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Annuities, public policy and demographic change in overlapping generations models

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Document Version

Publisher's PDF, also known as Version of record

Publication date:

2011

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Mierau, J. O. (2011). *Annuities, public policy and demographic change in overlapping generations models*. [Thesis fully internal (DIV), University of Groningen]. University of Groningen, SOM research school.

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Chapter 6

Capital accumulation and the sources of demographic change*

*This chapter is based on Mierau and Turnovsky (2011).

6.1 Introduction

In this chapter we return to the analysis of demographic change initiated in Chapter 4. However, rather than studying the moderating role of the pension system, we use this chapter to study how different types of demographic change affect the aggregate economy. How does a change in the birth rate affect the capital stock? How does a change in the mortality rate affect the capital stock? And what is the impact of a combined mortality and birth rate shock?

To analyse these issues, we construct a model similar to the one in Chapter 2 but it differs on a number of key issues. First of all, we step away from the analysis of an endogenous growth model and focus on an exogenous growth model in which all factor prices are endogenous. This set-up allows us to study how demographic changes affect the aggregate capital stock and gives us a basic idea of the dynamics governing the model. Second, we focus on a model featuring perfect annuity markets instead of imperfect annuity markets. As we saw in Chapter 2, annuity market imperfections, although very prevalent, have a mild impact if proper account is taken of the demographic structure and the redistribution of profits made by the annuity firms. In the same spirit as Chapter 2 we stick with the analysis of an age-dependent mortality rate.

In the first part of the analysis we study the theoretical aspects of the model. We build on the contribution of d'Albis (2007) by highlighting the mechanisms whereby the demographic structure impedes on the macrodynamic equilibrium. This is through the “generational turnover term”, which refers to the reduction in aggregate consumption due to the addition of newborn agents having no accumulated assets, together with the departure of agents with accumulated lifetime assets. Different demographic structures share the feature that they impact on the aggregate macrodynamic equilibrium through their effect on the aggregate consumption growth rate. Hence, differences among various demographic structures reduce to differences in the specification of the generational turnover term. By explicitly setting out the underlying dynamic system, we are able to establish that there are in fact two steady-state equilibria, rather than just the one identified in d'Albis (2007) and much of the remaining literature.

The two equilibria contrast sharply in how they are influenced by the demographic structure. In the first equilibrium (the one generally identified in the literature) demo-

graphic factors play an important role. They impede on equilibrium per capita consumption directly, through the impact of the mortality function on the discounting of future consumption. In contrast, in the second equilibrium we identify, demographic factors play no direct role, except insofar as they influence the overall population growth rate. The key feature of this equilibrium is that the equilibrium growth rate of consumption just equals the growth rate of population. As a result, the amount of consumption given up by the dying just equals that required to sustain the consumption of the growing population. Accordingly, steady-state consumption is sustainable, independent of the time profile of the underlying mortality function. However, we show that this latter steady-state is unsupported by any underlying dynamic path and is sustainable only in the presence of intergenerational transfer flows, much like the “bubble” steady-state in the Bommier-Lee (2003) model. We thus effectively dismiss it as a relevant equilibrium for the current analysis and focus on the “demographic” equilibrium for the remainder of the analysis.

To enhance our understanding of the dynamics of the model and to prepare for the numerical analysis, we must add more demographic structure, and we do so by adopting the Boucekkine, de la Croix, and Licandro (2002) (BCL) mortality function that we previously employed in Chapters 2 through 4. Using the BCL function we provide an explicit representation of the aggregate macrodynamic system. This turns out to be a highly nonlinear fifth order system involving not only capital and consumption, as in the standard representative agent economy, but also the dynamics of the various elements of the intergenerational turnover term. This model embeds the Blanchard model, the dynamics of which simplify dramatically due to the constant mortality assumption, which carries the implication that both human wealth and the marginal propensity to consume are independent of age. As it stands, the dynamic system cannot be solved explicitly and we focus on the steady-states in a numerical analysis. Naturally, the dynamics (and especially transitional dynamics) remains the obvious next step for future research.

In the numerical simulations we study the steady-state behavior of the model in response to both structural and demographic changes, illustrating their effects on aggregate quantities, as well as on the distributions of consumption and wealth across cohorts. Our numerical results show how the effects of a given increase in the popula-

tion growth rate contrast sharply – both qualitatively and quantitatively – depending upon whether it occurs through an increase in the birth rate or a decrease in mortality. Whereas in the former case an increase in the population growth rate is associated with a mild decline in the capital stock, in the latter case it leads to a substantial increase in the per capita stock of capital. These differences in turn carry over to other aspects of the aggregate economy.

This contrast echoes the results of Heijdra and Ligthart (2006) who study the two different types of demographic shocks in a perpetual youth overlapping generations model. Although the perpetual youth model gives more space to the analytical analysis of the dynamics, the magnitude of the results can be misleading due to the emphasis that the model puts on the elderly who are additionally endowed with too many assets (see also section 2.4.2 and 6.5.1). The contrast between the different types of demographic shocks is also consistent with empirical evidence obtained by Blanchet (1988) and by Kelley and Schmidt (1995). The latter summarize the difference in terms of children, having little accumulated wealth, being “resource users” and working adults with their accumulated capital being “resource creators”.¹ Our numerical results also confirm the empirical findings of Bloom, Canning, and Graham (2003) who find that increases in life expectancy leads to higher savings, as well as the consumption patterns obtained by Fair and Dominguez (1991), Attfield and Cannon (2003), and Erlandsen and Nymoer (2008).

As it stands, the current chapter studies mainly theoretical and quantitative issues pertaining to fertility and mortality in the neoclassical framework. However, just as the model in Chapter 2 served as a stepping stone to the analysis of taxation and pensions in Chapters 3 and 4, the current chapter will serve as a stepping stone for the analysis of public policy issues in future research. Naturally, revisiting the topics of taxation and pensions is a natural starting point for further analysis.

The remainder of the chapter is structured as follows. Section 2 lays out the components of the underlying analytical framework, while section 3 describes the corresponding macrodynamic equilibrium and steady state. Section 4 focuses on specific

¹ It is also consistent with the related evidence from cross-country studies of fertility and growth. These have typically found the correlations between economic growth and population growth to be negative for less developed economies, having higher birth rates, and positive for developed economies, with their lower mortality rates (Kelley, 1988).

demographic structures and section 5 performs the numerical simulations. The final section concludes and provides some suggestions for directions in which this research might be extended.

6.2 The analytical framework

The model developed and analyzed in this chapter shares many features with the model developed in Chapter 2. Hence, in what follows we keep the explanation brief and pay attention to the differences between the models rather than the similarities. The most important differences being that in this chapter we focus on perfect annuity markets and that we study an exogenous growth model instead of an endogenous growth model.

6.2.1 Individual household behavior

Discounted expected life-time utility of an individual newborn at time v is:

$$E\Lambda(v) = \int_v^{v+\bar{D}} U(C(v, t)) \cdot e^{-\rho(t-v)-M(t-v)} dt, \quad (6.1a)$$

where $C(v, t)$ denotes the consumption at time t of an individual born at time v , ρ is the pure rate of time preference, $M(t-v) \equiv \int_0^{t-v} \mu(s) ds$ is the cumulative mortality rate, $\mu(s)$ is the instantaneous probability of death and \bar{D} is the maximum attainable age. The agent supplies a unit of labor inelastically and is assumed to make his consumption and asset accumulation decisions to maximise his expected utility (6.1a) subject to his budget constraint:

$$A_t(v, t) \equiv \frac{\partial A(v, t)}{\partial t} = (r(t) + \mu(t-v)) A(v, t) + w(t) - C(v, t), \quad (6.1b)$$

where $A(v, t)$ are real assets held at time t of an individual born at time v , $w(t)$ is the wage rate, and $r(t)$ is the real interest rate (see below). Individuals are born without assets, have no bequest motive, and are not allowed to die indebted. Therefore, $A(v, v) = 0$, and individuals fully annuitize all their assets against the rate of return

$$r(t) + \mu(t-v)^2$$

Defining the present value Hamiltonian for an agent born at time v :

$$H \equiv e^{-\rho(t-v)-M(t-v)} \{U(C(v,t)) + \lambda(v,t) [(r(t) + \mu(t-v))A(v,t) + w(t) - C(v,t)]\}$$

and optimizing with respect to $C(v,t)$ and $A(v,t)$, we obtain:

$$U'(C(v,t)) = \lambda(v,t), \quad (6.2a)$$

$$\rho - \frac{\lambda_t(v,t)}{\lambda(v,t)} = r(t). \quad (6.2b)$$

Equation (6.2a) equates the marginal utility of consumption to the shadow value of financial wealth, while (6.2b) equates the rate of return on consumption, adjusted by the mortality hazard rate, to the rate of return on financial assets. In addition, the agent must satisfy the transversality condition: $A(v, v+D) = 0$.

For analytical convenience we assume an iso-elastic utility function:

$$U(C(v,t)) = \frac{C(v,t)^{1-1/\sigma} - 1}{1-1/\sigma},$$

where σ is the intertemporal elasticity of substitution. Combining (6.2a) and (6.2b) enables us to write the Euler equation as:

$$\frac{C_t(v,t)}{C(v,t)} = \sigma(r(t) - \rho), \quad (6.3)$$

which expresses how the agent's consumption changes with age. In particular, equation (6.3) implies that consumption of all agents grows at a common rate, independent of their age or level of wealth.

Solving (6.3) forward from time t , the agent's consumption at an arbitrary time $\tau > t$ is:

$$C(v,\tau) = C(v,t)e^{\sigma(R(t,\tau) - \rho(\tau-t))}, \quad (6.4)$$

where $R(t,\tau) \equiv \int_t^\tau r(s) ds$ is the cumulative interest rate over period (t,τ) . To express the agent's consumption in terms of financial resources, we integrate the budget

²In this chapter we focus on perfectly functioning annuity markets. See Chapters 2-4 for an analysis of imperfect annuity markets.

constraint (6.1b) forward from time t and impose the transversality condition. This procedure yields the agent's intertemporal budget constraint operative from time t :

$$\begin{aligned} & A(v, t) + e^{R(v, t) + M(t-v)} \int_t^{v+\bar{D}} w(\tau) e^{-R(v, \tau) - M(\tau-v)} d\tau \\ &= e^{R(v, t) + M(t-v)} \int_t^{v+\bar{D}} C(v, \tau) e^{-R(v, \tau) - M(\tau-v)} d\tau. \end{aligned} \quad (6.5)$$

Substituting (6.4) into (6.5) we obtain the following expression for $C(v, \tau)$:

$$C(v, t) = \frac{A(v, t) + \int_t^{v+\bar{D}} w(\tau) e^{-R(t, \tau) - (M(\tau-v) - M(t-v))} d\tau}{\int_t^{v+\bar{D}} e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau-t) - (M(\tau-v) - M(t-v))} d\tau} = \frac{A(v, t) + H(v, t)}{\Delta(v, t)}, \quad (6.6a)$$

where:

$$H(v, t) \equiv \int_t^{v+\bar{D}} w(\tau) e^{-R(t, \tau) - (M(\tau-v) - M(t-v))} d\tau, \quad (6.6b)$$

is discounted future labour income (human wealth) at time t of an individual born at time v , and:

$$\Delta(v, t) \equiv \int_t^{v+\bar{D}} e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau-t) - (M(\tau-v) - M(t-v))} d\tau, \quad (6.6c)$$

is the inverse of the marginal propensity to consume out of total wealth (i.e. financial wealth, $A(v, t)$, plus human wealth, $H(v, t)$) at age $t - v$. Expressions (6.6b) and (6.6c) show that an increase in the mortality rate leads to a decline in human wealth and an increase in the marginal propensity to consume, as agents will have a shorter expected lifespan over which to accumulate assets and to consume the income they generate. Setting $t = v$ yields the corresponding quantities at birth.

6.2.2 Aggregate household behavior

To obtain aggregate per capita quantities, we sum across cohorts by employing the following generic aggregator function:

$$x(t) \equiv \int_{t-\bar{D}}^t p(v, t) X(v, t) dv = \beta \int_{t-\bar{D}}^t e^{-n(t-v) - M(t-v)} X(v, t) dv, \quad (6.7)$$

where $p(v, t)$ denotes the relative size of the cohort born at time v that is still alive at

time t . Taking the time derivative of (6.7), the evolution of $x(t)$ is given by:

$$\dot{x}(t) = \beta X(t, t) + \int_{t-D}^t p(v, t) X_t(v, t) dv - nx(t) - \int_{t-D}^t \mu(t-v) p(v, t) X(v, t) dv, \quad (6.8)$$

where we have used the fact that (see Box 2.2 for details) that $p(t, t) = \beta$, and $p(t-D, t) = 0$.

Thus, aggregate consumption is:

$$c(t) \equiv \int_{t-D}^t p(v, t) C(v, t) dv. \quad (6.9)$$

Taking the time derivative of (6.9), the dynamics of per capita consumption is described by:

$$\dot{c}(t) = \frac{dc(t)}{dt} = (\sigma[r(t) - \rho] - n)c(t) + \beta C(t, t) - \int_{t-D}^t \mu(t-v) p(v, t) C(v, t) dv. \quad (6.10)$$

Combining (6.10) with (6.3) we see that:

$$\frac{\dot{c}(t)}{c(t)} = \frac{C_t(v, t)}{C(v, t)} - \frac{\Phi(t)}{c(t)} \quad (6.11a)$$

where:

$$\Phi(t) \equiv \int_{t-D}^t \mu(t-v) p(v, t) C(v, t) dv - \beta C(t, t) + nc(t) \quad (6.11b)$$

is the “generational turnover term”. That is, the reduction in aggregate per capita consumption (below the common consumption growth rate of each cohort) due to the addition of newborn agents with no accumulated assets and the departure of agents with assets. It depends upon: (i) total consumption given up by the dying relative to the average; and (ii) the difference between the consumption of a newborn and the overall average per capita consumption due to growth.

The expression in (6.11b) provides a very general specification that encompasses all of the standard demographic models. With zero population growth, the textbook infinitely-lived representative agent model is obtained by setting $\beta = \mu = 0$ (implying $D \rightarrow +\infty$). If there is (disembodied) population growth, we need to take account of the fact that at each instant each newborn is immediately endowed with the average

capital stock, part of which he must immediately set aside for the individuals born at the next instant. With the intertemporal elasticity of substitution σ , this reduces the per capita consumption growth rate by $\Phi(t)/c(t) = \sigma n$, so that (6.10) reduces to the familiar aggregate Euler equation $\dot{c}(t) = \sigma(r(t) - \rho - n)c(t)$. For a more realistic demographic process, one in which agents are born and eventually die, we get the more general aggregate Euler equation described in (6.11a). In that case the exact nature of the demographic process will determine the structure of Φ (see below).

Integrating by parts and simplifying, yields:

$$\begin{aligned}\Phi(t) &= -\beta \int_{t-D}^t e^{-n(t-v)-M(t-v)} [nC(v,t) + C_v(v,t)] dv + nc(t) \\ &= -\beta \int_{t-D}^t e^{-n(t-v)-M(t-v)} C_v(v,t) dv\end{aligned}\quad (6.11b')$$

where $C_v(v,t)$ represents the change in consumption across cohorts at a given point in time. Hence, using (6.12) in (6.10) the evolution of aggregate per capita consumption can be written as:

$$\dot{c}(t) = \sigma[r(t) - \rho]c(t) - \beta \int_{t-D}^t e^{-n(t-v)-M(t-v)} C_v(v,t) dv. \quad (6.12)$$

To determine the sign of $\Phi(t)$ we use the fact that at any instant in time, the rate of change of consumption of agents of age $t-v$ is $\dot{C}(v,t) = C_v(v,t) + C_t(v,t)$.³ Recalling (6.3), and letting $\gamma(v,t) \equiv \dot{C}(v,t)/C(v,t)$ denote the growth rate of consumption this implies:

$$C_v(v,t) = [\gamma(v,t) - \sigma[r(t) - \rho]]C(v,t).$$

Thus, a sufficient condition to ensure that $\Phi(t) > 0$ is that the growth rate of consumption with age exceeds the overall growth rate of consumption. In steady-state, $\gamma(v,t) = 0$, implying that $C_v(v,t) = -C_t(v,t) = -\sigma(r(t) - \rho)c(t)$ and we immediately derive $\Phi(t) = \sigma(r(t) - \rho)c(t)$.

Employing (6.7) again, aggregate per capita assets are:

$$a(t) \equiv \int_{t-D}^t p(v,t) A(v,t) dv. \quad (6.13)$$

³ Formally the rate of change of consumption of $t-v$ year old agents is $\lim_{h \rightarrow 0} \frac{C(v+h,t+h) - C(v,t)}{h}$.

Taking the time derivative of (6.13) and using (6.1b), per capita asset accumulation is determined by:

$$\begin{aligned} \dot{a}(t) &= \int_{t-\bar{D}}^t p(v, t) [(r(t) + \mu(t-v)) A(v, t) + w(t) - C(v, t)] dv \\ &\quad - \int_{t-\bar{D}}^t [n + \mu(t-v)] \cdot p(v, t) A(v, t) dv \end{aligned}$$

so that

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t), \quad (6.14)$$

where we have used the fact that $A(t, t) = 0$. The per capita rate of asset accumulation differs from the individual rate of asset accumulation, due to the fact that (i) the amount μA is a transfer by insurance companies from those who die to those who remain alive and thus does not add to aggregate wealth; and (ii) account has to be taken of the growing population.

6.2.3 Firms

Output is produced by a representative firm in accordance with the neoclassical production function having constant returns to scale and adhering to the Inada-conditions:

$$\begin{aligned} Y(t) &= F(K(t), L(t)), \quad F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0, F_{LK} > 0, \quad (6.15) \\ \lim_{K \rightarrow 0} F_K &= \lim_{L \rightarrow 0} F_L = \infty \quad \text{and} \quad \lim_{K \rightarrow \infty} F_K = \lim_{L \rightarrow \infty} F_L = 0, \end{aligned}$$

where $Y(t)$ is output, $K(t)$ is capital, and $L(t)$ is aggregate labor supply. In per capita terms this may be expressed as:

$$\frac{Y(t)}{L(t)} \equiv y(t) = F\left(\frac{K(t)}{L(t)}, 1\right) = f(k(t)). \quad (6.15')$$

Assuming that labor and capital are paid their marginal products, the equilibrium wage rate and return to capital are determined by:

$$w(t) = f(k(t)) - f'(k(t))k(t), \quad (6.16a)$$

$$r(t) = f'(k(t)) - \delta, \quad (6.16b)$$

where δ is the depreciation rate of capital.

6.3 General Equilibrium

In equilibrium, both the capital and the labor market must clear. Labor market clearance is reflected in the fact that all agents are fully employed so that the total population equals the total labor force. Capital market equilibrium is imposed by setting aggregate assets equal to total capital $A(t) = K(t)$, so that in aggregate per capita terms $a(t) = k(t)$, implying further that $\dot{a}(t) = \dot{k}(t)$.

Substituting the factor pricing relations (6.16) into (6.14) and (6.12) enables us to summarize the dynamics of the macroeconomic equilibrium in the form:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t) \quad (6.17a)$$

$$\dot{c}(t) = \sigma(f'(k(t)) - \delta - \rho)c(t) - \Phi(t) \quad (6.17b)$$

where

$$\Phi(t) = -\beta \int_{t-\bar{D}}^t e^{-n(t-v)-M(t-v)} C_v(v, t) dv. \quad (6.17c)$$

This pair of dynamic equations in \dot{k} and \dot{c} will be recognized as being a variant of the standard textbook neoclassical growth model. Equation (6.17a) is the standard aggregate per capita accumulation of capital relationship, where the normalization of individual labor supply at unity implies that that aggregate labor supply is equal to one, while (6.17b) is the aggregate Euler equation, determining the intertemporal allocation of consumption.

The key point to emphasize with regard to expressing the macroeconomic equilibrium in this way is that it highlights how the demographic structure impedes on the economy through the generational turnover term, $\Phi(t)$, and its impact on the aggregate Euler equation. It provides a very general representation in which various specifications of the demographic structure can be embedded. In the case of the pioneering Blanchard (1985) model, and variants such as those developed by Buiter (1988) and Weil (1989), the evolution of (6.17) is very straightforward and the full model can be described by a three dimensional dynamic system; see e.g. Blanchard (1985, p.234)

and below.

However, the fact that Φ depends upon how consumption at any instant of time varies across cohorts means that for more general demographic structures its dynamic evolution can be very complex. As we demonstrate in Section 4 below, a more realistic demographic structure leads to a much higher dynamic system, due to the fact that the marginal propensity to consume varies over the life-cycle. In general, in order to characterize the aggregate dynamics and to prevent them from being totally intractable it is necessary to impose some constraints on the demography.⁴

6.3.1 Steady-State

In the steady-state, consumption, asset accumulation, relative cohort size, survival and mortality no longer depend upon calendar time but only on age ($u \equiv t - v$). As a result, with no long-run per capita growth, per capita consumption, $c(t)$, per capita capital stock, $k(t)$, the wage rate, $w(t)$, the return to capital, $r(t)$, and the generational transfer term, $\Phi(t)$, are all constant over time. We shall denote all steady-state quantities by tildes.

Thus, when the aggregate economy is in steady state, consumption grows at the steady rate $\sigma(\tilde{r} - \rho)$ with age, so that the consumption level of an individual of age is equal to:

$$\tilde{C}(u) = \tilde{C}_0 e^{\sigma(\tilde{r} - \rho)u} \quad (6.18)$$

where, setting $t = v$ in (6.6a), consumption at birth, \tilde{C}_0 can be expressed as:

$$\tilde{C}_0 = \frac{\tilde{w} \int_0^{\bar{D}} e^{-\tilde{r}u - M(u)} du}{\int_0^{\bar{D}} e^{-(\tilde{r}(1-\sigma) + \sigma\rho)u - M(u)} du}. \quad (6.19)$$

In the steady-state $p(v, t) = p(u) = \beta e^{-nu - M(u)}$ implying that aggregate consumption per capita is:

$$\tilde{c} \equiv \int_0^{\bar{D}} p(u) \tilde{C}(u) du = \beta \tilde{C}_0 \int_0^{\bar{D}} e^{(\sigma(\tilde{r} - \rho) - n)u - M(u)} du. \quad (6.20)$$

⁴Having obtained $k(t)$, one can determine the time paths for the return to capital $r(t)$ and the wage rate $w(t)$. Having obtained these one can then derive the dynamics of consumption, savings, and capital accumulation across cohorts.

Defining the function:⁵

$$\Xi(\lambda) \equiv \int_0^{\bar{D}} e^{-\lambda s - M(s)} ds,$$

we can combine (6.18)-(6.20) to express the steady-state per capita consumption, (6.20) as:

$$\tilde{c} = \tilde{w} \frac{\Xi(\tilde{r})}{\Xi(\tilde{r}(1-\sigma) + \sigma\rho)} \frac{\Xi(n - \sigma(\tilde{r} - \rho))}{\Xi(n)}. \quad (6.21)$$

Finally, using the demographic steady-state condition,

$$\frac{1}{\beta} = \int_0^{\bar{D}} e^{-nu - M(u)} du = \Xi(n)$$

we can write:

$$\tilde{c} = \beta \tilde{w} \frac{\Xi(\tilde{r}) \cdot \Xi(n - \sigma(\tilde{r} - \rho))}{\Xi(\tilde{r}(1-\sigma) + \sigma\rho)} \quad (6.21')$$

as in d'Albis (2007, p.416).

Substituting for the steady-state factor prices, (6.16), the steady-state equilibrium values of per capita consumption, \tilde{c} , and capital, \tilde{k} , are jointly determined by:

$$\tilde{c} = f(\tilde{k}) - (\delta + n)\tilde{k} \quad (6.22a)$$

$$\tilde{c} = \beta [f(\tilde{k}) - \tilde{k}f'(\tilde{k})] \frac{\Xi(f'(\tilde{k} - \delta)) \cdot \Xi(n - \sigma(f'(\tilde{k}) - \delta - \rho))}{\Xi([f'(\tilde{k}) - \delta](1-\sigma) + \sigma\rho)} \quad (6.22b)$$

where the demographic characteristics are embedded in the Ξ -function. Letting $s(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/f(\tilde{k})$ denote the equilibrium share of capital, d'Albis (2007) shows that the pair of equations (6.22a) and (6.22b) have a unique solution as long as $\lim_{\tilde{k} \rightarrow 0} s(\tilde{k}) = [0, 1)$ and $\tilde{s} < \tilde{\epsilon}$, where $\tilde{\epsilon}$ is the elasticity of substitution in production and $\sigma < 1$.⁶ Both conditions are mild and hold for the Cobb-Douglas production function, for example. Figure 1 illustrates this equilibrium for the calibrated model specified in Section 6.5, where AA represents (6.22a), BB depicts (6.22b), and the two intersect at the point P.

⁵Our Ξ -function is very common in the overlapping generations literature and appears in one form or the other in d'Albis (2007), Heijdra and Romp (2008), Gan and Lau (2010) and Chapters 2-4 above.

⁶These conditions have been relaxed in subsequent work by Gan and Lau (2010), who show further that uniqueness is still obtained if $\sigma \geq 1$.

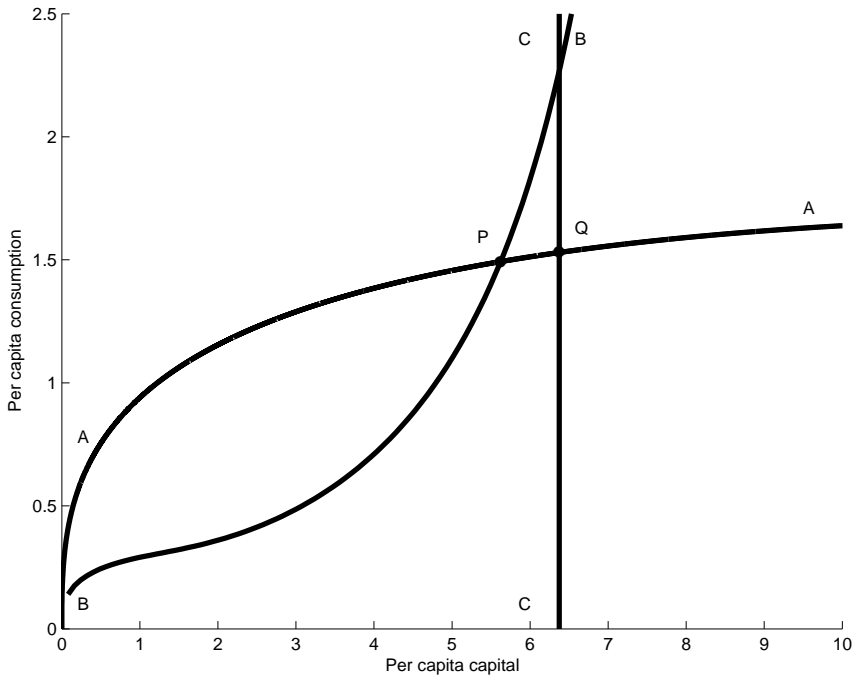


Figure 6.1. Steady-State Equilibrium

6.3.2 A ‘non-demographic’ steady-state

The steady-state equilibrium discussed in the previous section is the one identified by d’Albis (2007), Lau (2009), and Gan and Lau (2010). While they argue that the solution to (6.22a) and (6.22b) is unique, there is in fact a second steady-state equilibrium associated with the underlying dynamic system (6.17). This can be identified by taking the following steps: (i) substitute (6.12) into (6.12), (ii) use the result $\dot{C}(v, t) = C_v(v, t) + C_t(v, t)$, and (iii) recall (6.3), thereby enabling us to rewrite (6.12) as

$$\begin{aligned} \dot{c}(t) = & [\sigma(r(t) - \rho) - n] \left[c(t) - \beta \int_{t-\bar{D}}^t e^{-n(t-v)-M(t-v)} C(v, t) dv \right] \\ & + \beta \int_{t-\bar{D}}^t e^{-n(t-v)-M(t-v)} \dot{C}(v, t) dv. \end{aligned} \tag{6.12'}$$

Now, for notational convenience, let us define

$$X(t) \equiv c(t) - \beta \int_{t-\bar{D}}^t e^{-n(t-v)-M(t-v)} C(v, t) dv \text{ and, hence,}$$

$$\dot{X}(t) \equiv \dot{c}(t) - \beta \int_{t-\bar{D}}^t e^{-n(t-v)-M(t-v)} \dot{C}(v, t) dv$$

permitting us to express (6.23) in more compact form

$$\dot{X}(t) = [\sigma(r(t) - \rho) - n] X(t) \quad (6.23)$$

Recalling (6.18), equation (6.23) is seen to yield the *two* steady-state conditions

$$(i) \sigma(\bar{r} - \rho) = n \text{ and } (ii) \bar{c} = \beta \bar{C}_0 \int_0^{\bar{D}} e^{(\sigma(\bar{r}-\rho)-n)u-M(u)} du \quad (6.23')$$

Thus, in addition to (6.22a) and (6.22b), the pair of equations

$$f(\bar{k}) = \bar{c} + (\delta + n)\bar{k}, \quad (6.24a)$$

$$\sigma(f'(k) - \delta - \rho) - n = 0, \quad (6.24b)$$

define an alternative steady-state. This is illustrated in Fig. 1 for the calibrated model by the intersection of AA and the vertical line CC, corresponding to (6.24b), at the point, Q. The key point to observe is that this steady-state is independent of the demographic structure, except insofar as this determines the overall population growth rate through the demographic steady-state condition.

There is a sharp contrast between (6.24b) which characterizes the “non-demographic” steady-state and (6.20) [and (6.22b)], where the demographic structure plays an important role through the impact of the mortality function on the discounting of future consumption. Recalling (6.18) and (6.24b), the key feature of this ‘non-demographic equilibrium’ is that the steady-state growth rate of consumption across cohorts just equals the growth rate of the population. In that case, the amount of consumption given up by the dying just equals that required to sustain the consumption of the growing population.

The underlying dynamic equation (6.23) is similar in structure to equation (10)

of Bommier and Lee (2003), with (6.22b) corresponding to their “balanced” equilibrium and (6.24b) corresponding to their “bubble” equilibrium.⁷ Since we know that $X(t) = 0$, for all t , while $\sigma(r(t) - \rho) - n = 0$ holds only in the non-demographic steady-state, where $\sigma(\tilde{r} - \rho) = n$, equation (6.23) can help determine the relevance of the two steady-states. Thus, suppose the system starts out with an arbitrary aggregate capital stock, such that $\sigma(r(t) - \rho) - n \neq 0$. In this case, $X(t) = \dot{X}(t) = 0$ for all t , and eventually as the economy evolves we reach the demographic steady-state:

$$\bar{X} \equiv \bar{c} - \beta \int_0^{\bar{D}} e^{-nu - M(u)} C(u) du = 0$$

This occurs irrespective of the path of $[\sigma(r(t) - \rho) - n]$, making it clear that the demographic steady-state is in fact the relevant one.

Now consider what happens if the initial capital stock, \tilde{k}_N , yields $\sigma(\tilde{r} - \rho) = n$ and the economy is in the non-demographic steady-state. From (6.22a), the corresponding per capita consumption is \tilde{c}_N , while the implied return to capital and wage rate are respectively \tilde{r}_N and \tilde{w}_N . Given these steady-state values, the agent’s steady-state intertemporal budget constraint (6.19), and the steady-state aggregation of consumption across cohorts, (6.20) yield the following two solutions for consumption at birth, \tilde{C}_0 :

$$\tilde{C}_0 = \tilde{w} \frac{\int_0^{\bar{D}} e^{-\tilde{r}u - M(u)} du}{\int_0^{\bar{D}} e^{-(\tilde{r}-n)u - M(u)} du} \quad \text{and} \quad \tilde{C}_0 = \frac{\tilde{c}_N}{\beta \int_0^{\bar{D}} e^{-M(u)} du}. \quad (6.25)$$

However, these two solutions are, in general, inconsistent, and consequently the “non-demographic” steady-state is in general not viable.⁸

The question of viability of the bubble equilibrium is discussed by Tirole (1985) and Bommier and Lee (2003). They suggest that certain institutions, such as intergenerational transfers or money, may exist that assure imbalance in the capital market, which in their case lead to asset bubbles. In a similar vein, it may be possible to devise a suitable system of transfers that reconciles the two solutions for consumption at birth,

⁷The “bubble” steady-state was first identified by Tirole (1985) for a Diamond-Samuelson model and generalized to the continuous case by Bommier and Lee (2003, p. 146 ff.).

⁸However, the “non-demographic” steady-state does satisfy the transversality condition, so it cannot be ruled out as being unsustainable on the grounds of intertemporal insolvency.

\tilde{C}_0 implied by (6.25).⁹

Although we refrain from analyzing such transfers further, it is instructive to compare the two equilibria at P and Q, with the steady-state obtained in the infinitely-lived representative agent model. Denoting the corresponding steady-state per capita capital stocks by \tilde{k}_P , \tilde{k}_Q and \tilde{k}_R , these three quantities are determined respectively by:

$$\sigma (f'(\tilde{k}_P) - \delta - \rho) = \frac{\tilde{\Phi}}{\tilde{c}} \quad (6.26a)$$

$$\sigma (f'(\tilde{k}_Q) - \delta - \rho) = n \quad (6.26b)$$

$$\sigma (f'(\tilde{k}_R) - \delta - \rho) = \sigma n \quad (6.26c)$$

Recalling (6.12), (6.26) implies that if (i) the total consumption given up by the dying exceeds the consumption of the newborn, and if (ii) the intertemporal elasticity of substitution, $\sigma < 1$, that $\tilde{k}_P < \tilde{k}_Q < \tilde{k}_R$.¹⁰ Having established the basic properties of the “non-demographic” steady-state, for the remainder of the chapter we return to the analysis of the more relevant “demographic” steady-state.

6.3.3 Capital maximizing birth rate

Having established the steady-state characteristics of the model we now turn to the relationship between the aggregate capital stock and the underlying demographic structure. d’Albis (2007) argues that there exists a birth rate that maximizes the per capita capital stock. He defines the measure: $\alpha_x \equiv \int_0^{\bar{D}} up(u)x(u)du / \int_0^{\bar{D}} p(u)x(u)du$ where α_x measures the average of the quantity $x(u)$ across cohorts. He then shows that the capital stock-maximizing birth rate occurs where the average age of workers equals the average age of asset holders, i.e. $\alpha_W = \alpha_A$. In our case it is straightforward to show that:

$$\text{sgn}(\alpha_A - \alpha_W) = \text{sgn} \left(\int_0^{\bar{D}} u\tilde{A}(u)p(u)du - \int_0^{\bar{D}} up(u)du \cdot \int_0^{\bar{D}} \tilde{A}(u)p(u)du \right) \quad (6.27)$$

⁹In general, these transfer systems need to consist of an unproductive entity that transfers and collects resources from the agents in such a way that on aggregate the transferred and collected resources do not balance. The surplus or deficit of such a system may be due to capital flows to or from abroad in an open economy or, in a closed economy, an unbalanced pay-as-you-go pension system (see Bommier and Lee, 2003, p.150).

¹⁰In the sense that case all steady states are dynamically efficient, in that the capital stocks would be less than at the golden rule.

Using the fact that $p(u)$ may be interpreted as a probability density function we conclude that for $\alpha_W = \alpha_A$ the covariance between $\tilde{A}(u)$ and u must be zero.¹¹ For this to be so, assets either have to be constant over the life-cycle or their time profile has to be linearly independent of the age profile.

To see that assets are actually hump-shaped over the life cycle, rather than constant, note that in the steady state, agents accumulate assets according to:

$$\dot{\tilde{A}}(u) = (\tilde{r} + \mu(u)) \tilde{A}(u) + \tilde{w} - \tilde{C}(u), \quad (6.28a)$$

so that starting with a zero initial capital endowment, $\tilde{A}(0) = 0$, the agent's wealth at age u is:

$$\tilde{A}(u) = \int_0^u [\tilde{w} - \tilde{C}(u)] e^{-\tilde{r}u - M(u)} du, \quad (6.28b)$$

with the transversality condition implying:

$$\int_0^{\bar{D}} [\tilde{w} - \tilde{C}(u)] e^{-\tilde{r}u - M(u)} du = 0. \quad (6.28c)$$

Under weak conditions, d'Albis shows that in this steady-state $\tilde{r} > \rho$, so that agents' consumption grows uniformly over their lifetimes.

Using this fact, in conjunction with (6.18), (6.19), and (6.28a), one can show that because $\tilde{A}(0) = \tilde{A}(\bar{D}) = 0$, $\dot{\tilde{A}}(0) > 0$, $\dot{\tilde{A}}(\bar{D}) < 0$, and that the agent's assets reach a maximum at an age \hat{u} :

$$\tilde{A}(\hat{u}) = \frac{\tilde{C}(\hat{u}) - \tilde{w}}{\tilde{r} + \mu(\hat{u})}.$$

Thus, the time profile of the agent's wealth over the life-cycle is hump shaped as illustrated in Panel (iii) of Figures 3-5.

As the asset profile is hump shaped over the life-cycle it may be that there exists a unique value of the birth rate such that the asset profile and the age profile are not linearly dependent. In that case the average age of asset holders equals the average age of the workers. But as savings are primarily used to finance consumption later in life, it is fair to suppose that the average age of the capital owner is higher than the average age of the worker. Indeed, in our simulations we show that for a realistic mortality

¹¹ The key result that is being employed is that $E(xy) = E(x)E(y) + cov(x,y)$.

function $\alpha_A = 52.65$, $\alpha_W = 43.08$. Thus, we find that an increase in the population growth rate associated with an increase in the birth rate leads to a reduction in the per capita stock of capital. In contrast, our simulations also show that if the increase in the population growth rate is the result of a reduction in mortality it will result in an increase in the per capita capital stock; see Table 2 and Section 5.¹² This contrast in the two ways of increasing the growth rate of population is consistent with the empirical evidence on this issue obtained by, inter alia, Kelley and Schmidt (1995), Bloom, Canning and Graham (2003) and Erlandsen and Nymoer (2008).

6.4 Specific Demographic Models

Thus far, we have not imposed any restrictions on the exact form of the survival function. To proceed further, we focus on the functional form proposed by Boucekkine, de la Croix, and Licandro (2002) labeled BCL, which was used extensively in Chapter 2-4. As we saw there, it is very tractable, amenable to numerical simulations and fits the data well (for details see Box 2.2). For comparative purposes, and to show how it fits into our analytical framework, we also discuss the familiar demographic structure proposed by Blanchard (1985), Buiter (1988) and Weil (1989) labeled BBW.¹³

6.4.1 BCL demographic structure

While the general macrodynamic equilibrium is summarized by the system (6.17), the evolution of $\Phi(t)$ may in fact be complex, requiring one to consider the dynamics of its components. To this end it is practical to begin with the alternative definition of $\Phi(t)$, given in (20b), which for the BCL function becomes:

$$\Phi(t) = \frac{\beta\eta_1}{\eta_0 - 1} \int_{t-\bar{D}}^t e^{(\eta_1 - n)(t-v)} \cdot C(v, t) dv - \beta C(t, t) + nc(t).$$

¹²In an early contribution Sinha (1986) finds the same results in a numerical simulation of the Diamond-Samuelson (DS) model.

¹³Alternatively, Bruce and Turnovsky (2010) use the de Moivre function which has the advantage of including both the DS and BBW specifications as special cases, but is less tractable than the BCL function.

Using (6.4) and (6.6) we can write:

$$\Phi(t) = \Gamma(t) - \beta \frac{H_B(t)}{\Delta_B(t)} + nc(t), \quad (6.29)$$

where:

$$\Gamma(t) = \frac{\beta\eta_1}{\eta_0 - 1} \int_{t-\bar{D}}^t C(v, t) e^{(\eta_1 - n)(t-v)} dv \quad (6.30a)$$

$$\frac{H_B(t)}{\Delta_B(t)} = C(t, t) \quad (6.30b)$$

and

$$H_B(t) \equiv H(t, t) = \int_t^{t+\bar{D}} w(\tau) e^{-R(t, \tau) - M(\tau-t)} d\tau, \quad (6.30c)$$

$$\Delta_B(t) \equiv \Delta(t, t) = \int_t^{t+\bar{D}} e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau-t) - M(\tau-t)} d\tau. \quad (6.30d)$$

That is, $H_B(t)$ and $\Delta_B(t)$ are, respectively, the amounts of human wealth and the inverse marginal propensity to consume *at birth*.

Differentiating (6.30a)-(6.30d), imposing the factor prices (6.16a), (6.16b), and recalling the dynamics of consumption and capital (6.17a), (6.17b), the full dynamic system can then be expressed as:¹⁴

$$\dot{c}(t) = (\sigma(f'(k(t)) - \delta - \rho) - n)c(t) - \frac{\beta\eta_1}{\eta_0 - 1}\Gamma(t) + \beta\frac{H_B(t)}{\Delta_B(t)} \quad (6.31a)$$

$$\dot{k}(t) = f(k(t)) - (\delta + n)k(t) - c(t) \quad (6.31b)$$

$$\begin{aligned} \dot{\Gamma}(t) &= \frac{H_B(t)}{\Delta_B(t)} - \frac{H_B(t-\bar{D})}{\Delta_B(t-\bar{D})} e^{\sigma R(t-\bar{D}, t) + (\eta_1 - n - \sigma\rho)\bar{D}} \\ &\quad + (\sigma(f'(k(t)) - \delta - \rho) + \mu_1 - n)\Gamma(t) \end{aligned} \quad (6.31c)$$

$$\begin{aligned} \dot{\Delta}_B(t) &= -1 - ((\sigma - 1)(f'(k(t)) - \delta) - \sigma\rho + \mu_1)\Delta_B(t) \\ &\quad + \frac{\eta_1\eta_0}{\eta_0 - 1} \int_t^{t+\bar{D}} e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau-t)} d\tau \end{aligned} \quad (6.31d)$$

$$\begin{aligned} \dot{H}_B(t) &= -f(k(t)) + f'(k(t))k(t) + (f'(k(t)) - \delta - \mu_1)H_B(t) \\ &\quad + \frac{\eta_1\eta_0}{\eta_0 - 1} \int_t^{t+\bar{D}} w(\tau) e^{-R(t, \tau)} d\tau \end{aligned} \quad (6.31e)$$

¹⁴In determining (40d), (40e) we have used $e^{\mu_1(\tau-t)} = \mu_0 - e^{-M(\tau-t)}(\mu_0 - 1)$.

This comprises a fifth-order system in: (i) per capita consumption, (ii) per capita capital stock, (iii) the consumption given up by the dying, (iv) the initial human wealth of the new born, and (v) the (inverse) of the marginal propensity to consume out of wealth by the newborn. In principle, the dynamics can be analyzed using numerical simulations. We should note that with H_B and Δ_B being evaluated both at time t and at time $t - \bar{D}$ this involves the analysis of mixed differential-difference equations, which presents a computational challenge that is beyond the scope of the present chapter.¹⁵ Indeed, as d'Albis and Augeraud-Véron (2009) emphasize, the characterization of the dynamics in terms of a mixed differential-difference equation is essentially generic in continuous-time overlapping generations models, one of the few exceptions being the BBW model.¹⁶

Steady state

Defining $\varphi(x, \bar{D}) \equiv (1 - e^{-x\bar{D}}) / x$, the steady state can be summarized by the following system¹⁷

A. Demographic Variables

$$\frac{1}{\beta} = \frac{1}{\eta_0 - 1} [\eta_0 \varphi(n, \bar{D}) - \varphi(n - \eta_1, \bar{D})] \quad (6.32a)$$

$$\bar{D} = \frac{\ln \eta_0}{\eta_1} \quad (6.32b)$$

B. Economic Variables

$$\tilde{C}_0 = \frac{\tilde{w} [\eta_0 \varphi(\tilde{r}, \bar{D}) - \varphi(\tilde{r} - \eta_1, \bar{D})]}{[\eta_0 \varphi(\sigma\rho + (1 - \sigma)\tilde{r}, \bar{D}) - \varphi(\sigma\rho + (1 - \sigma)\tilde{r} - \eta_1, \bar{D})]} \quad (6.32c)$$

$$\tilde{c} = \frac{\beta \tilde{C}_0}{\sigma(\tilde{r} - \rho) - n} \left\{ \frac{\eta_1}{\eta_0 - 1} \varphi(\sigma(\rho - \tilde{r}) + n - \eta_1, \bar{D}) - 1 \right\} \quad (6.32d)$$

$$f(\tilde{k}) = \tilde{c} + (\delta + n)\tilde{k} \quad (6.32e)$$

¹⁵ In the special case of constant returns and a rectangular survival function it becomes possible to characterize the equilibrium dynamics; see, for instance, d'Albis and Augeraud-Véron (2009) and the references therein. As they point out, the representation of the equilibrium dynamics by a mixed differential-difference equation introduces oscillations into the transitional path.

¹⁶ The reason that the BBW model can be represented by a system of ordinary differential equations is because all individuals have the same life-expectancy independent of age.

¹⁷ We are focusing on the 'demographic equilibrium' at which $\sigma(\tilde{r} - \rho) \neq n$.

where \tilde{r} and \tilde{w} are defined in (6.16a) and (6.16b). Equations (6.32a) and (6.32b) define the demographic structure, summarized by the four parameters, β , η_0 , η_1 , and n . Given the demographic parameters and the definitions of \tilde{r} and \tilde{w} , equations (6.32c)-(6.32e) determine the economic variables, \tilde{C}_0 , \tilde{c} and \tilde{k} . By combining (6.32c) and (6.32d) this can be reduced to a pair of equations in \tilde{c} and \tilde{k} , which is analogous to (6.22a) and (6.22b). Having determined the aggregates, the steady-state age profiles of consumption and asset accumulations can be obtained by substituting (6.32c) into (6.18) and (6.28).

The system (6.32) provides the basis for our numerical simulations in Section 5. We use this system to examine the effects of a number of economic and demographic structural changes on both the aggregate behaviour of the economy and on the patterns of consumption and asset accumulation over the life cycle.

6.4.2 BBW demographic structure

For comparative purposes it is useful to show how the BBW model fits into this framework. Blanchard (1985) assumes the birth rate to be equal to the mortality rate ($\beta = \mu$), so that the net population growth rate is zero. Buiter (1988) relaxes this assumption and extends the model to the case where $\beta \neq \mu$, effectively combining the Blanchard model with that of Weil (1989).

The survival function is specified by:

$$S(t-v) \equiv e^{-M(t-v)} = e^{-\mu(t-v)}, \quad (6.33)$$

from which we immediately infer that the hazard rate, μ , is constant, while the relative cohort size is $p(v,t) = \beta e^{-\beta(v-t)}$. The demographic steady-state holds by definition, life-expectancy equals $1/\mu$ and is constant over the life cycle, while the average age of workers is $1/\beta$.

The key variable in the dynamics, the generational turnover term, $\Phi(t)$, now simplifies drastically to:

$$\begin{aligned} \Phi(t) &= \int_{-\infty}^t \mu \cdot \beta e^{\beta(v-t)} \cdot C(v,t) dv - \beta C(t,t) + nc(t) \\ &= (\mu + n)c(t) - \beta C(t,t). \end{aligned} \quad (6.34)$$

Introducing the BBW structure into (6.6a) leads to:

$$C(v, t) = \frac{A(v, t) + \int_t^\infty w(\tau) e^{-R(t, \tau) - \mu(\tau - t)} d\tau}{\int_t^\infty e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau - t) - \mu(\tau - t)} d\tau} = \frac{A(v, t) + H(t)}{\Delta(t)}. \quad (6.35)$$

The crucial characteristic that renders the model so tractable is that all agents have the same planning horizon (i.e., ∞) and mortality rate (i.e., μ). Therefore, human wealth, $H(t)$, (future discounted income from labour) is the same for all agents, irrespective of their age. The same applies to $\Delta(t)$, the (inverse of) the marginal propensity to consume out of human wealth:

$$\Delta(t) = \int_t^\infty e^{(\sigma-1)R(t, \tau) - \sigma\rho(\tau - t) - \mu(\tau - t)} d\tau. \quad (6.36)$$

Differentiating (6.36), its dynamics are governed by:

$$\dot{\Delta}(t) = -1 - ((\sigma - 1)r(t) - \sigma\rho - \mu)\Delta(t). \quad (6.37)$$

Aggregate per-capita consumption is:

$$\begin{aligned} c(t) &\equiv \int_{-\infty}^t p(v, t) C(v, t) dv = \int_{-\infty}^t p(v, t) [\Delta(t)]^{-1} (A(v, t) + H(t)) dv \\ &= [\Delta(t)]^{-1} (a(t) + H(t)) = [\Delta(t)]^{-1} (k(t) + H(t)). \end{aligned} \quad (6.38)$$

From (6.6a) consumption of a new born, $C(t, t)$, is:

$$C(t, t) = [\Delta(t)]^{-1} H(t) = c(t) - [\Delta(t)]^{-1} k(t). \quad (6.39)$$

Hence, using (6.34), and recalling that $n = \beta - \mu$, we can write the aggregate dynamic system as:

$$\dot{k}(t) = f(k(t)) - (\delta + n)k(t) - c(t) \quad (6.40a)$$

$$\dot{c}(t) = \sigma(f'(k(t)) - \delta - \rho)c(t) - \beta[\Delta(t)]^{-1}k(t) \quad (6.40b)$$

$$\dot{\Delta}(t) = -1 - ((\sigma - 1)(f'(k(t)) - \delta) - \sigma\rho - \mu)\Delta(t), \quad (6.40c)$$

thus reducing it to a tractable third order system; see also Blanchard (1985, p. 234). The

steady state follows readily by setting $\dot{k}(t) = \dot{c}(t) = \dot{\Delta}(t) = 0$.

6.5 Numerical Simulations

To obtain further insights, we simulate the steady-state demographic equilibrium using the BCL survival function. To do this, we first estimate its two parameters, η_0 and η_1 , by nonlinear least squares, using US cohort data for 2006.¹⁸ The estimation results reported in Table 1 highlight that we obtain a tight fit with highly significant parameter estimates. The resulting estimated survival function is illustrated in Fig. 2. Since we do not consider childhood and education, we normalize the function so that birth corresponds to age 18. As can be seen in the figure it tracks the actual survival data for the United States closely from age 18 until around 95. Beyond that age its concavity does not match the data particularly well. However, we do not view that as serious since only 1.5% of the US population exceeds 95 and these individuals are generally retired and are relatively inactive in the economy.¹⁹ For comparative purposes we also estimate and illustrate the BBW survival function in Table 1 and Fig. 2. Being convex, rather than concave, it does not match the data well.

Table 6.1. Estimated Survival Functions

$S(u) = I(u \leq D) \frac{\mu_0 - e^{\eta_1 u}}{\mu_0 - 1} + \varepsilon$ where $\varepsilon \sim i.i.d. (0, \sigma^2)$ ¹ $S(u) = e^{\eta u} + \varepsilon$ where $\varepsilon \sim i.i.d. (0, \sigma^2)$ ²		
Demographic function	BCL ¹	BBW ²
η_0 (st. dev.)	78.3618 (6.0193)	
$\eta_1(\mu)$ (st. dev.)	0.0566 (0.0011)	0.0112 (0.0011)
Adj. R^2	0.9961	0.6157

¹Boucekkine, de la Croix and Licandro (2002): $I(u \leq D)$ is an indicator function that is 1 for $u \leq D$ and 0 otherwise.

²Blanchard (1985)-Buiter (1988)-Weil (1989)

¹⁸Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 12/10/2010).

¹⁹With this in mind, it might be more appropriate to refer to \bar{D} as the maximum attainable economic age.

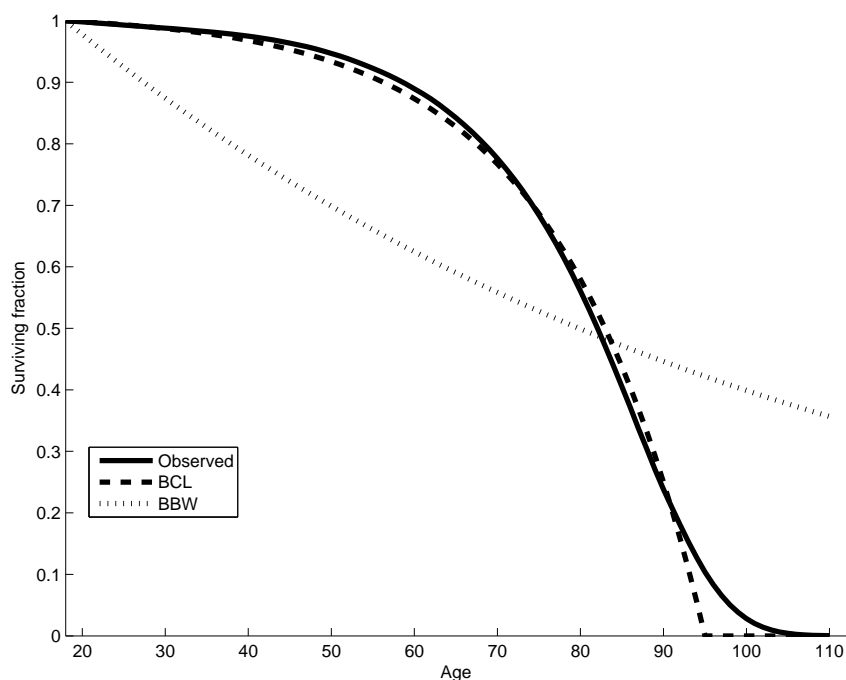


Figure 6.2. Demography

Table 2 summarizes the key structural parameters for the baseline economy, all of which are quite standard. Output is produced by a Cobb-Douglas function, $y = Ak^\alpha \bar{l}^{1-\alpha}$, where A is the exogenous technology index, \bar{l} denotes inelastically supplied labour, with the elasticity of capital $\alpha = 0.35$ and depreciation rate $\delta = 0.05$. With respect to preferences, we set the intertemporal elasticity of substitution to 0.5, consistent with the consensus estimates reported by Guvenen (2006). We take $\rho = 0.035$ to be the rate of time preference.

The baseline calibration adopts the demographic parameters estimated above. Thus, the estimates of the BCL function imply a maximum attainable age of 95.06 and life expectancy at age 18 of 78.38. These are a little low, reflecting the fact that, as Fig. 2 illustrates, the function fails to capture the outliers beyond age 90. We take the population growth rate to be 1.00% which given the survival function, implies a birth rate of 2.24%. This is a little high because the population growth rate also takes into

Table 6.2. Baseline Parameters and Benchmark Equilibrium

Baseline Model			
Structural Parameters		BCL ¹	BBW ²
Total factor productivity	A	1	1
Capital share of output	α	0.35	0.35
Depreciation rate	δ	5%	5%
Inter-temporal substitution elasticity	σ	0.35	0.35
Time preference rate	ρ	3.5%	3.5%
Demographic Parameters			
Youth mortality	μ_0	78.3618	N/A
Old age mortality	μ_1	0.0566	0.0112
Birth rate (implied)	β	2.24%	2.12%
Life-expectation at 18 (Age)	L_{18}	78.38	89.29
Average age of workers	α_W	43.08	32.76
Average age of asset holders	α_A	52.65	61.29
Maximum attainable age (implied)	\bar{D}	95.06	∞
Population growth rate	n	1.00%	1.00%
Implied Economic Variables			
Per capita capital stock	\tilde{k}	5.6226	7.4044
Per capita output	\tilde{y}	1.8301	2.0152
Capital/ Output ratio	\tilde{k}/\tilde{y}	3.0722	3.6742
Real interest rate	\tilde{r}	6.39%	4.52%
Wage rate	\tilde{w}	1.1896	1.3099
Average per capita consumption	\tilde{c}	1.4928	1.5710
Marginal propensity to consume at birth	$[\Delta_B]^{-1}$	0.053	0.051

¹Boucekkine, de la Croix and Licandro (2002)

²Blanchard (1985)-Buiter (1988)-Weil (1989)

account immigration. The implied equilibrium economic variables include an equilibrium capital-output ratio of 3.07 and a real net return on capital of 6.39%. The marginal propensity to consume at birth out of wealth is approximately 0.053%, and the each cohort's consumption grows at 1.45% with age. The corresponding parameters and implied equilibrium values for the BBW model are also reported in Table 2. It yields a much higher life expectancy, due to the fact that the maximum attainable age in that model is infinite.

From this initial baseline equilibrium we analyze the steady-state effects of two types of structural changes: (i) an increase in productivity; (ii) changes in the demographic structure.

6.5.1 Increase in productivity

We consider a neutral technological change, where A increases by 25% from 1 to 1.25. As seen from Row 2 in Table 3, this leads to a proportionate increase in capital and output, causing the capital-output ratio to remain unchanged.

Fig. 3.A illustrates the aggregate and the distributional effects for the BCL survival function. The locus BB in panel (i) depicts the pre-shock growth in consumption with age (eq. (6.3)). The increase in productivity raises the wage rate, while the rate of return on capital remains unchanged. This causes the BB locus to shift up to $B'B'$, implying a uniformly higher consumption level for all ages, but growing at the unchanged rate. The AA locus presents the average per capita consumption, which correspondingly jumps up to $A'A'$. Panel (ii) illustrates the long-run distributional changes across the cohorts. Its mildly hump-shaped locus reflects the fact that the increase in consumption with age is offset by the increasing mortality with age, leading to declining cohort-weighted consumption.

Panel (iii) illustrates the distribution of assets along the life cycle. Starting with zero assets at birth (18), agents accumulate wealth until around 70, after which they decumulate until assets run out at the maximum attainable age. This is reflected in the inverted-U locus EE which shifts out to $E'E'$ with the increase in productivity. The figures indicate that the greatest impact on wealth of the productivity increase accrues to individuals aged around 70. The upward shift in the distributional locus is also reflected in the horizontal line DD which illustrates the average per capita wealth, and which shifts up to $D'D'$ following the technological increase. Panel (iv) reflects assets weighted by the size of the cohorts. Due to the decline in survival with age the greatest share of the benefits is enjoyed by the 55 year old cohort.

Fig. 3.B illustrates the same exercise for the BBW demographic structure. It contrasts sharply, and is much less plausible, as a result of the convex survival function and the fact that agents may potentially live indefinitely (albeit with an arbitrarily low probability). For example, the perpetual upward slope of the assets accumulation locus EE in panel (iii) is unsatisfactory. However, with the dwindling cohort size the implications for distributions across cohorts, as illustrated in Panel (iv) is closer to the pattern implied by the more plausible BCL survival function.

Table 6.3. Structural Changes

		Demography		Economic Variables						
		L_{18}	n	\bar{k}	\bar{y}	\bar{k}/\bar{y}	\bar{r}	\bar{w}	\bar{c}	$[\Delta_B]^{-1}$
Baseline Model		78.38	1.00%	5.623	1.830	3.072	6.39%	1.190	1.493	0.053
Increase in productivity	$A \rightarrow 1.25$	78.38	1.00%	7.926	2.580	3.072	6.39%	1.677	2.104	0.053
Demographic Shocks										
Increase in the birth rate	$\beta \rightarrow 2.57\%$	78.38	1.50%	5.540	1.821	3.042	6.50%	1.184	1.461	0.054
Decrease in youth mortality	$\eta_0 \rightarrow 195$	93.98	1.50%	6.125	1.886	3.248	5.78%	1.226	1.488	0.049
Decrease in old age mortality	$\eta_1 \rightarrow 0.0443$	95.15	1.50%	6.211	1.895	3.278	5.68%	1.232	1.491	0.048
Off-setting change in birth rate and old age mortality rate	$\beta \rightarrow 2.53\%$ $\eta_1 \rightarrow 0.00551$	80.03	1.50%	5.623	1.830	3.072	6.39%	1.190	1.465	0.053

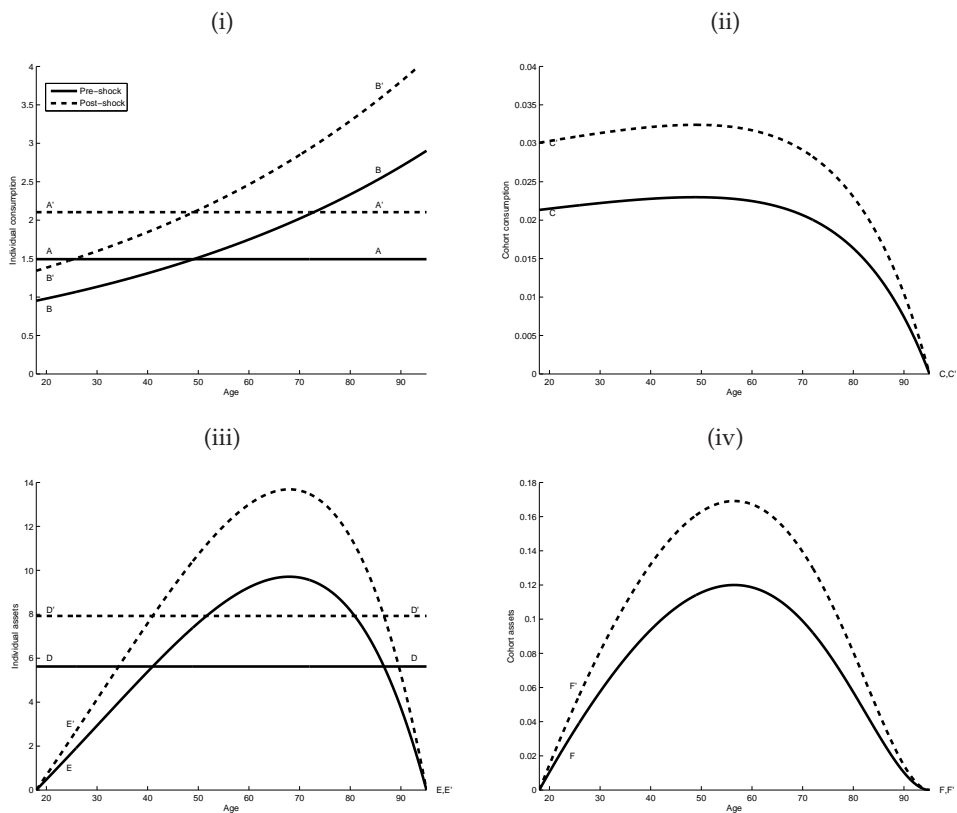


Figure 6.3. Increase in Productivity: BCL

6.5.2 Changes in the demographic structure

We contrast the impact of an increase in the population growth rate of 0.5 percentage points driven by either an increase in the birth rate, a decrease in mortality, or a combination of the two. Table 3 summarizes the various scenarios and shows how the economic consequences differ dramatically, depending on the source of the increase in the population growth rate.

Increase in the birth rate

In order to increase the population growth rate by 0.5 percentage point from 1.00% to 1.50% the birth rate must increase from 2.24% to 2.57%. Table 3, line 4 reveals that

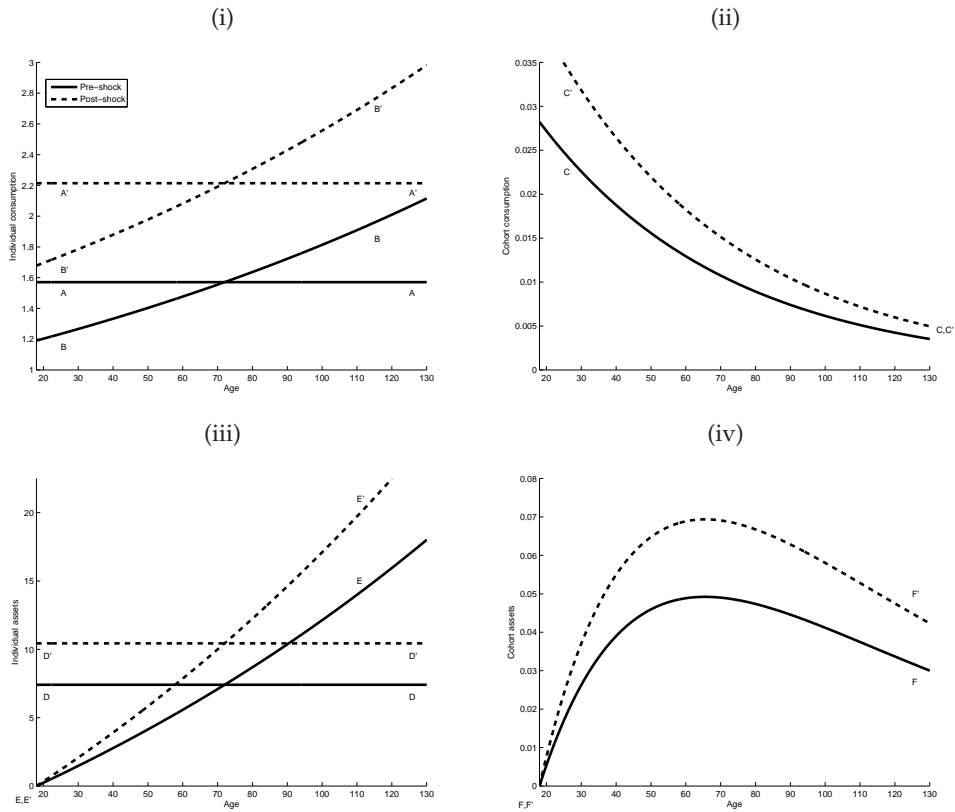


Figure 6.4. Increase in Productivity: BBW

this leads to a 1.48% reduction in the per capita capital stock (from 5.623 to 5.540). This is illustrated by the slight downward shift of the line DD in Fig. 5, Panel (iii). This response is consistent with the characterization of the steady state provided in Section 3 and the fact that the average age of wealth owners (52.65) exceeds that of workers (43.08). It is also consistent with the view emphasized by Kelley and Schmidt (1995) that an increase in the population growth rate resulting from a higher birth rate will have a negative effect on the level of economic activity. This is because it increases the relative number of young who have not accumulated any capital stock to contribute to the productive capacity of the economy. This reduction in aggregate assets accumulation has several consequences. It leads to a 0.5% reduction in the wage rate (from 1.190 to 1.184) and an increase in the rate of return on capital from 6.39% to

6.50%. It also leads to a 0.5% decline in per-capita output (from 1.830 to 1.821) and a 2.1% decline in per capita consumption (from 1.493 to 1.461), the latter being illustrated by the downward shift in the AA line in Fig. 5(i).

The distributional consequences are also modest, as Fig. 5 illustrates. The life cycle path for consumption, illustrated by BB in Panel (i), remains virtually unchanged. The slight reduction in the wage, with the anticipation of the future higher return to capital causes a very slight reduction in consumption at birth. However, the increase in the rate of return on capital increases the consumption growth rate over the life-cycle. Hence, toward the end of their life-cycle agents experience an increase in their consumption while average per capita consumption declines. The distributional consequences across cohorts are more substantial and in fact opposite to those experienced by individuals, as illustrated by the rotation of the CC curve to CC' in Panel (ii). Thus, the increase in the relative size of the younger cohorts, due to the higher birth rate, implies that they enjoy a larger share of the overall consumption, while the decline in the relative size of older cohorts means that their share of consumption declines, even though each surviving member's consumption level has increased.

The hump-shaped locus EE in Panel (iii), which reflects that the accumulation of assets over the life cycle shifts out, albeit slightly. This is a consequence of the increased rate of return on capital. Panel (iv) illustrates how, with the increase in the relative size of the young cohorts due to the higher birth rate, the share of wealth each existing cohort owns increases. This also explains why, even though at each age each individual has a slightly higher level of wealth, per capita wealth is nevertheless smaller. This is because with a higher birth rate a relatively larger share of the agents is young and as young agents possess relatively little capital, this leads to lower aggregate per-capita capital (see Panel (iii) locus DD and D'D').

6.5.3 Decrease in the mortality rate

The two alternative ways to increase the population growth rate from 1% to 1.5% are either to decrease youth mortality, η_0 , to 195 or old age mortality, η_1 , to 0.0443. As the economic consequences are similar, we restrict attention to the latter.

From Table 3 we see that this leads to a 10.46% increase in the per capita stock of

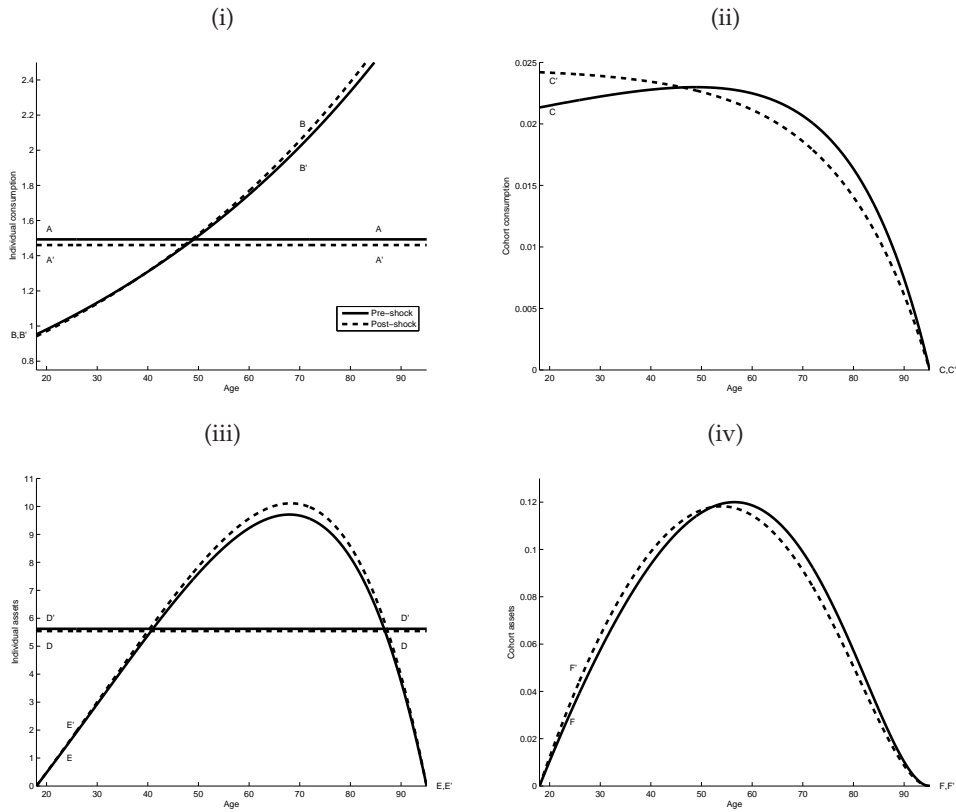


Figure 6.5. Increase in Birth Rate

capital (from 5.623 to 6.211). This is illustrated by the upward shift of the line DD in Fig. 6, Panel (iii). This response is consistent with the view emphasized by Kelley and Schmidt (1995) that an increase in the population growth rate resulting from a reduction in mortality will have a positive effect on the level of economic activity. This is because it increases the relative number of old people who have accumulated capital stock to contribute to the productive capacity of the economy. This increase in aggregate asset accumulation has several consequences. It leads to a 3.5% increase in the wage rate (from 1.190 to 1.232) and a decrease in the rate of return on capital from 6.39% to 5.68%. It also leads to a 3.5% increase in per-capita output (from 1.830 to 1.895) and a negligible (0.02%) decline in per capita consumption with the increased population, the latter being illustrated by the imperceptible shift in the AA line in Fig.

6(i).

The distributional consequences are illustrated in Fig. 6 and are seen to be non-monotonic. Panel (i) shows that the increase in the wage rate coupled with the anticipation of the future lower return to capital causes a slight increase in consumption at birth. However, the decrease in the rate of return on capital decreases the consumption growth rate over the life cycle. Hence, after a few years agents experience a decrease in their consumption and since this is the experience of most cohorts, average per capita consumption declines. In Panel (ii) we see that the increase in longevity and associated increase in old age cohorts, coupled with the upward shift and flattening of the BB curve, causes the CC curve to move out to $C'C'$. Thus, the increase in consumption of the very young causes their share of overall consumption to increase. However, the decline in the growth rate of consumption for people between around 30 and 80 causes their share of consumption to decline, while the increase in longevity leads to an increase in consumption share of the very old.

Panel (iii) reveals that the increase in longevity causes the EE locus to shift up and to the right. In early stages the life cycle the rate of asset accumulation declines very slightly, reflecting the decline in the rate of return on capital. As a result, the decline in mortality causes relatively young agents' wealth to decline slightly. However, the increase in longevity induces them to save for a longer period and to accumulate more assets in light of their increased longevity. Finally, Panel (iv) illustrates how the increase in the relative size of old cohorts tilts the share of wealth significantly in their direction.

These patterns are consistent with the empirical evidence. For example, the fact that consumption declines for all but the youngest cohorts, while the wealth of older agents increase is consistent with the empirical findings of Fair and Dominguez (1991), Attfield and Cannon (2003), and Erlandsen and Nymoen (2008) all of whom find that the effect of an aging population is to lead to a decline in overall per capita consumption for all equivalent income levels. The pattern we obtain of asset accumulation increasing with life expectancy agrees with the findings of Bloom, Canning, and Graham (2003).

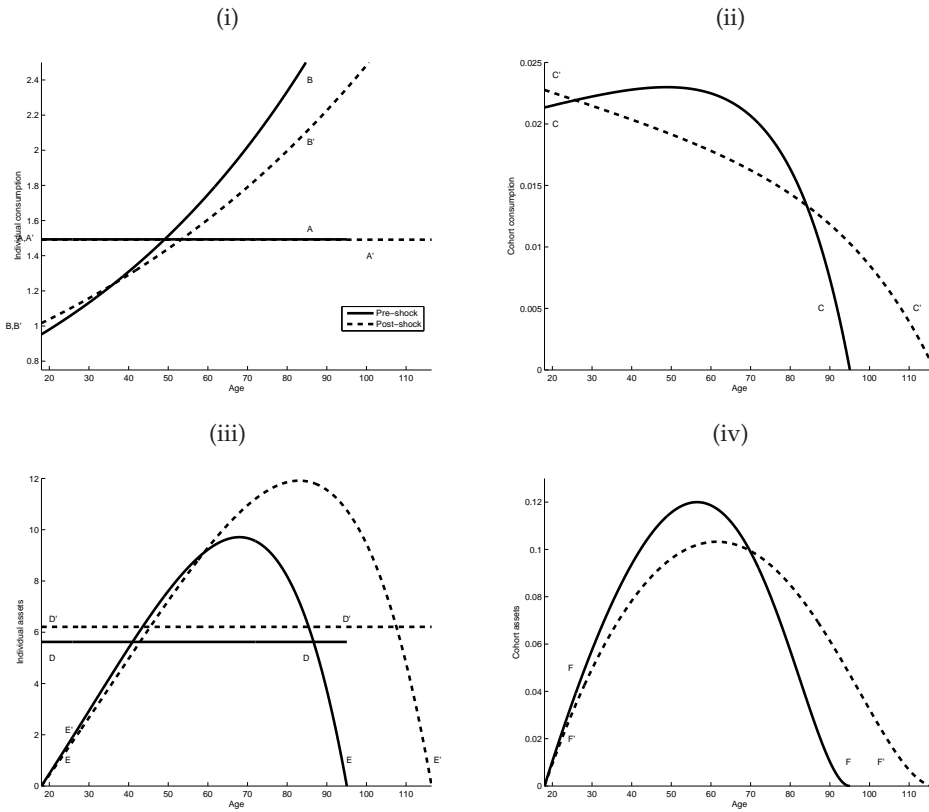


Figure 6.6. Decrease in Old Age Mortality

6.5.4 Increase in birth rate versus decrease in mortality

Comparing Figs. 4 and 5 we see that achieving a specified increase in the population growth rate by increasing the birth rate or decreasing the mortality rate has dramatically different consequences for the economy. First, whereas only a mild increase in the birth rate of 0.33% will raise the population growth rate by 0.5%, to achieve the same objective by reducing mortality would require increasing longevity by around 17 years, which would seem to be a much more formidable task. Second, whereas a 0.5 percentage point increase in the population growth rate resulting from an increase in births will have only a slight negative effect on the productive capacity of the economy (measured by its per capita capital stock), the same increase in the population

growth rate brought about by reduced mortality will have a significant expansionary effect. This contrast in magnitudes agrees exactly with the empirical results obtained by Blanchet (1988), thus emphasizing the importance of the form in which population growth occurs.

Finally, our results can be reconciled with the cross-country empirical evidence cited by Kelley and Schmidt (1995) who found that, whereas population growth had had a negligible effect on growth during the 1960s and 1970s, it had a negative effect in the 1980s. This can be explained by comparing line 7 of Table 3 with line 4. Increasing the birth rate to only 2.53% and reducing old age mortality to 0.0551 causes the economic effects to be largely offsetting so that the per capita capital stock, output, wage rate, return to capital all remain unchanged. In summary, the changing mix between increased birth rate and decreased mortality can very naturally account for the different empirically estimated long-run effects of population growth rates at different stages of development.

6.6 Conclusions

This chapter has introduced a realistic age-dependent demographic structure into a neoclassical growth model for a closed economy. In doing so, we have had two primary objectives. The first is to provide a general characterization of how the demographic structure impedes on the macrodynamic equilibrium. We show how this depends on the generational turnover term, which is an integral component of the intertemporal consumption allocation decision. Setting up the aggregate dynamics as a generalization of the conventional neoclassical growth model, provides two major insights. Not only does it enable us to view alternative demographic specifications in a unified way, but also we are able to identify two, rather than just one, steady-state equilibria. The first is highly sensitive to the demographic structure, whereas in the second equilibrium demographic factors play but a minor role. However, in the absence of intergenerational transfers, the latter is not relevant, and therefore has not been considered further.

The second objective is to analyze the effect of structural changes – most important demographic structural changes – on both the aggregate macro equilibrium, as well

as the distributional life-cycle implications. This is done numerically using the very general survival function proposed by Boucekkine, de la Croix, and Licandro (2002). The most striking result is the sharp contrast, both qualitatively and quantitatively, in the effects of changes in the population growth rate on the macro economy. Whether an increase in population occurs because of an increase in births or a decrease in mortality is crucially important, and in this regard our results corroborate the empirical findings obtained in the demographic literature.

While this chapter is mainly theoretical and quantitative, it clearly can be extended in various directions. First, the contrast between births and mortality in influencing the population growth rate and the resulting consequences for distribution across cohorts and for the aggregate economy raises interesting policy issues for a country seeking to influence its population growth rate. Second, it is straightforward to extend the framework to allow for retirement and to address issues pertaining to social security and retirement benefits, issues that are of crucial importance for the US and other countries with their ageing populations. Finally, while we have focused on the long-run (steady-state) implications of demographic structural changes, the nature of the transition from one steady-state to another is also important.