications. The entire course consisted of 17 lectures of 90 minutes, which took place during eight consecutive weeks. The course was taught online using the video conferencing software of the university. The lecturer used slides on which the text was highlighted or underlined, and sometimes, the lecturer added written comments to the slides while lecturing. The lecturer used slides for their courses regardless of the online or onsite format. The lecturer used a monologue while presenting the mathematical content in the proof-oriented course of this study. The students could attend the live streaming of the course and were able to ask questions through a chat function. Alternatively, the students could watch the recordings of the lectures that were available immediately after the live stream. In total, 103 students were registered for the course.

The case study focuses on the lectures of a mid-career male lecturer with ten years of teaching experience in analysis courses. At the time of data collection, the lecturer had taught the course six times, but it was the first time teaching it online. The main criterion for selecting the case of this lecturer’s teaching was the following: we considered that this case was typical of university mathematics teaching because the lecturer spoke over his written presentation of the mathematical content in a lecture format. The lecturer also had a reputation for being a clear and well-structured lecturer among the students at the university of this study. Investigating his teaching could thus provide valuable insights into teaching introductory proof-oriented analysis courses.

Table 1. Overview of lectures.

Lecture Number	Content
L1	absolute values, upper bounds, lower bounds
L3	injective and bijective functions, cardinality, countable and uncountable sets
L8	compactness, open covers
L12	pointwise and uniform convergence
L14	power series and their properties
L15	Taylor series, proof of their convergence using Lagrange's remainder theorem

3.2. Data collection and data analysis

The data was collected as part of the second author's doctoral project. For this study, we used data from the online lectures of the course. Thus, the second author observed, took field notes, and had access to the recordings of the 17 lectures. The shared slides were also collected during the course period.

Six lectures from the beginning, middle, and end of the course were selected for analysis for an overview of the development of the course, as presented in the table (see Table 1). The lectures were transcribed by the first author. After receiving audio and video recordings of the six lectures, automatic transcription software was used to transcribe the lectures verbatim. Next, all transcripts were compared to the recordings manually, correcting any mistakes still present in the transcripts. The corresponding slides were also inserted into the correct location in the transcripts (Creswell, 2011).

The lectures were analysed in two layers. The first one aimed at refining the adaptations made by Karavi et al. (2022). The second layer focused on connecting those to the proving routines in the mathematical discourse for the lecturer's teaching. The aim of the two layers was to characterise the lecturer's proof teaching.

In the first layer, the lectures were analysed using thematic analysis (Braun & Clarke, 2012). A combination of inductive and deductive thematic analysis was used to analyse the lectures. The code scheme for the deductive part of the analysis occurred from the proposed adaptation of the six changes by Lavie et al. (2019) as discussed in Karavi et al. (2022). First, we focused on lecture L1 and in particular, we started by coding the instances of bondedness in this lecture. After attaining a preliminary understanding of the code through constant comparisons (Creswell, 2011) of the coded instances, L1 was coded again by two coders. The codes were compared between the coders using constant comparisons. The same process was followed for those of the remaining five codes we identified in the data. After the coders agreed upon the codes in L1, the remaining lectures were also coded. When all the lectures were analysed with the codes, instances of the same codes were once more compared to one another. The codes of flexibility and objectification did not appear in the case under exploration. The layer was concluded with a grounded understanding of the codes, identifying them as *characteristics* in the mathematical discourse for proof teaching of the lecturer.

The second layer happened in parallel with the first, but this time, the focus was on the routines of the lecturer's mathematical discourse. We wanted to analyse further the instances we identified in the first layer by doing discourse analysis. For example, for the instances coded with bondedness, inductively this time, we focused on the type of connections the lecturer aimed to achieve. In particular, the instance 'But when a product is zero? If a product is zero, one of the two factors has to be zero.' was coded as *bondedness* and

Table 2. Example of the proving parts of an episode.

Proving parts	Transcript	Description
Introduction	Well, we can for instance prove the product rule. […] So, this rule says nothing else than: the absolute value of a product equals the product of the absolute values. Okay, so how to prove this? […]	The lecturer introduced the theorem by stating exactly what the theorem says. Then the lecturer outlined the key idea of the upcoming proof, which is the use of a case distinction between both x and y being either positive or negative.
Formal proof	Well, if either x is zero or y is zero, it should be obvious that this is true. Because if either x is zero or y is zero, both sides of the equality that I want to prove will be zero. […]	The lecturer continued with the proof presentation. Using the case distinction between both x and y being either positive or negative, he verified the product rule for each possible case, starting with the case where either x or y is equal to zero.
Meta-commentary	So here you see again that proofs that involve the absolute value typically involve a case distinction.	The lecturer looked back on the proving process and provided a tip for similar proofs.

further analysed *within mathematics: algebraic*. The latter description is a routine in the lecturer's discourse related to bondedness as a characteristic. We continued like this with the instances from the remaining codes, identifying routines. Then, we compared the identified routines using colour coding in a spreadsheet. The result was a list of routines in the lecturer's mathematical discourse for teaching. Those routines refined the characteristics of teaching from the first layer of analysis.

Last, the lectures were divided into episodes around proving. The episodes started with the introduction of a new theorem, followed by an explicit statement of the theorem that needed to be proven. The lecturer elaborated on this statement, explored the possible approaches for proving it and outlined the key ideas for the upcoming proof. Then, the lecturer began the proof presentation, starting with the assumptions in the theorem and finishing up with the repetition of the theorem's conclusion, producing a formal proof. Finally, the lecturer recapped the main ideas of the proof and explained the implications and future applications of the proof or theorem in a meta-commentary. Each episode had three proving parts in the lecturer's mathematical discourse for proof teaching. In Table 2, we illustrate the three parts by using excerpts from a proof of the product rule, $|x \cdot y| = |x| \cdot |y|$.

Given that our goal was to characterise proof teaching, a focus on the proof of the theorems was needed. A total of 44 episodes occurred in the six lectures we analysed. The episodes were compared to one another. We found a pattern of specific characteristics and routines per proving part: the introduction, the formal part (proof per se) and the meta-commentary on the proof.

4. Findings

The lectures of this course cannot be characterised as 'chalk and talk' (Artemeva & Fox, 2011) because the lecturer was not writing on a blackboard but used pre-written slides. However, the lectures were traditional and on a macro-level, they followed the DTP paradigm (Weber, 2004). In the following section, we explore the characteristics of proof teaching in the lecturer's mathematical discourse, as occurred from the first layer of analysis, and their function in proof teaching. The characteristics of flexibility and objectification do not appear in the case under investigation. Within each characteristic, we identify proving routines, which we explain by using three parts: the initiation and closure relating to the *when* of the routine and the procedure concerning the *how* of the routine (Sfard, 2008). Examples are presented below, using excerpts from the lectures. At the end of each excerpt, there is a tag (i.e. [L1: 34]) mentioning in which lecture (L1) and turn (34) of the transcript the quote appeared.

4.1. Bondedness

Bondedness gave rise to three proving routines in the lecturer's mathematical discourse for proof teaching. The first routine concerns the *connection between narratives,* where the lecturer explicitly connected two different narratives. The lecturer first proved the triangle inequality and then moved to the proof of the reverse triangle inequality connecting the two. The following excerpt is from the introduction of the proof of the reverse triangle inequality.

> Now, there's a related inequality, and this is called the reverse triangle inequality. So here you see an inequality for the absolute value that involves a sum. And, by analogy, you could ask yourself: is there an inequality for absolute values that is related to subtraction? And yes, there is. [L1:144]

For the initiation of the routine, the lecturer introduced the reverse triangle inequality as a related inequality. The procedure followed, where the lecturer explicitly addressed how the normal and reverse triangle inequality were related, asking whether the students could search for an analogy. The routine closed by determining the existence of an inequality with subtraction instead of addition. This routine was characterised by bondedness since the routine was underlined by a connection between two narratives (in this case, two theorems). The lecturer explicitly connected the normal and reverse triangle inequality in the excerpt by stating that they were analogous (an inequality for absolute value related to subtraction instead of addition).

The second routine where bondedness appeared as a characteristic was about the connections within the different parts of the proof, underlining the *structure of the proof*. The next excerpt is from the proof that a set A is countable if and only if there exists an injective function f from A to $\mathbb{N}$. The lecturer after closing the proof from left to right (proving that if a set A is countable, then there exists an injective function f from A to $\mathbb{N}$), continued with the proof from right to left (starting from the assumption that there exists an injective function f from A to $\mathbb{N}$).

> Okay, so that concludes the proof from left to right. What about from right to left? Well, it uses a sort of similar trick. [L3:157]

The initiation of the routine of this excerpt was the rhetorical question raised by the lecturer, followed by the procedure where he suggested using 'a sort of similar trick'. Bondedness, in this case, reflected the connection among the different parts of a proof, structuring the proof. By referring to the use of a similar trick, the lecturer compared the structure of the proof he would follow for proving from right to left with the one he discussed earlier while proving from left to right. The use of the word 'similar' appeared as a keyword in this excerpt, supporting the identification of a similar structure between these (sub)proofs.

The last routine, characterised by bondedness, concerned the connections *within mathematics*. Two types of connections within mathematics were identified. The first referred to logical connections because the lecturer arrived at conclusions by reasoning, while the second concerned algebraic connections, where the lecturer arrived at the conclusion by computing. As described in Karavi and Mali (2022), these types of connections appeared within computing routines where the lecturer performed computations to create connections within different arguments, promoting the equivalences among them.

The three routines underlined by bondedness as characteristic were about *the connections between narratives*, the *structure of the proof*, and the connections *within mathematics*. The purpose of bondedness as a characteristic of proof teaching was to show how certain connections within mathematical objects can be developed and inform proving. These connections between results were mostly found during the introduction of proofs while structuring the proof. Connections within mathematics were mostly found during the formal part of proofs. Given the identified routines, bondedness as a characteristic in the mathematical discourse of the lecturer was refined as *the enactment of intended connections of the narratives that appear within the proofs*.

4.2. Applicability

Two proving routines with applicability as a key characteristic were identified. Applicability as a characteristic of proving routines was coded only for the instances where the lecturer explicitly performed an application, not when the lecturer only mentioned that a theorem could be applied without him performing the application. The first proving routine was about a *narrative application*, where the lecturer applied a known narrative in a new context. In the following excerpt, the lecturer was proving that if two sets A and B are countable, then their union $A \cup B$ is also countable. He started by applying a lemma, A is countable if and only if there exists an injective function f from A to $\mathbb{N}$.

> How can I do that? Well, given the fact that I know that A and B are countable, I know by this lemma that I have the existence of a function f from A to $\mathbb{N}$, which is injective. [L3:204]

The lecturer initiated the routine by asking a rhetorical question highlighting the beginning of the proving process. Then, he moved to the formal part of the proof, where he applied a lemma. This lemma had been proven earlier in the same lecture. The lecturer emphasised the fact that because the statement of the theorem is that A and B are countable, he can apply the lemma as a narrative and justify the existence of the function f.

The second routine concerned a *technique application* where the lecturer applied a known technique in a new context. For our description, a technique is a particular method within a previously presented line of reasoning. The following excerpt is from the proof that a set $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded. The lecturer first proved the theorem from left to right. The excerpt concerns the proof from right to left.

> Now, the proof from right to left is quite similar to the proof I have given in an example, namely the example that showed that every closed and bounded interval was compact, and I can use exactly the same line of proof over here. [L8:40]

The excerpt is from the formal part of the proof. Before continuing with the proof from right to left, the lecturer mentioned that the line of proof he was about to use was similar to a line of proof used in an earlier example presented in the course. In this case, the lecturer was about to mimic a technique from a previously shown example and apply it in a new context. The technique he applied was to take any sequence in the set K. Since K was bounded the sequence must also be bounded, and thus, he could apply the Bolzano-Weierstrass theorem to show that this sequence had a convergent subsequence.

In conclusion, two of the lecturer's proving routines were identified with applicability as a characteristic: *a narrative application* and *a technique application*. Applicability mostly occurred during the formal part of the proof. In using applicability, the lecturer showed the use of previous narratives and techniques in new proving routines. Given the two identified routines, applicability as a characteristic was refined as *the enactment of the application of an available narrative in a new situation*.

4.3. Agentivity

Agentivity as a characteristic underlined four of the lecturer's proving routines. The first routine was concerned with the *sharing of the key idea* where the lecturer outlined the most important idea(s) of a proof in the introduction of a proof. The following excerpt is from the proof that a set $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded. The lecturer first gave an outline of the proving strategy before starting the formal proof.

> Well, let's do a proof by contradiction. Let's assume that a set K is not bounded. So, I assume K is compact and I assume K is not bounded, and then I want to force a contradiction. [L8:23]

In the initiation of the routine, the lecturer started directly by stating the proving approach, which was proof by contradiction. Then, during the procedure, he continued by assuming that K is compact but not bounded and tried to force a contradiction to show that K must be bounded. This excerpt was characterised by agentivity because the lecturer encouraged the core idea (proof by contradiction), focusing on explicitly detailing how the proof by contradiction would be achieved. However, although he used the phrases 'let's do' and 'let's assume', the initiation for agentivity was not open towards students (Christiansen et al., 2023) because the procedure of the routine was highlighted by assumptions made by the lecturer.

Agentivity also emerged from the next routine, which concerned the *use of available narratives*. When this routine was performed, the lecturer emphasised the importance of collecting the available narratives in the statement to be proven. The next excerpt is from the introduction of the proof of the statement: a set A is countable if and only if there exists an injective function f from A to $\mathbb{N}$. The available narratives here were the definitions of a countable set and an injective function. The lecturer discussed how to bridge the gap between the statement he wanted to prove and the proof itself.

> So, how do we prove this? Well, the only thing that I have is a definition, so I need to work with the definition. [L3:153]

The initiation of the routine was the rhetorical question where the lecturer asked how to continue with the proof. The procedure related to the statement that the only available narrative concerning the statement he wanted to prove was the definition of a countable set (A is countable if there exists a bijective function from A to $S \subseteq \mathbb{N}$). Consequently, he concluded that this definition must be used in this proof. In the excerpt, the lecturer wrote the definition of a countable set, and then he continued with the formal proof of the theorem.

The next routine was concerned with the *potential application of a narrative or technique* and is characterised by agentivity. The lecturer explained how a narrative or technique can be applied in future proving situations (without actually performing the application). The excerpt that follows is from the proof of the theorem that a function $f_n : A \to \mathbb{R}$ converges to f uniformly if and only if the limit of the supremum of $|f_n(x) - f(x)|$, when $x \in A$, is equal to 0. The proof of the theorem had been preceded. The following excerpt showcases the lecturer's reflection on proving.

> Now, this is a very useful characterisation because the supremum already suggests that in practical situations, you could, for instance, replace the supremum by a maximum. If the supremum were attained, you would have a maximum. And how would you compute the maximum? Well, there, you can use calculus tools because you know how to compute the maximum of a function. [L12:107]

The initiation of the routine included an acknowledgement of the usefulness of the characterisation for practical situations. This was followed by the procedure where the lecturer discussed how this characterisation for uniform convergence can generally be applied in future proving routines. The 'practical situations' of potential application included situations where the supremum was attained and can thus be replaced by a maximum. For the closure of the routine, the lecturer mentioned that the students could use 'calculus tools' to compute this maximum, which accounted as background knowledge for the students at this time of their studies. The closure of the routine (and the proof) highlighted the usefulness of this theorem.

The last proving routine that is characterised by agentivity is related to the *emphasis on the importance of self-study*. In these proving routines, the lecturer urged the students not only to attend the lectures and read the course material but also to actually practice doing proofs by themselves because this is the only way they can learn how to do proofs. The last excerpt is from the meta-commentary of a proof of the reverse triangle inequality.

> So, what I really recommend is that you try to reproduce these proofs yourself because it takes a little bit of practice to get used to these kinds of tricks and manipulations with absolute values. Now you have seen how I did the proof, try to do the proof yourself and see if you can reproduce all these steps. [L1:161]

The excerpt is concerned with the closure of the routine where the lecturer encouraged the students to reproduce the proof of the reverse triangle inequality by themselves. The emphasis was on the need to practice to learn how to do these kinds of proofs. The lecturer specified that students should try to 'reproduce all these steps', thus mimicking his proving procedure, suggesting a ritualised participation from the students as an initial step of the analysis course (Lavie et al., 2019). This routine was characterised by agentivity as the emphasis on the importance of self-study was encouraged for the development of students' agency.

In conclusion, four proving routines related to agentivity were identified and were concerned with *sharing a key idea*, the *use of the available narratives*, the *potential application of a narrative or technique*, and the *emphasis on the importance of self-study*. Agentivity routines appeared as characteristic in the introduction or meta-commentary of a proof. The purpose of agentivity routines during the introduction of the proof was to show how to bridge the gap from the available narratives to the arguments of the proof. Accordingly, during the introduction of proofs, using available narratives and sharing a key idea were most often encountered. The purpose of agentivity routines during the meta-commentary was to offer a reflection on future proving routines referring to potential applications and emphasise the importance of *self-study*. Given the four identified routines, agentivity was refined as the *development of reasoned decisions for the emergence of the proof*.

4.4. Substantiability

A proving routine of the lecturer was characterised by substantiability when the emphasis was on the importance of the proof. The following excerpt occurred during the meta-commentary of the proof of the Lagrange remainder theorem. The lecturer went over the steps of the proof once more, highlighting the most important parts of the proof.

> So again, this was the quickest possible proof, not necessarily the most intuitive one. It all relied on this little lemma, and then choosing a very specific function that satisfies the condition of that lemma. And then by a simple rearrangement, the Lagrange formula simply pops up. So, that proves the Lagrange remainder formula. [L15:66]

The initiation of the routine focused on a reflection about the proof, which included that the proof is ‘the quickest possible’ but not ‘the most intuitive’ one. Next, the procedure was about summarising the main steps of the proof. Last, the routine concluded that implementing the specific main steps proves the Lagrange remainder formula. This example was characterised by substantiability because the lecturer looked back on the proving process and highlighted some key narratives or recapped the main steps of the proof. Using substantiability, the lecturer’s routines justified and discussed the central proof ideas. Substantiability as a characteristic was refined as *the justification of the reasons for endorsement of a narrative*.

4.5. Specification

Last, in this section, we discuss the lecturer’s proving routines that had specification as a characteristic. Specification is an additional characteristic that emerged from the data of this study; it is about decreasing the level of abstraction of a narrative. Two of the lecturer’s proving routines had specification as a characteristic, and we exemplify them below.

The first proving routine was about *discussing a narrative intuitively* by using an intuitive example at the introduction of the proof. The following excerpt is from the proof that the interval $(0, 1)$ is uncountable. In the introduction of the proof, the lecturer provided an intuitive understanding of the narrative.

> So, my claim is: the interval $(0, 1)$ is uncountable. So, you do not have sufficiently many natural numbers to label all the points in the interval $(0, 1)$. That’s what it means from an intuitive point of view. [L3:220]

The routine started with the lecturer’s claim, while the procedure included an intuitive explanation of the claim. The closure verified the intuitive meaning. The routine was characterised by specification as the lecturer translated the narrative to be proved into a more intuitive statement for the beginning of the proving process.

Table 3. Summary of the proving routines and characteristics.

Proving parts	Characteristic	Proving routines	Expression of the characteristic
Introduction	**Bondedness**	*Between narratives*	Connection between narratives
	Agentivity	*Sharing a key narrative*	Outline of the most important ideas of a proof
		Using available narratives	Emergence of the use of the available narratives
	Specification	*Discussing a narrative intuitively*	Presentation of an example
Formal proof	**Bondedness**	*Structuring the proof*	Connection between different steps of a proof
		Within mathematics (logical or algebraic)	Connection between mathematical objects
	Applicability	*Narrative application*	Application of a known result
		Technique application	Application of a known technique
Meta-commentary	**Substantiability**	*Looking back on the proving process*	Highlights of some key ideas or the main steps of the proof
	Specification	*Summarising the key narratives of the proving process*	Summaries with morals
	Agentivity	*Potential applicability*	Explanation of how a narrative or technique can be applied in future proving situations (without actually applying it)
		Emphasising the importance of practice	Explanation that learning proofs involves practicing proofs

The second proving routine that was characterised by specification was about *summarising the key narratives of the proving process.* The excerpt below is from the closure of the proof that the set of natural numbers $\mathbb{N}$ has the same cardinality as the set of even numbers E.

> So, the moral of this example is: there are as many even numbers as natural numbers. [L3:81]

The excerpt is from the routine procedure where the lecturer specified what this example meant in a less abstract way, calling it moral. Thus, the moral stated that there are as many even numbers as natural numbers. The lecturer translated a statement about cardinalities in terms of a statement about sizes, which was a less abstract notion when working with finite sets. The morals may facilitate the participants of the discourse to designate a familiar situation and then generate new routines on their own, familiar to the ones of previous participation.

The proving routines, which were characterised by specification at the initiation of the proof, were used to explain intuitively what a theorem means. During the meta-commentary of a proof, they were used to distil the moral of a proof. Thus, specification can be refined as *decreasing the level of abstraction of a mathematical object.*

Summarising the findings of this section, the characteristics of the first layer of analysis were identified within the episodes and were refined. Table 3 provides a pattern of proving specific to our case study, summarising the three proving parts of an episode and their characteristics.

5. Discussion

In this study, we focused on the identification of characteristics in the mathematical discourse for a lecturer's teaching of a proof-oriented course on mathematical analysis. We also identified proving routines and related them to the aforementioned characteristics, examining the characteristics in the lecturer's mathematical discourse. The analysis of routines offered a micro-level description of proof teaching in lectures. Commognition allowed a detailed description of a lecturer's mathematical discourse for proof teaching.

The starting point of our analysis was the six codes from Karavi et al. (2022), and some of those eventually formed characteristics of teaching. Specifically, we were able to identify characteristics related to four of the six codes and an additional one (specification) that was added to the list inductively. The findings contributed to the refinement of the codes that formed the characteristics. Bondedness is *the enactment of intended connections of the narratives that appear within the proofs*, applicability is *the enactment of the application of an available narrative in a new situation*, agentivity is *the development of reasoned decisions for the emergence of the proof*, substantiability is *the justification of the reasons for endorsement of a narrative*, and specification is about *decreasing the level of abstraction of a mathematical object.* These characteristics of teaching and the corresponding proving routines offered a detailed characterisation of proof teaching in university lectures. In the episodes, in particular, the analysis of the characteristics and their proving routines suggested a pattern of proving that gave coherence to the structure of proof teaching of the lecturer. Another characterisation of teaching in university lectures can be achieved by addressing metarules (rules that define routines) that model lecturers' teaching (Viirman, 2021).

The characteristics of teaching and the corresponding proving routines added a new analytical process to the discussion about the DTP paradigm, which appears as a collection of teaching styles (Weber, 2004). Indeed, looking at the proof teaching as a whole and using Weber's (2004) description of three styles of proof teaching, the teaching explored in this study cannot be characterised by one single style, because it uses all three styles. Each of the characteristics of proof teaching the lecturer implemented in his proving routines related to a style. In our case, these styles can be viewed as lying on a spectrum of teaching styles. Thus, the introduction of the proof appeared to match a logico-structural teaching style, the formal part related to the procedural teaching style, whereas the metacommentary of the proof included characteristics suitable for the semantic style in terms of an intuitive explanation of the gist of the proof and the proving process. Indeed, lecturers can navigate within this spectrum of styles depending on what they choose to emphasise in their teaching. Identifying the characteristics of teaching along with their proving routines promotes the identification of different styles and an understanding of when and why lecturers sometimes move across styles.

Since only one particular lecturer's teaching was analysed, the discussion of the proving routines along with the underlined characteristics might not generalise to other courses or lecturers. As Viirman (2014) noted, 'although the overall form of the lectures is similar, with teachers using 'chalk talk', (…) there are in fact significant differences in the way the teachers present and do mathematics in their lectures' (p. 512). For example, in our study, objectification was not identified within the body of proofs and proving. Analysing another case of teaching may illuminate this issue. Moreover, the case we chose to investigate relates to a lecturer with experience in teaching the specific course and to an online format. Those may not be typical in other countries and contexts outside the Netherlands.

Overall, the identified characteristics and proving routines can be a powerful tool for analysing proof teaching at a micro-level. Future studies can benefit from this refinement for addressing and exploring the metarules in lecturers' discourse, getting access to fine aspects of lecturing. Identifying metarules would explain why specific proving routines and characteristics appear in the lecturers' discourse. The initial studies from Viirman (2021) and Pinto (2019) provide the starting point for such exploration. Thus, the characteristics of proof teaching and their proving routines can support future researchers who aim to investigate proof teaching further.

Regarding education, the characteristics can support lecturers' reflection on their teaching of proof and proving while exploring with a researcher what characteristics they use, when, and why. Furthermore, a discussion about the introduction of 'new' characteristics in their mathematical discourse for proof teaching can contribute to the lecturers' development of teaching. The types of characteristics the lecturers prepare to promote eventually provide indications of the changes students may make in their discourse towards the de-ritualisation of proving routines, thereby supporting students' experience with exploration routines.

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ORCID

Thomais Karavi http://orcid.org/0000-0002-0227-6663
Angeliki Mali http://orcid.org/0000-0002-6985-2848

References

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication*, *28*(4), 345–379. https://doi.org/10.1177/0741088311419630

Braun, V., & Clarke, V. (2012). *Thematic analysis*. American Psychological Association.

Christiansen, I. M., Corriveau, C., & Pettersson, K. (2023). Hybrids between rituals and explorative routines: Opportunities to learn through guided and recreated exploration. *Educational Studies in Mathematics*, *112*(1), 49–72. https://doi.org/10.1007/s10649-022-10167-z

Creswell, J. W. (2011). *Educational research: Planning, conducting, and evaluating quantative and qualitative research*. Pearson Education International.

Hanna, G. (2020). Mathematical proof, argumentation, and reasoning. *Encyclopedia of Mathematics Education*, 561–566. https://doi.org/10.1007/978-3-030-15789-0_102

Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics*, *75*(3), 271–291. https://doi.org/10.1007/s10649-010-9256-3

Karavi, T., & Mali, A. (2022). Investigation of the connections within proof in complex analysis lecturing. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 2429–2436). ERME and Free University of Bozen-Bolzano.

Karavi, T., Mali, A., & Avraamidou, L. (2022). Commognition as an approach to studying proof teaching in university mathematics lectures. *Eurasia Journal of Mathematics, Science and Technology Education*, *18*(7), em2132. https://doi.org/10.29333/ejmste/12173

Lai, Y., & Weber, K. (2014). Factors mathematicians profess to consider when presenting pedagogical proofs. *Educational Studies in Mathematics*, *85*(1), 93–108. https://www.jstor.org/stable/43589808.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Lew, K., Fukawa-Connelly, T. P., Mejia-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162–198. https://doi.org/10.5951/jresematheduc.47.2.0162

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *The Journal of Mathematical Behavior*, *67*, 100986. https://doi.org/10.1016/j.jmathb.2022.100986

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271. https://doi.org/10.1007/s10649-018-9848-x

Österling, L. (2022). Operationalising de-ritualisation for the analysis of teaching-as-usual. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 3229–3236). ERME and Free University of Bozen-Bolzano.

Pampaka, M., Williams, J., Hutcheson, G., Wake, G., Black, L., Davis, P., & Hernandez-Martinez, P. (2012). The association between mathematics pedagogy and learners' dispositions for university study. *British Educational Research Journal*, *38*(3), 473–496. https://doi.org/10.1080/01411926.2011.555518

Petropoulou, G., Jaworski, B., Potari, D., & Zachariades, T. (2020). Undergraduate mathematics teaching in first year lectures: Can it be responsive to student learning needs? *International Journal of Research in Undergraduate Mathematics Education*, *6*(3), 347–374. https://doi.org/10.1007/s40753-020-00111-y

Pinto, A. (2019). Variability in the formal and informal content instructors convey in lectures. *The Journal of Mathematical Behavior*, *54*, 100680. https://doi.org/10.1016/j.jmathb.2018.11.001

Pritchard, D. (2010). Where learning starts? A framework for thinking about lectures in university mathematics. *International Journal of Mathematical Education in Science and Technology*, *41*(5), 609–623. https://doi.org/10.1080/00207391003605254

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*(1), 4–36. https://doi.org/10.2307/30034698

Sfard, A. (2008). *Thinking as communicating: Human development, development of discourses, and mathematizing*. Cambridge University Press.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different?—Early numerical thinking revisited. *Cognition and Instruction*, *23*(2), 237–309. https://doi.org/10.1207/s1532690xci2302_3

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive development of proof. In G. Hanna & M. deVilliers (Eds.), *Proof and proving in mathematics education* (pp. 13–49). Springer.

Viirman, O. (2014). The functions of function discourse – University mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, *45*(4), 512–527. https://doi.org/10.1080/0020739X.2013.855328

Viirman, O. (2015). Explanation, motivation and question posing routines in university mathematics teachers' pedagogical discourse: A commognitive analysis. *International Journal of Mathematical Education in Science and Technology*, *46*(8), 1165–1181. https://doi.org/10.1080/0020739X.2015.1034206

Viirman, O. (2021). University mathematics lecturing as modelling mathematical discourse. *International Journal of Research in Undergraduate Mathematics Education*, *7*(3), 466–489. https://doi.org/10.1007/s40753-021-00137-w

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), 115–133. https://doi.org/10.1016/j.jmathb.2004.03.001

Weber, K. (2012). Mathematicians' perspectives on their pedagogical practice with respect to proof. *International Journal of Mathematical Education in Science and Technology*, *43*(4), 463–482. https://doi.org/10.1080/0020739X.2011.622803