Strategic interactions in environmental economics

Heijnen, Pim

Chapter 6

On the probability of breakdown in participation games

6.1 Introduction

The aim of this Chapter is to analyze public good games, where the amount of contribution is a binary variable. This means that contributing can be interpreted as participating in a program or joining a group to do a certain amount of volunteering. I refer to this as a participation game (cf. Anderson and Engers, 2005). Participation is assumed to be costly and since the good is non-excludable everyone benefits from someone participating. The symmetric (Nash) equilibrium in these kind of games is a mixed equilibrium; each member of the group participates with a certain probability between zero and one. This implies that, in equilibrium, there is a certain probability that no one participates. Surprisingly, this probability tends to be increasing in group size. For the basic model and the derivation of this paradoxical result, see Dixit and Skeath (1999, pp.388–392), Harrington (2001) or Rasmusen (2001, pp.77–79).

An example is the murder of Kitty Genovese, who was killed in New York City in 1964 while 38 neighbors looked on without calling the police. This story is consistent with evidence from social psychology which suggests that people are less likely to help someone in need if they are surrounded by other people (Dixit and Skeath, 1999, p. 389).

Mukhopadhyay (2003) extends the basic model with incomplete information. Mukhopadhyay examines the Condorcet jury theorem which states that a larger jury makes a more accurate verdict. Mukhopadhyay argues that if paying attention in court is costly (i.e. boring) and every juror wants to make an accurate verdict (for which they have to pay attention), then in a large jury there is an increasingly large probability that no one will pay attention. The Condorcet jury theorem does not hold. This conclusion is not altered when the jurors receive a signal, whether the defendant is likely to be guilty, prior to the court case.

In an interesting paper, Johnson (2002) studies the development of open source software. Open-source software is a public good since once it is developed other software producers can just “steal” the code. Both the valuation of the end-product and the cost of development are drawn from some distribution creating *ex post* asymmetry. Although Johnson does not derive it, it is straightforward to show that the probability that software is not developed increases in the number of software producers.

Palfrey and Rosenthal (1983) offer one of the first systematic analyses of binary public good games. They consider the following variant: there are two groups of individuals. The group with the most individual contributions wins. A real-life example would be a presidential election. Given the behavior of the other group, there is an incentive to free ride. Haan and Kooreman (2003) show that even if one group has an arbitrarily large potential majority (if everyone in their group will vote), it will not always win. This is due to the fact there is a subgame where members of the majority group have to decide whether they are going to contribute to the public good (i.e. voting to help their group win).

Here a more general game is analyzed which encompasses Harrington (2001) and Mukhopadhyay (2003) as special cases. Although variants of this game have been studied extensively, previous work takes the benefit of provision of the public good to be independent of the number of players that contribute. I identify which payoff structures[1] can be considered characteristic for participation games and then proceed to analyze these games. I present sufficient conditions under which a breakdown (i.e. no one participates) is more likely if the group becomes larger. While my model cannot be seen as a direct generalization of most of the papers mentioned in the previous paragraphs, at heart the paradoxical results they all report are due to the presence of a participation (sub)game. By generalizing the payoff structure, the robustness of these results is checked. I also provide three economics ex-

[1]By payoff structure I mean the utility a player receives depending on how many other players participate and whether the player himself participates.

amples: the probability that a patent is produced, debt overhang (based on Kaneko and Prokop, 1993) and oil spills.

Anderson and Engers (2005) present a different generalization: they examine the effect of different thresholds and of positive and negative externalities of participating. A threshold is the minimal number of contributions that are needed for the benefits of the public good (per contributor) to exceed the cost of participation. By positive and negative externalities I mean that if more of the other people participate, then the incentive to participate respectively increases and decreases. This is basically a monotonicity requirements on the payoff function. They examine the different kind of equilibria that can occur and some comparative statics such as the effects of increasing the payoff of participating. I impose no monotonicity restrictions on the payoff function, but only examine a threshold of one.

The Chapter is organized as follows: in Section 6.2 I introduce the participation game, derive all its equilibria and show that the probability of participating for each player is decreasing in group size. Section 6.3 discusses the probability of breakdown. Section 6.4 gives some additional results. Section 6.5 gives three economic examples. Section 6.6 concludes.

6.2 The participation game

There are $N+1$ players.[2] Denote the group of players with $\aleph = \{1, \dots, N+1\}$. Each player $i \in \aleph$ can choose to participate ($a_i = 1$) or not participate ($a_i = 0$). For player i, define $A_i = \sum_{j \neq i} a_j \in \{0, \dots, N\}$ as the total number of other players who participate. The payoff of player i depends only on a_i and A_i; players do not have preferences about *who* participates. The game is anonymous. Denote the payoff function by $F : \{0, 1\} \times \{0, \dots, N\} \to \mathbb{R}$.

Assumption 6.1 *Participation is costly if at least one other player participates: for each player i, for all $A_i \in \{1, \dots, N\}$ we have $F(1, A_i) < F(0, A_i)$.*

Assumption 6.2 *If nobody participates, the best-response is to participate: for each player i we have $F(0, 0) < F(1, 0)$.*

I refer to the game defined above as the participation game.

Example 6.1 (Harrington, 2001) In Harrington's model, a group of $N + 1$ people stand around a lake and someone is drowning in this lake. Each person has the opportunity to rescue this person. Each person receives a payoff of

[2] In a mixed equilibrium the players are indifferent given the actions of the other players. It is convenient to set the number of other players equal to N.

b if the drowner is rescued, but the person who actually rescues the person incurs a cost c. The payoff function for each player i is:

$$\begin{aligned} F(0,0) &= 0, \\ F(1,0) &= b - c, \\ F(0,A_i) &= b, \quad \text{for all } A_i > 0, \\ F(1,A_i) &= b - c, \quad \text{for all } A_i > 0, \end{aligned}$$

where $b, c > 0$ and $b > c$. This is a special case of the participation game. □

Proceeding, there is no symmetric equilibrium in pure strategies in this game. To see this, suppose that everyone else participates. Then by Assumption 6.1 it is optimal to not participate. Suppose that no one else participates. Then by Assumption 6.2 it is optimal to participate.

In general, the symmetric mixed equilibrium can be derived as follows. Suppose that N people participate with probability p. The $(N+1)$-th person should be indifferent between participating and not participating. The expected payoff of participating is:

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} F(1,k), \tag{6.1}$$

and the expected payoff of not participating is:

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} F(0,k). \tag{6.2}$$

In equilibrium, these two payoffs should be equal:

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \theta_k = 0, \tag{6.3}$$

where $\theta_k \equiv F(0,k) - F(1,k)$. This parameter signifies the incentive to not participate if k people are already participating. From Assumptions 6.1 and 6.2, it follows that $\theta_0 < 0$ and $\theta_k > 0$ (for $k > 0$). Define the function

$$g_N(p) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \theta_k. \tag{6.4}$$

The following proposition characterizes all possible equilibria.

Proposition 6.1 *There are $2^{N+1}-1$ equilibria. In each of these equilibria, each member of a set $\mathcal{N} \subseteq \aleph$ (with the exception of the empty set) participates with probability $p > 0$ and each member of $\aleph - \mathcal{N}$ does not participate. Moreover:*

1. *If $\#\mathcal{N} = 1$, then $p = 1$ (equilibrium in pure strategies).*

2. *If $\#\mathcal{N} > 1$, then p is the unique root of $g_{\#\mathcal{N}-1}(\cdot)$, $0 < p < 1$.*

Proof. See Appendix 6.A. ■

Corrolary 6.1 *There is a unique symmetric mixed equilibrium, where everybody participates with probability p_N. This probability is the unique root of the function $g_N(\cdot)$.*

From now on my focus will be on the symmetric equilibrium. This equilibrium has the following property:

Proposition 6.2 *In the symmetric equilibrium of the participation game we have $p_{N+1} < p_N$: the probability of participation declines if group size increases.*

Proof. See Appendix 6.B. ■

This proposition shows the free-riding problem that is inherent to public good games.

6.3 The probability that no one participates

Let $S_N \equiv (1-p_N)^{N+1}$. This is the probability that no one participates. I am interested in how S_N depends on N. For instance, Harrington (2001) showed that for his specification S_N is increasing in N. I first note that if $S_{N+1} > S_N$, then:

$$(1-p_{N+1})^{N+2} > (1-p_N)^{N+1} \Leftrightarrow p_{N+1} < 1-(1-p_N)^{\frac{N+1}{N+2}} \approx p_N, \qquad (6.5)$$

where the approximation holds if N large. This can be seen as an indication that it might be difficult to find examples where $S_{N+1} \leq S_N$ for N large. I am able to prove the following results.

Proposition 6.3 *If $|\theta_0| < |\theta_1|$, then*

(i) $S_{N+1} > S_N$ *for all* $N \in \mathbb{N}$,

(ii) $S_\infty = \lim_{N\to\infty} S_N \geq e^{\frac{\theta_0}{\theta_1}}$,

(iii) the expected number of participants is less than one.

Proof. See Appendix 6.C. ■

If we solve (6.3) for $N = 1$, then we see that the condition that $|\theta_0| < |\theta_1|$ is equivalent to $p_1 < 1/2$. It also implies that S_∞ is always bigger than $e^{-1} \approx 0.3679$. So, in all participation games satisfying the condition $|\theta_0| < |\theta_1|$, the probability of breakdown is increasing and always more than 37% in the limit. It is interesting to notice that if $|\theta_0| < |\theta_1|$, then the values of $\theta_2, \ldots, \theta_N$ are completely irrelevant: the probability of breakdown will increase.

Proposition 6.3 shows that, under a fairly weak condition only dependent on the ratio of θ_0 and θ_1, the probability of breakdown in binary public good games is increasing in group size. This paradoxical result (not only will people participate less, the group as a whole will participate less) appears very robust to different payoff specifications. The Proposition also shows that on average less than one person participates and the public good is underprovided (more than one being socially optimal).[3]

Example 6.2 (Harrington, 2001) In this game $\theta_0 = -(b-c)$ and $\theta_1 = c$. Using Proposition 6.3: if $b < 2c$ and thus $|\theta_0| < |\theta_1|$, then the probability that the drowner is not saved is increasing in group size. However, in this specific example I can make a stronger claim. Note that $g_N(\cdot)$ can be rewritten as:

$$(c-b)(1-p)^N + c[1-(1-p)]^N = 0 \tag{6.6}$$

From this we find:

$$p_N = 1 - \left(\frac{c}{b}\right)^{\frac{1}{N}} \text{ and } S_N = \left(\frac{c}{b}\right)^{\frac{N+1}{N}}, \tag{6.7}$$

where S_N is always increasing in N. □

Example 6.3 (The hero-bonus) As an extension of the game in the previous example, suppose that being recognized as a rescuer has merits of its own.[4] If there are k rescuers, then each obtains an extra payoff of H/k. Now, $\theta_0 = -[b+H-c]$ and $\theta_1 = c - H/2$, where I assume that $H < 2c$ and $b+H-c > 0$. The condition $|\theta_0| < |\theta_1|$ corresponds to $H < \frac{4}{3}c - \frac{2}{3}b$. So, if the hero bonus

[3] The socially optimal outcome maximizes $F(0,k)(N+1-k) + F(1,k)k$ with respect to k.

[4] As Nietzsche (1878/1988) remarked: *"Man springt einem Menschen, der in's Wasser fällt, noch einmal so gern nach, wenn Leute zugegen sind, die es nicht wagen."*

is sufficiently small, then the probability that no one rescues the drowner is still increasing in group size. Notice that for $N > 2$ the equilibrium strategy will in general not be solvable without resorting to numerical methods. □

Remark 6.1 The condition $|\theta_0| < |\theta_1|$ is a sufficient but not a necessary condition. Take for instance the game where $\theta_0 = -2$ and $\theta_k = 1$ for all $k > 0$. Now (6.3) can be rewritten as:

$$[1 - (1-p)^N] - 2(1-p)^N = 0, \tag{6.8}$$

and subsequently $p_N = 1 - \left(\frac{1}{3}\right)^{\frac{1}{N}}$. Note that $S_N = \left(\frac{1}{3}\right)^{\frac{N+1}{N}}$, of which the derivative is:

$$\frac{\mathrm{d}S_N}{\mathrm{d}N} = -\log(3) \times \frac{-1}{N^2} \times S_N > 0. \tag{6.9}$$

Hence, S_N is increasing in N. △

Remark 6.2 One can construct examples in which S_N is decreasing. Take the following game: $\theta_0 = -1$, and $\theta_k = \alpha\beta^k$ for all $k \geq 1$. Assume that $\alpha > 0$ and $0 < \beta < 1$. It can be shown that (see Appendix 6.D):

$$S_N = \left[\frac{\beta}{\sqrt[N]{(1+\alpha)/\alpha} - (1-\beta)}\right]^{N+1}. \tag{6.10}$$

Take $\beta = \frac{1}{25}$ and $\alpha = \frac{25}{4}$. Numerical calculations show that $S_1 = 0.0400$, $S_2 = 0.0399$, $S_3 = 0.0378$, $S_4 = 0.0359$. So, $S_4 < S_3 < S_2 < S_1$. In this example, θ_2, θ_3 etcetera are very close to zero and, hence, $F(0,k) \approx F(1,k)$ for $k \geq 2$: that is, players are basically indifferent between participating and not participating. △

6.4 Further explorations

In this section I will focus on monotone payoff functions. This corresponds to negative and positive externalities (if the θ_k's are respectively increasing or decreasing in k). The purpose of this is threefold: (1) it will yield another sufficient condition for the probability of breakdown to be increasing; (2) it will give some insight into when the probability of breakdown is decreasing; and (3) the sufficient condition of Proposition 6.3 does not always yield 'nice' conditions as the application in Section 6.5.2 shows whereas the condition for a negative externality is easier to interpret.

Suppose we have a game with $N+1$ players such that $\theta_0 = -1$ (just a normalization) and, for all $k = 1, \ldots, N$, $\theta_{k+1} > \theta_k$. If the θ_k's are increasing, then $F(0,k) - F(1,k)$ is increasing in k. As more players participate the incentive to not participate becomes stronger; this is the case of a negative externality (cf. Anderson and Engers (2005) who focus solely on such monotonic payoff functions).

Construct now an auxiliary game with parameters $\theta_0^C = -1$ and for all $k > 0$, $\theta_{k+1}^C = \theta_k^C$ such that this game with $N+1$ players has the same symmetric equilibrium as the game above with increasing θ_k's. Note that for every N such an auxiliary game can be uniquely constructed. Observe that for the auxiliary game it is known that the probability of breakdown is increasing in N (since it is a game similar to the one analyzed in Example 6.2). Denote the equilibrium of the auxiliary game by p_N^C. By construction $p_N^C = p_N$, where p_N is the probability of participation in the game with a negative externality. If we can show that $p_{N+1} < p_{N+1}^C$, then $(1-p_{N+1})^{N+2} > (1-p_{N+1}^C)^{N+2} > (1-p_N^C)^{N+1} = (1-p_N)^{N+1}$. Then the probability of breakdown in a game where, for all $k > 0$, $\theta_{k+1} > \theta_k$, is increasing in group size.

I will now show that $p_{N+1} < p_{N+1}^C$. Examining the function $g_N(p)$, I observe that it is a weighted average of the θ_k's. The most weight is given to $k = \lfloor (N+1)p \rfloor$ since in a binomial distribution this is the most likely outcome.[5] An equilibrium is then a p which shifts the weights in such a way that the weighted average is zero. In the auxiliary game, there exists an integer $m \in \{1, \ldots, N\}$ such that for $0 < k < m$ we have $\theta_k^C > \theta_k$ and for $k \geq m$ we have $\theta_k^C \leq \theta_k$. This implies that $\theta_{N+1}^C < \theta_{N+1}$. So, in determining the equilibrium in the original game p_{N+1} has to be smaller than p_{N+1}^C to compensate.

The following proposition is thus obtained.

Proposition 6.4 *If, for all $k > 0$, $\theta_{k+1} > \theta_k$, then S_N is increasing in N.*

Suppose we have a game such that $N = 2$, $\theta_0 = -1$, $\theta_1 = \alpha > 0$, $\theta_2 = \beta$ (where $\beta < \alpha$) and $S_2 < S_1$ From Proposition 6.3 we then know that $\alpha < 1$. In fact an even stricter bound for α can be obtained. It is sufficient to show that $S_2 < S_1$ for $\beta = 0$. Straightforward calculations show that:

$$S_1 = \left(\frac{\alpha}{1+\alpha}\right)^2 \tag{6.11}$$

$$S_2 = \left(\frac{2\alpha}{1+2\alpha}\right)^3. \tag{6.12}$$

If $\alpha = 0$ or $\alpha = \sqrt{20}/8 - 1/4$, then $S_1 = S_2$. Since for $\alpha \geq 1$ it is known that $S_1 < S_2$, it must be that, for $\alpha \in (0, \sqrt{20}/8 - 1/4)$, $S_2 < S_1$. Note that

[5] The truncation function is denoted by $\lfloor \cdot \rfloor$.

$\sqrt{20}/8 - 1/4$ is approximately 0.3090.

Proposition 6.5 *For $\alpha \in (0, 0.3090)$, there exists $\beta < \alpha$ such that, in the game characterized by $\theta_0 = -1$, $\theta_1 = \alpha$ and $\theta_2 = \beta$, we have $S_2 < S_1$.*

Games in which $S_2 < S_1$ are limited to a small subset of the parameter space.

Proposition 6.4 shows that if the parameters θ_k are increasing, then this is also a sufficient condition for the probability of breakdown to be increasing. Increasing θ_k's can be interpreted as negative externalities. As Remark 6.2 shows, if the parameters decrease sufficiently fast, then the probability of breakdown can decrease. Hence, sufficiently strong positive externalities in the payoff (*and* $|\theta_0| > |\theta_1|$) might mitigate the paradox.

6.5 Applications

6.5.1 R&D and teamwork

An economic application is teamwork. Suppose that the R&D-department of a company consists of $N+1$ employees. Each employee has to decide whether to exert effort or not. The effort is interpreted as contributing to the research project. If an employee exerts effort, then some innovation is created and the employee bears a cost c. The employee values this innovation, but the value only depends on the number of employees that have exerted effort (including the employee himself). Denote this value by $V(k)$, where k is the number of participants, and assume, without loss of generality, that $V(0) = 0$. The payoff function, using the notation of Section 6.2, is $F(a_i, A_i) = V(a_i + A_i) - a_i c$. Notice that

$$\theta_k = V(k) - V(k+1) + c. \tag{6.13}$$

To fit into my framework, the following restrictions are needed:

$$\theta_0 = -V(1) + c < 0 \tag{6.14}$$

$$\theta_k = V(k) - V(k+1) + c > 0 \text{ for all } k \geq 1 \tag{6.15}$$

The condition $|\theta_0| < |\theta_1|$ yields (from Proposition 6.3):

$$V(2) < 2c. \tag{6.16}$$

A possible interpretation of (6.16) is the following: If you could put in twice the effort, then this is not profitable. While you may benefit if others exert effort, it does not pay to do the extra effort yourself. This ensures that the probability that the patent is not discovered is increasing in the number of co-workers.

If we assume that $V(\cdot)$ is concave, then the θ_k's are increasing in k, i.e.

$$\frac{\partial \theta_k}{\partial k} = V'(k) - V'(k+1) > 0. \tag{6.17}$$

Under concavity of $V(\cdot)$, the probability that the patent is not discovered is also increasing in the number of co-workers. Concavity can be interpreted in the following manner: the effort of each employee increases the value of the innovation, but the additional value diminishes. However, this lack of 'synergy' does not imply that team work is necessarily a bad thing. In a cooperative equilibrium, that maximizes $NV(k) - ck$ with respect to k, the optimal number of employees that exert effort can be larger than one.

6.5.2 International debt overhang

Another application, due to Kaneko and Prokop (1993), is debt overhang. Kaneko and Prokop (1993, p.2) define it as follows:

> The term 'debt overhang' expresses a situation where a sovereign country has borrowed money from foreign banks and has not succeeded in fulfilling the scheduled repayments for some period. The existence of the debt overhang is a serious problem for the debtor country, which keeps the country in a bad economic situation.

I discuss a symmetric version of their model. In this game there are $N+1$ creditors and one country that is in debt. The debtor country is treated as part of the environment and not as an active player.

Each creditor holds a debt of size D. I assume that total outstanding debt is constant, i.e. $(N+1)D = \bar{D}$. If the number of creditors increases, then they also hold a smaller share of total debt. This assumption is needed to ensure that below the parameters θ_k do not depend on N.

The creditors can sell off their debt at a secondary market price. The market price is given by the pricing function $P : \mathbb{R}^+ \to \mathbb{R}^+$ whose domain is total outstanding debt. For instance, if a creditor has an outstanding debt of D and total outstanding debt is $\bar{D}$, then it can sell of this debt at $DP(\bar{D})$.[6] The larger the debt (and the higher the probability of default), the lower the price of buying back the debt.

[6] This price is often far below 1; in the case of Bolivia in 1985 the price of buying back one dollar of debt was 5 cent (cf. Kaneko and Prokop, 1993, p.2).

I follow Kaneko and Prokop (1993) and suppose that each creditor can either sell its debt now at a price $P(\bar{D})$ or wait for one period and then sell it. If the creditor postpones, then its debt will rise to βD, where $\beta > 1$ is one plus the interest rate. However, the present value of the debt will stay D. The price of buying back debt in the next period is $P(\beta(N+1-k)D)$, where k is the number of other creditors that have not postponed. Each creditor has two actions: sell or postpone.

For the moment, and contrary to Kaneko and Prokop (1993), assume that $P(\cdot)$ is a decreasing and differentiable function. Since $P(\cdot)$ is a decreasing function, we have that:

$$DP(\bar{D}) > DP(\beta\bar{D}), \tag{6.18}$$

where the present value of debt stays constant at D, but the price of buying back debt does change. From (6.18) it follows that it is optimal for the creditors to sell if none of the other creditors have sold. I will show that this game can be considered as a participation game. In terms of the participation game selling is equal to participating and (6.18) is equal to Assumption 6.2. In order to qualify as a participation game, it must also be optimal to postpone if one other creditor sells:

$$DP(\beta(\bar{D} - D)) > DP(\bar{D}), \tag{6.19}$$

which boils down to $\beta < \frac{N+1}{N}$. This is equal to Assumption 6.1 if either the number of creditors is small or the interest rate is not too large. From Corollary 6.1, it follows that there is a unique symmetric equilibrium in this game.

Since the present value of debt, D, is constant for a creditor, the creditor is just interested in the price at which the debt is sold off. Hence, the payoff function is can be simplified to this price. The parameters of the model are defined as follows:

$$\theta_k = P(\beta(\bar{D} - kD)) - P(\bar{D}). \tag{6.20}$$

If $\beta < \frac{N+1}{N}$ and $P(\cdot)$ is decreasing, then $\theta_0 < 0$ and $\theta_k > 0$ for all $k \geq 1$ ensuring that we have a participation game. Notice that:

$$\frac{\partial\theta_k}{\partial k} = -\beta DP'(\beta(\bar{D} - kD)) > 0, \tag{6.21}$$

and thus θ_k is increasing in k. Hence, it follows directly from Proposition 6.4, that an increase in the number of creditors implies that the probability that no one sells is increasing. Therefore having more creditors has a detrimental effect on debt overhang. This result is, unlike the results of Kaneko and Prokop (1993), valid for a small number of creditors.

Kaneko and Prokop (1993) do not assume that $P(\cdot)$ is decreasing: they merely assume that $P(0) \geq P(\bar{D}) > P(\beta\bar{D})$. In this case, Proposition 6.3 can still be used to formulate sufficient conditions for the probability that no one sells to increase. First, to have $\theta_k > 0$, for all $k \geq 1$, we need:

$$P(\beta(\bar{D} - kD)) > P(\bar{D}). \tag{6.22}$$

The condition $|\theta_0| < |\theta_1|$ (from Proposition 6.3) yields

$$2P(\bar{D}) < P(\beta\bar{D}) - P(\beta(\bar{D} - D)). \tag{6.23}$$

If (6.22) and (6.23) both hold, then the probability that no one sells increases in the number of creditors. Remark that the sufficient condition of Proposition 6.3 does not yield easily interpretable conditions. The reasonable assumption of a decreasing pricing function, however, implies negative externalities and gives the desired result immediately.

6.5.3 Oil spills

Suppose an oil tanker has had an accident in the middle of a sea and there are $N+1$ countries surrounding this sea. As long as the oil does not hit the coast, the effects on the environment and the cost of cleaning the oil are relatively small. After the oil spill has reached the coast, the cost of cleaning the oil and the effects on the wildlife are large. The oil spill effects each country equally. Let $c_L(k)$ denote the cost of cleaning the oil before it reaches the coastline. This cost is made dependent on k, the number of countries that decide to clean the oil immediately. Let c_H denote the cost of cleaning the oil afterward. Each country must either clean immediately at a cost $c_L(k)$ or wait and risk to clean the oil at a cost c_H. Assume that, for all $k \geq 1$, $c_H > c_L(k) > 0$. This reflects the consideration that when countries take action early, then the cost for each country is likely to be lower. We have a participation game with the following parameters:

$$\theta_0 = F(0,0) - F(1,0) = c_L(1) - c_H < 0, \tag{6.24}$$

$$\theta_k = F(0,k) - F(1,k) = c_L(k) > 0 \qquad \text{for all } k \geq 1. \tag{6.25}$$

Now the probability that the oil spill is not cleaned immediately, is increasing in the number of countries if $|\theta_0| < |\theta_1|$. Here this boils down to $c_H < 2c_L(1)$, i.e. the probability is definitely increasing as long as the cost of waiting is not too high.

6.6 Conclusion

It was shown in Proposition 6.3 that, under a fairly weak condition only dependent on the ratio of θ_0 and θ_1, the probability of breakdown in binary public good games is increasing in group size. This paradoxical result (not only will people participate less, the group as a whole will participate less) appears very robust to different payoff specifications. Proposition 6.3 also showed that on average less than one person participated and the public good is underprovided (more than one player participating being socially optimal). For very large groups, the probability of breakdown always exceeds $\mathrm{e}^{-1} \approx 37\%$.

Proposition 6.4 showed that if the parameters θ_k are increasing, then this is also a sufficient condition for the probability of breakdown to be increasing. Increasing θ_k's could be interpreted as negative externalities.

The examples in Section 6.5 showed that, in a wide variety of economic applications, Propositions 6.3 and 6.4 can be used to find sufficient conditions under which the probability of breakdown is increasing. Moreover, these conditions tend to be weak.

Further research, however, should focus on necessary conditions. A promising avenue for research is hinted by Remark 6.2. As this Remark showed, if the parameters decrease sufficiently fast, then the probability of breakdown could decrease. Hence, sufficiently strong positive externalities in the payoff (*and* $|\theta_0| > |\theta_1|$) might mitigate the paradox.

6.A Proof of Proposition 6.1

It is convenient to present the following Lemmata first.

Lemma 6.1 *There exists no symmetric pure strategy equilibrium.*

Proof. Suppose all players participate: $a_i = 1$ and $A_i = N$ for all i. Each player then receives a payoff $F(1, N)$. However, by Assumption 6.1 the best-response is to not participate since $F(0, N) > F(1, N)$. Suppose all players do not participate: $a_i = 0$ and $A_i = 0$ for all i. However, by Assumption 6.2 the best-response is to participate. ■

Lemma 6.2 *There are $(N + 1)$ non-symmetric pure strategy equilibria.*

Proof. Let $A_i \geq 1$ be the number of other people who participate. Those who participate have a payoff $F(1, A_i) < F(0, A_i)$ (Assumption 6.1). Clearly, if $A_i \geq 1$, this cannot be an equilibrium. This does not hold for $A_i = 0$, since $F(1, 0) > F(0, 0)$ (Assumption 6.2). Only one player participates is an equilibrium. ■

Lemma 6.3 *There exists a unique symmetric mixed-strategy equilibrium where everyone participates with probability $p \in (0, 1)$.*

Proof. We know that p must be a real positive root of $g_N(\cdot)$ defined in (6.4). First, it has to be shown that it exists. Second, it has to be shown that it is unique.

It is straightforward to show that there is at least one root on the unit interval. Note that by Assumption 6.1 and 6.2:

$$g_N(0) = F(0, 0) - F(1, 0) = \theta_0 < 0 \tag{6.26}$$

$$g_N(1) = F(0, N) - F(1, N) = \theta_N > 0, \tag{6.27}$$

and remark that $g_N(\cdot)$ is a continuous function. This also implies that the number of roots is positive and odd.

To proof unicity, notice that, for $p \neq 1$, $g_N(\cdot)$ can be rewritten as:

$$g_N(p) = (1 - p)^N \sum_{k=0}^{N} \binom{N}{k} r^k \theta_k, \tag{6.28}$$

where $r \equiv p/(1 - p)$. This allows us to look at the roots of $\sum_{k=0}^{N} \binom{N}{k} r^k \theta_k$. By using Descartes' sign rule (see Atkinson, 1989, p. 95), we now easily obtain that the root is unique. ■

In determining all asymmetric equilibria, it is useful to think of a person joining the group. Suppose that a group of $N+1$ players has already decided to participate with the equilibrium probability of p_N. If an extra person would join this group, what would this person do? The next Lemma shows that this person becomes a bystander: someone who does not participate.

Lemma 6.4 (The bystander) *The following is an equilibrium: in a group of size $N+2$, let $N+1$ players participate with probability p_N and the remaining player (the bystander) will not participate.*

Proof. For the players who participate this is obviously an equilibrium strategy. For the bystander, the benefit of not participating over participating is given by:

$$\sum_{k=0}^{N+1} \binom{N+1}{k} p_N^k (1-p_N)^{N-k}\theta_k = p_N^{N+1}\theta_{N+1} + \sum_{k=0}^{N} \binom{N}{k} p_N^k (1-p_N)^{N-k}\theta_k \times \left[\frac{(N+1)(1-p_N)}{N+1-k}\right]. \quad (6.29)$$

Note that (i) the first term on the RHS is positive and (ii) the second term on the RHS is also positive. In order to see the latter recall that $\sum_{k=0}^{N}\binom{N}{k}p_N^k(1-p_N)^{N-k}\theta_k = 0$. Observe that this sum consists of one negative term at $k=0$ followed by exclusively positive terms. The new weighing factor $(N+1)/(N+1-k)$ puts more emphasis on these positive terms since it increases as k increases. Hence, the bystander will not participate. ■

Lemma 6.5 *Any non-empty subset of the group of players can play the symmetric mixed equilibrium for a subset of that size with the rest of the group being bystanders.*

Proof. Follows from Lemma 6.4. ■

Lemma 6.6 *There exists no non-symmetric strictly mixed equilibrium.*

Proof. Suppose that each member of the group in an arbitrary mixed equilibrium participates with probability p_i, where $i = 1, \ldots, N+1$. Suppose the focus is on a subset of the group without members $n_1, \ldots, n_m$. Call this subset $\mathcal{N} \equiv \aleph - \{n_1, \ldots, n_m\}$. The probability that k players in this subset participate is:

$$\sum_{\mathfrak{N}\subseteq\mathcal{N}, \#\mathfrak{N}=k} \left\{ \left(\prod_{j\in\mathfrak{N}} p_j\right) \left(\prod_{\ell\notin\mathfrak{N}} (1-p_\ell)\right) \right\}. \quad (6.30)$$

This expression is obtained by summing over all subsets of $\mathcal{N}$ of size k and calculating the probability that for each of these subsets the k people in this subset participate. Let $\Pr(k|n_1,\ldots,n_m)$ denote this probability. If $k<0$ or $k>\#\aleph-m$, then $\Pr(k|n_1,\ldots,n_m)$ is defined to be zero. Notice that $\Pr(k|i)=\Pr(k|i,j)\times(1-p_j)+\Pr(k-1|i,j)\times p_j$.

For any $i,j\in\aleph$, in a strictly mixed equilibrium the following should hold:

$$\sum_{k=0}^{N}\Pr(k|i)\theta_k=0 \tag{6.31}$$

$$\sum_{k=0}^{N}\Pr(k|j)\theta_k=0. \tag{6.32}$$

Both player i and j should be indifferent between participating and not participating. This is equivalent with:

$$\sum_{k=0}^{N}[\Pr(k|i,j)\times(1-p_j)+\Pr(k-1|i,j)\times p_j]\theta_k=0 \tag{6.33}$$

$$\sum_{k=0}^{N}[\Pr(k|i,j)\times(1-p_i)+\Pr(k-1|i,j)\times p_i]\theta_k=0, \tag{6.34}$$

which implies that in equilibrium:

$$p_i=p_j=\frac{\sum\Pr(k|i,j)\theta_k}{\sum\Pr(k|i,j)\theta_k-\sum\Pr(k-1|i,j)\theta_k}. \tag{6.35}$$

Since $p_i=p_j$ holds for any i and any j, this concludes the proof. ■

Proposition 6.1 follows easily from Lemmata 6.1–6.6. Observe that for each subset of $\aleph$ (with the exception of the empty set), there is an equilibrium where the members of the subset participate with some probability and the other players do nothing. The number of subsets is 2^{N+1}.

6.B Proof of Proposition 6.2

Note that $g_{N+1}(p)$ can be rewritten as:

$$g_{N+1}(p)=p^{N+1}\times\theta_{N+1}+(1-p)\times\sum_{k=0}^{N}\frac{N+1}{N+1-k}\binom{N}{k}p^k(1-p)^{N-k}\theta_k. \tag{6.36}$$

To see that $g_{N+1}(p_N)>0$, note that:

1. The first term on the RHS of (6.36) is positive,

2. The second term on the RHS of (6.36) is also positive in p_N. It is known that $\sum_{k=0}^{N} \binom{N}{k} p_N^k (1-p_N)^{N-k}\theta_k = 0$. Observe that this sum consists of one negative term at $k = 0$ followed by exclusively positive terms. The new weighing factor $(N + 1)/(N + 1 - k)$ puts more emphasis on the latter terms since it increases as k increases.

It is known from Lemma 6.3 that $g_{N+1}(\cdot)$ has a unique positive real root at p_{N+1}. Hence, all p for which $g_{N+1}(p) > 0$ must be either larger or smaller than p_{N+1}. Notice that $g_{N+1}(1) > 0$ and therefore $p_N > p_{N+1}$.

6.C Proof of Proposition 6.3

The symmetric Nash-equilibrium is the root of $g_N(\cdot)$. Remark that if $p \neq 1$ we can write

$$g_N(p) = (1-p)^N \sum_{k=0}^{N} \binom{N}{k} r^k \theta_k, \tag{6.37}$$

where $r = p/(1 - p)$. Note that r is a monotonic transformation from $[0, 1)$ to $[0, \infty)$. Since the root of $g_N(\cdot)$ is never equal to 1, it is possible to focus on the roots of:

$$h_N(r) = \sum_{k=0}^{N} \binom{N}{k} r^k \theta_k. \tag{6.38}$$

Let r_N denote the unique positive real root of $h_N(\cdot)$. Note that $p_N = \frac{r_N}{1+r_N}$. The assumption that $p_1 < \frac{1}{2}$ is equivalent to $r_1 < 1$. It is convenient to present the following two Lemmata first.

Lemma 6.7 *For every $N \in \mathbb{N}$, we have $r_{N+1} < \frac{N}{N+1} r_N$.*

Proof. If $h_{N+1}(Nr_N/(N + 1)) > 0$, then $r_{N+1} < \frac{N}{N+1} r_N$. This follows from $h_{N+1}(0) < 0$ and $h_{N+1}(\cdot)$ continuous. It has to be shown that $h_{N+1}(Nr_N/(N + 1)) > 0$ given that $h_N(r_N) = 0$. Let $r \equiv r_N$ and $s \equiv \frac{N}{N+1} r$ to simplify notation. Observe that:

$$\binom{N}{0}\theta_0 + \binom{N}{1} r\theta_1 = \binom{N}{0}\theta_0 + \frac{N+1}{N}\binom{N}{1}\left(\frac{N}{N+1} r\right)\theta_1 \tag{6.39}$$

$$= \binom{N+1}{0}\theta_0 + \binom{N+1}{1} s\theta_1 \tag{6.40}$$

and, since $h_N(r) = 0$ that,

$$\binom{N}{0}\theta_0 + \binom{N}{1}r\theta_1 = -\binom{N}{2}r^2\theta_2 - \ldots - \binom{N}{N}r^N\theta_N. \tag{6.41}$$

Combining these two results gives:

$$h_{N+1}\left(\frac{N}{N+1}r\right) > 0 \tag{6.42}$$

$$\Leftrightarrow -\binom{N}{2}r^2\theta_2 - \ldots - \binom{N}{N}r^N\theta_N + \binom{N+1}{2}s^2\theta_2 + \ldots + \binom{N+1}{N}s^N\theta_N + s^{N+1}\theta_{N+1} > 0 \tag{6.43}$$

$$\Leftrightarrow -\binom{N}{2}r^2\theta_2 - \ldots - \binom{N}{N}r^N\theta_N + \frac{N+1}{N-1}\binom{N}{2}s^2\theta_2 + \ldots + \frac{N+1}{1}\binom{N}{N}s^N\theta_N + s^{N+1}\theta_{N+1} > 0 \tag{6.44}$$

$$\Leftrightarrow \sum_{i=2}^{N}\left\{\binom{N}{i}\theta_i\left[\frac{N+1}{N+1-i} \times \left(\frac{N}{N+1}\right)^i - 1\right]r\right\} + s^{N+1}\theta_{N+1} > 0. \tag{6.45}$$

Since $s_{N+1}\theta_{N+1} > 0$, it suffices to show that the following is true for every N and for all $i = 2, \ldots, N$:

$$\frac{N+1}{N+1-i} \times \left(\frac{N}{N+1}\right)^i - 1 \geq 0. \tag{6.46}$$

Rearrange this to obtain:

$$\frac{N^i}{(N+1-i)(N+1)^{i-1}} - \frac{(N+1-i)(N+1)^{i-1}}{(N+1-i)(N+1)^{i-1}} = \frac{N^i - (N+1-i)(N+1)^{i-1}}{(N+1-i)(N+1)^{i-1}} \geq 0, \tag{6.47}$$

which is equivalent with:

$$N^i - (N+1-i)(N+1)^{i-1} \geq 0. \tag{6.48}$$

The proof is by induction. Let $i = 2$. Now (6.48) boils down to $N^2 - (N-1)(N+1) = N^2 - N^2 + 1 = 1 \geq 0$. Next, assume that (6.48) is true for an $i > 2$ and show that it also is true for $i+1$. Assume that $N^i - (N+1-i)(N+1)^{i-1} \geq 0$ is true. It has to be shown that $N^{i+1} - (N-i)(N+1)^i \geq 0$ is true. Rewrite this as:

$$\begin{aligned}
&N^{i+1} - (N-i)(N+1)^i \\
&= N \times N^i - (N-i)(N+1)^{i-1}(N+1) \\
&= N \times N^i - (N+1-i)(N+1)^{i-1}(N+1) + (N+1)^{i-1}(N+1) \\
&= N \times N^i - (N+1-i)(N+1)^{i-1}N \\
&\quad + (N+1)^{i-1}(N+1) - (N+1-i)(N+1)^{i-1} \\
&= N[N^i - (N+1-i)(N+1)^{i-1}] + i(N+1)^{i-1} > 0,
\end{aligned} \tag{6.49}$$

since $N^i - (N+1-i)(N+1)^{i-1} \geq 0$ by assumption and since $i(N+1)^{i-1} > 0$. ■

Lemma 6.8 *If $r_{N+1} < \frac{N}{N+1} r_N$ and $r_1 < 1$, then for all N we have $(1 + r_N)^{N+1} > (1 + r_{N+1})^{N+2}$.*

Proof. It is sufficient to show that this results holds for the limit case of $r_{N+1} = \frac{N}{N+1} r_N$. This defines a recursive relation and therefore r_N can be rewritten as an explicit function of N, i.e. $r_N = \frac{r_1}{N}$. It has to be shown that:

$$\left(1 + \frac{r_1}{N}\right)^{N+1} > \left(1 + \frac{r_1}{N+1}\right)^{N+2}. \tag{6.50}$$

Let $T_N \equiv \left(1 + \frac{r_1}{N}\right)^{N+1}$. Note that $\lim_{N\to\infty} T_N = \mathrm{e}^{r_1}$ since $\mathrm{e}^x = \left(1 + \frac{x}{N}\right)^N$. Using the logarithmic derivative rule, it follows that:

$$\frac{\mathrm{d}T_N}{\mathrm{d}N} = \left[\log\left(1 + \frac{r_1}{N}\right) - \frac{N+1}{N+r_1}\frac{r_1}{N}\right]\left(1 + \frac{r_1}{N}\right)^{N+1}, \tag{6.51}$$

which has to be strictly negative. Equivalently:

$$\log\left(1 + \frac{r_1}{N}\right) < \frac{N+1}{N+r_1}\frac{r_1}{N}. \tag{6.52}$$

Since $r_1/N < 1$ by assumption, we can use the following power function (Apostol, 1967: pp. 388–391) to replace $\log\left(1 + \frac{r_1}{N}\right)$ with:

$$\frac{r_1}{N} + \sum_{k=1}^{\infty}\left[-\frac{1}{2k}\left(\frac{r_1}{N}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{N}\right)^{2k+1}\right]. \tag{6.53}$$

It is easily checked that $\frac{r_1}{N} < \frac{N+1}{N+r_1}\frac{r_1}{N}$ if $r_1 < 1$. Remark further that:

$$-\frac{1}{2k}\left(\frac{r_1}{N}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{N}\right)^{2k+1} < 0 \tag{6.54}$$

$$\Leftrightarrow -\frac{1}{2k} + \frac{1}{2k+1} \times \frac{r_1}{N} < 0 \tag{6.55}$$

$$\Leftrightarrow \frac{r_1}{N} < \frac{2k+1}{2k}, \tag{6.56}$$

which is true if $r_1 < 1$. These two facts imply that

$$\log\left(1+\frac{r_1}{N}\right) = \frac{r_1}{N} + \sum_{k=1}^{\infty}\left[-\frac{1}{2k}\left(\frac{r_1}{N}\right)^{2k} + \frac{1}{2k+1}\left(\frac{r_1}{N}\right)^{2k+1}\right] < \frac{r_1}{N} < \frac{N+1}{N+r_1}\frac{r_1}{N}. \quad (6.57)$$

Therefore, T_N is decreasing in N if $r_1 < 1$, which concludes the proof. ■

Now we are able to provide the proof of Proposition 6.3.

Proof of part *(i)* We have to show that:

$$(1-p_N)^{N+1} < (1-p_{N+1})^{N+2} \quad (6.58)$$

$$\Leftrightarrow \left(\frac{1}{1+r_N}\right)^{N+1} < \left(\frac{1}{1+r_{N+1}}\right)^{N+2} \quad (6.59)$$

$$\Leftrightarrow (1+r_N)^{N+1} > (1+r_{N+1})^{N+2}. \quad (6.60)$$

Since $p_1 < \frac{1}{2}$ implies that $r_1 < 1$ and, from Lemma 6.7, we know that $r_{N+1} < \frac{N}{N+1}r_N$, the result follows from Lemma 6.8.

Proof of part *(ii)* Let $\hat{r}_N \equiv \frac{r_1}{N}$. Note that $r_N < \hat{r}_N$ and $(1+r_N)^{N+1} < (1+\hat{r}_N)^{N+1}$. Taking the limit we obtain:

$$\lim_{N\to\infty}(1+r_N)^{N+1} < \mathrm{e}^{r_1}. \quad (6.61)$$

Since $\lim_{N\to\infty}(1+r_N)^{N+1} = \left[\lim_{N\to\infty}(1-p_N)^{N+1}\right]^{-1}$ and substituting $r_1 = p_1/(1-p_1)$ leads to:

$$\lim_{N\to\infty}(1-p_N)^{N+1} > \mathrm{e}^{-p_1/(1-p_1)}, \quad (6.62)$$

where the RHS is equal to $\mathrm{e}^{\theta_0/\theta_1}$.

Proof of part *(iii)* The expected number of participants is the probability of participation, p_N, times group size, $N+1$ (since the number of participants follows a binomial distributions). We have to proof that $(N+1)p_N < 1$. Substituting $p_N = r_N/(1+r_N)$ yields:

$$r_N < \frac{1}{N}. \quad (6.63)$$

It is known that $r_1 < 1$ and $r_{N+1} < \frac{N}{N+1}r_N$. Therefore $r_N < \frac{1}{N}r_1 < \frac{1}{N}$.

6.D Derivation of the probability of breakdown in Remark 6.2

The probability of participation is implicitly defined by:

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \theta_k = 0, \tag{6.64}$$

which may be rewritten as:

$$-(1-p)^N + \alpha \sum_{k=1}^{N} \binom{N}{k} (\beta p)^k (1-p)^{N-k} = 0. \tag{6.65}$$

If $\beta < 1$, then some straightforward algebra yields:

$$\begin{aligned} &-(1-p)^N + \alpha[\beta p + (1-p)]^N \\ &\qquad \times \sum_{k=1}^{N} \binom{N}{k} \left(\frac{\beta p}{\beta p + (1-p)}\right)^k \left(\frac{1-p}{\beta p + (1-p)}\right)^{N-k} = 0. \end{aligned} \tag{6.66}$$

Now the part behind the summand is a proper binomial again and represents 'the probability of at least one success'. Exploiting this fact, we obtain:

$$-(1-p)^N + \alpha[\beta p + (1-p)]^N \left(1 - \left(\frac{1-p}{\beta p + (1-p)}\right)^N\right), \tag{6.67}$$

which is equivalent to:

$$-(1-p)^N + \alpha[\beta p + (1-p)]^N - \alpha(1-p)^N = 0, \tag{6.68}$$

which is easily solved for p. This yields:

$$p_N = \frac{\sqrt[N]{(1+\alpha)/\alpha} - 1}{\sqrt[N]{(1+\alpha)/\alpha} - (1-\beta)}. \tag{6.69}$$

Applying $S_N = (1-p_N)^{N+1}$ gives the desired result.