The microeconomics of strategic activism

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Chapter 3

Information Provision by Interest Groups

3.1 Introduction

Strategic activism has become a nonnegligible factor in many markets. Interest groups are increasingly able to influence consumers' behaviour via various means. Informing consumers about the detrimental aspects of certain production practices has proven to be a widespread instrument of such *private politics* (Baron, 2001). The use of tropical timber, child labour, lab animals, and toxic substances are just a few examples of externalities interest groups bring to the attention of consumers.

Such information provision by interest groups (IGs) raises a number of questions. Which industries are susceptible to this practice? How does an IG's information acquisition and dissemination choices compare to the socially optimal amount of information acquisition and dissemination? To what extent affect the IG's actions the level of competition on a market?

We develop a framework which sheds light on these issues. In our model an IG, whose objective is to minimize the total 'damage' associated with a given industry (say the total amount of pollution caused by the industry), can exert costly efforts to investigate the production practices of firms. It subsequently shares its findings with consumers. The IG's incentives to provide this product information originate from the fact that consumers cannot determine the level of pollution associated with a specific brand when they consume this good, i.e. pollution is a *credence* attribute of the products supplied by the industry. Because consumers do care about pollution, any information provided by the IG regarding pollution associated with a given product affects the attractiveness of this product relative to the attractiveness of the products of competing firms. Crucially, because the number of firms is limited and products are horizontally differentiated, firms wield some market power. Firms can thus try to mitigate any negative impact the IG's actions might have on profits by adjusting the prices they charge.

A relatively 'dirty' firm's equilibrium pricing adjustment only partially offsets the loss in sales caused by the IG's revelation of the firm's damaging practices. Consequently, the IG's information provision does reduce the damage caused by this firm. In contrast, a relatively 'clean' firm cannot fully reap the rewards from the revelation of its best practices by the IG because of those price adjustments. Of course, the extent of such price adjustments depends on the market power of individual firms. The IG is therefore most inclined to investigate firms in industries with many firms and a low level of product heterogeneity.

If the IG's investigations were to reveal only small differences in pollution levels across firms, then the impact of the IG's actions on aggregate industry pollution would be limited. The variability of the pollution levels across firms in a given industry therefore plays an important role in the IG's decision whether or not to investigate firms in this industry: a larger variability makes the industry a more attractive target.

Clearly, consumers benefit from any product information supplied by the IG. More surprisingly, expected industry profits increase should the IG investigate some firms. The reason is that the IG's information effectively creates vertical product differentiation, thereby relaxing competition and hence increasing profits (compare Shaked and Sutton, 1982). In other words, if consumers do not receive any information regarding the pollution levels of individual firms, then consumers cannot distinguish clean from dirty products and firms are unable to make use of such hidden quality differences. The IG thus reduces the level of competition on the market.

The results discussed in the previous paragraph suggest that the IG's information provision improves total welfare. Indeed, in most cases it does. Only if the IG's costs of information acquisition and dissemination are substantial can total welfare deteriorate due to the IG's actions and do the choices of the IG diverge from the choices of a social planner who has the same information disclosure technology at her disposal. Since aggregate pollution reacts stronger to information regarding pollution levels than consumers' surplus and industry profits combined does, the IG overprovides information if the total costs of investigations are close to the total benefits of investigations as perceived by the IG. Whether or not one should simply add consumers' surplus, industry profits, and the IG's utility (commensurate to monetary units) to arrive at the 'right' measure of total welfare is debatable. It could be that the IG represents (some of) the consumers. In that case, adding the three payoffs would entail counting the benefits of having less pollution accruing to those consumers more than once. Alternatively, it could be that the IG represents the interests of people who are not active on the market, but who are affected by the pollution caused by the industry. To circumvent these issues we allow the weight attached to aggregate pollution in our total welfare measure to vary. It turns out that total welfare unambiguously increases if the overlap between the IG and the consumers is not too large. Since our setting can be readily translated to situations in which a public party contemplates imposing a quality labelling system, e.g. eco-labels, on an imperfectly competitive industry, these welfare results provide clear policy recommendations regarding the use of such quality labelling.

Information provision by IGs is not confined to environmental groups informing consumers about pollution. For instance, consumers can visit the website of the American environmental organization Environmental Defense Fund to learn about the health risks associated with eating various species of fish.[1] Consumer groups, to give another example, regularly test a wide range of products. See, for instance, the website of "International Consumer Research & Testing".[2] Lastly, we mention the labels on product packages indicating energy usage, eco-friendliness, 'fairness of trade', and numerous other attributes of the products involved. A significant fraction of these labels are approved by interest groups. Our analysis readily applies to these instances of information provision.

Disclosure of product information by sellers of those goods has been extensively studied, starting with Grossman (1981) and Milgrom (1981). These authors show, using an unravelling argument, that sellers disclose the quality of their product (even if quality is low) if this is costless. Jovanovic (1982) shows that sellers endowed with a single unit of a good disclose their good's quality only if this quality is sufficiently high if information provision is costly. Contrary to the present work, Jovanovic's analysis does not incorporate competition among sellers.[3] With (Bertrand) competition a similar partial unravelling result obtains as shown by Cheong and Kim (2004). Cheong and Kim also show that the probability that a given seller discloses its quality in equilibrium decreases as the number of sellers increases. This contrasts with our result that information provision by an IG becomes *more* likely as the number of firms increases. We borrow from this strand of the literature the assumption that any information provided is truthful and credible. However, we assume that

[1] See `http://www.oceansalive.org/eat.cfm?subnav=healthalerts`.
[2] See `http://www.international-testing.org/index.html`.
[3] The number of potential buyers is large compared to the number of sellers in his model.

firms do not send any messages. Equivalently, messages of firms regarding damage levels are not credible. The firms are consequently unable to alleviate the market failure stemming from imperfect information.

Several authors have studied the impact information provision by third parties has on market outcomes. The literature on eco-labelling indicates when labels issued by third parties can induce firms to invest in cleaner technologies. See, for instance, Mason (2006) and its references. Kennedy *et al.* (1994), using a framework with perfect competition, were among the first to argue that public information provision can correct market failures caused by pollution. Petrakis *et al.* (2005), who study pollution externalities created in a duopoly context, compare public information provision with corrective taxes. They find that a policy which combines these two instruments is optimal. More closely related to the present work are Feddersen and Gilligan (2001) and Heijnen (2007). In Feddersen and Gilligan (2001) the prospect of an activist informing consumers about their production practices can induce duopolists to choose a 'clean' technology instead of a 'dirty' technology. Heijnen (2007) studies the signalling equilibria which can arise if an environmental group tries to reveal the environmental attributes of a monopolist's product. Since signalling is costly, the environmental group would be better off if it were able to commit to inactivity, although its interference can increase social welfare. Except for Feddersen and Gilligan (2001) and Petrakis *et al.* (2005), who study duopolies, either a monopoly or a perfectly competitive market is investigated in the above work. We, however, link the incidence of third party information provision to market structure. To our knowledge, this topic has not yet been examined in the literature.

Crawford and Sobel (1982) consider strategic information transmission from an informed party to an uninformed party whose subsequent action determines the payoffs of both parties. They find that the informed party only partially reveals her private information should the interests of the parties not be perfectly aligned. We extend our basic model to allow for such imprecise information provision on the part of the IG. This does not affect our results qualitatively. In fact, from an *ex ante* perspective all parties involved prefer more precise information to less precise information. This result is in line with the welfare implications discussed above.

We use a model which allows for both horizontal and vertical product differentiation. This model was introduced by Economides (1993) and builds on the circular competition model of Salop (1979). Economides (1993) derives the symmetric equilibria of games in which the firms choose their location (the horizontal product characteristic) before qualities and prices are chosen.[4] Symmetry ensures that,

[4]Economides (1993) considers two different games. In the first variant, qualities and prices are

in equilibrium, neither locations nor qualities affect prices. As a consequence, the equilibrium prices are the familiar Salop prices. In our setting, firms offer different qualities in equilibrium if the IG has informed consumers about damages. We thus have to study the properties of asymmetric equilibria. In a recent paper, Vogel (2008) models endogenous horizontal and vertical differentiation with heterogeneous firms. Vogel (2008) shows that more efficient firms are more isolated in product space. In contrast to our model, Vogel considers a model with complete information.

The remainder of this chapter is organized as follows. The basic model is presented in Section 3.2. We derive the market equilibrium (i.e. equilibrium prices and quantities) in Section 3.3. The IG's equilibrium strategy can be found in Section 3.4. Section 3.5 contains welfare results. The extension of the model in which the IG provides consumers with imprecise information about damage levels is given in Section 3.6. Section 3.7 concludes. Proofs are relegated to the appendix.

3.2 The Model

We consider a market in which $n \geq 2$ firms offer horizontally differentiated products to a continuum of consumers with total mass M. A firm j offers precisely one product, which we call good j, $j \in \{1, \ldots, n\} =: N$. Each good is located on a circle of unit length as in Salop (1979). Just as in the aforementioned work, we assume that the goods are equally spaced around the circle, whereas the consumers are distributed uniformly on this circle.[5] Production of each good exhibits constant returns to scale, allowing us to normalize the costs of all firms to zero.

Each consumer has a reservation price v for his ideal brand. A consumer either buys one unit of his most preferred good or refrains from buying. We normalize the utility of the latter option to zero. The willingness-to-pay for a certain good decreases linearly at a rate t with the distance between a consumer and that good. Moreover, a consumer's utility derived from buying good j depends on the production of some bad associated with this product. This bad might entail purely privately experienced disutility, as is the case with health risks associated with the consumption of certain goods. Alternatively, the bad might be a public bad, for instance the emission of pollutants during production. In the latter case, the consumers' disutility associated with the bad represents the level of guilt felt by consumers for (indirectly) causing the externality. Of course, a mixture of these two extremes also fits the current framework. We denote consumers' expectation of the amount of the bad produced

chosen simultaneously. In the second variant, qualities are chosen prior to the pricing stage.

[5]So, the distance on the circle between good j and the adjacent good $j+1$ is $\frac{1}{n}$.

per unit of production of good j with s_j. We assume that s_j enters a consumer's utility linearly. The net utility a consumer located at distance x from good j derives from buying good j therefore reads[6]

$$u(x, j) = v - tx - p_j - s_j, \tag{3.1}$$

where p_j is the price firm j charges for its good. Note that differences between the various s_j's effectively create vertical product differentiation between the associated goods.

The true amount of the bad firm j produces per unit of good j, d_j, is unknown to consumers. However, the d_j's are known to be independent random draws from some continuous distribution with support $[0, D]$, where $D > 0$. The mean of this distribution is μ and the variance is σ^2. Thus, without any information regarding d_j, s_j equals μ. For ease of exposition we refer to d_j as the damage associated with good j and to s_j as the expected damage associated with good j. Importantly, the damage level d_j is exogenous to firm j.

For the sake of tractability we assume that the parameters of the model are such that, in equilibrium, the market is covered (i.e. all consumers buy) and that all firms are active, irrespective of consumers' expectations regarding damages. This implies that one can find numbers $x_j \in [0, \frac{1}{n}]$, $j \in N$, which indicate the locations of the *indifferent consumers*. Specifically, x_j is the distance between good j and the consumer located in between good j and good $j+1$ who is indifferent between buying good j and buying good $j+1$.[7] The indifferent consumer x_j (weakly) prefers buying one of these two goods to not buying any good. Of course, x_j depends on prices and expected damages. We present conditions which ensure that the market is covered and that all firms are active in the next section.

An interest group (IG) can investigate the production practices of the firms. If the IG opts to investigate k firms, then it learns the true amount of damages caused by these k firms. Investigating k firms costs $c(k)$, where $c(\cdot)$ is an increasing, concave function with $c(0) = 0$. Concavity of $c(\cdot)$ means that investigating one more firm becomes less costly the larger the number of firms that have already been investigated. After having concluded its investigations (if any) and prior to the market stage the firm informs the consumers about these damages via some campaign. For instance, the IG contacts several media outlets which subsequently

[6] One can easily incorporate consumer heterogeneity with respect to the disutility associated with the bad by multiplying s_j with a consumer-specific preference parameter θ. As long as differences in the θ's of the various consumers are not too large and the distribution of θ is identical at each point on the circle, the results will not be affected qualitatively.

[7] To save on notation we sometimes call firm 1 firm $n+1$ and firm n firm 0.

pay attention to the IG's findings. We suppose that this campaign is such that it reaches every single consumer and that the IG credibly and truthfully disseminates all information it has acquired during its investigations. The firms also observe the IG's messages. The IG thus informs consumers and firms about the true d_j's of investigated firms. Because the IG's campaign is credible, $s_j = d_j$ if the IG has investigated firm j. If the IG has not investigated firm j, then consumers do not receive any information pertaining to d_j and thus $s_j = \mu$. In Section 3.6 we generalize the IG's advertising technology, allowing it to be imprecise about its findings.[8]

The IG's objective is to minimize the expected aggregate damage plus its total costs by choosing which firms, if any, to investigate, i.e. it solves[9]

$$\min_{K \subseteq N} G(K) = \min_{K \subseteq N} E\Big(\sum_{j=1}^{n} d_j q_j(\boldsymbol{s}_K) + c(|K|) \Big) = \min_{K \subseteq N} E(\boldsymbol{d}'\boldsymbol{q}(\boldsymbol{s}_K)) + c(|K|), \quad (3.2)$$

where q_j, the quantity sold by firm j, depends on prices as well as on the vector of consumers' expectations $\boldsymbol{s}_K$. The j^{th} entry of this vector equals the true value d_j if $j \in K$ and it equals μ if $j \notin K$. The elements of K are the firms which are investigated/targeted by the IG, $|K|$ is the number of firms investigated by the IG. We call K the IG's investigation strategy. Because the IG has to decide on K before it learns any d_j's, the expectation is taken with respect to all damages d_j, $j \in N$.

We stress that choosing K does not boil down to simply choosing a number. There are various ways in which the IG can target, say, three firms. It can choose three adjacent firms, three firms equally spaced around the circle, two neighbouring firms and one firm at the other side of the circle, and so forth. So, the program (3.2) also entails choosing a configuration of firms around the circle.

Note that since we assume that marginal costs are independent of the level of damage, there is no scope for signalling. Thus whether or not firms know their own d_j is inconsequential for our results. These assumptions regarding firms are presumably met in the Environmental Defense fund case regarding fish mentioned in the introduction as well as various other cases in which the bad can be identified with a health risk.[10] If the bad stems from only one out of a large number of inputs, then these assumptions are reasonable: the one input that is causing damage accounts for only a small part of the production costs.[11] Note that, since the firms observe

[8] We sometimes call the IG's campaign an advertising campaign and the associated technology an advertising technology to fix ideas.

[9] We use bold symbols to indicate vectors and a prime ($'$) to indicate the transpose of a vector or matrix. For instance, $\boldsymbol{d} = (d_1, \ldots, d_n)'$. The dimension(s) of vectors and matrices are omitted if there is no risk of confusion.

[10] Most fisherman will not know that spotted seatrouts are likely to contain quite some PCBs and mercury, as the Environmental Defense Fund claims.

[11] Think of toys coated with paint containing lead. The paint accounts for only a fraction of the

the IG's advertisements, they know consumers' expectations $\boldsymbol{s}_K$ regarding $\boldsymbol{d}$.

Our assumptions regarding the advertising technology avoid signalling by the IG. If the IG were allowed to include any possible message in its advertisements (i.e. any element of $[0, D]^n$), then it would have an incentive to embroider its findings by understating the damages of relatively 'clean' products and exaggerating the damages of relatively 'dirty' firms. However, in most countries legal restrictions pertaining to advertising are such that an IG is not allowed to misrepresent its findings. Similarly, an IG cannot post misleading information about a firm without facing legal consequences. Lastly, an IG often relies on independent research when informing consumers about the production practices of firms. So, the assumptions of truthful and credible information provision are reasonable.

We study the following three-stage game of incomplete information. After Nature has drawn the vector of damages $\boldsymbol{d}$, the IG chooses which firms to investigate during stage one. After learning the true damages associated with the investigated firms, it informs consumers (and firms) about its findings. In stage two the firms engage in price competition knowing $\boldsymbol{s}_K$. In the last stage each consumer decides which good to buy.

We look for *subgame perfect Nash equilibria* of this game (simply called 'equilibria'). Equilibrium strategies must be best responses given the history of play and given expectations. An equilibrium consists of a first period investigation strategy K^*, a pricing rule $\boldsymbol{p}^*(\boldsymbol{s})$, and a vector of indifferent consumers $\boldsymbol{x}^*(\boldsymbol{p}, \boldsymbol{s})$.

The pair $(\boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{x}^*(\boldsymbol{p}, \boldsymbol{s}))$ constitutes a *market equilibrium* (associated with the expectations $\boldsymbol{s}$). We determine such market equilibria in the next section before we discuss the equilibrium behaviour of the IG in Section 3.4.

3.3 Market Equilibria

We calculate the equilibrium choices $\boldsymbol{p}^*(\boldsymbol{s})$ and $\boldsymbol{x}^*(\boldsymbol{p}, \boldsymbol{s})$ assuming that every consumer buys, i.e. the market is covered, and every firm sells a nonnegative quantity at a nonnegative price, i.e. all firms are active. Below we present conditions (see Condition 3.1 and Condition 3.2) which ensure that the market equilibria do indeed exhibit these properties. Since the derivation of the equilibrium with two firms is standard, the calculations pertaining to the duopoly case are relegated to the appendix.

Let $n \geq 3$ be given. We first derive the demand function for a brand $j \in N$ given its own price, the prices of its two neighbouring brands, and the expectations s_{j-1}, s_j, and s_{j+1}. The location of the indifferent consumer x_j solves $v - p_j - tx_j - s_j =$

total production costs of the toys.

$v - p_{j+1} - t\left(\frac{1}{n} - x_j\right) - s_{j+1}$. This leads to:

$$x_j = \frac{1}{2n} + \frac{1}{2t}\left(p_{j+1} - p_j + s_{j+1} - s_j\right). \tag{3.3}$$

Since $q_j = M\left(x_j + (\frac{1}{n} - x_{j-1})\right)$, the demand function for good j reads

$$q_j = \frac{M}{n} + \frac{M}{2t}\left(p_{j+1} - 2p_j + p_{j-1} + s_{j+1} - 2s_j + s_{j-1}\right). \tag{3.4}$$

Firm j thus considers the following first order condition to maximize its profit ($\pi_j = p_j q_j$):

$$p_{j+1} - 4p_j + p_{j-1} + s_{j+1} - 2s_j + s_{j-1} + \frac{2t}{n} = 0.$$

Gathering the first order conditions of all firms results in the system of equalities which determines the second stage equilibrium prices $\boldsymbol{p}^* = \boldsymbol{p}^*(\boldsymbol{s})$:

$$-(I - A)\boldsymbol{p}^* + \frac{t}{2n}\boldsymbol{\iota} - \frac{1}{2}(I - 2A)\boldsymbol{s} = 0, \tag{3.5}$$

where $\boldsymbol{\iota}$ is a vector of ones of length n, I is the $n \times n$ identity matrix, and

$$A := \begin{pmatrix} 0 & \frac{1}{4} & 0 & \dots & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \dots & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \dots & \frac{1}{4} & 0 \end{pmatrix}$$

is a symmetric $n \times n$ matrix. The subdiagonal and superdiagonal entries of this matrix as well as its top-right and bottom-left entry are $\frac{1}{4}$. All other entries of A are 0. To get more concise expressions for the equilibrium values we introduce the following $n \times n$ matrix:[12]

$$\Gamma = \Gamma(n) := (I - A)^{-1}A = A(I - A)^{-1}. \tag{3.6}$$

We now have the tools to present the strategies that support the market equilibrium in a concise manner:

Proposition 3.1 *Suppose the relevant covered market condition and active firms condition (see below) hold. Then the unique equilibrium prices are:*

$$\boldsymbol{p}^*(\boldsymbol{s}) = \frac{t}{n}\boldsymbol{\iota} - \frac{1}{2}(I - \Gamma)\boldsymbol{s}. \tag{3.7}$$

The associated (second stage) equilibrium quantities equal:

$$\boldsymbol{q}^*(\boldsymbol{s}) = \frac{M}{n}\boldsymbol{\iota} - \frac{M}{2t}(I - \Gamma)\boldsymbol{s} = \frac{M}{t}\boldsymbol{p}^*(\boldsymbol{s}), \tag{3.8}$$

[12]Existence of $(I-A)^{-1}$ follows from the fact that $I-A$ has a dominant diagonal (see for instance Takayama, 1985, pp. 381-382). Γ inherits symmetry from A.

where $\Gamma(n)$, $n \geq 3$, *is defined in (3.6) and* $\Gamma(2) := \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Proof. See the appendix. ■

If the IG investigates none of the firms, then $\boldsymbol{s} = \mu\boldsymbol{\iota}$ and consumers thus do not perceive the goods to be vertically differentiated. Without vertical product differentiation the standard Salop (1979) equilibrium prevails:

Corollary 3.1 *Suppose* $v \geq \frac{3t}{2n}$. *If the IG investigates none of the firms* ($K = \emptyset$)*, then each firm sells* $\frac{M}{n}$ *units at a price* $\frac{t}{n}$ *in the market equilibrium.*

Proof. See the appendix. ■

We now provide conditions which ensure that, in equilibrium, the market is covered and all firms are active. See the appendix for proofs of these claims.

Condition 3.1 (Covered Market) *The gross willingness-to-pay is sufficiently large, specifically* $v \geq \frac{3t}{2n} + D$.

Condition 3.2 (Active Firms) *The maximal damage per unit of production is not too large, specifically* $D \leq \frac{t}{n}$.

In the proof associated with Condition 3.2 we show that the diagonal elements of $I - \Gamma$ are positive, whereas its off-diagonal elements are negative, implying that firm j's equilibrium price and quantity depend negatively on s_j, but positively on s_i, $i \neq j$. Thus the IG's actions induce shifts in demand from relatively damaging, targeted firms to less damaging or untargeted firms. In the next section we show that expected aggregate damage decreases in the number of investigated firms. Moreover, since the smallest amount of expected aggregate damage is associated with perfect information (i.e. $\boldsymbol{s} = \boldsymbol{d}$), the IG has an incentive to investigate as many firms as possible.

3.4 The Interest Group's Strategy

Fix an investigation strategy $K \subseteq N$. *Ex post* the true aggregate damage associated with K equals

$$\boldsymbol{d}'\boldsymbol{q}^*(\boldsymbol{s}_K) = \frac{M}{n}\boldsymbol{d}'\boldsymbol{\iota} - \frac{M}{2t}\boldsymbol{d}'(I - \Gamma)\boldsymbol{s}_K. \tag{3.9}$$

In the first stage the IG thus minimizes

$$E\Big(\frac{M}{n}\boldsymbol{d}'\boldsymbol{\iota} - \frac{M}{2t}\boldsymbol{d}'(I - \Gamma)\boldsymbol{s}_K\Big) + c(|K|)$$

with respect to K. The first part of this expression is the *ex ante* expected aggregate damage. This expectation equals

$$\frac{M}{n}n\mu-\frac{M}{2t}E\Big(\sum_{j\in K}d_j^2+\sum_{j\notin K}\mu d_j\Big)+\frac{M}{2t}E(\boldsymbol{d}'\Gamma\boldsymbol{s})=M\mu-\frac{M}{2t}(n\mu^2+|K|\sigma^2)+\frac{M}{2t}E(\boldsymbol{d}'\Gamma\boldsymbol{s}).$$

In the appendix we calculate the last part of the above expression, leading to the following result:

Proposition 3.2 *Suppose Condition 1 and Condition 2 hold. If the IG chooses investigation strategy K, then the expected equilibrium aggregate damage equals* [13]

$$M\mu-\frac{M|K|\sigma^2(1-\gamma)}{2t}, \tag{3.10}$$

where $\gamma=\gamma(n)\in(0,\frac{1}{2})$ is the generic diagonal element of $\Gamma(n)$.

Proof. See the appendix. ■

Surprisingly, the reduction in expected aggregate damage caused by the IG's actions (that is, the amount $\frac{M|K|\sigma^2(1-\gamma)}{2t}$) does only depend on the number of investigated firms, not on the distribution of the investigated firms around the circle. For instance, investigating two neighbouring firms has the same impact on expected aggregate damage as targeting two firms which are far apart. This result stems from some key assumptions of the model: expected damages enter the utility function (3.1) linearly, the damages are independent and identically distributed draws, and the covered market assumption. Because utility is linear in damages, the equilibrium sales of firm j decrease linearly in s_j and increase linearly in the damages of its competitors. Consequently, the realized aggregate damage is linear in $\boldsymbol{s}_K$ as can be gathered from (3.9). Since damages are uncorrelated, this in turn implies that expected aggregate damage is linear in the number of investigated firms $|K|$. Intuitively, for prices held fixed the *shift* in the location of the indifferent consumer x_j caused by the revelation of d_j is not affected by whether or not consumers learn d_{j+1} (see (3.3)). So, with fixed prices the reduction in aggregate damage associated with informing consumers about d_j does not depend on information regarding d_{j+1} or any other damage level. Because equilibrium prices are linear in the vector of damages, this result remains valid with endogenous prices.

It should be noted that the fact that a consumer's utility depends linearly on the damage associated with the product he buys is not a strong assumption. As long as

[13] Of course, the expected aggregate damage must be positive. In the appendix we demonstrate that this quantity is indeed positive.

the distribution from which the d_j's are drawn has full support (i.e the associated density is strictly positive on $(0, D)$), any monotonic relation between damage and utility fits the current framework. An appropriate transformation of the damages leads to the specification (3.1).

Since the reduction in expected aggregate damage is linear in the number of firms the IG investigates while the cost function $c(\cdot)$ is concave, the IG cannot find an interior solution to its minimization problem (3.2). Instead, the IG uses the following all-or-nothing strategy:

Proposition 3.3 *Suppose Condition 1 and Condition 2 hold. Then in equilibrium the IG opts to investigate every single firm in the industry ($K^* = N$) if and only if*

$$\frac{M\sigma^2(1-\gamma)}{2t} \geq \frac{c(n)}{n}. \tag{3.11}$$

If this inequality fails to hold, then the IG investigates none of the firms ($K^* = \emptyset$).

Proof. Obvious. ■

Proposition 3.3 implies that only two kinds of equilibria can emerge. If (3.11) fails to hold, then the IG refrains from investigations and the standard Salop equilibrium obtains. However, if (3.11) does hold, i.e. the reduction in expected aggregate damage per investigated firm exceeds the average cost per firm if all firms are investigated, then the IG inspects the production practices of each firm and subsequently informs consumers about its findings. This leads to a reduction in expected aggregate damage of $\frac{Mn\sigma^2(1-\gamma)}{2t}$.

A closer look at the expression $\frac{Mn\sigma^2(1-\gamma)}{2t}$ reveals the following comparative statics results:

Proposition 3.4 *Suppose Condition 1 and Condition 2 hold. Then the IG becomes more inclined to inform the consumers (i.e. condition (3.11) is more easily met) if:*

- *the total mass of consumers M increases,*
- *the IG can investigate firms more efficiently (i.e. $c(\cdot)$ is replaced by some $\tilde{c}(\cdot)$ such that $\tilde{c}(k) \leq c(k)$, $k = 1, \ldots, n$),*
- *the variance of the per-firm damage σ^2 increases,*
- *the level of product heterogeneity t decreases,*
- *the number of firms n increases.*

Proof. See the appendix. ■

Clearly, if more consumers intend to buy one of the products or if investigating firms becomes easier, then investigating firms becomes more lucrative for the IG. As the variance of the per-firm damage σ^2 increases, it becomes more likely that the IG encounters an 'extreme' damage level (either one close to 0 or one close to D) when it investigates a firm. Informing consumers about an extreme damage level leads to a significant shift in demand and consequently to a considerable reduction in aggregate damage, whereas informing consumers about a damage level close to the mean μ has only a small impact on demand and thus on aggregate damage. The IG is thus more inclined to investigate firms if σ^2 is relatively large. As the level of product heterogeneity t decreases, the competition between firms becomes fiercer. Specifically, firms will exploit any differences in damage levels more aggresively (a glance at (3.7) reveals that the part of $\boldsymbol{p}^*$ which depends on $\boldsymbol{s}$ becomes relatively more important as t decreases), leading to larger differences in equilibrium sales between 'clean' and 'dirty' firms and thus a stronger incentive of the IG to investigate firms. As n increases the supply side of the (circular) market becomes more crowded and an individual firm consequently wields market power over a smaller part of the market. This implies that a relatively dirty firm loses a larger fraction of its customers to adjacent, relatively clean firms the larger the number of firms is. The reduction in expected aggregate damage per firm $\frac{M\sigma^2(1-\gamma(n))}{2t}$ therefore increases in n.

3.5 Welfare Analysis

Assuming risk neutrality of all parties involved, allows us to focus on ex ante expected surpluses. Simply adding consumers' surplus, industry profits, and the IG's disutility $-G(K)$ might not lead to a reasonable measure of total welfare. The reason is that the IG might to some extent represent (some of) the consumers. If so, simply adding the aforementioned surpluses would amount to putting too much weight on the expected disutility $E(\boldsymbol{d}'\boldsymbol{q}(\boldsymbol{s}_K))$ experienced by consumers. However, if the IG represents outsiders, i.e. agents not participating in the market, then this problem would not arise. The first situation is more likely if production of the bad results in purely privately experienced disutility, whereas the second possibility is more relevant for situations with public bads. We incorporate both possibilities and any intermediate situation in the subsequent analysis by varying the weight attached to the disutility component. Specifically, we look at the following total expected welfare measures $E(W(K,\alpha))$:

$$E(W(K,\alpha)) = E\big(CS(K) + \Pi(K) - \alpha \boldsymbol{d}'\boldsymbol{q}(\boldsymbol{s}_K)\big) - c(|K|), \tag{3.12}$$

where CS is aggregate consumers' surplus, Π is total industry profits, E is the appropriate expectations operator, and $\alpha \in [0,1]$ measures the extent of overlap between consumers and the IG. A larger α means less overlap: the less overlap the more weight must be put on the term $E(\boldsymbol{d}'\boldsymbol{q}(\boldsymbol{s}_K))$. We calculate the various components of $W(K,\alpha)$ in turn, starting with aggregate consumers' surplus. Again, we focus on situations with at least three firms. The duopoly analysis can be found in the appendix.

Consider the surplus of the consumers who buy good j. Consumers located at distance 0 of good j enjoy a surplus of $v - p_j - s_j =: w_j$ from consuming this good. Consumers who are indifferent between buying good j and buying good $j+1$ enjoy a surplus of $w_j - tx_j$, whereas consumers who are indifferent between buying good j and buying good $j-1$ enjoy a surplus of $w_j - t\big(\frac{1}{n} - x_{j-1}\big)$. The total consumers' surplus generated by sales of good j therefore equals

$$\begin{aligned}&M\Big(\frac{w_j + w_j - tx_j}{2} \times x_j + \frac{w_j + w_j - t(\frac{1}{n} - x_{j-1})}{2} \times (\tfrac{1}{n} - x_{j-1})\Big) = \\ &M\Big(w_j\big(x_j + \tfrac{1}{n} - x_{j-1}\big) - \tfrac{t}{2}x_j^2 - \tfrac{t}{2}\big(\tfrac{1}{n} - x_{j-1}\big)^2\Big).\end{aligned}$$

Adding the consumers' surpluses generated by sales of all goods yields

$$\begin{aligned}CS &= M\sum_{j=1}^{n}\Big(w_jx_j - w_jx_{j-1} + \tfrac{1}{n}w_j - \tfrac{t}{2}x_j^2 - \tfrac{t}{2}x_{j-1}^2 + \tfrac{t}{n}x_{j-1} - \tfrac{t}{2n^2}\Big)\\ &= M\big(\boldsymbol{w}'(2I - B')\boldsymbol{x} + \tfrac{1}{n}\boldsymbol{\iota}'\boldsymbol{w} - t\boldsymbol{x}'\boldsymbol{x} + \tfrac{t}{n}\boldsymbol{\iota}'\boldsymbol{x} - \tfrac{t}{2n}\big),\end{aligned}$$

where

$$B := \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{3.13}$$

is an $n \times n$ matrix. The diagonal and superdiagonal entries of this matrix as well as its bottom-left entry are 1. All other entries of B are 0.

Total profits accruing to the industry equal

$$\Pi = \sum_{j=1}^{n} M\big(x_j + \tfrac{1}{n} - x_{j-1}\big)p_j = \tfrac{M}{n}\boldsymbol{\iota}'\boldsymbol{p} + M\boldsymbol{p}'(2I - B')\boldsymbol{x}.$$

In the appendix we have a closer look at the various components of $W(K,\alpha)$. There we also compute the expected total welfare in equilibrium. These results are summarized in the following:

Lemma 3.1 *Suppose Condition 1 and Condition 2 hold. If the IG uses investigation strategy K, then expected aggregate consumers' surplus equals:*[14]

$$E\big(CS(K)\big) = \frac{M\sigma^2}{8t}\,\mathrm{trace}\left(\Delta(I-\Gamma^2)\right) + M\big(v - \mu - \tfrac{5t}{4n}\big),$$

where $\Delta = \Delta(K)$ is a diagonal matrix whose j^{th} diagonal entry is 1 if $j \in K$ and 0 if $j \notin K$. Total expected industry profits are:

$$E\big(\Pi(K)\big) = \frac{Mt}{n} + \frac{M\sigma^2}{4t}\Big(\mathrm{trace}\left(\Delta(I+\Gamma^2)\right) - 2|K|\gamma\Big).$$

Expected total welfare is:

$$\begin{aligned} E(W(K,\alpha)) =& M(v - \tfrac{t}{4n}) - \frac{M\sigma^2}{8t}\,\mathrm{trace}\left(\Delta(I-\Gamma^2)\right) \\ &- (1+\alpha)\Big(M\mu - \frac{M|K|(1-\gamma)\sigma^2}{2t}\Big) - c(|K|). \end{aligned}$$

Proof. See the appendix. ∎

Recall that in equilibrium the IG investigates each firm if (3.11) holds, but refrains from any investigations if this condition is not met. We thus only need to compare the various expected surpluses in case $K = N$ with their counterparts in case $K = \emptyset$ to assess the impact on equilibrium expected total welfare of the IG's presence:

Proposition 3.5 *Suppose Condition 1 and Condition 2 hold. The IG's (equilibrium) investigations strictly increase both expected consumers' surplus and expected industry profits if (3.11) holds. Expected total welfare unambiguously increases if (3.11) holds as long as $\alpha \geq \frac{n-1}{2n}$.*

Proof. See the appendix. ∎

Consumers are better off if the IG is present, because they can base their purchasing decisions on better information. Surprisingly, firms are also better off with an IG. The reason is that the IG's information provision creates vertical product differentiation. This differentiation relaxes price competition and hence increases profits (compare with Shaked and Sutton, 1982). As the number of firms grows, firms will be less able to internalize quality differences and the competition-reducing effect subsides. As a consequence, consumers will be able to reap the benefits of improved information to a larger extent. The increase in expected consumers' surplus combined with the increase in expected aggregate profits implies that expected

[14] The trace of a square matrix is the sum of its diagonal elements.

total welfare increases as long as the overlap between the consumers and the IG is not too large. However, because the total increase in expected consumers' surplus and expected industry profits caused by the disclosure of damage levels is smaller than the expected reduction in aggregate damage, a social planner is less inclined to investigate firms than the IG is:[15]

Proposition 3.6 *Suppose Condition 1 and Condition 2 hold. Consider a social planner capable of investigating k firms at a cost $\tilde{c}(k)$ whose objective is to maximize $E\big(CS(K) + \Pi(K)\big) - \tilde{c}(|K|)$, where $\tilde{c}(\cdot)$ is an increasing, concave function with $\tilde{c}(0) = 0$. Then this social planner investigates every single firm in the industry if and only if*

$$\frac{M(1-\gamma)\sigma^2}{2t} - \frac{M\sigma^2}{8t} \times (1 - \tfrac{1}{n}\,\mathrm{trace}\,\Gamma^2) \geq \frac{\tilde{c}(n)}{n}.$$

If this inequality fails to hold, then the social planner investigates none of the firms.

Proof. See the appendix. ∎

Because expected profits per firm are larger if the IG is present, one might wonder whether firms have incentives to establish their own interest group capable of informing consumers. A firm/industry-sponsored group would presumably be met by consumer skepticism: firms have strong incentives to downplay the damages they cause. As has been argued in Section 3.2, the IG we consider also faces credibility issues. Yet, because the consumers, just like the IG present in our model, care about the damages caused by the firms, whereas the firms are solely interested in profits, the divergence in preferences between the IG we consider and consumers is small compared to the divergence in preferences between a firm/industry-sponsored group and consumers. It is therefore much more difficult for an interest group with overt ties to a (damaging) industry to credibly inform consumers. This is not to say that such groups do not exist. Lyon and Maxwell (2004) provide examples in which firms subsidize the communication efforts of IGs.[16] Constructions where firms can choose to being scrutinized by an interest group which subsequently discloses the firm's conduct also occur. The Carbon Disclosure Project is an example of such a hybrid.[17]

[15] It could be that outsiders are also affected by the damages and that the social planner therefore uses a total welfare measure akin to expression (3.12), i.e. a total welfare measure with more weight on aggregate damage. This does not alter the qualitative result that the social planner is less inclined to carry out investigations than the IG is. Of course, an immediate corollary to Proposition 3.5 is that a social planner opts to investigate each firm if the additional weight attached to aggregate damage is at least $\frac{n-1}{2n}$.

[16] Lyon and Maxwell (2004) coin this strategy the *bear hug*. They indeed argue that it reduces the informativeness of the subsidized IG's messages.

[17] See `http://www.cdproject.net/`.

3.6 Imprecise Information Provision

Information supplied by interest groups does usually not take the form of the kind of precise, quantitative statement we have considered so far. Rather, IGs often indicate whether a damage level belongs to a certain category. For example, the Environmental Defense Fund uses three categories to classify fish species: "eco-best", "eco-ok", and "eco-worst". Moreover, consumers might find it difficult to interpret the IG's information, effectively using a noisy variant of the IG's messages in their purchasing decisions. In the present setting, such imprecise information dissemination (or interpretation) implies that the IG does not inform consumers about the true damage d_j associated with production of good j, but only tells them that d_j belongs to some subinterval of $[0, D]$. Below we discuss the impact on equilibrium and welfare if the IG resorts to such an *imprecise advertising technology.*

Such imprecise information provision on the part of the IG can have a strategic rationale, as is argued in Crawford and Sobel (1982). These authors consider a setting in which an informed party can send a possibly noisy signal to an uninformed party. The latter subsequently makes a move that determines the payoffs of both players. Crawford and Sobel show that the sender's equilibrium signal is imprecise as long as the interests of the two players are not perfectly aligned. Moreover, the larger the conflict in interests, the coarser equilibrium signals can be.[18] We do not attempt to endogenize the IG's choice of advertising technology. Instead, we determine the (unique) equilibrium of the, by now familiar, three-stage game for any *given* advertising technology.

Consider an exogenously given imprecise advertising technology with $r > 1$ possible signals characterized by some partition $(Z_k)_{k \in \{1,\dots,r\}}$ of $[0, D]$.[19] If the IG finds out that $d_j \in Z_k$, then the consumers receive signal k regarding good j.[20] The mean and variance of d_j conditional on $d_j \in Z_k$ play an important role in the subsequent analysis. We therefore introduce the following notation:

$$E\big(d_j | d_j \in Z_k\big) =: \mu_k =: \mu + \nu_k, \quad E\big((d_j - \mu_k)^2 | d_j \in Z_k\big) =: \sigma_k^2, \quad k \in \{1, \dots, r\}. \tag{3.14}$$

Just as in the model presented in section 3.2, the consumers' expectation of d_j equals the unconditional mean μ if firm j is not investigated. However, if firm j is

[18] The model of Crawford and Sobel (1982) is plagued by a multiplicity of equilibria. Formally, the most informative equilibrium becomes weakly less informative as the conflict in interests grows. Strictly speaking, our setting deals with a situation with two uninformed audiences: the firms and the consumers. See Farrell and Gibbons (1989) for a model of strategic information transmission with two audiences.

[19] $(Z_k)_{k \in \{1,\dots,r\}}$ is a partition of $[0, D]$ iff $\cup_{k=1}^{r} Z_k = [0, D]$ and $Z_k \cap Z_{k'} = \emptyset$ as long as $k \neq k'$.

[20] Whether or not the IG itself learns the precise value of d_j is immaterial to our results.

investigated, then consumers do not learn the true value of d_j, but they do update their expectation to μ_k if they receive signal k regarding good j. Each entry of the vector of consumers' expectation of damages $\boldsymbol{s}$ thus equals either μ or one of the μ_k's. For each $\boldsymbol{s}$ equilibrium prices and quantities are again given by (3.7) and (3.8) respectively. The expression for equilibrium ex post aggregate damage can be found in (3.9). Ex ante expected aggregate damage equals

$$M\mu - \frac{M}{2t}E(\boldsymbol{d}'(I-\Gamma)\boldsymbol{s}).$$

The second part of this expression can be calculated using the iterated expectation formula. This is done in the appendix. These calculations yield the following insights.

Proposition 3.7 *Suppose Condition 1 and Condition 2 hold. If the IG uses imprecise advertising technology* $\mathcal{Z} := (Z_k)_{k\in\{1,\ldots,r\}}$ *and opts for investigation strategy* K, *then the expected aggregate damage equals*

$$M\mu - \frac{M|K|(1-\gamma)}{2t}\Phi_{\mathcal{Z}}, \tag{3.15}$$

where

$$\Phi_{\mathcal{Z}} := \sum_{k=1}^{r} \nu_k^2 \Pr(d_j \in Z_k). \tag{3.16}$$

In equilibrium, the IG investigates every single firm in the industry if and only if

$$\frac{M(1-\gamma)}{2t}\Phi_{\mathcal{Z}} \geq \frac{c(n)}{n}. \tag{3.17}$$

If this inequality does not hold, then the IG investigates none of the firms.

Proof. See the appendix. ∎

Compared to the situation in which the IG fully informs consumers, the imprecise advertising technology $\mathcal{Z}$ leads to a smaller reduction in expected aggregate damage. The intuition behind this observation can be most easily seen by decomposing the variance σ^2 as follows:

$$\begin{aligned}\sigma^2 =& E(d_j^2) - \mu^2 = \sum_{k=1}^{r} E(d_j^2|d_j \in Z_k)\Pr(d_j \in Z_k) - \mu^2 \\ =& \sum_{k=1}^{r} \big((\mu+\nu_k)^2 + \sigma_k^2\big)\Pr(d_j \in Z_k) - \mu^2 = \Phi_{\mathcal{Z}} + \sum_{k=1}^{r}\sigma_k^2 \Pr(d_j \in Z_k).\end{aligned}$$

One sees that the imprecise advertising technology $\mathcal{Z}$ fails to 'attack' the intra-interval variances σ_k^2. This is a straightforward implication of the very definition of this advertising technology. Because the IG cannot achieve as much reduction in aggregate damage with an imprecise technology as with the precise technology, the IG is less inclined to investigate firms if it can subsequently only provide imprecise information (compare (3.11) with (3.17)).

We now investigate the welfare implications of the presence of an IG which employs an imprecise advertising technology. The following result generalizes Lemma 3.1:

Lemma 3.2 *Suppose Condition 1 and Condition 2 hold. If the IG uses advertising technology $\mathcal{Z}$ and opts for investigation strategy K, then expected aggregate consumers' surplus equals* [21]

$$E\big(CS(K)\big) = M\big(v - \tfrac{5t}{4n} - \mu\big) + \frac{M\,\mathrm{trace}\,\big(\Delta(I-\Gamma^2)\big)}{8t}\Phi_{\mathcal{Z}}.$$

Total expected industry profits are:

$$E(\Pi(K)) = \frac{Mt}{n} + \frac{M\Big(\mathrm{trace}\,\big(\Delta(I+\Gamma^2)\big) - 2|K|\gamma\Big)}{4t}\Phi_{\mathcal{Z}}. \tag{3.18}$$

Expected total welfare reads:

$$\begin{aligned} E(W(K,\alpha)) =& M(v - \tfrac{t}{4n}) - \frac{M}{8t}\,\mathrm{trace}\,\big(\Delta(I-\Gamma^2)\big)\Phi_{\mathcal{Z}} \\ & - (1+\alpha)\Big(M\mu - \frac{M|K|(1-\gamma)}{2t}\Phi_{\mathcal{Z}}\Big) - c(|K|). \end{aligned}$$

Proof. See the appendix. ■

One sees that the advertising technology's imprecision impacts the various components of welfare in a straightforward fashion. Both the change in expected consumers' surplus and the change in expected aggregate profits due to the IG's investigations are proportional to the inter-interval variation $\Phi_{\mathcal{Z}}$. As the advertising technology becomes more precise, the intra-interval variances vanish and the changes in the aforementioned surpluses resemble those mentioned in Lemma 3.1 to a larger extent. The consequences of the imprecision on equilibrium total welfare follows immediately from comparing Lemma 3.1, Lemma 3.2, and Proposition 3.5:

Proposition 3.8 *Suppose Condition 1 and Condition 2 hold. The IG's equilibrium investigations strictly increase both expected consumers' surplus and expected industry profits if (3.17) holds. Expected total welfare unambiguously increases if (3.17) holds as long as $\alpha \geq \frac{n-1}{2n}$.*

[21] Recall that Δ is a diagonal matrix whose j^{th} diagonal entry is one if $j \in K$ and zero otherwise.

Note that all parties involved always prefer a fine partition to a coarse partition from an ex ante point of view. This observation is reminiscent of the comparative statics results in Crawford and Sobel (1982). In their game of strategic information transmission both the sender of information and the receiver prefer *ex ante* to end up in the most informative equilibrium even if the preferences of sender and receiver differ significantly.

3.7 Conclusions

We have linked an interest group's incentives to inform consumers about the damaging production practices of firms to a number of important market characteristics. In short, an IG becomes more inclined to investigate firms and inform consumers about its findings as the market size increases, the number of firms increases, products become less heterogeneous, or the variability of the damage levels increases. The welfare analysis indicate that, except for situations in which the IG is mainly sponsored by consumers, the IG's actions are welfare enhancing, even if the IG's information is imprecise. Since our model can be readily reinterpreted as dealing with a public party contemplating issuing quality labels, our analysis quantifies the (short run) efficacy of such labels.

Of course, our analysis is in some respects limited. The model abstracts away from issues which would potentially lead to signalling, either by firms or by the IG. As has already been argued, the assumptions which preclude signalling are often reasonable. Yet, with the present framework we cannot study situations in which the IG strategically withholds information or situation in which firms *greenwash* (Kirchhoff, 2000) their products (i.e exaggerate the environmental performance of their products). It would therefore be interesting to consider a richer framework in which damages are correlated with costs, are spatially correlated, or in which disseminating information is costly. A second limitation concerns our covered market assumption. We conjecture that expected industry profits need not increase due to IG's actions if this assumption is relaxed. Lastly, the firms' damage levels are exogenous. It would be interesting to allow the firms to adjust their production practices in response to the information revealed by an interest group. We hope to deal with the above issues in future work.

3.A Appendix

Proof of Proposition 3.1

Using (3.5) it follows that for $n \geq 3$:

$$\boldsymbol{p}^* = \frac{t}{2n}(I-A)^{-1}\boldsymbol{\iota} - \frac{1}{2}(I-A)^{-1}(I-2A)\boldsymbol{s} = \frac{t}{2n}(I-A)^{-1}\boldsymbol{\iota} - \frac{1}{2}(I-\Gamma)\boldsymbol{s}.$$

We have to show that the entries of each column and the entries of each row of $(I-A)^{-1}$ sum to 2. By symmetry of $(I-A)$ it suffices to prove that the entries of each column add up to 2. Denote the $(i,j)^{\text{th}}$ entry of $(I-A)^{-1}$ with α_{ij}. Writing out the j^{th} column of $(I-A)(I-A)^{-1}$ yields

$$\begin{pmatrix} 1 & -\frac{1}{4} & 0 & \dots & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & \dots \\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{4} & 0 & \dots & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \alpha_{3j} \\ \cdot \\ \alpha_{nj} \end{pmatrix} = \begin{pmatrix} \alpha_{1j} - \frac{1}{4}\alpha_{2j} - \frac{1}{4}\alpha_{nj} \\ -\frac{1}{4}\alpha_{1j} + \alpha_{2j} - \frac{1}{4}\alpha_{3j} \\ -\frac{1}{4}\alpha_{2j} + \alpha_{3j} - \frac{1}{4}\alpha_{4j} \\ \dots \\ -\frac{1}{4}\alpha_{1j} - \frac{1}{4}\alpha_{n-1,j} + \alpha_{nj} \end{pmatrix} = \boldsymbol{e}_j, \quad (3.19)$$

where $\boldsymbol{e}_j$ is the j^{th} unit vector. The last equality implies that

$$\sum_{i=1}^{n} \alpha_{ij} - \frac{1}{4}\sum_{i=1}^{n} \alpha_{ij} - \frac{1}{4}\sum_{i=1}^{n} \alpha_{ij} = 1, \quad (3.20)$$

yielding $\sum_{i=1}^{n} \alpha_{ij} = 2$.

Next we calculate the equilibrium quantities. Equation (3.4) reveals that:

$$\begin{aligned} \boldsymbol{q}^* &= \frac{M}{n}\boldsymbol{\iota} - \frac{2M}{2t}(I-2A)(\boldsymbol{p}^* + \boldsymbol{s}) \\ &= \frac{M}{n}\boldsymbol{\iota} - \frac{M}{t}(I-2A)\Big(\frac{t}{n}\boldsymbol{\iota} + \frac{1}{2}\big(I + (I-A)^{-1}A\big)\boldsymbol{s}\Big) \\ &= \frac{M}{n}\boldsymbol{\iota} - \frac{M}{n}(I-2A)\boldsymbol{\iota} - \frac{M}{2t}(I-2A)(I-A)^{-1}\boldsymbol{s} \\ &= \frac{M}{n}\boldsymbol{\iota} - \frac{M}{2t}\big(I - A(I-A)^{-1}\big)\boldsymbol{s}, \end{aligned}$$

where the third equality follows from the fact that $I + (I-A)^{-1}A = (I-A)^{-1}$ and the last equality from the fact that $(I-2A)\boldsymbol{\iota} = 0$.

We now calculate the prices duopolists charge. The two indifferent consumers x_1 and x_2 solve

$$\begin{aligned} v - p_1 - tx_1 - s_1 &= v - p_2 - t(\tfrac{1}{2} - x_1) - s_2, \\ v - p_2 - tx_2 - s_2 &= v - p_1 - t(\tfrac{1}{2} - x_2) - s_1. \end{aligned}$$

Consequently, $x_1 = \frac{1}{4} + \frac{1}{2t}\big(p_2 - p_1 + s_2 - s_1\big)$ and $x_2 = \frac{1}{4} + \frac{1}{2t}\big(p_1 - p_2 + s_1 - s_2\big)$. This leads to the following profit expressions:

$$\pi_1 = M\big(\tfrac{1}{2} + \frac{p_2 - p_1 + s_2 - s_1}{t}\big)p_1, \quad \pi_2 = M\big(\tfrac{1}{2} + \frac{p_1 - p_2 + s_1 - s_2}{t}\big)p_2.$$

The associated first order conditions are solved by $p_1^* = \frac{t}{2} + \frac{s_2 - s_1}{3}$ and $p_2^* = \frac{t}{2} + \frac{s_1 - s_2}{3}$. The equilibrium quantities read $q_i^* = \frac{Mp_i^*}{t}$, $i = 1, 2$. ∎

Proof of Corollary 3.1

Equality (3.20) informs us that $(I - A)^{-1}\boldsymbol{\iota} = 2$. The claims now follow from the fact that $A\boldsymbol{\iota} = \frac{1}{2}\boldsymbol{\iota}$, which implies that $(I - \Gamma)\mu\boldsymbol{\iota} = \mathbf{0}$. One easily checks that the indifferent consumers enjoy a nonnegative surplus if each firm charges $\frac{t}{n}$ as long as $v \geq \frac{3t}{2n}$. ∎

Proof of Covered Market Claim

We have to prove that if $v \geq \frac{3t}{2n} + D$, then for every $j \in N$ the following two inequalities hold for every realization $\boldsymbol{s} \in [0, D]^n$:

$$v - p_j^* - tx_j^* - s_j \geq 0, \tag{3.21}$$

$$v - p_{j+1}^* - t(\tfrac{1}{n} - x_j^*) - s_{j+1} \geq 0, \tag{3.22}$$

where $x_j^* = \frac{1}{2n} + \frac{1}{2t}\big(p_{j+1}^* - p_j^* + s_{j+1} - s_j\big)$. This claim obviously holds if $n = 2$. Fix an $n \geq 3$. Note that, since

$$v - p_j^* - tx_j^* - s_j = v - \tfrac{t}{2n} - \tfrac{1}{2}p_j - \tfrac{1}{2}p_{j+1} - \tfrac{1}{2}s_j - \tfrac{1}{2}s_{j+1} = v - p_{j+1}^* - t(\tfrac{1}{n} - x_j^*) - s_{j+1},$$

we only have to look at condition (3.21). Using the matrix B (see equation (3.13)) we can write the LHS of (3.21) for all firms jointly:

$$\begin{aligned} v\boldsymbol{\iota} - \frac{t}{2n}\boldsymbol{\iota} - \frac{1}{2}B\boldsymbol{p}^* - \frac{1}{2}B\boldsymbol{s} &= v\boldsymbol{\iota} - \frac{t}{2n}\boldsymbol{\iota} - \frac{1}{2}B\Big(\frac{t}{n}\boldsymbol{\iota} + \frac{1}{2}\big(I + (I - A)^{-1}A\big)\boldsymbol{s}\Big) \\ &= v\boldsymbol{\iota} - \frac{3t}{2n}\boldsymbol{\iota} - \frac{1}{4}B\big(I + (I - A)^{-1}A\big)\boldsymbol{s}, \end{aligned} \tag{3.23}$$

where the last equality follows from the fact that $B\boldsymbol{\iota} = 2\boldsymbol{\iota}$. By Theorem 4.C.9 in Takayama (1985), each entry of $(I - A)^{-1}$ is positive, implying

$$B\big(I + (I - A)^{-1}A\big)\boldsymbol{s} \leq B\big(I + (I - A)^{-1}A\big)(D\boldsymbol{\iota}) = 2D\boldsymbol{\iota} + DB(I - A)^{-1}A\boldsymbol{\iota},$$

for every realisation of $\boldsymbol{s}$. Combining the above inequality with (3.23) yields that for arbitrary $\boldsymbol{s} \in [0, D]^n$ one has:

$$v\boldsymbol{\iota} - \frac{t}{2n}\boldsymbol{\iota} - \frac{1}{2}B\boldsymbol{p}^* - \frac{1}{2}B\boldsymbol{s} \geq v\boldsymbol{\iota} - \frac{3t}{2n}\boldsymbol{\iota} - \frac{D}{4}\big(2\boldsymbol{\iota} + B(I - A)^{-1}A\boldsymbol{\iota}\big) = v\boldsymbol{\iota} - \frac{3t}{2n}\boldsymbol{\iota} - D\boldsymbol{\iota} \geq \mathbf{0},$$

where we have used the fact that $A\boldsymbol{\iota} = \frac{1}{2}\boldsymbol{\iota}$ and equation (3.20). ∎

Proof of Active Firms Claim

We first show that $x_j^* \geq 0$, $j = 1, \ldots, n$ if $D \leq \frac{t}{n}$ holds. If x_j^* were negative, then firm j would be undercut by firm $j+1$ and firm j would consequently be inactive. With the aid of the matrix B (see (3.13)) one can write the equilibrium vector of indifferent consumers as follows:

$$\boldsymbol{x}^* = \frac{1}{2n}\boldsymbol{\iota} + \frac{1}{2t}(B-2I)(\boldsymbol{p}^*(\boldsymbol{s})+\boldsymbol{s}) = \frac{1}{2n}\boldsymbol{\iota} + \frac{1}{4t}(B-2I)(I+\Gamma)\boldsymbol{s},$$

where the second equality follows from the fact that $(B-2I)\boldsymbol{\iota} = \mathbf{0}$. Let us focus for ease of exposition on x_1^*. We have to find the vector $\boldsymbol{s}_1$ that minimizes $x_1^* = x_1^*(\boldsymbol{s})$. Because $(I+\Gamma) = (I-A)^{-1}$, one has:

$$x_1^* = \frac{1}{2n} + \frac{1}{4t}\boldsymbol{e}_1'(B-2I)(I-A)^{-1}\boldsymbol{s} = \frac{1}{2n} + \frac{1}{4t}\Big(- \sum_{j=1}^{n}\alpha_{1j}s_j + \sum_{j=1}^{n}\alpha_{2j}s_j\Big).$$

Vogel (2008) shows that the *circulant matrix* $(I-A)^{-1}$ is such that:[22]

$$\alpha_{11} > \alpha_{12} > \ldots > \alpha_{1,\lceil\frac{n+1}{2}\rceil}, \quad \alpha_{1,\lceil\frac{n+1}{2}\rceil} \leq \alpha_{1,\lceil\frac{n+1}{2}\rceil+1} \leq \ldots \leq \alpha_{1n}, \tag{3.24}$$

where $\lceil y \rceil$ is the smallest integer which is at least y. By circularity of $(I-A)^{-1}$ one has:

$$-\sum_{j=1}^{n}\alpha_{1j}s_j + \sum_{j=1}^{n}\alpha_{2j}s_j = -(\alpha_{11}-\alpha_{12}s_1) + \sum_{j=1}^{n-1}(\alpha_{1j}-\alpha_{1,j+1})s_{j+1}.$$

Combining this result with the inequalities (3.24) one sees that x_1^* (weakly) decreases in s_j if and only if $j \in \{1\} \cup \{\lceil\frac{n+1}{2}\rceil, \ldots, n\}$. Consequently, $\boldsymbol{s}_1$'s j^{th} entry is D if $j \in \{1\} \cup \{\lceil\frac{n+1}{2}\rceil, \ldots, n\}$ and 0 otherwise. Evaluating x_1^* at this vector of expectations yields

$$\begin{aligned} x_1^*(\boldsymbol{s}_1) =& \frac{1}{2n} + \frac{1}{4t}\Big(-(\alpha_{11}-\alpha_{12})D + \sum_{j=\lceil\frac{n+1}{2}\rceil-1}^{n-1}(\alpha_{1j}-\alpha_{1,j+1})D\Big) \\ =& \frac{1}{2n} - \frac{1}{4t}(\alpha_{11}-\alpha_{1,\lceil\frac{n-1}{2}\rceil})D, \end{aligned}$$

where the last equality follows from the telescoping property of the above series. Since $\alpha_{11} - \alpha_{1,\lceil\frac{n+1}{2}\rceil} < 2$, we conclude that $x_1^*(\boldsymbol{s}) \geq 0$ for every possible $\boldsymbol{s}$ if $D \leq \frac{t}{n}$.

Next, we prove that for $n \geq 3$ the vector of equilibrium prices $\frac{t}{n}\boldsymbol{\iota} - \frac{1}{2}(I-\Gamma)\boldsymbol{s}$ is nonnegative for every possible $\boldsymbol{s}$ if $D \leq \frac{2t}{n(1-\gamma)}$ (γ is the generic diagonal element of Γ), a condition which is less demanding than $D \leq \frac{t}{n}$. The facts that each entry of A is nonnegative and each entry of $(I-A)^{-1}$ is positive implies that each entry

[22] A matrix is circulant if each row vector is rotated one element to the right relative to the preceding row vector. Our matrix $(I-A)^{-1}$ equals four times Vogel's matrix H with $\tau = 0$. See pp. 451-455 of Vogel (2008).

γ_{ij}, $(i,j) \in N^2$, of Γ is positive. More precisely, since $\gamma_{ij} = \frac{1}{4}\alpha_{i,j-1} + \frac{1}{4}\alpha_{i,j+1} < \frac{1}{4}\sum_{k=1}^{n} \alpha_{kj} = \frac{1}{2}$, we know that $\gamma_{ij} \in (0, \frac{1}{2})$. This observation implies that $I - \Gamma$ has positive diagonal elements and negative off-diagonal elements. Therefore:

$$\boldsymbol{e}_j'(I-\Gamma)\boldsymbol{s} \leq \max_{\boldsymbol{s}\in[0,D]^n} \boldsymbol{e}_j'(I-\Gamma)\boldsymbol{s} = D\boldsymbol{e}_j'(I-\Gamma)\boldsymbol{e}_j = (1-\gamma)D.$$

As a consequence, $p_j^* \geq \frac{t}{n} - \frac{D}{2}(1-\gamma)$. This lower bound is nonnegative as long as $D \leq \frac{2t}{n(1-\gamma)}$.

If $n = 2$ firm j is active only if $p_j^* = \frac{t}{2} + \frac{s_{3-j}-s_j}{3} \geq 0$, $j = 1, 2$. These prices are always nonnegative if $D \leq \frac{3t}{2} = \frac{2t}{2(1-\frac{1}{3})}$. One easily checks that $x_j^* \geq 0$, $j = 1, 2$, if $D \leq \frac{t}{2}$. ■

Proof of Proposition 3.2

Let Δ be a diagonal matrix whose j^{th} diagonal entry, δ_j, is 1 if $j \in K$ and 0 if $j \notin K$ and let $\boldsymbol{\epsilon} = \boldsymbol{d} - \mu\boldsymbol{\iota}$. Note that $\boldsymbol{\epsilon}$ is a random vector with mean $\mathbf{0}$. By construction $\boldsymbol{s} = \mu\boldsymbol{\iota} + \Delta\boldsymbol{\epsilon}$ whereas $\boldsymbol{d}'\Gamma\boldsymbol{s}$ can be written as follows:

$$\boldsymbol{d}'\Gamma\boldsymbol{s} = (\mu\boldsymbol{\iota}' + \boldsymbol{\epsilon}')\Gamma(\mu\boldsymbol{\iota} + \Delta\boldsymbol{\epsilon}) = \mu^2\boldsymbol{\iota}'\Gamma\boldsymbol{\iota} + \mu\boldsymbol{\iota}'\Gamma\Delta\boldsymbol{\epsilon} + \mu\boldsymbol{\epsilon}'\Gamma\boldsymbol{\iota} + \boldsymbol{\epsilon}'\Gamma\Delta\boldsymbol{\epsilon}.$$

Because $\Gamma\boldsymbol{\iota} = 2A\boldsymbol{\iota} = \boldsymbol{\iota}$, this expression reduces to

$$\boldsymbol{d}'\Gamma\boldsymbol{s} = n\mu^2 + \mu\boldsymbol{\iota}'\Delta\boldsymbol{\epsilon} + \mu\boldsymbol{\epsilon}'\boldsymbol{\iota} + \boldsymbol{\epsilon}'\Gamma\Delta\boldsymbol{\epsilon}.$$

Since $E\boldsymbol{\epsilon} = 0$, the expected value of $\boldsymbol{d}'\Gamma\boldsymbol{s}$ reads

$$E(\boldsymbol{d}'\Gamma\boldsymbol{s}) = n\mu^2 + E(\boldsymbol{\epsilon}'\Gamma\Delta\boldsymbol{\epsilon}).$$

We write out $\boldsymbol{\epsilon}'\Gamma\Delta\boldsymbol{\epsilon}$:

$$\boldsymbol{\epsilon}'\Gamma\Delta\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'\begin{pmatrix}\gamma_{11}\delta_1 & \dots & \gamma_{1n}\delta_n \\ \dots & \dots & \dots \\ \gamma_{n1}\delta_1 & \dots & \gamma_{nn}\delta_n\end{pmatrix}\boldsymbol{\epsilon} = \sum_{i=1}^{n}\sum_{j=1}^{n} \epsilon_i\gamma_{ij}\delta_j\epsilon_j.$$

Taking expectations yields:

$$E\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} \epsilon_i\gamma_{ij}\delta_j\epsilon_j\Big) = \sum_{j=1}^{n} \delta_j\gamma_{jj}\sigma^2 = |K|\gamma\sigma^2,$$

where the first equality follows from the fact that $E(\epsilon_i\epsilon_j) = 0$ if $i \neq j$. Therefore $E(\boldsymbol{d}'\Gamma\boldsymbol{s}) = n\mu^2 + |K|\gamma\sigma^2$, implying

$$E(\boldsymbol{d}'\boldsymbol{q}^*(\boldsymbol{s}_K)) = M\mu - \frac{M}{2t}(n\mu^2 + |K|\sigma^2) + \frac{M}{2t}(n\mu^2 + |K|\gamma\sigma^2) = M\mu - \frac{M|K|\sigma^2(1-\gamma)}{2t}.$$

We now demonstrate that this expectation is always positive. It is sufficient to show this for $|K| = n$. Since Condition 3.2 implies that $\frac{n\sigma^2(1-\gamma)}{2t} \leq \frac{\sigma^2}{D}$, it suffices to prove that $\mu > \frac{\sigma^2}{D}$. Denote the c.d.f. of d_j with F. Then:

$$\sigma^2 = \int_0^D (t-\mu)^2 \, \mathrm{d}F(t) < \int_0^D Dt \, \mathrm{d}F(t) - \int_0^D 2\mu t \, \mathrm{d}t + \mu^2 = D\mu - \mu^2.$$

The inequality $\mu > \frac{\sigma^2}{D}$ holds *a forteriori*. ■

Proof of Proposition 3.4

All claims except the one pertaining to n are obvious. To prove that the reduction in aggregate damage increases in n, it suffices to show that $\gamma(n+1) < \gamma(n)$, $n \geq 2$. Straightforward calculations reveal that $\gamma(3) = \frac{1}{5} < \frac{1}{3} = \gamma(2)$. To prove the claim for $n \geq 3$, we first reiterate (3.19) for $j = 1$:

$$\alpha_{11} - \tfrac{1}{4}\alpha_{21} - \tfrac{1}{4}\alpha_{n1} = 1, \tag{3.25}$$

$$-\tfrac{1}{4}\alpha_{k1} + \alpha_{k+1,1} - \tfrac{1}{4}\alpha_{k+2,1} = 0, \quad k \in \{1, 2, \ldots, n-2\}, \tag{3.26}$$

$$-\tfrac{1}{4}\alpha_{11} - \tfrac{1}{4}\alpha_{n-1,1} + \alpha_{n1} = 0. \tag{3.27}$$

The equalities (3.26) imply that the elements $\alpha_{11}, \ldots, \alpha_{n1}$ of $(I - A)^{-1}$ solve the second order difference equation

$$x(k+2) - 4x(k+1) + x(k) = 0. \tag{3.28}$$

The general solution to this difference equation reads

$$x(k) = \beta\lambda_1^k + \kappa\lambda_2^k, \quad k \geq 1,$$

where $\lambda_1 = 2 - \sqrt{3}$, $\lambda_2 = 2 + \sqrt{3}$, and $\beta, \kappa \in \mathbb{C}$. The equalities (3.25) and (3.27) provide boundary conditions ($x(1) - \frac{1}{4}x(2) - \frac{1}{4}x(n) = 1$ and $-\frac{1}{4}x(1) - \frac{1}{4}x(n-1) + x(n) = 0$) which peg the values of β and κ at $\beta = \beta(n)$ and $\kappa = \kappa(n)$, as is shown below. Since $\lambda_1 - \frac{1}{4}\lambda_1^2 = \lambda_2 - \frac{1}{4}\lambda_2^2 = \frac{1}{4}$, the first boundary condition can be rewritten as follows:

$$(1 - \lambda_1^n)\beta + (1 - \lambda_2^n)\kappa = 4. \tag{3.29}$$

Evaluating (3.28) at $k = n - 1$ and combining with the general solution leads to:

$$4x(n) - x(n-1) = \beta\lambda_1^{n+1} + \kappa\lambda_2^{n+1}.$$

At the same time, the second boundary condition must hold, i.e. $4x(n) - x(n-1) = x(1) = \beta\lambda_1 + \kappa\lambda_2$. Therefore:

$$\beta\lambda_1 + \kappa\lambda_2 = \beta\lambda_1^{n+1} + \kappa\lambda_2^{n+1}. \tag{3.30}$$

Solving (3.29) and (3.30) simultaneously yields

$$\beta(n) = \frac{4\lambda_2}{(1-\lambda_1^n)(\lambda_2-\lambda_1)}, \quad \kappa(n) = \frac{4\lambda_1}{(\lambda_2^n-1)(\lambda_2-\lambda_1)}.$$

This implies:

$$\begin{aligned}\alpha_{k1} &= \frac{4\lambda_1\lambda_2}{(1-\lambda_1^n)(\lambda_2-\lambda_1)}\lambda_1^{k-1} + \frac{4\lambda_1\lambda_2}{(\lambda_2^n-1)(\lambda_2-\lambda_1)}\lambda_2^{k-1} \\ &= \frac{2}{\sqrt{3}}\frac{\lambda_1^{k-1}}{1-\lambda_1^n} + \frac{2}{\sqrt{3}}\frac{\lambda_2^{k-1}}{\lambda_2^n-1}, \quad k \in N,\end{aligned} \tag{3.31}$$

where we have used the facts that $\lambda_1\lambda_2 = 1$ and $\lambda_2 - \lambda_1 = 2\sqrt{3}$. We are now in a position to present an explicit expression for $\gamma(n)$:

$$\begin{aligned}\gamma(n) =& \frac{1}{4}\alpha_{21} + \frac{1}{4}\alpha_{n1} = \frac{1}{2\sqrt{3}}\frac{\lambda_1+\lambda_1^{n-1}}{1-\lambda_1^n} + \frac{1}{2\sqrt{3}}\frac{\lambda_2+\lambda_2^{n-1}}{\lambda_2^n-1} \\ =& \frac{1}{\sqrt{3}}\frac{\lambda_2^{n-1}-\lambda_1^{n-1}+\lambda_2-\lambda_1}{\lambda_1^n+\lambda_2^n-2},\end{aligned}$$

where the last equality follows from repeated usage of $\lambda_1\lambda_2 = 1$. So, the difference between $\gamma(n+1)$ and $\gamma(n)$ reads:

$$\begin{aligned}\gamma(n+1)-\gamma(n) =& \frac{(\lambda_2^n-\lambda_1^n+\lambda_2-\lambda_1)(\lambda_1^n+\lambda_2^n-2)}{\sqrt{3}T(n)} \\ &- \frac{(\lambda_2^{n-1}-\lambda_1^{n-1}+\lambda_2-\lambda_1)(\lambda_1^{n+1}+\lambda_2^{n+1}-2)}{\sqrt{3}T(n)} \\ =& \frac{\left(\lambda_1^{n+2}-\lambda_1^{n+1}+\lambda_1^n-\lambda_1^{n-1}-\lambda_1^2\right)}{\sqrt{3}T(n)} \\ &- \frac{\left(\lambda_2^{n+2}-\lambda_2^{n+1}+\lambda_2^n-\lambda_2^{n-1}-\lambda_2^2\right)}{\sqrt{3}T(n)},\end{aligned}$$

where $T(n) = (\lambda_1^{n+1}+\lambda_2^{n+1}-2)(\lambda_1^n+\lambda_2^n-2) > 0$. Since $\lambda_1^3-\lambda_1^2+\lambda_1-1 = 20-12\sqrt{3}$, $\lambda_2^3-\lambda_2^2+\lambda_2-1 = 20+12\sqrt{3}$, and $\lambda_2^2-\lambda_1^2 = 8\sqrt{3}$, it suffices to show that

$$(20-12\sqrt{3})\lambda_1^{n-1} - (20+12\sqrt{3})\lambda_2^{n-1} + 8\sqrt{3} < 0,$$

for every $n \geq 3$. Because $\lambda_1 < 1 < \lambda_2$, $(20-12\sqrt{3})\lambda_1^{n-1} - (20+12\sqrt{3})\lambda_2^{n-1}$ is strictly decreasing in n. The fact that $(20-12\sqrt{3})\lambda_1 - (20+12\sqrt{3})\lambda_2 + 8\sqrt{3} = -80\sqrt{3} < 0$ therefore completes the proof. ■

Proof of Lemma 3.1

Below we make frequently use of the following easily verifiable facts:

$$\Gamma\boldsymbol{\iota}=\boldsymbol{\iota},\quad (I-\Gamma)\boldsymbol{\iota}=\mathbf{0},\quad (B-I)\boldsymbol{\iota}=(B-I)'\boldsymbol{\iota}=\boldsymbol{\iota},\quad (B-2I)\boldsymbol{\iota}=\mathbf{0},$$
$$B+B'=BB'=2I+4A,\quad (B-I)'(B-2I)=2I-B',$$
$$(B-2I)'(B-2I)=2(I-2A),\quad I+\Gamma=(I-A)^{-1},\quad (I-2A)(I+\Gamma)=I-\Gamma.$$

The following equalities will also be useful (recall that $\boldsymbol{\epsilon}=\boldsymbol{d}-\mu\boldsymbol{\iota}$):

$$\begin{aligned} E\big(\boldsymbol{\epsilon}'\Delta(I-\Gamma)^2\Delta\boldsymbol{\epsilon}\big) &= |K|\sigma^2-2|K|\gamma\sigma^2+E\big(\boldsymbol{\epsilon}'\Delta\Gamma^2\Delta\boldsymbol{\epsilon}\big)\\ &= |K|(1-2\gamma)\sigma^2+\sum_{j\in K}\sum_{i=1}^{n}\gamma_{ij}^2\sigma^2, \end{aligned}$$

$$E\big(\boldsymbol{\epsilon}'\Delta\Gamma(I-\Gamma)\Delta\boldsymbol{\epsilon}\big)=|K|\gamma\sigma^2-\sum_{j\in K}\sum_{i=1}^{n}\gamma_{ij}^2\sigma^2.$$

Both equalities follow from the facts that $E(\epsilon_i\epsilon_j)=0$ if $i\neq j$ and the j^{th} diagonal entry of Γ^2 equals $\sum_{i=1}^{n}\gamma_{ij}^2$.

Consider first the case $n\geq 3$. In the market equilibrium one has:

$$\begin{aligned} \boldsymbol{w}^* &= v\boldsymbol{\iota}-\boldsymbol{p}^*-\boldsymbol{s}=v\boldsymbol{\iota}-\tfrac{t}{n}\boldsymbol{\iota}-\frac{1}{2}(I+\Gamma)(\mu\boldsymbol{\iota}+\Delta\boldsymbol{\epsilon})\\ &= \big(v-\tfrac{t}{n}-\mu\big)\boldsymbol{\iota}-\frac{1}{2}(I+\Gamma)\Delta\boldsymbol{\epsilon},\\ \boldsymbol{x}^* &= \frac{1}{2n}\boldsymbol{\iota}+\frac{1}{4t}(B-2I)(I+\Gamma)(\mu\boldsymbol{\iota}+\Delta\boldsymbol{\epsilon})\\ &= \frac{1}{2n}\boldsymbol{\iota}+\frac{1}{4t}(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}. \end{aligned}$$

Let us analyze the various components of $CS(K)$:

$$\begin{aligned} \boldsymbol{w}^{*\prime}(2I-B')\boldsymbol{x}^* =&\frac{1}{2n}\big(v-\tfrac{t}{n}-\mu\big)\boldsymbol{\iota}'(2I-B')\boldsymbol{\iota}\\ &+\frac{1}{4t}\big(v-\tfrac{t}{n}-\mu\big)\boldsymbol{\iota}'(2I-B')(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\\ &-\frac{1}{4n}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(2I-B')\boldsymbol{\iota}\\ &-\frac{1}{8t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(2I-B')(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\\ =&\frac{1}{4t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-2A)(I+\Gamma)\Delta\boldsymbol{\epsilon}=\frac{1}{4t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\epsilon}, \end{aligned} \tag{3.32}$$

$$\boldsymbol{\iota}'\boldsymbol{w}^*=n\big(v-\tfrac{t}{n}-\mu\big)-\boldsymbol{\iota}'\Delta\boldsymbol{\epsilon}, \tag{3.33}$$

$$\begin{aligned} \boldsymbol{x}^{*\prime}\boldsymbol{x}^* =&\Big(\frac{1}{2n}\boldsymbol{\iota}+\frac{1}{4t}(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\Big)'\Big(\frac{1}{2n}\boldsymbol{\iota}+\frac{1}{4t}(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\Big)\\ =&\frac{1}{4n}+\frac{1}{8t^2}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-2A)(I+\Gamma)\Delta\boldsymbol{\epsilon}=\frac{1}{4n}+\frac{1}{8t^2}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\epsilon}, \end{aligned} \tag{3.34}$$

$$\boldsymbol{\iota}'\boldsymbol{x}^* = \boldsymbol{\iota}'\Big(\frac{1}{2n}\boldsymbol{\iota} + \frac{1}{4t}(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\Big) = \frac{1}{2}. \tag{3.35}$$

With the aid of (3.32)-(3.35) we can compute $CS(K)$:

$$\begin{aligned}\frac{1}{M}CS(K) =& \frac{1}{4t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\epsilon} + \big(v - \tfrac{t}{n} - \mu\big) - \frac{1}{n}\boldsymbol{\iota}'\Delta\boldsymbol{\epsilon} - \frac{t}{4n} \\ & - \frac{1}{8t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\epsilon} + \frac{t}{2n} - \frac{t}{2n} \\ =& \big(v - \tfrac{5t}{4n} - \mu\big) - \frac{1}{n}\boldsymbol{\iota}'\Delta\boldsymbol{\epsilon} + \frac{1}{8t}\boldsymbol{\epsilon}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\epsilon}.\end{aligned}$$

Taking expectations yields:

$$\begin{aligned}E\big(CS(K)\big) =& M\Big(\frac{|K|\sigma^2}{8t} - \frac{1}{8t}\sum_{j\in K}\sum_{i=1}^{n}\gamma_{ij}^2\sigma^2 + v - \mu - \tfrac{5t}{4n}\Big) \\ =& \frac{M\sigma^2}{8t}\,\text{trace}\,\big(\Delta(I-\Gamma^2)\big) + M\big(v-\mu-\tfrac{5t}{4n}\big).\end{aligned}$$

We now calculate $\Pi(K)$:

$$\begin{aligned}\frac{1}{M}\Pi(K) =& \frac{1}{n}\boldsymbol{\iota}'\Big(\frac{t}{n}\boldsymbol{\iota} - \frac{1}{2}(I-\Gamma)(\mu\boldsymbol{\iota}+\Delta\boldsymbol{\epsilon})\Big) \\ & + \Big(\frac{t}{n}\boldsymbol{\iota} - \frac{1}{2}(I-\Gamma)(\mu\boldsymbol{\iota}+\Delta\boldsymbol{\epsilon})\Big)'(2I-B')\Big(\frac{1}{2n}\boldsymbol{\iota} + \frac{1}{4t}(B-2I)(I+\Gamma)\Delta\boldsymbol{\epsilon}\Big) \\ =& \frac{t}{n} - \frac{\mu}{2n}\boldsymbol{\iota}'(I-\Gamma)\boldsymbol{\iota} - \frac{1}{2n}\boldsymbol{\iota}'(I-\Gamma)\Delta\boldsymbol{\epsilon} \\ & + \frac{1}{4t}(\mu\boldsymbol{\iota}+\Delta\boldsymbol{\epsilon})'(I-\Gamma)(I-2A)(I+\Gamma)\Delta\boldsymbol{\epsilon} \\ =& \frac{t}{n} + \frac{1}{4t}\boldsymbol{\epsilon}'\Delta(I-\Gamma)^2\Delta\boldsymbol{\epsilon}.\end{aligned}$$

Expected industry profits thus read:

$$\begin{aligned}E(\Pi(K)) =& \frac{Mt}{n} + \frac{M|K|(1-2\gamma)\sigma^2}{4t} + \frac{M\sigma^2}{4t}\sum_{j\in K}\sum_{i=1}^{n}\gamma_{ij}^2 \\ =& \frac{Mt}{n} + \frac{M\sigma^2}{4t}\big(\text{trace}(\Delta(I+\Gamma^2)) - 2|K|\gamma\big).\end{aligned}$$

Using the above equalities one sees that expected total welfare equals:

$$\begin{aligned}\frac{1}{M}E(W(K,\alpha)) =& \frac{\sigma^2}{8t}\,\text{trace}\,\big(\Delta(I-\Gamma^2)\big) + \big(v-\mu-\tfrac{5t}{4n}\big) + \frac{t}{n} - \frac{c(|K|)}{M} \\ & + \frac{\sigma^2}{4t}\big(\text{trace}(\Delta(I+\Gamma^2)) - 2|K|\gamma\big) - \alpha\big(\mu - \frac{|K|(1-\gamma)\sigma^2}{2t}\big) \\ =& \big(v - \tfrac{t}{4n}\big) - \frac{\sigma^2}{8t}\,\text{trace}\,\big(\Delta(I-\Gamma^2)\big) - \frac{c(|K|)}{M} \\ & - (1+\alpha)\big(\mu - \frac{|K|(1-\gamma)\sigma^2}{2t}\big),\end{aligned}$$

which is the desired expression for $n \geq 3$.

If $n = 2$ aggregate consumers' surplus is

$$CS(K) = M\Big(\frac{w_1 + w_1 - tx_1}{2}x_1 + \frac{w_2 + w_2 - tx_2}{2}x_2 + \frac{w_1 + w_1 - t(\frac{1}{2} - x_2)}{2}\big(\tfrac{1}{2} - x_2\big)\Big) + M\frac{w_2 + w_2 - t(\frac{1}{2} - x_1)}{2}\big(\tfrac{1}{2} - x_1\big),$$

where $w_j = v - p_j - s_j$ is the maximal surplus enjoyed by consumers buying good j, $j = 1, 2$. Inserting the equilibrium values $w_j^* = v - \frac{t}{2} - \frac{s_{3-j} - s_j}{3} - s_j$ and $x_j^* = \frac{1}{4} - \frac{s_j - s_{3-j}}{6t}$ yields

$$CS(K) = M\Big(v - \frac{5t}{8} - 2\frac{(s_1 - s_2)^2}{9t} - \frac{s_1 + s_2}{2} + \frac{(s_1 - s_2)^2}{3t} - \frac{(s_1 - s_2)^2}{18t}\Big).$$

Taking expectations results in

$$E\big(CS(K)\big) = M\Big(v - \frac{5t}{8} - \mu + \frac{\delta_1\sigma^2 + \delta_2\sigma^2}{18t}\Big) = M\big(v - \tfrac{5t}{8} - \mu\big) + \frac{M|K|\sigma^2}{18t}.$$

Because $\operatorname{trace}\big(\Delta(I - \Gamma(2)^2)\big) = \frac{4|K|}{9}$, this is indeed the required expression. Expected aggregate industry profits, $E(\Pi(K)) = \frac{Mt}{2} + \frac{2M|K|\sigma^2}{9t} = \frac{Mt}{2} + \frac{\operatorname{trace}\big(\Delta(I+\Gamma(2)^2)\big) - 2\gamma(2)|K|}{4}\frac{M\sigma^2}{t}$, follow directly from taking the expectation of

$$\pi_1 + \pi_2 = \frac{M}{t}\Big(\frac{t}{2} - \frac{s_1 - s_2}{3}\Big)^2 + \frac{M}{t}\Big(\frac{t}{2} - \frac{s_2 - s_1}{3}\Big)^2.$$

The expression for expected aggregate damage follows from Proposition 3.2. ■

Proof of Proposition 3.5

We first investigate the impact of the investigations on consumers' surplus. Note that:

$$E\big(CS(N)\big) - E\big(CS(\emptyset)\big) = \frac{M\sigma^2}{8t}\operatorname{trace}(I - \Gamma^2) = \frac{M\sigma^2}{8t}\big(n - \operatorname{trace}\Gamma^2\big).$$

In the proof of the active firms claim it has been shown that $\gamma_{ij} \in (0, \frac{1}{2})$ for each $(i, j) \in N^2$. We have also seen that the entries of a row or a column of Γ sum to 1. Thus $(\gamma_{1j}, \dots, \gamma_{nj}) \in S := \{\boldsymbol{z} \in (0, \frac{1}{2})^n : \sum_{i=1}^n z_i = 1\}$, $j \in N$. This implies that

$$\operatorname{trace}\Gamma^2 = \sum_{j=1}^n \sum_{i=1}^n \gamma_{ij}^2 \leq n \sup_{\boldsymbol{z} \in S} \sum_{i=1}^n z_i^2.$$

Since the gradient of $\boldsymbol{z}'\boldsymbol{z}$ is strictly positive on S, the above supremum is attained on ∂S, the boundary of S. One easily verifies that $\max_{\boldsymbol{z} \in \partial S} \boldsymbol{z}'\boldsymbol{z}$ is attained in the extremal points of S where two out of the n z_i's are $\frac{1}{2}$ and the others are 0. Therefore $\operatorname{trace}\Gamma^2 \leq \frac{n}{2}$ and consequently

$$E\big(CS(N)\big) - E\big(CS(\emptyset)\big) \geq \frac{M\sigma^2}{8t}\frac{n}{2} > 0.$$

The impact of the IG on industry profits reads:

$$E\big(\Pi(N)\big) - E\big(\Pi(\emptyset)\big) = \frac{M\sigma^2}{4t}\big(\operatorname{trace}(I+\Gamma^2) - 2n\gamma\big) = \frac{M\sigma^2}{4t}\big(n(1-2\gamma) + \operatorname{trace}\Gamma^2\big).$$

To get a lower bound for this expression, we need to calculate $\inf_{\boldsymbol{z}\in S} \boldsymbol{z}'\boldsymbol{z}$. Obviously, $\boldsymbol{z}'\boldsymbol{z}$ attains its minimum on the closure $\bar{S}$ of S in the point of $\bar{S}$ closest to the origin, i.e. the point $(\frac{1}{n},\dots,\frac{1}{n})$. Combining with the fact that $\gamma \in (0,\frac{1}{2})$ yields

$$E\big(\Pi(N)\big) - E\big(\Pi(\emptyset)\big) > \frac{M\sigma^2}{4t} n \inf_{\boldsymbol{z}\in S} \boldsymbol{z}'\boldsymbol{z} = \frac{M\sigma^2}{4t}.$$

The change in welfare due to the IG's investigations is

$$E\big(W(N,\alpha)\big) - E\big(W(\emptyset,\alpha)\big) = (1+\alpha)\frac{Mn(1-\gamma)\sigma^2}{2t} - \frac{M\sigma^2}{8t}\operatorname{trace}(I-\Gamma^2) - c(n).$$

Invoking (3.11) leads to:

$$E\big(W(N,\alpha)\big) - E\big(W(\emptyset,\alpha)\big) \geq \alpha\frac{Mn(1-\gamma)\sigma^2}{2t} - \frac{M\sigma^2}{8t}\operatorname{trace}(I-\Gamma^2).$$

Welfare thus (strictly) increases if $4n(1-\gamma)\alpha - \operatorname{trace}(I-\Gamma^2) > 0 \iff \alpha > \frac{n-\operatorname{trace}\Gamma^2}{4n(1-\gamma)}$. Using the facts that $\gamma \in (0,\frac{1}{2})$ and $\operatorname{trace}\Gamma^2 \geq 1$ yields the desired inequality. ■

Proof of Proposition 3.6

Note that:

$$E\big(CS(K)+\Pi(K)\big) - E\big(CS(\emptyset)+\Pi(\emptyset)\big) = \frac{M|K|(1-\gamma)\sigma^2}{2t} - \frac{M\sigma^2}{8t}\operatorname{trace}\big(\Delta(I-\Gamma^2)\big).$$

It thus suffices to show that the expression

$$|K|(1-\gamma) - \tfrac{1}{4}\operatorname{trace}\big(\Delta(K)(I-\Gamma^2)\big)$$

increases in the cardinality $|K|$ of K. We know from the proof of Proposition 3.5 that $\sum_{i=1}^{n}\gamma_{ij}^2 \geq \frac{1}{n}$ for all $j \in N$. This implies that for any $K, K' \subseteq N$ such that $K \cap K' = \{j\}$ for some $j \in N$ one has:

$$(1-\gamma) - \tfrac{1}{4}\operatorname{trace}\big(\Delta(K')(I-\Gamma^2)\big) + \tfrac{1}{4}\operatorname{trace}\big(\Delta(K)(I-\Gamma^2)\big) =$$
$$(1-\gamma) - \tfrac{1}{4}(1-\sum_{i=1}^{n}\gamma_{ij}^2) \geq (1-\gamma) - \tfrac{1}{4}(1-\tfrac{1}{n}) \geq \tfrac{2}{3} - \tfrac{1}{4}(1-\tfrac{1}{n}) > 0,$$

where the second inequality follows from the fact that $\gamma(n) \leq \gamma(2) = \frac{1}{3}$ (see the proof of Proposition 3.4). ■

Proof of Proposition 3.7

We first calculate $E(\boldsymbol{d}'\boldsymbol{s})$. Recall that $\Delta = \Delta(K)$ is the diagonal matrix whose j^{th} diagonal entry, δ_j, is 1 if $j \in K$ and 0 if $j \notin K$. Then $\boldsymbol{s} = \mu\boldsymbol{\iota} + \Delta\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is a random vector such that $\tau_j = \nu_k$ precisely if $d_j \in Z_k$, $j \in N$. Observe that

$$E(d_j\tau_j) = \sum_{k=1}^{r} E\big(d_j\tau_j | d_j \in Z_k\big)\Pr(d_j \in Z_k) = \sum_{k=1}^{r}(\mu + \nu_k)\nu_k \Pr(d_j \in Z_k) = \Phi_{\mathcal{Z}},$$

where the last equality follows from the fact that τ_j has mean zero. Therefore:

$$E(\boldsymbol{d}'\boldsymbol{s}) = \mu E(\boldsymbol{d}'\boldsymbol{\iota}) + E(\boldsymbol{d}'\Delta\boldsymbol{\tau}) = n\mu^2 + |K|\Phi_{\mathcal{Z}}.$$

Next we tackle $E(\boldsymbol{d}'\Gamma\boldsymbol{s})$. Since the various d_j's are independent draws,

$$E(\boldsymbol{d}'\Gamma\boldsymbol{s}) = n\mu^2 + E\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_i\gamma_{ij}d_j\tau_i\Big)$$

reduces to

$$n\mu^2 + E\Big(\sum_{j=1}^{n}\delta_j\gamma_{jj}d_j\tau_j\Big) = n\mu^2 + |K|\gamma\Phi_{\mathcal{Z}}.$$

The expression (3.15) immediately follows. Comparison of the expected aggregate damage with the total investigation costs reveals that the IG adopts the all-or-nothing strategy characterized by (3.17). ■

Proof of Lemma 3.2

We omit the proof for $n = 2$. Fix an $n \geq 3$. Recall from Section 3.5 the expression for total consumers' surplus:

$$CS(K) = M\Big(\boldsymbol{w}'(2I - B')\boldsymbol{x} + \tfrac{1}{n}\boldsymbol{\iota}'\boldsymbol{w} - t\boldsymbol{x}'\boldsymbol{x} + \tfrac{t}{n}\boldsymbol{\iota}'\boldsymbol{x} - \tfrac{t}{2n}\Big),$$

where the (second stage) equilibrium values of both $\boldsymbol{w}$ and $\boldsymbol{x}$ ($\boldsymbol{w}^*$ respectively $\boldsymbol{x}^*$) depend on $\boldsymbol{s} = \mu\boldsymbol{\iota} + \Delta\boldsymbol{\tau}$:

$$\begin{aligned}\boldsymbol{w}^* &= \big(v - \tfrac{t}{n} - \mu\big)\boldsymbol{\iota} - \frac{1}{2}(I + \Gamma)\Delta\boldsymbol{\tau},\\ \boldsymbol{x}^* &= \frac{1}{2n}\boldsymbol{\iota} + \frac{1}{4t}(B - 2I)(I + \Gamma)\Delta\boldsymbol{\tau}.\end{aligned}$$

Using these expressions and various facts established in the proof of Lemma 3.1 one can verify that:

$$\begin{aligned}\boldsymbol{w}^{*\prime}(2I - B')\boldsymbol{x}^* =& \frac{1}{4t}\boldsymbol{\tau}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\tau},\\ \boldsymbol{\iota}'\boldsymbol{w}^* =& n\big(v - \tfrac{t}{n} - \mu\big) - \boldsymbol{\iota}'\Delta\boldsymbol{\tau},\\ \boldsymbol{x}^{*\prime}\boldsymbol{x}^* =& \frac{1}{4n} + \frac{1}{8t^2}\boldsymbol{\tau}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\tau},\\ \boldsymbol{\iota}'\boldsymbol{x}^* =& \frac{1}{2}.\end{aligned}$$

Using these equalities one sees that expected consumers' surplus equals

$$E(CS(K)) = M\big(v - \tfrac{5t}{4n} - \mu\big) - \frac{M}{n}E(\boldsymbol{\iota}'\Delta\boldsymbol{\tau}) + \frac{M}{8t}E\big(\boldsymbol{\tau}'\Delta(I+\Gamma)(I-\Gamma)\Delta\boldsymbol{\tau}\big).$$

Since $E(\boldsymbol{\iota}'\Delta\boldsymbol{\tau}) = 0$ and $E\big(\boldsymbol{\tau}'\Delta(I-\Gamma^2)\Delta\boldsymbol{\tau}\big) = \text{trace}\,\big(\Delta(I-\Gamma^2)\big)\Phi_{\mathcal{Z}}$, one has:

$$E\big(CS(K)\big) = M\big(v - \tfrac{5t}{4n} - \mu\big) + \frac{M\,\text{trace}\,\big(\Delta(I-\Gamma^2)\big)}{8t}\Phi_{\mathcal{Z}}.$$

Aggregate industry profits equal

$$\boldsymbol{p}^{*\prime}\boldsymbol{q}^* = \frac{M}{t}\boldsymbol{p}^{*\prime}\boldsymbol{p}^* = \frac{Mt}{n} + \frac{M}{4t}\boldsymbol{s}'(I-\Gamma)^2\boldsymbol{s},$$

where we have used the fact that $(I-\Gamma)\boldsymbol{\iota} = \boldsymbol{0}$. Substituting $\boldsymbol{s} = \mu\boldsymbol{\iota} + \Delta\boldsymbol{\tau}$, taking expectations, and using the fact that $E\big(\boldsymbol{\tau}'\Delta(I-\Gamma)^2\Delta\boldsymbol{\tau}\big) = \text{trace}\,\big(\Delta(I-\Gamma)^2\big)\Phi_{\mathcal{Z}}$ yields

$$E(\Pi(K)) = \frac{Mt}{n} + E\big(\boldsymbol{\tau}'\Delta(I-\Gamma)^2\Delta\boldsymbol{\tau}\big) = \frac{Mt}{n} + \frac{M\Big(\text{trace}\,\big(\Delta(I+\Gamma^2)\big) - 2|K|\gamma\Big)}{4t}\Phi_{\mathcal{Z}}.$$

The derivation of total expected welfare is straightforward and therefore omitted. ■