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Formulating a conjecture through an identification of robust invariants with a dynamic geometry system

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ABSTRACT

Conjecturing has been considered to inspire the need for proof, enhance the understanding of proofs, and construct a valid proof. This study describes students' processes of formulating a Euclidean geometry conjecture in the form of a conditional statement through constructing a geometric figure and using measuring and dragging modalities of a dynamic geometry system (DGS). To accomplish this aim, we adapted the existing conjecturing model by Baccaglini-Frank and Mariotti ([2010]. Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–253. https://doi.org/10.1007/s10758-010-9169-3) and used it to analyse students' conjecturing processes. Our participants were prospective mathematics teachers (PMTs) during their first year at an Indonesian university, but the findings can be useful for secondary school students in other countries. We selected and categorized episodes from task-based interviews with eight PMTs a week after a teaching intervention. We interpreted their identification of robust invariants during constructing, dragging, and measuring. Our findings indicated that the adapted model was appropriate to describe PMTs' processes of conjecturing, which emerged through an exploration that involved robust invariants. We found that PMTs determined these invariants as premises or conclusion of the conjecture by observing the measure of parts of the constructed figure during dragging. The findings indicated that the measuring and dragging modalities of DGS supported PMTs in conjecturing.

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KEYWORDS
Conjecturing; measuring modalities; dragging modalities; robust invariant; dynamic geometry

1. Introduction

Proving is internationally recognized as an essential skill for mathematics and mathematics education. A proof is a mathematical argument which is a connected sequence of assertions for or against a mathematical claim with three characteristics: a set of accepted

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statements, modes of argumentations and modes of argument representations (Stylianides, 2007). Proving, as an activity in search for a proof, starts by a precursor activity such as conjecturing (Weber et al., 2017). Conjecturing has been considered to inspire the need of proof, to enhance understanding of proof, and to construct a valid proof (Yang & Lin, 2012). In addition, the conjecturing activity is critically important for students because this activity can support and give meaning to students' engagement with proof (G. J. Stylianides, 2008).

There has been a considerable number of studies conducted on conjecturing in mathematics education, particularly in recent years (e.g. Fernández-León et al., 2020; Lin et al., 2012; Lynch & Lockwood, 2019; Mariotti & Pedemonte, 2019; Pawlaschyk & Wegner, 2020). This literature has established the essential role that conjecturing can play in teaching proof and proving. Specifically, there are several studies on the use of dynamic geometry systems (DGS) supporting conjecturing (e.g. Baccaglini-Frank & Mariotti, 2010; Botana et al., 2015; Christou et al., 2004; De Villiers & Heidemen, 2014; Mariotti, 2012). Some of these studies focused on proposing theoretical principles of task design for conjecturing (Lin et al., 2012) and developing models of the conjecturing process (Baccaglini-Frank & Mariotti, 2010). Other studies experimentally investigated the use of DGS in conjecturing and proving, particularly in high school classrooms (Mariotti, 2012) and in a small case study of three prospective primary teachers (Christou et al., 2004).

In our Indonesian context, students who enrol in mathematics or mathematics education programmes at the university start to learn formal mathematical proof, including geometric proof, in the first year of study (Abadi & Chairani, 2020; Anwar et al., 2021). One of the topics in geometry proof relates to congruence and we assume that the use of measuring modalities of DGS suggested by Olivero and Robutti (2007) could support students in identifying congruencies. Therefore, in this study, we investigate how prospective mathematics teachers formulate a conjecture through an identification of congruencies using dragging and measuring modalities of a DGS. All our participants did not learn geometry proof at secondary school. So, the geometrical proof background of these university students may be comparable to secondary school students in other countries. A difference in geometrical proof between Indonesian universities and schools in other countries relates to the formal notation used to introduce geometrical ideas. Thus, our findings might contribute, internationally, to the teaching and learning of proof, particularly to conjecturing, at secondary school level.

2. Theoretical background

2.1. Conjecturing, proving and their inter-relationship

Fernández-León et al. (2020) asserted that conjecturing and proving are interrelated mathematical practices in developing new mathematical knowledge. Lannin et al. (2011) defined conjecturing as an act to develop a conjecture, which is a statement that is tentatively thought to be true but is not known by students to be true. Fernández-León et al. (2020) argued that conjecturing activities, encouraging students to formulate a conjecture through an open problem, may help them produce a proof of that conjecture. Research findings by Boero et al. (2007) revealed that combining conjecturing and proving may allow grade 8 students (13–14 years) to use arguments from the exploration process of formulating a conjecture to construct proofs. Lin et al. (2012) contended that asking students to formulate their own conjecture and to prove it will engage students in understanding multiple functions of proof and internalizing proof schemes (i.e. three ways of thinking: external conviction, empirical and deductive). They stated that asking students to prove the proposition that has already been shown to be true might enable students to conceive proving as a ritual.

Stylianides (2009) argued that conjecturing leads to the development of proofs and supports the development of proof competency. In addition, studies by Baccaglini-Frank and Mariotti (2010, 2011) found that conjecturing helped students understand a conditional link between premises and a conclusion as a conditional statement (specifically, a geometric proposition). This understanding of the conditional link is part of understanding the structure of proof; in writing-oriented tasks it relates to understanding of the chaining elements of proof (Miyazaki et al., 2017), and in reading-oriented tasks it relates to understanding of the logical relation between premises and a conclusion (Yang & Lin, 2008).

2.2. Dynamic geometry system in formulating conjectures

The dragging feature makes the DGS environment different from the traditional paper-and-pencil environment. What makes the DGS pedagogically more powerful is not only the construction facility but also the direct manipulation of the constructed figure. The DGS is programmed such that figures have an intrinsic logic that places the figure elements in a hierarchy that corresponds to a relationship of the logical conditionality (Arzarello et al., 2011). Botana et al. (2015) stated that a DGS could help students identify interesting properties of the constructed figure. For instance, when students construct the circumcentre, the centroid and the orthocentre of a triangle, the DGS will 'notify' the students that these points are collinear.

Hanna and de Villiers (2008) stated that a DGS can serve as a context for making conjectures about geometric figures and then lead to proof construction. They also stressed that the dragging modalities provided by the DGS help users see logical relationships between geometric objects. Aligned with this, Baccaglini-Frank and Mariotti (2011) argued that the dragging tools can be exploited to support the formulation of conjectures.

Another feature offered by the DGS is the measuring modality. Olivero and Robutti (2007) conducted a study aimed at investigating how measuring tools affect formulating conjectures and the proving process. Their results indicated that the modalities of measuring helped grade 8 students (13–14 years) connect the graphical observation of the figure to Euclidean geometry which enabled them to formulate a conjecture.

In this study, we introduced the use of a DGS to support PMTs to (1) construct dynamic figures step-by-step by using its construction tools, and (2) formulate a conjecture in the form of a conditional statement (if–then statement) by using the dragging and measuring modalities of the DGS. The use of dragging modalities helps students identify invariant properties (functioning as premises in an if–then statement), which are determined directly by the construction steps, and invariant properties (functioning as conclusions) as a consequence through the dragging experience. These constructions and dragging activities were expected to support students not only to perceive the invariants but also to perceive the conditional link between the invariants. We also introduced the measuring features of the DGS to help students identify certain properties of the invariants such as congruence of line segments and angles.

2.3. Dynamic exploration in a dynamic geometry system: invariants, dragging and measuring modalities

In a DGS, geometric figures and their properties can be presented in a dynamic format so that any constructed figure can be acted upon, dragged, by using a mouse or by touching a screen. A point used to construct a GeoGebra-figure and upon which other elements depend, is referred to as a *base point*. After construction of the figure, the solver drags base points and observes the movement of the figure on the screen, noticing the changes and stability of properties.

Baccaglini-Frank and Mariotti (2010) considered invariants for the development of the Maintaining Dragging (MD)-conjecturing model and tested it in their experimental classroom. The model was used as a framework to describe aspects of the cognitive processes leading to the formulation of a conjecture based on a DGS-constructed figure; these aspects may be motivated by tasks that participants engage in. Furthermore, this model offered a protocol for teachers and researchers to design learning sequences to support students in formulating conjectures and to hypothesize students' thinking during the conjecturing process.

In the MD-conjecturing model, dragging modalities involved during conjecturing are random dragging, maintaining dragging, dragging with trace activation, and dragging test. The random dragging refers to the dragging of a base point on the screen, randomly, to look for interesting configurations or constant properties of the constructed GeoGebra figure. The maintaining dragging refers to the dragging of a base point to maintain a certain property of the GeoGebra figure. The dragging with trace activation concerns the dragging of a base point by activating the trace tool to observe the movement path of the dragged base point. Finally, the dragging test relates to the dragging of base points to check whether the constructed figure, representing the movement path, maintains the desired properties.

The stable conditions of the figure are called *invariants*. There are two types of invariants, namely direct and indirect invariants. *Direct invariants* are identified by the geometrical relations determined by the construction steps, and *indirect invariants* by the consequence of the construction (Baccaglini-Frank & Mariotti, 2011). Solver's experience of construction and dragging enables him/her not only to distinguish between direct and indirect invariants but also to notice the logical dependency between them. Baccaglini-Frank and Mariotti (2010) called this logical dependence a *conditional link*. There are three main steps in the MD-conjecturing model, namely:

(1) Determining a first invariant while randomly dragging base points of the constructed figure.
(2) Searching for a second invariant while dragging a base point, ensuring that the first invariant is maintained by using maintaining dragging with trace activated. The second invariant is interpreted from the trace path of the dragged base point.
(3) Establishing and checking a Conditional Link (CL) between the first and second invariants by using a dragging test.

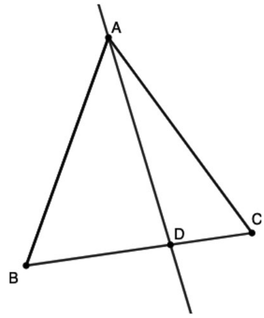


Figure 1. A, B, C, D are base points; a line is constructed as a line $\overleftrightarrow{AD}$ through the point A and intersects segment $\overline{BC}$ at point D.

In the context of the appearance of invariants in literature, two types are distinguished: *robust* invariants and *soft* invariants (Healy, 2000; Mariotti, 2014). These two types of invariants relate to dragging modalities used to perceive them. The robust invariant is perceived by random dragging of a base point of the figure, and the soft invariant can be perceived by a specific way of dragging a base point. So, the soft invariant is only invariant of the dynamic figure under certain conditions. In this study, we follow Baccaglini-Frank (2012) in their use of the term *direct robust invariant (DRI)* to refer to a robust invariant determined directly by the step-by-step construction, and *indirect robust invariant (IRI)* as a robust invariant corresponding to geometrical properties that are consequences of the construction. For example, consider the following construction steps, as shown in Figure 1.

#1. Construct an isosceles triangle ΔABC with $\overline{AB} \cong \overline{AC}$
#2. Construct a line $\overleftrightarrow{AD}$ which is a line through the point A and intersects segment $\overline{AB}$ at point D For numbered lists

Upon dragging, the constructed triangle ABC remains an isosceles triangle, because the condition '$\overline{AB}$ congruent to $\overline{AC}$' is conserved. Also, the property 'angle $\angle ABD$ congruent to angle $\angle ACD$' is conserved because it is a logical consequence. All these conditions '$\overline{AB}$ congruent to $\overline{AC}$' and 'angle $\angle ABD$ congruent to angle $\angle ACD$' are robust invariants for random dragging of base points. In this case, '$\overline{AB}$ congruent to $\overline{AC}$' is a direct robust invariant (DRI) and 'angle $\angle ABD$ congruent to angle $\angle ACD$' is an indirect robust invariant (IRI). In contrast, invariants like '$\overleftrightarrow{AD}$ perpendicular to $\overline{BC}$', '$\overleftrightarrow{AD}$ is a bisector of angle $\angle BAC$', and 'D is a midpoint of segment $\overline{BC}$' are soft invariants, as they can be induced only by particular ways of dragging the base points. So, they are not always invariant for any random dragging. However, these invariants cannot be induced automatically through observing the figural configuration (shape) of the constructed GeoGebra figure. Students could perceive the properties of congruence of line segments and angles by using additional information if they use the GeoGebra tool that measures angles and segments (Olivero & Robutti, 2007).

2.4. Our modified maintaining dragging-conjecturing model

In the MD-conjecturing model, it is required that the solver chooses the first soft invariant as a prospective conclusion through observation and interpretation of a figural configuration. Next, the solver drags a chosen point such that the first soft invariant is conserved. Thus, the solver is expected to notice a path of the point movement as the second invariant in which the first soft invariant is maintained. This expectation requires expert use of dragging modalities because the solver needs to be precise to drag the point and observe the figural configuration carefully with limited information.

However, the characteristic of the soft invariants used in this conjecturing process might also bring the solver some obstacles in perceiving invariance. The solver needs to perform certain actions (e.g. maintaining dragging) to perceive this kind of invariants, because the properties of the constructed figure are not always stable. To do so, the solver needs precise dragging to see the trace as a representation of the movement of a certain part of the constructed figure in order to perceive the second invariant as a premise of the conjecture. This performance is also crucial for perceiving the logical connection (i.e. conditional link) between the second invariant as a premise and the first invariant as a conclusion. The requirement had been cautioned by Baccaglini-Frank and Mariotti (2010) about the 'expert use' of dragging modalities which play an essential role in determining the invariants. They argued that determining both soft invariants was not an easy task. This claim was supported by their finding that 70% of the participants (grade 10–11 high school students, aged 16 and 17) was not able to make conjectures through the conjecturing process following their model (Baccaglini-Frank & Mariotti, 2011).

Considering some obstacles and required expert use of dragging modalities in the MD-conjecturing model, we propose in this study a modification of the original model by Baccaglini-Frank and Mariotti (2010). The modification refers to the types of invariants determined by given construction steps in a task, the types of DGS modalities used, and the order of steps during conjecturing. In the task, the commands of construction of the figure are designed to construct direct robust invariants (DRIs) as the premises of the conjecture and indirect robust invariants (IRIs) as the conclusion of the conjecture. As our model involves robust invariants, all invariants are simultaneously perceived through dragging. We argue that identifying a DRI as the first step is a better choice as a starting point for conjecturing, because DRIs are determined by the construction steps. Then, as a second step, IRIs can be searched by observing the figural configuration through the dragging of base points. After having observed both robust invariants, the solver can perceive the logical dependence between these robust invariants through observing the stability of both invariants during dragging. So, as opposed to the MD-conjecturing model, in our model, the solver starts to determine DRIs as the first invariant which will be the premises of the conjecture, and then searches for the IRI as the second invariant, which will be the conclusion of the conjecture. Consequently, in our model the order of the steps follows the natural flow of a conditional statement, if p then q, and gives students opportunities to engage in a forward-thinking process from premises and consecutive propositions, relating these to premises and conclusion, and connecting them logically.

Another consideration for our modification of the model comes from the geometric context in which the conjecture to be formulated by our students is embedded: congruent triangles. For instance, we expect that the solver formulates a conjecture such as 'If P

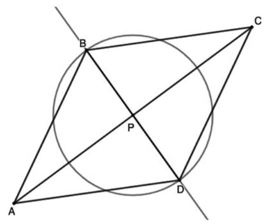


Figure 2. ABCD as a dynamic figure constructed in DGS.

is a midpoint of segment $\overline{AC}$, and $\overline{PB}$ is congruent to $\overline{DP}$, then $\overline{CB}$ is congruent to $\overline{AD}$' (see Figure 2). In this case, besides dragging modalities, the measuring modalities play an essential role in inducing these congruencies as invariants. We also used a robust approach to support students' conjecturing as recommended by Healy (2000) to avoid tedious and time-consuming activities as the students receive immediate feedback from the dragging.

Our new conjecturing model has two roles namely: (1) designing a learning sequence to support students in formulating a conjecture through an exploration in the DGS environment, and (2) describing student's conjecturing process in the context of open problems, particularly, where the DRIs are given by a step-by-step construction of a geometric figure and lead to the formulation of a conjecture. The term 'open' refers to a situation that solvers can find a variety of conjectures that may be true.

There are three main steps in this modified model, namely.

2.4.1. Step 1: determining direct robust invariants

Initially, the solver determines the DRIs by observing the conditions of the geometric figure determined by the construction steps. To determine the DRIs of the figure, the solver uses measuring tools of the DGS to perceive observable stable conditions of the figure (e.g. congruence of angles or line segments) and random dragging to see the stability of the observed conditions.

The solver can also connect the observed conditions (i.e. DRIs), which are determined by the construction steps and perceived while dragging, to the concepts of Euclidean geometry (e.g. geometric properties, definitions, axioms). For instance, the solver may determine that two segments are congruent because the two segments are radii of the same circle. Identifying the conditions of parts of the figure determined by the construction steps may not be simple for students; the solver may observe properties of parts of the figure which are not determined by the steps. In this situation, the solver needs to focus on the components of the constructed figure (e.g. segments, midpoint of segment, etc.) mentioned explicitly in the construction steps.

2.4.2. Step 2: searching for indirect robust invariant caused by the direct robust invariants

After finding the DRIs, the solver may start using the DGS measures of parts of the constructed figure (e.g. segments and/or angles), which are not given directly by the construction steps. Then, the solver may drag base points and observe the stability of the equal measures. By dragging s/he perceives not only the equality of the observed parts but also the stability of the determined DRIs. The solver may use random dragging with trace activation to see the movement of a certain part of the constructed figure. While the solver is dragging base points, s/he may perceive the stable condition of DRIs and IRI simultaneously.

2.4.3. Step 3: establishing and verifying a conditional link between direct robust invariants and indirect robust invariant

At this moment, the solver has already identified two stable conditions (invariants): the DRIs and the IRI. The solver might articulate the link, as a logical dependence, between these two invariants using phrases like 'every time I drag a point, this condition is true [the condition of the DRIs] and this condition is true [the condition of IRI]' or 'when the construction determines that this condition is true [DRI], this condition is always true [IRI]'. To become convinced of the existence of a link between these conditions, the solver may drag base points (i.e. perform a dragging test), in which the trace of the point is activated to see the stability of these conditions. The use of dragging with trace activation gives visual feedback that the dragging of the base point maintains these two invariants, DRIs and IRI, simultaneously for any movement of the base points.

The key requirement for perceiving logical dependence (i.e. a conditional link (CL)) between DRIs as premises and IRI as conclusion is the solver's experience of (1) direct and indirect control, that is control over the direct and indirect construction of the robust invariants, and (2) simultaneity while dragging. So, after finding DRIs and choosing IRI, the solver can observe their simultaneity and perceive the stability of properties of an indirect robust invariant, IRI, as a consequence of the direct robust invariants, DRIs.

3. Aim and research question

This study aims to describe how prospective mathematics teachers (PMTs) from an Indonesian university formulate a conjecture in the form of a conditional statement in interaction with GeoGebra, particularly, while constructing figures and using the measuring and dragging modalities during the intervention designed based on the modified MD-conjecturing model. The research question of the study is:

> How do the measuring and dragging modalities of a DGS support students to formulate a geometry conjecture?

4. Research method

This study is a qualitative research study that uses our conjecturing model as a tool to analyse data collected from task-based interviews after they followed an introductory course on geometry proof. Before we describe the task-based interviews, we will describe the intervention designed to enable students to formulate a conjecture and to prove it subsequently, but this paper focuses on the conjecturing process.

4.1. Teaching intervention

Our classroom intervention consisted of six lessons in an introductory course on geometry proof. The first and second lessons focused on defining geometrical definitions (e.g. segments, angles, vertical angles, polygon, triangles, etc.) through an exploration with the use of GeoGebra. The third lesson aimed to support students in (1) understanding how to construct a geometric figure by reference to given step-by-step constructions using GeoGebra, and (2) formulating conjectures, which were related to axioms of congruent triangles (e.g. axioms of Side-Angle-Side and Angle-Side-Angle), with the use of dragging modalities of GeoGebra. The fourth until sixth lessons focused on the comprehension and the construction of a geometric proof. In this current study, we focused on the third lesson aiming to support students in formulating conjectures. While the findings related to the effect of the intervention were described in our previous publication (Anwar et al., 2021).

4.2. Participants

Sixty prospective mathematics teachers (PMTs) followed the six geometry lessons during their first year at a public university in Indonesia. Students (aged 18–19) were 11 males and 49 females. Eight of these students participated in interviews based on their willingness to participate, their ability to use the GeoGebra app (construction tools and dragging modalities) and their communication skills (who could speak up their thinking during working the task).

4.3. Data collection

The data of this study were collected in individual task-based interview sessions with eight PMTs conducted a week after the sixth lesson of the teaching intervention to an introductory geometry proof course. Each of the eight individual interviews lasted for about 60 min. Before interviews, the interviewer explained his interest in understanding PMTs' thinking and how participants could help him achieve this goal by speaking out loud and explaining as much as they could. During interviews, the PMTs were asked to work on the interview task of Figure 3; in particular, they were asked to construct a GeoGebra figure and formulate a geometric proposition as a conjecture.

While the participants were solving the interview task, the interviewer observed and sometimes gave technical guidance only when the responses were essential for subsequent interview questions. For instance, the interviewer reminded a participant of a specific tool

Construct the geometric figure using GeoGebra by following these steps:

#1. Construct a segment $\overline{AB}$
#2. Draw a midpoint C on the segment $\overline{AB}$
#3. Draw a line $\overleftrightarrow{DC}$
#4. Construct a circle with center in C and radius CD
#5. Construct an intersecting point E between the circle (#4) and line $\overleftrightarrow{CD}$
#6. Draw a segment $\overline{ED}$
#7. Construct a polygon ADBE

Make conjectures based on the constructed figure. Write the conjectures including the geometric figure.

Figure 3. Conjecturing task given in interview session.

that can be used to construct a certain part of figure (e.g. midpoint of segments and intersection of segments). Sometimes, the interviewer asked students for clarification without an indication of correctness (Goldin, 1997; Maher & Sigley, 2014) to get insights into their conjecturing processes. In the interview session, we collected (1) audio recordings, which were transcribed verbatim, (2) participants' written answers, and (3) video recordings of PMTs' explorations created by screen video-capturing software running in the background on the interview iPads.

The task was chosen for the following reasons: (1) the expected conjectures were in the form of 'if then' statements, which were seen and practiced by the participants during the intervention, (2) the topic of the task (i.e. congruent triangles) was relevant to the course, and (3) the steps (#1 through #7) for the construction of the figure of the task were similar with those from Baccaglini-Frank and Mariotti (2010).

4.4. Data analysis

The data analysis consisted of two phases: categorization and interpretation. During the ***first phase (selection and categorization of episodes)***, we identified the relevant episodes of the conjecturing process from the transcript. Then, the episodes were divided into categories, which were predefined sets of components of the conjecturing model from Baccaglini-Frank and Mariotti (2010): (1) Determining direct robust invariants (DRIs), (2) Searching for an indirect robust invariant (IRI), and (3) Establishing and checking a Conditional Link (CL) between DRIs and IRI.

In the ***second phase (interpretation)***, the first author interpreted students' contributions in each episode based on the transcripts extracted from audio recordings, the screen-captured video of the construction and dragging process, and the students' written answers. The first author used these multiple data sources for triangulation purposes. Then, the three authors, as a research team, examined these interpretations to test their credibility (Miles & Huberman, 1994). They determined that an interpretation was credible only after weighing alternative interpretations and searching for potential disconfirming evidence.

5. Findings

In this section, we describe how PMTs formulated conjectures in the form of conditional statements (i.e. if–then statements) with the use of GeoGebra Geometry application while performing the construction steps of a geometrical figure. We categorized three main processes of conjecturing: determining DRIs, searching for IRI, and establishing and verifying a conditional link. In this section, we present each of these processes. To avoid redundance, we used PMT26 as a paradigmatic case and supplied this with other participants when needed. In the transcripts, the letter 'R' refers to the teacher-researcher (interviewer), and the texts between [] refer to descriptions of physical activities (i.e. behaviour, body language).

5.1. Determining direct robust invariants

In this part, we describe the process of determining DRIs by exemplifying five episodes from two participants, which represent all eight participants. During construction, PMTs were expected to determine the following two DRIs: (1) the equality of the measure of segments $\overline{AC}$ and $\overline{BC}$, and (2) the equality of the measure of segments $\overline{CD}$ and $\overline{CE}$.

We found three different ways, from three PMTs, to determine these DRIs. The first way, used by all eight participants, was identifying the properties of the constructed figure directly from the construction steps of the task. For instance, in Episode 1, PMT2 stated 'the second step constructs a condition that C is a midpoint of line segment $\overline{AB}$' (turn 3) and wrote 'C is a midpoint of segment $\overline{AB}$'. So, it seems to us that PMT2 determined the first DRI 'C is a midpoint of $\overline{AB}$' (without making use of the GeoGebra modalities) because that was literally stated by a construction step.

Episode 1	
1	PMT2: the first step constructs a condition: a line segment $\overline{AB}$
2	R: then,
3	PMT2: the second step constructs a condition that C is a midpoint of line segment $\overline{AB}$

The second way to determine DRIs was connecting a condition of a constructed figure (determined directly by the construction steps) to the associated Euclidean geometry (e.g. definition of a concept). This was found to be used by all participants. For instance, in Episode 2, PMT26 stated that 'I get a given condition' (turn 2) by the construction step. The condition referred to '$\overleftrightarrow{DC}$ is a bisector of line segment $\overline{AB}$' (turn 6) which was given from 'the first until third steps' (turn 6). So, PMT26 determined that the first DRI is '$\overleftrightarrow{DC}$ is a bisector of line segment $\overline{AB}$' based on the information given by the first, second and third steps (see construction steps in Figure 4). In our interpretation, PMT26 connected the figural information (see the figure he constructed in Figure 4) from the three construction steps of the task to the definition of a bisector of a line segment from Euclidean geometry.

Episode 2	
1	R: what do you get from the construction steps?
2	PMT26: I get a given condition [condition determined by construction steps]
3	R: how do you get it?
4	PMT26: from this [pointing to step-by-step construction]
5	R: okay, please explain how you find them.
6	PMT26: here, from the first until third step, I found a condition that $\overleftrightarrow{DC}$ is a bisector of AB [writing '#1–#3 $\overleftrightarrow{DC}$ is bisector of line-segment $\overline{AB}$' in his answer sheet]

Likewise, PMT26 determined the second DRI, '$\overline{CD}$ and $\overline{CE}$ are congruent' (turn 2) in Episode 3 through the figural information that line segments $\overline{CD}$ and $\overline{CE}$ are radii of

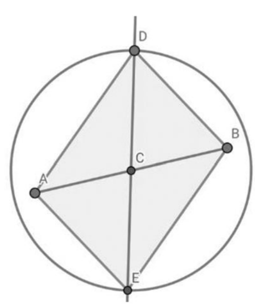


Figure 4. GeoGebra figure constructed by PMT26.

the circle with centre C (Figure 3) from steps 4 and 5 of the tasks. Our interpretation is that PMT26 determined that '$\overline{CD}$ and $\overline{CE}$ are congruent' as the DRI by connecting the figural information to the properties of a circle, particularly that all radii of a circle are congruent. This interpretation was supported by PMT26's geometrical explanation about how he concluded that $\overline{CD}$ is congruent to $\overline{CE}$ (turn 4) when the interviewer asked for a clarification.

Episode 3

1	R: so, what is condition defined by these steps [the fourth and fifth construction steps]?
2	PMT26: $\overline{CD}$ and $\overline{CE}$ are congruent [writing '#4–#5 determining point E, $\overline{CD} \cong \overline{CE}$' in his answer sheet]
3	R: okay, I want to ask you about this: how do you know that $\overline{CD}$ is congruent to $\overline{CE}$?
4	PMT26: this is a circle with centre C and radius $\overline{CD}$ and so the radius of the circle is $\overline{CD}$ and this the intersection point [pointing to point E] this [point $\overline{CE}$] equals $\overline{CD}$ then.

In a similar episode, we found that PMT2 used the second way to determine DRI but then she verified it by using measuring and dragging modalities of DGS. She made the measures of the line segments visible and dragged the points E, D, B and D. At that time, PMT2 did not say why she used dragging and measuring modalities and whether she focused on the equality of the measures of segments CD and CE. However, indications that this was used by PMT2 for establishing the correctness of the DRIs were based on the fact that using measurement or dragging was not requested by the task or the interviewer.

The third way was connecting information acquired by the measurement of parts of the constructed figure and random dragging to Euclidean geometry (i.e. definition, axioms, or theorems). The third way is different from the second way, because PMTs did not use dragging and measuring for verification, but used these for determining DRIs. We found six participants used dragging modalities to verify the DRIs. For example, in Episode 4, PMT51 made the measures of segments visible in the constructed figure (see Figure 5). During the random dragging, PMT51 perceived a stable property (equality of measures of $\overline{AC}$ and $\overline{CB}$), which was evident in his response that 'there are two segments having equal measure' (turn 2) and 'line segments $\overline{AC}$ and $\overline{CB}$ are congruent' (turn 4) as a property given by the first two steps. So, our interpretation is that PMT51 determined 'line segments $\overline{AC}$ and $\overline{CB}$ are congruent' as the first DRI by connecting the information acquired by measuring and dragging to the definition of a congruent segment in Euclidean geometry he learned during the course.

Episode 4

1	R: can you identify a condition determined by the construction steps, for instance the first two steps?
2	PMT51: [pointing to line segments $\overline{AC}$ and $\overline{CB}$] there are two segments having equal measure
3	R: the second step determines the condition?
4	PMT51: line segments $\overline{AC}$ and $\overline{CB}$ are congruent [writing 'given condition: $\overline{AC} \cong \overline{CB}$' on his answer sheet]

5.2. Searching for an indirect robust invariant

In this part, we describe the process of determining the indirect robust invariant (IRI) by exemplifying an episode from one participant, which is typical for all participants. PMTs determined IRI during random dragging by observing the measures of parts of the figure and connecting this to Euclidean geometry. For instance, in Episode 5, after determining

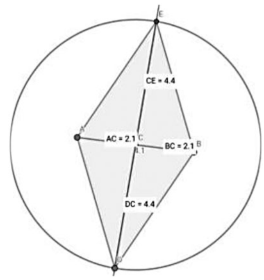


Figure 5. PMT51 made the measures of segments $\overline{AC}$, $\overline{CB}$, $\overline{CD}$ and $\overline{CE}$visible.

the DRIs, PMT26 dragged the base points D, B, D, A and observed the equality of the measures of the segments $\overline{DB}$ and $\overline{AE}$, and $\overline{AD}$ and $\overline{BE}$ (turn 2), see Figure 4. Then, he stated '$\overline{DB}$ is congruent to $\overline{AE}$', '$\overline{AD}$ is congruent to $\overline{BE}$' (turns 2, 4). Our interpretation is that PMT26 connected the equality of the measures during dragging to the definition of congruent segments (i.e. segments having equal measure) to determine that '$\overline{DB}$ is congruent to $\overline{AE}$' is an IRI.

Episode 5	
1	R: okay, what is next?
2	PMT26: looking for conditions maintained … [dragging base point D, B, D, A with traced activated and observing the measure of segments $\overline{AD}$, $\overline{DB}$, $\overline{BE}$, and $\overline{AE}$] … a condition maintained is that $\overline{DB}$ is congruent to $\overline{AE}$, then …
3	R:anything else?
4	PMT26: $\overline{AD}$ is congruent to $\overline{BE}$

In line with PMT26, to search the IRI, PMT2 started to hide a figure of a circle with centre C and line CD (Episode 6, turn 2). Here, PMT2 seemed to focus on the main figure representing the given conditions (DRIs). Then, she added the measures of segments CD, CE, CB, CA, AE, DB, AD, and EB and dragged point D (turn 4). In this situation, PMT2 used measuring and dragging modalities of GeoGebra. Our interpretation is that PMT2 found conditions of the constructed figure as a prospective of the IRI by observing the stability of the measures of two pairs of segments (i.e. DB and AE, and EB and AD) during dragging, as shown in Figure 6. So, here, PMT2 chose certain properties of the constructed figure that were not defined directly by the construction steps as candidates of IRI, but based on the stability of the property (e.g. the measure of segments and angles) observed during dragging of any base point. She also connected the measured equality of line segments to the definition of congruent line segment to express the IRI 'DB is congruent to AE; EB is congruent to AD' (turn 10) as IRI.

Further, we found that only three of eight PMTs activated the trace when dragging base points. However, all eight PMTs determined the IRI. This indicates that the use of dragging with trace activated did not play an essential role in determining the IRI. For some cases, the dragging with trace activated might help students to perceive the relational connection between DRI and IRI from the occurrence of their movement. But for others, trace on could hide observed properties.

Episode 6	
1	R: so, it means from the steps you found the given condition, what next?
2	PMT2: find other condition [hiding the figure of circle with centre C and line CD, adding the measure of segment CD and CE, dragging points E, D, B, and D]
3	R: what did you find?
4	PMT2: [adding the measure segments CB and CA, dragging point B, adding the measure of angles ACE and DCB, dragging point B]
5	R: did you find any special configuration?
6	PMT2: the measure of angle ACE and DCB is equal
7	R: so, do you mean this condition is caused by the given condition?
8	PMT2: oh, no sir
9	R: if not, so which one?
10	PMT2: [adding the measure of segments AE, DB, AD, EB, dragging point D] there are two side are congruent namely DB is congruent to AE, EB is congruent to AD

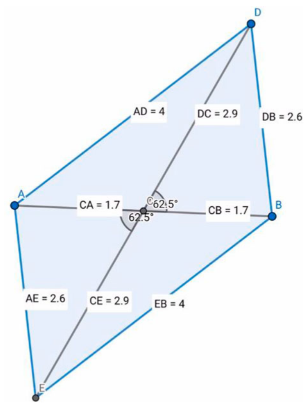


Figure 6. PMT2's GeoGebra figure showing the use of measuring to search the IRI.

5.3. Establishing and verifying a conditional link

In this part, we describe the process of establishing a conditional link between DRIs and IRI by PMTs. We exemplify three episodes from two participants, which reflect the processes of all eight participants.

5.3.1. Perceiving a conditional link

We found one way used by PMTs to eventually identify the conditional link between DRIs and IRI. That was by observing the stable properties of DRIs and IRI during a dragging test of base points that move simultaneously. During the dragging test, PMT26 perceived two DRIs '$\overline{CD}$ equals $\overline{CE}$' and equality of '$\overline{AC}$ and $\overline{BC}$' (Episode 7, turns 1 and 5); an IRI '$\overline{AD}$ is equal to $\overline{BE}$' (turn 7); and the stability of both DRIs and IRI simultaneously (turns 7 and 9), see Figure 7.

In our interpretation, PMT26 perceived this stability as an indication for a link between the DRIs and IRI. During the process of searching for IRI, PMT26 focused on observing the segments $\overline{AD}$, $\overline{DB}$, $\overline{BE}$, and $\overline{EA}$ (Episode 7, turn 2), whose properties were caused by the construction of the figure determined by the construction steps (i.e. DRIs). This indicates that PMT26 realized that the IRI was a condition caused by the DRIs. At the time of perceiving the conditional link and causality between DRIs and IRIs, PMT26 saw the DRIs as premises and the IRIs as conclusion of a conditional statement. So, it can be concluded that PMT26 perceived the conditional link between DRIs and IRI by observing the stability of both invariants and its link during the dragging test.

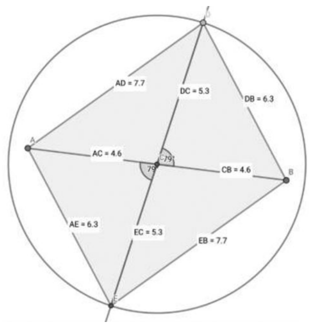


Figure 7. PMT26's GeoGebra figure with the measures of line segments.

Episode 7	
1	PMT26: $\overline{CD}$ equals $\overline{CE}$
2	R: $\overline{CD}$ equals $\overline{CE}$?
3	PMT26: They [$\overline{CD}$ is congruent to $\overline{CE}$] are maintained [by dragging] while I am dragging the measure is equal.
4	R: maintained by what?
5	PMT26: by these steps [pointing to step-by-step construction], also $\overline{AC}$ and $\overline{BC}$
6	R: okay, anything else you observed
7	PMT26: $\overline{AD}$ is equal to $\overline{BE}$
8	R: always?
9	PMT26: [dragging points A and D] yes

5.3.2. *A 'false' conditional link*

During dragging, some of the PMTs perceived a 'false' conditional link when they found an inappropriate IRI that is not caused by the DRIs. For instance, in Episode 8, PMT32 made the measures of segments and angles visible, and dragged points B and A. Then, PMT32 observed that the measures of vertical angles are equal (turn 3) and chose this congruence of vertical angles as a stable property and an IRI candidate (turn 5). However, the congruence of vertical angles (e.g. $\angle ACD$ and $\angle BCE$ in Figure 5) is a property of the constructed figure. So, PMT32 could have seen the existence of the vertical angles as another DRI, and that the congruence of vertical angles was caused by the existence of the vertical angles. So, this congruence of vertical angles was an invariant, caused by only one of the DRIs (i.e. the existence of vertical angles), so an inappropriate IRI.

After the interviewer asked PMT32 to observe the case of the vertical angles '$\angle GJI$ and $\angle FJH$' (turn 8), as shown in Figure 8, PMT32 concluded that the stability of congruence of vertical angles was not caused by the condition of equality of line segments in the figure (turn 13). A counterexample helped PMT32 realize this 'false' conditional link. Next, PMT32 identified the stability of the equality of 'segments $\overline{DB}$ and $\overline{EA}$' (turn 17) during dragging B, D, and A (turn 15) as an appropriate IRI.

5.3.3. *Formulating a conjecture*

After seeing the DRIs as premises, the IRI as a conclusion, and the conditional link between DRI and IRI, PMTs started formulating a conjecture as the connection between premises and conclusion, in the form of a conditional proposition. In writing the conjecture, all eight participants used similar structures of a conjecture including pen-and-pencil figures as a representation (e.g. Figure 8). For instance, in Episode 9, PMT26 wrote a conjecture 'If $\overline{DC}$

Episode 8

1	PMT32: [adding the measure of segments $\overline{AC}$, $\overline{CB}$, $\overline{CD}$, $\overline{CE}$, and angles $\angle ACD$, $\angle DCB$, $\angle BCE$ and $\angle ACE$, dragging points B and A]
2	R: what is condition you see that they are stable?
3	PMT32: angles
4	R: which angles?
5	PMT32: the vertical angles [pointing to two pairs of vertical angles $\angle ACD$ and $\angle BCE$, and $\angle ACE$ and $\angle DCB$]
6	R: do you mean this condition was caused by the given condition [pointing to segments $\overline{AC}$, $\overline{CB}$ and $\overline{CE}$ and $\overline{CD}$]
7	PMT32: yes, they are congruent
8	R: okay, let's see this figure [constructing a pair of vertical angles $\angle GJI$ and $\angle FJH$, adding the measure of angles $\angle GJI$ and $\angle FJH$, dragging point G] what do you see?
9	PMT32: the measure of angles is equal
10	R: how about the segments?
11	PMT32: they are not equal [the measure is not equal]
12	R: what does it mean?
13	PMT32: oops … these [pointing to vertical angles] are not caused by these [pointing to the segments $\overline{CD}$, $\overline{CE}$ and $\overline{AC}$ and $\overline{CB}$]
14	R: so, which condition?
15	PMT32: [adding the measure of sides $\overline{DB}$, $\overline{AE}$, $\overline{BE}$ and $\overline{AD}$, dragging points B, D, A] the segments [the measure of segments is equal]
16	R: which segment do you observe?
17	PMT32: these [pointing to segments $\overline{DB}$ and $\overline{AE}$]

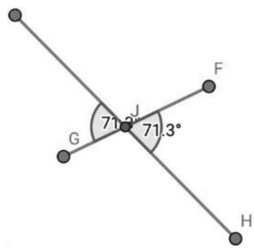


Figure 8. Vertical angles GJI and FJH constructed by the interviewer as a counterexample.

is a bisector of segment $\overline{AB}$, and $\overline{CD} \cong \overline{CE}$ on a polygon ADBE, then $\overline{AD} \cong \overline{BE}$and $\overline{BD} \cong \overline{AE}$' (turn 2).

Then, in Episode 9 PMT26 revised the initial conjecture by deleting the IRI '$\overline{AD} \cong \overline{BE}$' (turn 2) and drew a figure by hand representing the conjecture (turn 6), as shown in Figure 9. PMT26 wrote the conjecture as a geometric proposition (which is familiar to students from textbooks or from presentations in the classroom in the form of 'If X, then Y' combined with a figure) together with the geometric figure representing the proposition.

We found that all eight participants formulated correct geometric propositions, both in terms of premises and conclusions and in formulation as an 'If X, then Y' statement. However, we saw some variation in the propositions in the way they wrote the equivalence of the premises (e.g. the measure of segments $\overline{CA}$ and $\overline{CB}$ is equal, $\overline{CA} \cong \overline{CB}$, point C is a midpoint of line segment $\overline{AB}$, or $\overline{DC}$ is a bisector of line segment $\overline{AB}$) and the conclusions (e.g. congruence of line segments $\overline{AD}$ and $\overline{BE}$ or congruence of angles $\angle CAE$ and $\angle CBD$),

Episode 9	
1	R: so, now you are writing the conjecture
2	PMT26: [writing the conjecture: 'If DC is a bisector of segment $\overline{AB}$, and $\overline{CD} \cong \overline{CE}$ on a polygon ADBE, then $\overline{AD} \cong \overline{BE}$'] this $\overline{CD}$ equals to $\overline{CE}$ might be confusing [deleting the statement '$\overline{BD} \cong \overline{AE}$'] is it enough?
3	R: any other information needs to be added [highlighting the conjecture by covering other texts in the paper]
4	PMT26: ehm … the figure
5	R: okay, please complete it
6	PMT26: [drawing the figure]

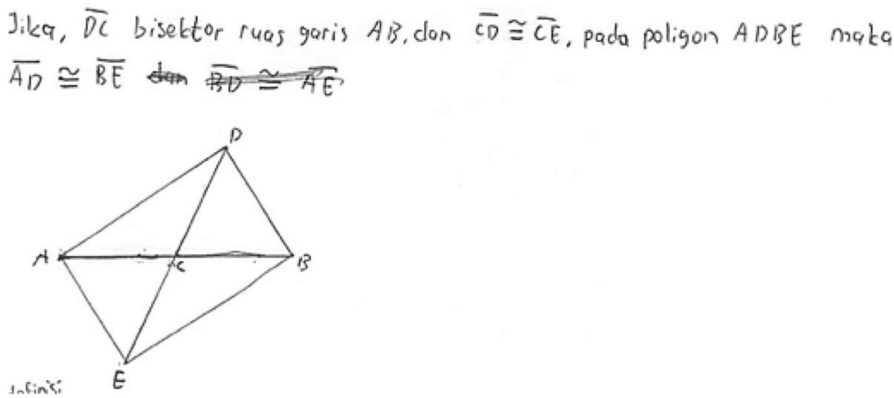


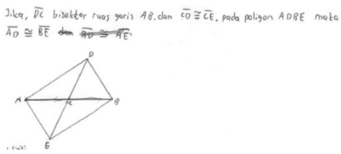
Figure 9. PMT26's conjecture 'If, $\overline{DC}$ is a bisector of line segment *AB*, and $\overline{CD} \cong \overline{CE}$, on the polygon *ADBE* then $\overline{AD} \cong \overline{BE}$' with the pen-and-pencil drawing.

see Figure 10. This variety in geometric propositions can be explained by the open nature of the interview task.

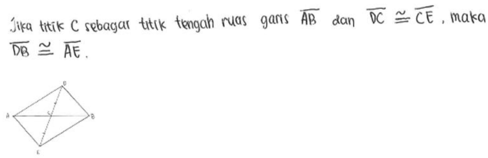
6. Discussion and conclusions

Literature indicates that conjecturing supports the development of proof competency, particularly proof construction, which plays an essential role in mathematics and in mathematics education (Boero et al., 2007; Stylianides, 2009). In this study, we presented a conjecturing model which is a modification of the original MD-conjecturing model developed by Baccaglini-Frank and Mariotti (2010). We modified the MD-conjecturing model based on a literature review and our own teaching experience. The modifications were twofold: we adapted theoretical constructs and conceptualized these as direct robust invariants (DRIs) and indirect robust invariant (IRI), and we included measuring modalities and the associated dragging modalities of the DGS. We included only robust invariants, because soft invariants remain stable only when the solvers perform special actions (e.g. maintaining dragging) (Arzarello et al., 2011; Baccaglini-Frank & Mariotti, 2010, 2011; Mariotti, 2012).

In this study, the modified model was used to design a teaching intervention of 150 min in a geometry course. The intervention designed based on the modified MD-conjecturing model introduced construction tools and the use of dragging and measuring modalities of a DGS, aimed at supporting PMTs in constructing GeoGebra figures and in conjecturing a geometric proposition. The data of this paper was collected during individual



10(a)



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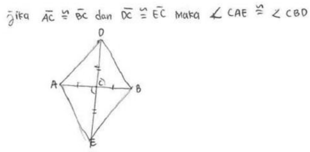

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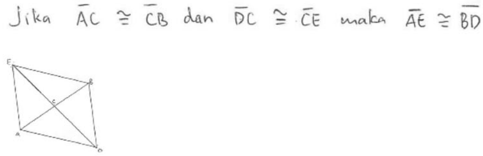

10(d)

Figure 10. Variation of conjectures formulated by the PMTs. (a) PMT26' conjecture 'If, $\overline{DC}$ is a bisector of line segment AB and $\overline{CD} \cong \overline{CE}$ in a polygon ADBE, then $\overline{AD} \cong \overline{BE}$'. (b) PMT2's conjecture 'If point C is a midpoint of line segment $\overline{AB}$ and $\overline{CD} \cong \overline{CE}$, then $\overline{DB} \cong \overline{AE}$' which was equivalent with PMT28 and PMT33's conjectures. (c) PMT32's conjecture 'If $\overline{AC} \cong \overline{CB}$ and $\overline{DC} \cong \overline{CE}$, then $\angle CAE \cong \angle CBD$' which was equivalent with PMT60's conjecture. (d) PMT51's conjecture 'If $\overline{AC} \cong \overline{CB}$ and $\overline{DC} \cong \overline{CE}$, then $\overline{AE} \cong \overline{BD}$' which was equivalent with PMT40's conjecture.

interview sessions with eight PMTs after the intervention. During the interviews, PMTs formulated conjectures of a geometric proposition in the form of a conditional statement. Our data analysis (interview transcripts and PMTs' written conjectures) indicated that the DGS activities supported PMTs in formulating a geometric conjecture.

At the first step of conjecturing, PMTs used construction steps to determine the DRIs, connected those to Euclidean geometry, and/or used measuring and dragging modalities of the DGS. The measurement information along with dragging was used to manifest certain stable properties (i.e. invariants), such as congruence. Determining the DRI by connecting properties to Euclidean geometry involved universal instantiation. In universal instantiation, a geometric property is true for a specific geometric object because that property is known to be true for all geometric objects of that kind (Durand-Guerrier et al., 2012; Velleman, 2006). For instance, PMTs used the property of a circle that all radii are congruent to instantiate the congruence of two segments. Miyazaki et al. (2017) stated that the understanding of universal instantiation is part of structural understanding which is needed to construct a geometric proof. So, a conclusion is that determining a DRI may provide opportunities for PMTs to apply universal instantiation (Durand-Guerrier et al., 2012).

In the next step, while searching for the indirect robust invariant (IRI), PMTs focused on certain elements of the constructed figure (e.g. segments and angles) that were not directly determined as DRIs by the construction steps within Euclidean Geometry (in which the algorithm of GeoGebra was programmed). PMTs identified the properties of those elements during dragging. In particular, the dragging of base points changed their position and the measures of parts of the figure as opposed to what remained invariant (e.g. congruence of segments or angles). Specifically, PMTs identified the IRIs by using the measuring modalities and random dragging modalities of GeoGebra. Olivero and

Robutti (2007) showed that measurements helped students connecting the figural information to a part of theory and enabled them to formulate a conjecture. In line with Olivero and Robutti, our data indicated that all PMTs identified more than one possible IRI that could be chosen as a conclusion of the conjecture. The PMTs spent only a few minutes in using dragging to ultimately find an IRI. Also Healy (2000) found that the use of robust invariants helped solvers find an invariant more effectively in terms of time consumption.

At the last step of conjecturing, after finding DRIs and IRIs, PMTs checked the stability of these invariants simultaneously through a dragging test. In other words, PMTs observed the stability of the two invariants and the connection between them. Leung et al. (2013) asserted that in order for draggers to formulate a conjecture in the form of an 'if premise, then conclusion' statement, they need to perceive level-1 invariants (e.g. DRI and IRI in our study) and level-2 invariants (i.e. the invariant relationship between level-1 invariants). In our study, the invariant relationship refers to the stable condition of logical dependence between DRI and IRI.

Baccaglini-Frank and Mariotti (2011) contended that robust invariants remain stable for any dragging of base points. In our study, DRI and IRI are both robust invariants and searching for them only needs random dragging and does not require more advanced dragging modalities, such as maintaining dragging (Baccaglini-Frank & Mariotti, 2010). Another advantage of involving robust invariants only is that it provided PMTs with immediate feedback on the stability of both the invariants and their relationship (Botana et al., 2015).

Our findings indicated that searching for an appropriate IRI is key to establishing a valid conditional link between DRI and IRI. We found that a 'false' conditional link was identified when PMTs identified a false IRI and simultaneously perceived the stability between this IRI and DRI (due to the robustness of invariants). The pitfall was that a false IRI has similar characteristics as an appropriate IRI. But to search for the appropriate IRI, a solver needs to check whether the IRI is caused by all DRIs (thus determined by all construction steps). In our case, a counterexample functioned as a means to probe a 'false' conditional link between DRI and IRI. However, in the context of proof construction in Euclidean geometry, such a false IRI may help PMTs realize that the associated property can be used as a premise or an intermediate proposition in a proof. In a DGS exploration, both appropriate or false IRIs can be perceived using the measuring and dragging modalities of DGS. So, introducing students to measuring and dragging modalities might provide them with support in moving from conjecturing toward constructing the proof of the conjecture. The measuring tool helps students to formulate a conjecture about the geometrical proposition. The use of measurement might give students the incorrect idea that the measurement is allowed to be used to justify the truth of the proposition. In our case, however, measurement information was used to find a conjecture as a 'clue' to the truth of a proposition. Students should be well informed by the teacher about the role of the measurement tool before they learn about geometry proof.

We used the modified MD-conjecturing model to describe PMTs' use of measuring and dragging modalities, particularly the use of random dragging while searching for the direct and the indirect robust invariants (DRI, IRI) and the use of a dragging test while establishing the conditional link between DRI and IRI towards the development of a conjecture. We

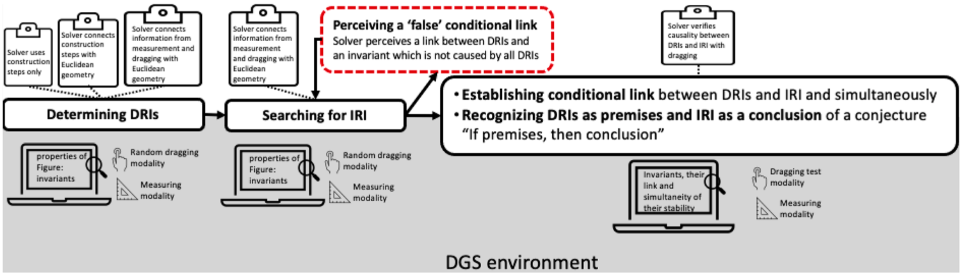

Figure 11. PMTs' conjecturing with the use of dragging and measuring modalities of the DGS.

thus provided insight into how PMTs formulate a conjecture using the DGS modalities. Figure 11 visualizes the conjecturing process in the DGS environment.

The modified MD-conjecturing model proved to be an appropriate analytical tool of PMTs' conjecturing process through the dynamic exploration of robust invariants. The modified model presented three steps of the solver's conjecturing process. Our data analysis validated the modified model and indicated that the intervention, whose design was based on the model, supported students' conjecturing process. Our analysis with the modified MD-conjecturing model, also shed light on the conjecturing steps and the support of the measuring and dragging modalities of the dynamic geometry system (DGS) in identifying the invariants. By discerning students' conjecturing process into steps, this model also turned our gaze to students' difficulties, particularly 'false' conditional links, and the role of counterexamples in helping students overcome that obstacle.

There may be some possible limitations of this study that could point to the need for further research. First, a small number of students (i.e. eight PMTs) were involved in the task-based interview session. This small number might limit the variation of conjecturing processes among students, and, therefore, the validation of our model. Also, testing the intervention in other contexts with other students and different conjecturing problems may contribute to knowledge about the application of our model in designing conjecturing tasks and about students' reasoning during these tasks. Second, in the individual interview setting, the communication between the participants and the interviewer may have influenced the process and the results of conjecturing. In fact, during the interview session, the interviewer asked students for clarifications without an indication of correctness, encouraged them speak up their thinking (to promote the thinking-aloud protocol) and sometimes gave technical support only where the responses were essential for subsequent interview questions to be meaningful. In future research, we will investigate the conjecturing process in a natural setting, i.e. during classroom activities and group discussions, to gain further insights into the learners' processes of formulating conjectures with the use of DGS.

In this study, our findings showed that our modified conjecturing model was appropriate to describe PMTs' conjecturing processes and indicated that the measuring and dragging modalities of DGS supported PMTs in conjecturing. Specifically, our findings also indicated that determining invariants by connecting properties to Euclidean geometry involved universal instantiation, which is part of the structural understanding needed to construct a geometric proof. It might be an indication of the existence of continuity between the argumentation produced to support conjecturing (e.g. universal instantiation) and the proof construction of the conjecture. This continuity is defined as cognitive unity (Boero et al., 1996). It might be interesting to investigate students' thinking process in transition from argumentation used during conjecturing, particularly the conjecturing that involves exploration in a DGS environment, to proof construction.

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