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Nonlocal Field theories: Theoretical and Phenomenological Aspects

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UNIVERSITÀ DEGLI STUDI DI SALERNO

Nonlocal Field Theories: Theoretical and Phenomenological Aspects

PhD thesis

to obtain the degree of PhD at the
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and in accordance with
the decision by the College of Deans
and

to obtain the degree of PhD at the
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and in accordance with
the decision by the faculty board
of the Physics Department

Double PhD degree

This thesis will be defended in public on
Tuesday 5 November 2019 at 9.00 hours
by

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ABSTRACT

Einstein's theory of general relativity (GR) has been tested to a very high precision in the infrared (IR) regime, i.e. at large distances and late times. Despite its great achievements, there are still open questions which suggest that GR is incomplete in the ultraviolet (UV) regime. From a classical point of view GR suffers from the presence of black hole and cosmological singularities; while from a quantum point of view GR lacks of predictability in the UV regime, being not perturbatively renormalizable.

One of the most straightforward attempt aimed to complete Einstein's GR in the ultraviolet (or short-distance) regime was to introduce quadratic curvature terms in the gravitational action besides the Einstein-Hilbert term, as for example $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$. Such an action turns out to be power counting renormalizable, but suffers from the presence of a massive spin-2 ghost degree of freedom, which causes classical Hamiltonian instabilities and breaks the unitarity condition at the quantum level.

Recently, it has been pointed out that a possible way to ameliorate the issue of ghost is to go beyond finite order derivative theories, and to modify the Einstein-Hilbert action by introducing differential operators made up of *infinite* order covariant derivatives, thus giving up the locality principle. In fact, by generalizing the Einstein-Hilbert action with quadratic curvature terms made up of nonlocal (i.e. non-polynomial) operators, one can formulate a quantum theory of gravity which is unitary and that shows an improved ultraviolet behaviour. The nonlocal differential operators are required to be made up of *exponential of entire functions* in order to avoid the presence of ghost-like degrees of freedom in the graviton propagator and preserve the unitarity condition.

In this Thesis, we investigate some fundamental aspect of nonlocal (infinite derivative) field theories, like causality, unitarity and renormalizability. We also show how to define and compute scattering amplitudes for a nonlocal scalar quantum field theory, and how they behave for a large number of interacting particles. Subsequently, we discuss the possibility to enlarge the class of symmetries under which a local Lagrangian is invariant by means the introduction of non-polynomial differential operators.

Furthermore, we move to the gravity sector. After showing how to construct a ghost-free higher derivative theory of gravity, we will find a linearized metric solution for a (neutral and charged) point-like source, and show that it is nonsingular. By analysing all the curvature tensors one can capture and understand the physical implications due to the nonlocal nature of the gravitational interaction. In particular, the Kretschmann invariant turns out to be non-singular, while all the Weyl tensor components vanish at the origin meaning that the metric tends to be conformally-flat at $r = 0$. Similar features can be also found in the case of a Delta Dirac distribution on a ring for which no Kerr-like singularity appears. Therefore, nonlocality can regularize singularities

by smearing out point-like objects. At the full non-linear level, we show that the Schwarzschild metric cannot be a full metric solution valid in the entire spacetime, but it can be true only in some subregion, for instance in the large distance regime where there is vacuum.

Finally, we also discuss some phenomenological implications in the context of ultra-compact objects (UCOs), in which ghost-free infinite derivative gravity can be put on test and constrained.

All the obtained results appear to be relevant for the follow up research. We also emphasize that, besides the conceptual significance of our results, we also developed new frameworks in which testability of nonlocal interaction might become more feasible in future experiments.

Publication List

This is the list of the publications on which the present PhD Thesis is based on.

- P1** L. BUONINFANTE, A. S. Koshelev, G. Lambiase and A. Mazumdar
Classical properties of non-local, ghost- and singularity-free gravity
JCAP **1809**, no. 09, 034 (2018)
arXiv:1802.00399
- P2** L. BUONINFANTE, A. S. Koshelev, G. Lambiase, J. Marto and A. Mazumdar
Conformally-flat, non-singular static metric in infinite derivative gravity
JCAP **1806**, no. 06, 014 (2018)
arXiv:1804.08195
- P3** L. BUONINFANTE, G. Lambiase and A. Mazumdar
Ghost-free infinite derivative quantum field theory
Nucl. Phys. B **944**, 114646 (2019)
arXiv:1805.03559
- P4** L. BUONINFANTE, G. Harmsen, S. Maheshwari and A. Mazumdar
Nonsingular metric for an electrically charged point-source in ghost-free infinite derivative gravity
Phys. Rev. D **97**, no. 8, 104006 (2018)
arXiv:1804.09624
- P5** L. BUONINFANTE, A. S. Cornell, G. Harmsen, A. S. Koshelev, G. Lambiase, J. Marto and A. Mazumdar
Towards nonsingular rotating compact object in ghost-free infinite derivative gravity
Phys. Rev. D **98**, no. 8, 084009 (2018)
arXiv:1807.08896
- P6** L. BUONINFANTE, A. Ghoshal, G. Lambiase and A. Mazumdar
Transmutation of nonlocal scale in infinite derivative field theories
Phys. Rev. D **99**, no. 4, 044032 (2019)
arXiv:1812.01441
- P7** L. BUONINFANTE, G. Lambiase and M. Yamaguchi
Nonlocal generalization of Galilean theories and gravity
Phys. Rev. D **100**, no. 2, 026019 (2019)
arXiv:1812.10105
- P8** L. BUONINFANTE and A. Mazumdar
Nonlocal star as blackhole mimicker

Phys. Rev. D **100**, no. 2, 024031 (2019)
arXiv:1903.01542

P9 L. BUONINFANTE, A. Mazumdar and J. Peng
Nonlocality amplifies echoes
Submitted, 2019
arXiv:1906.03624

P10 L. BUONINFANTE
Linearized metric solutions in ghost-free nonlocal gravity
J. Phys. Conf. Ser. **1275**, no. 1, 012042 (2019)
DOI: 10.1088/1742-6596/1275/1/012042

During my PhD I also published or submitted the following papers, whose contents are not part of this Thesis.

P11 L. BUONINFANTE and G. Lambiase,
Cosmology with bulk viscosity and the gravitino problem
Eur. Phys. J. C **77**, no. 5, 287 (2017)
arXiv:1610.01827

P12 L. BUONINFANTE, G. Lambiase and A. Mazumdar
Quantum solitonic wave-packet of a meso-scopic system in singularity free gravity
Nucl. Phys. B **931**, 250 (2018)
arXiv:1708.06731

P13 L. BUONINFANTE, G. Lambiase and A. Mazumdar
Quantum spreading of a self-gravitating wave-packet in singularity free gravity
Eur. Phys. J. C **78**, no. 1, 73 (2018)
arXiv:1709.09263

P14 L. BUONINFANTE, G. Lambiase, L. Petrucciello and An. Stabile
Casimir effect in quadratic theories of gravity
Eur. Phys. J. C **79**, no. 1, 41 (2019)
arXiv:1811.12261

P15 L. BUONINFANTE, G.G. Luciano and G. Petrucciello
Generalized Uncertainty Principle and Corpuscular Gravity
Eur. Phys. J. C **79**, no. 8, 663 (2019)
arXiv:1903.01382

P16 L. BUONINFANTE, G. G. Luciano, L. Petrucciello and L. Smaldone
Neutrino oscillations in extended theories of gravity

Submitted, 2019
arXiv:1906.03131

P17 L. BUONINFANTE, G. Lambiase and An. Stabile
Testing fundamental physics with photon frequency shift
Submitted, 2019
arXiv:1907.09542

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This is the first time I write a section on "Acknowledgements" in a Thesis work or anywhere else, indeed I did it neither for my master's thesis nor for my bachelor's one. So, please, try to understand that I am not an expert and, therefore, I might follow unconventional ways.

Ah by the way, here I assume that you are reading this now 5th November 2019, the day of my PhD defence...So, let's start!

First of all, I will start and end my Acknowledgements by expressing my gratitude and my infinite love to the most important people in my life: My Dad Remo, My Mum Giuseppina and My Brother Carletto. I have so much to say that a standard lifetime would not be enough, but there are a couple of things (maybe more) which I really need to and want to tell you. If Nature would give me the opportunity to select and choose two parents and a brother, I would not be able to find anything better than you in this or any other Universe. You are not just parents and brother for me, but my Best Friends. You have been always present, always available and ready to help and give me any kind of support and unique advices. From you I have learned and will keep learning so much about life. Papà and Mamma, You gave me the Best Thing any human being can dream of: FAMILY. Not only you thought me what the meaning and the importance hidden in the word "family" are, but you also showed me and gave me the possibility to fully embrace and live such a supreme gift!

Carletto (Fratmo), sometime it's difficult to express in words a real feeling, because it can be so strong that one would feel too emotional and lose the capacity to do it. Well, this is the right moment and right place to tell you that you occupy the most special part of my heart. Since you were born a very powerful connection was also established between you and me. I really don't have any words to explain this concept, probably no existing words can do so. What I can say is that when you are sad I am sad, when you are happy I am happy. I don't know if I have been a good older brother so far, but I want you to know that I have always tried to do my best and I would give my life for you.

One can think that I have been very lucky for having lived a life surrounded by such beautiful people...but wait, what my parents also thought me is that FAMILY has no limit, and yes I have been even more lucky than what you can expect...almost four years ago I met you, my love Elizabeth. Like my parents are not just parents, you are not just my girlfriend but you can be everything as you have shown in any kind of moments since I met you: you have been my best love, my best friend, my best enemy, my best half, my best partner in crime, my best guidance, my best trusted person, my best pain in the a...(neck), my best inspiration, my best Spanish and also English teacher, my best whatever you

can add... In so little time you became part of my FAMILY, everyone around me likes you. I have already told you this many times but you never believe me, so I will repeat it again: from you I have learned so much, you are one of the strongest and smartest person I have ever met, you are a real fighter, and I respect you so much. PhD life sometime can be very stressful and a lot of patience is required, not only from the PhD students but also from the people close to them. And yes, you have been the only person who has been really present in each moment and always understood my feelings and tried to help me offering all your best. You know what I mean so I don't need to say anything else about this but just THANKS! I promise that I will do my best to make all our dreams to come true. Also for you I have so many other things to say, but I will continue to tell you more things with calm tonight...

Before going ahead I need to say that in my family nest there is another lovely person which has contributed to make my life more complete, she is Giusy, my brother's love. Giusy, first of all I would like to thank you for making my brother happy, because this makes happy me too, you cannot even imagine how valuable it is. Secondly, I want to tell you that wherever I go I will always bring with me that special gift you and my brother gave me. It is not just a normal gift, but it contains a very deep meaning from which I get strength in certain moments, especially when I am far away from you. Thanks Cognatina!

Nonno e Nonna, of course I haven't forgotten about you...but since people here might get fed up with so much love coming out of this page, I will come back to you very soon...

So far I have mentioned the word "PhD" a couple of times, but if I had the possibility to learn about this word and work as a PhD student is especially thanks to my two supervisors Prof. Gaetano Lambiase and Prof. Anupam Mazumdar.

Gaetano, I met you during my first year of master's, and just attending your lectures on Theoretical Physics I fell even more in love with physics; basically, you naturally pushed me towards you. I would really like to tell you that you have been not just a supervisor but also a friend from which I had the possibility to learn and understand so many things, not only a lot of physics but also about life in general. You have been an excellent guidance in many difficult moments, always up for very good advices. You made my PhD life very easy and enjoyable, giving me all the freedom I wanted, thus teaching me that the first rule in order to be a researcher is to be free and open minded. Many thanks for sharing your deep knowledge, your intuition and your wisdom with me during these three years of PhD.

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for the first time aspects of quantum gravity. I would like to thank you for giving me the possibility to realize many things about scientific research in our physics community. Moreover, by working with you I had the possibility to meet many physicists, and also collaborators, which I would have probably never met without you.

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I have already said that sometime PhD life can be hard, tiring, boring and so on. But fortunately, I have always had several sources of strength which have helped me to relax and enjoy life as usual. One of the best source comes from my Old Friends: Luigi, Paolo, Raffaele and Stefano. Amici, I really want you to know that like my parents are not just parents, and my girlfriend is not just girlfriend, you are not just friends but I consider you Brothers. With you I feel so comfortable, I can do, behave and say anything I want, I never feel judged and I know that anytime you tell me off is because there is a real reason. Everyday I can learn new things from you, and I really like the way we behave with each other. We are very solid and tight to each other and always up to defend any of us if some difficulty arise. I hope with all my heart that Our Friendship will never end because my life wouldn't be the same without you.

Of course, it is my duty to reserve a special treatment for you Raffaele. Yes, I have to...you know? You have been the only friend who has been always present during all my thesis's defences: bachelor's, master's and now PhD's. I still remember that for the other two you were the one coming to pick me up with your car, you cannot even imagine how valuable it was for me and how valuable it is the fact that you are here in Groningen. Bro, there is something special between you and me, we know, it doesn't matter how far we are, or how

often we speak, nothing can change our relationship. With one big THANK YOU I would like to express my strong feelings and all my gratitude to you.

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Let us now go through my experience at University of Salerno and University of Groningen.

Salerno has been the place where I started my studies, I did my bachelor's and master's, and I am now also finishing my PhD. Of course, it is my duty to thank all my professors and university's friends who have contributed a lot in these unforgettable 8 years. A special thank goes to my General Relativity's professor, Prof. Gaetano Vilasi. From you I have learned few interesting things, and one sentence I will never forget is: "You don't need to know a lot, what is important is to know little but extremely well."

Before jumping to the PhD experience, let me say that during my studies I met beautiful people with whom I shared so many special moments: Luciano and Francesco (bestia and zuzzus, when are going to work on a project together?), Francesca, Melly, Simone, Marco (Prsso), Mariateresa and Ofelia (ragazze, we had so much fun together, I miss those moments so much!).

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other PhD students, Post-Docs and professors I have met during my experience in Groningen.

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Nonna Assunta, for obvious reasons we haven't been able to communicate so much, but I would really like to tell you that anytime you smile to me, you really make my day!

Nonno Mario and Nonna Felicetta, you are the wisest people I have ever met in all my life. What I have learned from your words, your stories and your advices has no price; none in Earth could have taught me so profound and precious things. I cannot use the word "thanks" because it would not be enough to pay back what you have done for me in all these years. Nonno, in this three years of PhD not only I have learned something about "nonlocal field theories"..., but thanks to you I finally learned how to make real good wine and this is something I wouldn't have been able to learn without you. Moments in which we play cards, watch football, discuss about your past experiences are so enjoyable that I would replace them with nothing else in this world. Nonna, you have been a second mum for me, you gave and are still giving me a very good education, and taught me how to behave since I was a kid. The only thing I regret is that I have never tried to learn how to cook the way you do, not even one percent of it. You, together with my mum, cook the best food I have ever eaten, it's AMAZING!

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I don't know if you managed to read all of it and reached to this point, but I truly believe that now it is time to end this long "section," and of course I will do it with the people who have been with me since I was a kid, that is my FAMILY. Papà and Mamma, you are my Idols, thanks for everything you did since I was born! Carlé, I admire you so much, and I want you to know that you are the most important person in my life. GRAZIE FAMIGLIA!

*Sono sempre andato a letto cinque minuti più tardi degli altri,
per avere cinque minuti in più da raccontare*

Franco Califano

List of Acronyms

AdS anti-de Sitter

dS de Sitter

EOM Equation(s) of motion

GR General relativity

IDG Infinite derivative gravity

IR Infrared

QCD Quantum chromodynamics

QFT Quantum field theory

QNF Quasi-normal frequencies

QNM Quasi-normal mode

SFT String field theory

SUGRA Supergravity

UV Ultraviolet

Conventions and Notations

In this Thesis we adapt all our conventions to the “mostly plus” metric signature $(-+++)$. Moreover, unless otherwise specified, we work in Natural units: $c = \hbar = 1$. We shall use the index 0 for the temporal coordinate, and the other indices 1, 2, 3 for the spatial coordinates. Then, latin indices i, j, k, l etc generally run over three spatial coordinate labels, usually, 1, 2, 3 or x, y, z . Greek indices μ, ν, ρ, σ etc generally run over the four coordinate labels in a general coordinate system.

Note that in this thesis we shall frequently suppress the indices, especially when we work with the spin projector operators. Thus, for instance, $\mathcal{P}_{\mu\nu\rho\sigma}^2$ will be just written as $\mathcal{P}^2$, and in the same way also in the formulas that contain the spin projector operators there will be a suppression of the indices.

Let us introduce a notation for the expressions containing either symmetric or antisymmetric terms. The indices enclosed in parentheses or brackets satisfy, respectively, the properties of symmetry or antisymmetry defined by the following rules:

$$T_{(\mu\nu)} = \frac{T_{\mu\nu} + T_{\nu\mu}}{2} \quad \text{and} \quad T_{[\mu\nu]} = \frac{T_{\mu\nu} - T_{\nu\mu}}{2} .$$

The adopted conventions for the curvature tensors are the following. The Christoffel symbol is defined as:

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2}g^{\rho\lambda}(\partial_{\mu}g_{\lambda\nu} + \partial_{\nu}g_{\mu\lambda} - \partial_{\lambda}g_{\mu\nu});$$

the Riemann tensor components read:

$$\mathcal{R}_{\mu\rho\nu}^{\alpha} = \partial_{\rho}\Gamma_{\mu\nu}^{\alpha} - \partial_{\nu}\Gamma_{\rho\mu}^{\alpha} + \Gamma_{\rho\rho}^{\alpha}\Gamma_{\mu\nu}^{\rho} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\rho\mu}^{\rho};$$

the Ricci tensor, $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\alpha\nu}^{\alpha} = g^{\alpha\rho}\mathcal{R}_{\alpha\mu\rho\nu}$:

$$\mathcal{R}_{\mu\nu} = \partial_{\alpha}\Gamma_{\mu\nu}^{\alpha} - \partial_{\nu}\Gamma_{\mu\alpha}^{\alpha} + \Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha\beta}^{\beta} - \Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta};$$

while, the Ricci scalar $\mathcal{R} = \mathcal{R}_{\mu}^{\mu} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$.

By lowering the upper index with the metric tensor we can obtain the completely covariant Riemann tensor:

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_{\nu}\partial_{\rho}g_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}g_{\nu\rho} - \partial_{\sigma}\partial_{\nu}g_{\mu\rho} - \partial_{\mu}\partial_{\rho}g_{\nu\sigma}) \\ &\quad + g_{\alpha\beta}(\Gamma_{\nu\rho}^{\alpha}\Gamma_{\mu\sigma}^{\beta} - \Gamma_{\sigma\nu}^{\alpha}\Gamma_{\mu\rho}^{\beta}).\end{aligned}$$

Moreover, the d'Alembertian operator is defined as $\square = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$.

Let us now introduce the linearized forms of the above curvature tensors around Minkowskias as we shall frequently use them. By perturbing around the Minkowski background,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

the curvature tensors up to linear order read

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_{\nu}\partial_{\rho}h_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}h_{\nu\rho} - \partial_{\sigma}\partial_{\nu}h_{\mu\rho} - \partial_{\mu}\partial_{\rho}h_{\nu\sigma}), \\ \mathcal{R}_{\mu\nu} &= \frac{1}{2}(\partial_{\rho}\partial_{\nu}h_{\mu}^{\rho} + \partial_{\rho}\partial_{\mu}h_{\nu}^{\rho} - \partial_{\mu}\partial_{\nu}h - \square h_{\mu\nu}), \\ \mathcal{R} &= \partial_{\mu}\partial_{\nu}h^{\mu\nu} - \square h.\end{aligned}$$

Moreover, we can also defined the traceless Riemann tensor also known as Weyl tensor:

$$\begin{aligned}\mathcal{C}_{\mu\nu\rho\sigma} &= \mathcal{R}_{\mu\nu\rho\sigma} + \frac{\mathcal{R}}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &\quad - \frac{1}{2}(g_{\mu\rho}\mathcal{R}_{\nu\sigma} - g_{\mu\sigma}\mathcal{R}_{\nu\rho} - g_{\nu\rho}\mathcal{R}_{\mu\sigma} + g_{\nu\sigma}\mathcal{R}_{\mu\rho}).\end{aligned}$$

The usual form of the field equation for General Relativity is given by

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \kappa^2 T_{\mu\nu},$$

where $G_{\mu\nu} = \mathcal{R}_{\mu\nu} + g_{\mu\nu}\mathcal{R}/2$ is the Einstein tensor and $T_{\mu\nu}$ the energy-momentum (or stress-energy) tensor. In SI units the coupling is given by $\kappa = 8\pi G/c^4$, where the value of the Newton constant is $G = 6.67 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ s}^{-2}$. In Natural units, since $c = 1 = \hbar$, one has $\kappa^2 = 8\pi G$. Often it is useful to display the Planck mass in the gravitational field equations. Indeed, the Planck mass is defined as

$$m_p := \sqrt{\frac{\hbar c}{G}} \simeq 1.2 \times 10^{19} \text{ GeV}/c^2 = 2.2 \times 10^{-8} \text{ kg}$$

and in natural units $G = 1/M_p^2$. To get rid of the 2π factor is useful to introduce the reduced Planck mass that is defined as

$$M_p := \sqrt{\frac{\hbar c}{8\pi G}} \simeq 2.4 \times 10^{18} \text{ GeV}/c^2 = 4.3 \times 10^{-9} \text{ kg}.$$

Therefore, the coupling constant in Natural units turns out to be equal to

$$\kappa = \frac{1}{M_p},$$

and the Einstein equations turn out to be expressed in terms of the reduced Planck mass.

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Introduction

Albert Einstein's General Relativity (GR) since 1916 has become the widely accepted theory of gravity and has been tested to a very high precision in the infrared (IR) regime, i.e. at large distances and time scales. A vast amount of observational data [1] have made GR the best current theory to describe classical aspects of the gravitational interaction. Remarkably, the recent observation of gravitational waves (GW) emission from merging of compact objects has given an additional powerful confirmation of its predictions, even after one hundred years from its formulation. Despite the great success, there still remain fundamental questions with no answer. At the classical level, Einstein's theory is plagued by the presence of cosmological and black hole singularities which make the theory incomplete at short distances [2, 3]. Moreover, at the quantum level GR lacks of predictability in the ultraviolet (UV), indeed it turns out to be perturbatively non-renormalizable. In 1972, 't Hooft and Veltman [4] calculated the one-loop effective action of Einstein's theory and found that gravity coupled to a scalar field is non-renormalizable, but also showed how to introduce counter-terms to make pure GR finite at one-loop. The crucial result was only obtained several years later by Goroff and Sagnotti [5] and van de Ven [6], who showed the existence of a two loops divergent term cubic in the Riemann tensor.

The theory of GR is described by the simple Einstein-Hilbert action:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \mathcal{R},$$

where $\mathcal{R}$ is the Ricci scalar and the coupling $\kappa = \sqrt{8\pi G} = 1/M_p$ has the dimension of mass inverse. One can prove that such an action is non-renormalizable

by making a power-counting of the coupling constant dimension in front of the interaction terms. First, since we want to work in the realm of standard perturbative quantum field theory (QFT), we need to expand the action in Eq.(1) around Minkowski:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x),$$

so that we obtain

$$S = \frac{1}{4} \int d^4x h_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} h_{\rho\sigma} + \mathcal{O}(\kappa h^3),$$

where $\mathcal{O}^{\mu\nu\rho\sigma}$ is the kinetic operator, while $\mathcal{O}(\kappa h^3)$ takes into account higher order interaction terms in the perturbation. We can immediately notice that all the interaction terms are multiplied by powers of κ which has the dimension of a mass inverse, and this implies that such an action is non-renormalizable by power-counting.

This feature of the theory is also reflected on the UV behavior of loop-integrals. Indeed, if we compute the superficial degree of divergence in four dimensions we obtain [7]:

$$D = 2L + 2,$$

which tells us that the degree of divergence increases with the number of loops. In fact, one can always implement a renormalization prescription but an infinite number of counter-terms are needed, namely an infinite number of couplings. However, any experiment can never determine the value of an infinite number of parameters, therefore GR's predictability is spoiled at high energy.

In the past there have been several attempts aimed to resolve this problem. Some of them are based on standard tools of QFT, while others attempts are rely on different physical principles and alternative mathematical frameworks. Let us list some of them.

- The most straightforward and conservative attempt is to generalize the Einstein-Hilbert action by introducing local operators made up of higher order terms in the curvatures and to use tools of standard perturbative QFT. In 1977, Stelle proved that a theory described by an action including the Einstein-Hilbert term plus quadratic curvature terms like $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$,¹

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (\mathcal{R} + \alpha \mathcal{R}^2 + \beta \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu})$$

¹Note that the Riemann tensor squared does not appear in Stelle's action because it can be rewritten in terms of the Ricci scalar squared and Ricci tensor squared by means the so called Euler characteristic:

$$\mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} + \mathcal{R}^2 = \text{div},$$

where div stands for total covariant derivative.

is perturbatively renormalizable [8, 9]. However, the same action is plagued by the presence of an additional massive spin-2 ghost degree of freedom which causes classical Ostrogradsky instabilities by making the Hamiltonian unbounded from below [10], while at the quantum level it breaks the unitarity condition. Indeed, by expanding the action in Eq.(1) around Minkowski, we can compute the propagator whose spin-2 part reads

$$\Pi_{\mu\nu\rho\sigma}(k) = \Pi_{\text{GR},\mu\nu\rho\sigma} - \frac{\mathcal{P}_{\mu\nu\rho\sigma}^2}{k^2 + m_2^2},$$

where Π_{GR} is the massless spin-2 graviton propagator of GR, while the second term is the so called Weyl ghost with mass m_2^2 . Hence, at the perturbative level, there is a conflict between unitarity and renormalizability which seem to be incompatible in both Einstein's GR and quadratic curvature gravity. Despite the presence of such an unhealthy degree of freedom, such a theory can be still considered predictive as an effective field theory whose validity is accurate at energy scales lower than the cut-off represented by the mass of the ghost [11, 12]. Another important achievement of quadratic gravity can be found in the Starobinski-model of inflation [13, 14], which is able to suitably explain the current data; differently from the model of Stelle, here only the term $\mathcal{R}^2$ shows up in the quadratic part of the action. It is also worthwhile to highlight that gravitational actions with quadratic curvature corrections were taken into account in several different frameworks (see for example Refs. [15, 16, 17, 18]).

- An alternative class of attempts was based on the introduction of new particles and new symmetries, going beyond GR and the standard model of particles. The most important examples in this class are supergravity theories (SUGRA), whose pioneers are Freedman, Ferrara and van Nieuwenhuizen [19]. Supersymmetric theories are very special because the balance of bosonic and fermionic degrees of freedom leads to cancellation of divergences in loop diagrams and indeed even the simplest SUGRAs do not have the two-loop divergence that is present in GR; in particular, $N = 8$ SUGRA has been shown to be finite up to five loops [20, 21]. However, besides the improved quantum behavior, these theories have other kind of either theoretical and experimental difficulties that thwarted this hope.
- A third possibility is that the non-renormalizability is an intrinsic pathology of the perturbative approach, and not of gravity itself. There have been more than one way of implementing this idea. The Hamiltonian approach to quantum gravity can be viewed as falling in this broad category,

²See Appendix A for more details on ghosts and unitarity and Appendix C for the definition of the spin projector operator $\mathcal{P}^2$.

like *loop quantum gravity* [22, 23, 24]. A more recent non-perturbative approach to quantize gravity is based on the *asymptotic safety* scenario in which it has been argued that there exist a UV fixed point in a region of the parameters space where the couplings are not small [25, 7]. Therefore, GR would be renormalizable from a non-perturbative point of view. Also for this approach a lot of work is still needed, indeed the unitarity problem is still open.

- Furthermore, a very popular attempt which is not based on the principles of quantum field theory is given by String Theory [26, 27], whose main aim is to construct an unified quantum framework of all interaction. The quantum aspect of the gravitational field only emerges in a certain limit in which the different interactions can be distinguished from each other. All particles have their origin in excitations of fundamental strings. The fundamental scale is given by the string length which is supposed to be of the order of the Planck length $L_p = 1/M_p$.

In this Thesis we mainly focus on the first category of attempts and try to attack the problem of unitarity and renormalizability by questioning the main principles of standard QFT. Any standard QFT (like the standard model of particles) is based on the principles of locality (polynomial Lagrangians), causality, unitarity and renormalizability. Moreover, the quantization of the theory and the definition of quantum scattering amplitudes is based on the Feynman *iε*-prescription. We may ask whether we can give up some of these key ingredients and be able to formulate a consistent theory of quantum gravity compatibly with the standard model. There is still no definite answer to this fundamental question, but there have been very interesting recent works along this direction, which we now list.

- *Lee-Wick theories of gravity*: are a class of higher (than four) derivative theories of gravity which have been shown to be both super-renormalizable and unitary. The simplest case is sixth order gravity whose Lagrangian contains sixth order differential operators, like for instance $\mathcal{R}\Box\mathcal{R}$ and $\mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu}$, besides the Einstein-Hilbert term; see Refs.[28, 29, 30]. In this case, one can easily notice that all the couplings have the dimension of a mass which means they are super-renormalizable by power-counting. Moreover, the spin-2 part of the graviton propagator does not have any extra real massive pole, but a pairs of complex conjugate poles. In Refs.[31, 32, 30, 33] a new quantization prescription alternative to the Feynman one was introduced, and it was shown that the optical theorem holds true at all order in perturbation theory. However, although the *S*-matrix seems to be well defined at the quantum level, it is still not clear whether the presence of this kind of higher derivatives, with complex conjugate poles, can cause Hamiltonian instabilities at the classical level.

- *Fourth order gravity with fakeons*: this is the only example of strictly renormalizable and unitary theory of quantum gravity [30, 34, 35, 36]. Indeed, as already mentioned above, fourth order gravity is renormalizable by power-counting but non-unitarity if the standard Feynman quantization prescription is implemented. However, the authors in Refs.[30, 34, 35, 36] have shown that it is still possible to make the theory unitary by implementing a new quantization prescription under which the Weyl ghost is converted into a fake degree of freedom (fakeon), so that the optical theorem can be preserved to all orders in perturbation theory. Also in this case, through this new prescription, one is able to construct a well defined S -matrix for quantum gravity, but at the classical level is still not completely clear how to avoid the Ostrogradsky instability without using perturbative tools.
- ***Nonlocal field theories***: are based on giving up one of the key principles of standard QFT, i.e. locality. In fact, it consist in generalizing the Einstein-Hilbert action by including higher order curvature terms made up of nonlocal (i.e. non-polynomial) analytic differential operators whose peculiar form is crucial in order to make the graviton propagator ghost-free around any background and ameliorate the UV behavior of amplitudes and loop integrals [37, 38, 39, 40, 41, 42]. These ghost-free nonlocal theories are also known as *infinite derivative theories of gravity* (IDG), since non-polynomial operators are usually made up of infinite order derivatives. This alternative approach can be useful not only at the quantum level but also classically, indeed no extra unhealthy degree of freedom is present in the physical spectrum.

In this Thesis we focus on this last approach to quantum gravity and discuss many of its aspects not only in relation to the gravitational sector, but also in a more general context of QFT. Let us now go through the history of nonlocal (infinite derivative) and introduce its main ingredients.

From local to nonlocal Lagrangians

In standard local field theory, Lagrangians are constructed in terms of polynomials of fields and polynomials of derivatives of fields since one is interested in observables at low energies, therefore, the order of derivatives is always finite:

$$\mathcal{L} \equiv \mathcal{L}(\phi, \partial\phi, \partial^2\phi, \dots, \partial^n\phi),$$

where n is a positive finite integer and $\phi(x)$, in principle, can be any kind of tensorial field. Instead, a nonlocal Lagrangian is a function which can be also made up of non-polynomial differential operators, like for instance

$$\mathcal{L} \equiv \mathcal{L}\left(\phi, \partial\phi, \partial^2\phi, \dots, \partial^n\phi, \frac{1}{\square}\phi, \ln(\square/M_s^2)\phi, e^{\square/M_s^2}\phi, \dots\right),$$

where the non-polynomial operators contain infinite order covariant derivatives; M_s is the energy scale of nonlocality beyond which new physics should manifest and observables at high energy can be computed, and it is mathematically needed to make the arguments of logarithm and exponential dimensionless. For example, in the case of the exponential of the d'Alembertian, we can write the operator:

$$e^{\Box/M_s^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\Box}{M_s^2} \right)^n,$$

where the derivative order n goes up to infinity. In terms of Taylor expansions it seems that we have to provide the full function (and thus nonlocal information) when an infinite number of derivatives is present. This is in contrast with a standard two derivative theory for which only the field and its first derivative (and thus rather local information) are needed. By thinking in terms of discrete derivatives one can explicitly see why infinite order derivative operator are nonlocal in nature: to define the discrete version of the first derivative we need to know the function on two adjacent lattice points, to define the second derivative on three and to define the n -th derivative on $n + 1$ lattice points, while for infinite order derivatives one needs to know the function on an infinite set of lattice points. Thus, the higher the derivative order is, the more nonlocal the information required to know the system is. However, although the derivative order is infinite, one can show that for some specific choice of the nonlocal operator the number of independent solutions of an infinite order differential equation can be still finite [43, 44].

Generally, nonlocality can be thought *at least* in two different ways: (i) as discretization of the spacetime; (ii) or purely related to the interaction in systems defined in a continuum spacetime. In the case (i) there would be a minimal length scale given by the size of the unit cell in such a discrete background, and it is often identified with the Planck length, $L_p = 1/M_p$. As for (ii), the nonlocality does not affect the kinematics at the level of free theory, but it becomes relevant only when dynamics is considered. In other words, in the free-theory this kind of nonlocality would not play any role, but it would become relevant as soon as the interaction is switched on. In this regard, we will be investigating the latter scenario, where we will consider a continuum spacetime and introduce nonlocality through non-polynomial differential operators into either the kinetic operator and/or the interaction vertex.

First attempts along (ii) trace back in the fifties, when people were still facing the problem of UV divergences in QFT and renormalization was still not very well understood. Thus, an alternative possibility to deal with divergences was the introduction of nonlocal interactions with the aim to regularize the theory and make it finite in the UV [45, 46, 47]. Subsequently, they were also studied from a pure axiomatic point of view [48]. In 1987 Krasnikov constructed a nonlocal Lagrangian for gauge theories and made some progress towards a

super-renormalizable and unitary nonlocal theory of quantum gravity. In 1989 Kuz'min [38] continued and extended the previous works and computed the one-loop effective action for a nonlocal quadratic theory of gravity.³ In Refs.[49, 50, 51, 39] further studies were made in relation to both finiteness of loop integrals and unitarity of higher derivative theories. It was noticed that by working with infinite order derivatives, and in particular using *exponentials of entire functions*, one could construct a ghost-free propagator as, for instance,

$$\Pi(k) = \frac{e^{\gamma(-k^2)}}{k^2 + m^2},$$

where $\gamma(-k^2)$ is an entire function, i.e. a function with no poles in the complex plane. Note that this kind of nonlocality refers to *analytic* differential operators for which a Taylor expansion around $\square = 0$ can be defined, but in literature other possibilities involving non-analytic operators, like $1/\square$ and $\ln(\square)$, have been explored; see for instance Refs.[52, 53, 54, 55, 56, 57]. Nonlocal field theories constructed in terms of analytic differential operators are often called *infinite derivative field theories*.

In 2005 the authors in Ref.[40] used this kind of analytic non-polynomial operators and made a more detailed investigation of nonlocal actions in the context of gravity. In particular, they noticed that nonlocality not only can help to make the theory unitary and to ameliorate high energy behavior of loop integrals, but also to resolve classical singularities. Indeed, exact non-singular bouncing solutions were found; see also Refs.[58, 59, 60, 61] for further progresses along this direction. In a gravitational context, the graviton propagator undergoes a similar generalization as in Eq.(1), for example the simplest IDG action can give a propagator around Minkowski which is the GR propagator multiplied by an exponential of an entire function. Further relevant studies were made in Refs.[41, 42], in which non-singular spherically symmetric solutions were obtained in the linearized regime around Minkowski. Very interestingly, in Refs.[41, 62, 63] the most general quadratic ghost-free actions were constructed around any maximally symmetric backgrounds, i.e. Minkowski, de Sitter (dS) and anti-de Sitter (AdS). IDG in three dimensions has been recently studied, in both massless and massive cases [64]; see also Refs.[65, 66, 67] for details on three-dimensional local massive gravity.

One of the main physical implications due to nonlocal interactions is the resolution of singularities. For instance, one can straightforwardly show that infinite order differential operators acting on a Delta source can yield a non-point support, i.e. can map a point-like object into an extended one whose size is given by the nonlocal length scale $L_s = 1/M_s$. Indeed, at the classical level it was shown that in IDG full non-linear bouncing solutions can be found in the context

³It is still not clear to the physics community whether the computations in Ref.[38] are correct, indeed further investigations are still required.

of cosmology [40, 58, 59, 60, 61]. Moreover, nonlocal gravitational interaction may also be useful to solve black hole singularities; so far exact non-singular spherically symmetric solutions have been found in the linearized regime both in static [68, 69, 41] and dynamical [70, 71, 72] scenarios. Further investigations have been made in Refs.[42, 73, 63, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83]. Moreover, IDG can also address the horizon problem in black hole physics: it has been argued that nonlocal effects can spread out up to the horizon scale so that the size of the nonlocal region always engulfs the Schwarzschild radius, thus preventing the formation of any horizon [75, 78, 84]. In fact, recently horizonless compact solutions have been discussed in IDG, where the gravitational system is assumed to be made up of a very large number of constituents interacting nonlocally [84]. Such systems are known as *nonlocal stars*.

At a quantum level there are hints that the UV behavior of the theory is ameliorated by the presence of exponential of entire functions [38, 39, 42, 85, 86, 87, 88]; however a lot of work has to be done especially in the gravity sector. In fact, by making a simple power counting, one can straightforwardly show that the superficial degree of divergence of loop integrals in IDG is given by [42, 86]

$$D = 1 - L,$$

which would seem to imply that for $L > 1$ all loop integrals should be finite. However, the power counting argument is not sufficient since we now have to deal with nonlocal Lagrangian for which the structure of the counter-terms may be very complicated and not simply given by polynomials. Therefore, in IDG we cannot apply the usual theorems on renormalizability of standard local QFT and further investigations are needed. It is worthwhile mentioning that in Refs.[42, 85, 89] it has been claimed that by choosing peculiar nonlocal operators which behave polynomially in the UV a super-renormalizable IDG theory can be proven to exist. Moreover, in Ref.[86] it was proposed an alternative approach to renormalization where all bare propagators were replaced by the dressed ones.

There have been attempts to construct a model with infinite derivative Higgs [90] and fermions [87], which indeed ameliorates the UV aspects by making the β -function to vanish at high energies [91, 90]. It has also been argued that in presence of multi-particle interaction, the nonlocal scale M_s can be transmuted from the UV to the IR depending on the number of particles involved in the scattering process [92]. Infinite derivative Lagrangians were also studied in the context of thermal field theory [93, 94, 95, 96], inflationary cosmology [97, 98, 99, 100, 101, 102], supersymmetry [103, 104] and applied to the study of the Casimir effect in curved background [105]. Moreover, a nonlocal modification of the Schrödinger equation has been analyzed in Refs.[106, 107, 108, 109].

Furthermore, the appearance of nonlocality in string theory is very well known. In particular, in string field theory (SFT), which has lots of similarity with a nonlocal QFT, vertices of the following form arise [110, 111, 112, 68, 69,

113]:

$$V(\phi) \sim \left(e^{c\alpha' \square} \phi \right)^3$$

where $c \sim \mathcal{O}(1)$ is a dimensionless constant that can change depending on whether one considers either open or closed string [113], and α' is the so called universal Regge slope. In fact, by discretizing the string functional integral as a sum over all lattices and using the large- N expansion to define surfaces on these lattices, one can define a continuum limit [114] which produces a nonlocal scalar field action with a Gaussian behavior ($e^{-\alpha' k^2}$) either in the propagator or in the vertex [115, 116]. Such a method in string theory was also thought as a non-perturbative approach to confinement in quantum chromodynamics (QCD) [114]; see Ref.[117] for a recent review on QCD.

This Thesis is motivated by the success of infinite derivative field theories until now and by the possibility to solve fundamental open questions, like a consistent formulation of a quantum theory of the gravitational interaction.

Organization of the Thesis

The goal of this Thesis is to study different aspects of infinite derivative field theories ranging from fundamental field theoretical problems to aspects of infinite derivative gravity. After a deep and quite complete theoretical analysis, we will discuss some phenomenological aspects of nonlocal interaction.

Chapter 1: We make a very general study of Lorentz-invariant infinite derivative field theories by working with a real scalar field for simplicity. We analyze fundamental aspects like propagator, causality, unitarity and renormalizability. In particular, we explicitly show that the presence of nonlocality implies violation of microcausality. Moreover, we show that the scale of nonlocality depends on the number of interacting particles, meaning that the size of the region on which nonlocal interaction takes place is not fixed but dynamical. We also investigate the possibility to enlarge the class of symmetries under which a local Lagrangian is invariant, by using non-polynomial differential operators.

Chapter 2: We introduce infinite derivative gravity and show how the presence of nonlocal operators in the gravitational action can make the propagator ghost-free and preserve perturbative unitarity. We study the linear regime around Minkowski background and find spacetime solutions for a neutral and an electrically charged point-like source, and for a rotating ring, showing how nonlocality can regularize the singularities from which Einstein's GR suffers. Subsequently, we move towards the non-linear regime and discuss the main feature of nonlocality. In particular, we show that the Schwarzschild metric cannot be a full solution in IDG.

Chapter 3: We study the possibility to extract some phenomenology in relation with the physics of horizonless ultra-compact objects. First, we discuss how nonlocal interaction can help to prevent the formation of a horizon in astrophysical scenarios and to form very compact objects whose radius is slightly larger than the Schwarzschild radius. Secondly, we analyze the interaction between waves and potential barriers in presence of nonlocality. As an example, we consider a double delta potential which can mimic the two photon spheres of a wormhole, or the two potential barriers at the photon sphere and at the surface of a an ultra-compact object. Especially, we show how nonlocality modify the behavior of quasi normal modes and echoes.

Chapter 4: We summarize the main results obtained in this work and discuss the outlook.

Appendix: In Appendix A we shortly review the concepts of unitarity and ghost, and theirs main implications. We show why higher derivative field theories, like fourth order gravity, are pathological. In Appendix B we show explicitly the steps of the computation of some Green functions in infinite derivative field theory. In Appendix C we review the formalism of the spin projector operators which are very useful to compute the graviton propagator and find the relevant degrees of freedom of the theory around Minkowski background.

Infinite derivative field theories

In this Chapter we investigate classical and quantum aspects of Lorentz-invariant infinite derivative field theories whose Lagrangians contain *analytic* form factors made up of infinite order derivatives. We will treat the simplest case of a scalar field.

This chapter is mainly based on **P3**, **P6**, **P7** and is organized as follows. In Section 1.1, we will introduce the action for a real scalar field and analyze into details the structure of the propagator, and emphasize that nonlocality is important only when the interactions are switched on. We will see how to perform calculations with operators involving derivatives of infinite orders. In Section 1.2, we will show that nonlocality leads to a violation of causality in a space-time region whose size is given by the scale of nonlocality $L_s = 1/M_s$. We will show that the retarded Green function becomes acausal due to nonlocality and as a consequence also local commutativity is violated. In Section 1.3, we will discuss quantum scattering amplitudes also in relation to unitarity. We will consider some simple computations of correlators and amplitudes in Euclidean space and how to analytically continue back to Minkowski. In particular, we will show that the Euclidean 2-point correlation function is non-singular at the Euclidean origin, unlike the local case. In Subsection 1.3.5 we will discuss the behavior of scattering amplitudes when a very large of interacting particles are involved. In Section 1.4 we will consider the possibility to enlarge the class of symmetries under which a local Lagrangian is invariant by means nonlocal generalizations.

1.1 Infinite derivative actions

We now wish to introduce a Lorentz-invariant infinite derivative field theory for a real scalar field $\phi(x)$ by an action:

$$S = \frac{1}{2} \int d^4x d^4y \phi(x) \mathcal{K}(x-y) \phi(y) - \int d^4x V(\phi(x)), \quad (1.1)$$

where the operator $\mathcal{K}(x-y)$ in the kinetic term makes explicit the dependence on the field variables at finite distances $x-y$, signaling the presence of a nonlocal nature; the second contribution to the action is a standard local potential term. We can rewrite the kinetic term as follows

$$\begin{aligned} S_K &= \frac{1}{2} \int d^4x d^4y \phi(x) \mathcal{K}(x-y) \phi(y) \\ &= \frac{1}{2} \int d^4x d^4y \phi(x) \int \frac{d^4k}{(2\pi)^4} F(-k^2) e^{ik \cdot (x-y)} \phi(y) \\ &= \frac{1}{2} \int d^4x d^4y \phi(x) F(\square) \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} \phi(y) \\ &= \frac{1}{2} \int d^4x \phi(x) F(\square) \phi(x), \end{aligned} \quad (1.2)$$

where $F(-k^2)$ is the Fourier transform of $\mathcal{K}(x-y)$, and we have used the integral representation of the Dirac delta, $\int \frac{d^4k}{(2\pi)^4} e^{ik \cdot (x-y)} = \delta^{(4)}(x-y)$. From Eq.(1.2) note that the operator $\mathcal{K}(x-y)$ has the following general form [118]:

$$\mathcal{K}(x-y) = F(\square) \delta^{(4)}(x-y). \quad (1.3)$$

Note that the action in Eqs.(1.1,1.2) is manifestly *Lorentz invariant*, thus it is possible to define a divergenceless stress-energy momentum tensor [119]. Note that $\square$ is dimensionful, and strictly speaking we should write $\square/M_s^2$. For brevity, we will suppress M_s in the definition of the form factors from now on. Further note that the action without the potential has *no* nonlocality. The homogeneous solution obeys the local equations of motion.

In what follows we will not refer to the operator $\mathcal{K}(x-y)$ anymore, but we will speak in terms of $F(\square)$.

1.1.1 Choice of form factor and degrees of freedom

So far we have not required any property for the form factor $F(\square)$, other than being Lorentz invariant; however it has to satisfy special conditions in order to define a consistent quantum field theory, in particular absence of *ghosts* at the tree level; see also Appendix A for a general discussion on unitarity and ghosts. We will restrict the class of operators by demanding $F(\square)$ to be an *entire*

*analytic function*¹. We can now apply the Weierstrass factorization-theorem for entire functions, so that we can write:

$$F(\square) = e^{-\gamma(\square)} \prod_{i=1}^N (\square - m_i^2), \quad (1.5)$$

where $\gamma(\square)$ is also an entire function, N can be either finite or infinite and it is related to the number of zeros of the entire function $F(\square)$. From a physical point of view, $2N$ counts the number of poles in the propagator that is defined as the inverse of the kinetic operator in Eq.(1.5). The exponential function does not introduce any extra degrees of freedom and it is suggestive of a cut-off factor that could improve the UV-behavior of loop-integrals in perturbation theory, moreover it contains all information about the infinite-order derivatives:

$$e^{-\gamma(\square)} = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \square^n, \quad (1.6)$$

where $\gamma_n \equiv \partial^{(n)} e^{-\gamma(\square)} / \partial \square^n|_{\square=0}$. By inverting the kinetic operator in Eq.(1.5), we obtain the propagator that in momentum space reads ²

$$\Pi(k) = e^{\gamma(-k^2)} \prod_{i=1}^N \frac{-i}{k^2 + m_i^2}. \quad (1.7)$$

One can immediately notice that if $N > 1$ ghosts appear. Indeed, we can decompose the propagator in Eq.(1.7) as

$$e^{\gamma(-k^2)} \prod_{i=1}^N \frac{1}{k^2 + m_i^2} = e^{\gamma(-k^2)} \sum_{i=1}^N \frac{c_i}{k^2 + m_i^2}, \quad (1.8)$$

where the coefficients c_i contain the sign of the residues of the propagator at each pole; then by multiplying with $e^{-\gamma(-k^2)} k^2$, and taking the limit $k^2 \rightarrow \infty$,

¹Let us remind that an entire function is a complex-valued function that is holomorphic at all finite points in the whole complex plane. It is worthwhile to mention that in literature there are also examples of field theory where the operator is a non-analytic function. For instance, from quantum correction to the effective action of quantum gravity non-analytic terms like $\mathcal{R}(\mu^2/\square)\mathcal{R}$ and $\mathcal{R}\ln(\square/\mu^2)\mathcal{R}$ emerge [52, 53, 54, 55]. Moreover, in causal-set theory [56, 57], the Klein-Gordon operator for a massive scalar field is modified as follows

$$F(\square + m^2) = \square + m^2 - \frac{3L_p^2}{2\pi\sqrt{6}}(\square + m^2)^2 \left[3\gamma_E - 2 + \ln \left(\frac{3L_p^4(\square + m^2)^2}{2\pi} \right) \right] + \dots, \quad (1.4)$$

where $\gamma_E = 0.57721\dots$ is the Euler-Mascheroni constant and L_p is the appropriate length scale; note also the presence of branch cuts once analyticity is given up.

²We adopt the convention in which the propagator in the Minkowski signature is defined as the inverse of the kinetic term times the imaginary unit "i".

we obtain

$$0 = \sum_{i=1}^N c_i, \quad (1.9)$$

which means that at least one of the coefficients c_i must be negative in order to satisfy the equality in Eq.(1.9), i.e., at least one of the degrees of freedom must be *ghost* like. We will focus on the case $N = 1$, so that tree level unitarity will be preserved and *no ghosts* whatsoever will be present in the physical spectrum of the theory.

Let us now fix the function $\gamma(\Box)$ in the exponential. As we have already mentioned, it has to be an entire function, moreover it has to recover the local Klein-Gordon operator, i.e. two-derivatives differential operator, in the IR regime, $\Box/M_s^2 \rightarrow 0$. In this Thesis we will mainly consider polynomial functions of $\Box$, in particular we will study the simplest operator

$$\gamma(\Box) = \frac{(-\Box + m^2)^n}{M_s^{2n}} \implies F(\Box) = e^{\frac{(-\Box + m^2)^n}{M_s^{2n}}} (\Box - m^2), \quad (1.10)$$

where n is a positive integer and we have explicitly reinstated M_s .

In literature also other kind of entire functions have been deeply studied, see for instance Refs.[39, 42, 74]³. In particular, in Refs.[39] was introduced the following entire function

$$\gamma(\Box) = \Gamma(0, p^2(\Box)) + \gamma_E + \log(p^2(\Box)), \quad (1.11)$$

where $\Gamma(0, z)$ is the incomplete gamma-function and $p(\Box)$ is any polynomial of $\Box$. This last operator is very suitable to construct renormalizable gauge theories since it assumes a polynomial behavior in the UV, such that the counter-terms are still local operators and all renormalization techniques of standard local quantum field theory can be still applied [39, 42, 85, 91, 89]. However, working with the entire functions in Eq.(1.10) will be sufficient to understand which are the main features and physical implications due to nonlocality.

1.1.2 Field redefiniton and nonlocal interaction

The infinite derivative field theory introduced in Eqs.(1.1) and (1.2) shows a modification in the kinetic term. However, note that we can also define an infinite derivative field theory where the kinetic operator corresponds to the usual local Klein-Gordon operator by making the following field re-definition:

$$\tilde{\phi}(x) = e^{-\frac{1}{2}\gamma(\Box)}\phi(x) = \int d^4y \mathcal{F}(x-y)\phi(y), \quad (1.12)$$

³In Section 1.4 we will study a slightly different type of form factors which can admit extra complex conjugate poles but still giving a ghost-free propagator at the tree level.

where $\mathcal{F}(x - y) := e^{-\frac{1}{2}\gamma(\square)}\delta^{(4)}(x - y)$ is the kernel of the differential operator $e^{-\frac{1}{2}\gamma(\square)}$. By inserting such a field redefinition into the action in Eq.(1.1), we obtain an equivalent action that we can still name by S :

$$S = \frac{1}{2} \int d^4x \tilde{\phi}(x)(\square - m^2)\tilde{\phi}(x) - \int d^4x V\left(e^{\frac{1}{2}\gamma(\square)}\tilde{\phi}(x)\right). \quad (1.13)$$

From Eq.(1.13) it is evident that now the form factor $e^{\frac{1}{2}\gamma(\square)}$ appears in the interaction term and that nonlocality only plays a crucial rule when the interaction is switched on as the free-part is just the standard local Klein-Gordon kinetic term. Such a feature of nonlocality is relevant only at the level of interaction, this will become more clear below, when we will discuss homogeneous (without interaction-source), and inhomogeneous (with interaction-source) field equations.

1.1.3 Homogeneous solution: Wightman function

We can now determine the field equation for a free massive scalar field by varying the kinetic action in Eq.(1.2) in the case of $N = 1$ degree of freedom and we obtain

$$F(\square)\phi(x) = 0 \iff e^{-\gamma(\square)}(\square - m^2)\phi(x) = 0, \quad (1.14)$$

that is a homogeneous differential equation of infinite order. One of the first question one needs to ask is how to formulate the Cauchy problem corresponding to Eq.(1.14) or, in other words, whether we really need to assign an infinite number of initial conditions in order to find a solution; if this is the case we would lose physical predictability as we would need an infinite amount of information to uniquely specify a physical configuration. Fortunately, as pointed out in Ref.[43, 44], what really fixes the number of independent solutions is the pole structure of the inverse operator $F^{-1}(\square)$. For instance, as for Eq.(1.14) we have two poles solely given by the Klein-Gordon operator $\square - m^2$, which implies that the number of initial conditions and independent solutions is also two.

In particular, note that the equality $(\square - m^2)\phi(x) = 0$ also solves Eq.(1.14), namely the two independent solutions of Eq.(1.14) are given by the same two independent solutions of the standard local Klein-Gordon equation ⁴:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{ik \cdot x} + a_{\vec{k}}^* e^{-ik \cdot x} \right), \quad (1.15)$$

⁴The normalization factor $\frac{1}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}$ in the field-decomposition Eq.(1.15) is consistent with the following conventions for the creation operator $a_{\vec{k}}^\dagger|0\rangle = \frac{1}{\sqrt{2\omega_{\vec{k}}}}|\vec{k}\rangle$, for the states-product $\langle\vec{k}|\vec{k}'\rangle = 2\omega_{\vec{k}}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')$ and for the identity in the Fock space $\mathbb{I} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} |\vec{k}\rangle\langle\vec{k}|$. With such conventions, the canonical commutation relation for free-fields reads $[\phi(x), \pi(y)]_{x^0=y^0} = \delta^{(3)}(\vec{x} - \vec{y})$, where $\pi(y)$ is the conjugate momentum to $\phi(y)$.

where $k \cdot x = -\omega_{\vec{k}} x_0 + \vec{k} \cdot \vec{x}$, with $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. The coefficients $a_{\vec{k}}$ and $a_{\vec{k}}^*$ are fixed by the initial conditions and once a quantization procedure is applied they become the usual creation and annihilation operators satisfying the following commutation relations:

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'), \quad [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] = 0 = [a_{\vec{k}}, a_{\vec{k}'}]. \quad (1.16)$$

Furthermore, let us remind that the *Wightman function* is defined as a solution of the homogeneous differential equation Eq.(1.14), thus from the above considerations it follows that it is not affected by the infinite derivative modification.

Indeed, in a local field theory the Wightman function is found by solving the homogeneous Klein-Gordon equation, and reads ⁵

$$W_L(x - y) = \int \frac{d^4 k}{(2\pi)^3} \theta(k^0) \delta^{(4)}(k^2 + m^2) e^{ik \cdot (x - y)}. \quad (1.17)$$

The corresponding infinite derivative Wightman function would be defined by acting on Eq.(1.17) with the operator $e^{\gamma(\square)}$. However, because of the Lorentz-invariance of the operator $e^{\gamma(\square)}$, with $\gamma(\square)$ being an entire analytic function, Eq.(1.17) will only depend on k^2 in momentum space. Therefore, given the on-shell nature of $W_L(x - y)$ through the presence of $\delta^{(4)}(k^2 + m^2)$, one has⁶

$$\begin{aligned} W(x - y) &= e^{\gamma(\square)} W_L(x - y) \\ &= e^{\gamma(m^2)} \int \frac{d^4 k}{(2\pi)^3} \theta(k^0) \delta^{(4)}(k^2 + m^2) e^{ik \cdot (x - y)}. \end{aligned} \quad (1.18)$$

The exponential operator only modifies the local Wightman function by an overall constant factor $e^{\gamma(m^2)}$ that can be appropriately normalized to 1: $e^{\gamma(m^2)} = 1$. For instance, in the case of exponential of polynomials, as in Eq.(1.10), one has $e^{-(k^2 - m^2)^n / M_s^{2n}} = 1$, once we go on-shell, $k^2 = -m^2$. Thus, infinite derivatives *do not* modify the Wightman function. It is also clear that the commutation relations between the two free-fields evaluated at two different space-time points will not change:

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= W(x - y) - W(y - x) \\ &= W_L(x - y) - W_L(y - x). \end{aligned} \quad (1.19)$$

Let us remind that for a massive scalar field, one has:

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = -\frac{i}{2\pi^2} \frac{1}{r} \int_0^\infty d|\vec{k}| \frac{|\vec{k}| \sin(\sqrt{\vec{k}^2 + m^2} t) \sin(|\vec{k}| r)}{\sqrt{\vec{k}^2 + m^2}} \equiv i\Delta(t, r) \quad (1.20)$$

⁵Whenever there is a confusion, we will label the local quantities with a subscript L .

⁶Note that Wightman function for the free-theory can get modified in field theories with non-analytic form factors, see Refs. [56, 57], in our scenario this is not the case.

where we have defined $t = x^0 - y^0$ and $\vec{r} = \vec{x} - \vec{y}$; $\Delta(t, r)$ is called Pauli-Jordan function. The above integral can be calculated, and in the massive case this is given by [120]:

$$\Delta(t, r) = -\frac{1}{2\pi}\varepsilon(t)\delta(\rho) + \frac{m}{4\pi\sqrt{\rho}}\theta(\rho)\varepsilon(\rho)J_1(m\sqrt{\rho}), \quad (1.21)$$

where $\rho := t^2 - r^2$, $\varepsilon(t) = \theta(t) - \theta(-t)$, and J_1 is the Bessel function of the first kind. It is clear that $\Delta(t, r)$ has support only within the past and future lightcones, indeed it vanishes for space-like separations ($\rho < 0$). When $m = 0$, one has

$$\begin{aligned} \Delta(t, r)|_{m=0} &= -\frac{1}{2\pi}\varepsilon(t)\delta(\rho) \\ &= \frac{1}{4\pi r} [\delta(t+r) - \delta(t-r)], \end{aligned} \quad (1.22)$$

which has support *only* on the *lightcone surface*. By defining the lightcone coordinates $u = t - r$ and $v = t + r$, the massless fields are parametrized by $u = 0 = v$, as indicated by the Dirac deltas in Eq.(1.22), so it follows that the commutation relations in Eqs.(1.21),(1.22) define the lightcone structure of the theory, which is not modified by infinite derivatives ⁷.

1.1.4 Inhomogeneous solution: propagator

From the previous considerations it is very clear that nonlocality in infinite derivative theories is not relevant at the level of free-theory, but it will play a crucial role when interactions are included. In fact, in presence of the potential

⁷It is worth mentioning that there are examples of field theories where the commutation relations for free-fields are modified by the presence of a minimal length-scale. For instance, it happens in non-commutative geometry [121] and causal-set theory [56, 57]. In the latter the form factor $F(k)$, not only depends on the invariant k^2 , but also on the sign of k^0 signaling the presence of branch-cuts in the Wightman function due to non-analyticity. Furthermore, modified commutation relations may emerge in theories where Lorentz-invariance is broken. Let us consider a very simple pedagogical example where the form factor explicitly breaks Lorentz-invariance: $F(\nabla^2) = e^{-\nabla^2/M_s^2}$, where $\nabla^2 \equiv \delta_{ij}\partial^i\partial^j$ is the spatial Laplacian, or in momentum space $F(\vec{k}^2) = e^{\vec{k}^2/M_s^2}$. In such a case it is easy to show that the commutator between two free massless scalar fields assumes the following form:

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= -\frac{i}{2\pi^2} \frac{1}{r} \int_0^\infty d|\vec{k}| e^{-\vec{k}^2/M_s^2} \sin(|\vec{k}|t) \sin(|\vec{k}|r) \\ &= \frac{iM_s}{8\pi^{3/2}} \left[e^{-\frac{1}{4}M_s^2(r+t)^2} - e^{-\frac{1}{4}M_s^2(r-t)^2} \right]. \end{aligned} \quad (1.23)$$

It is evident from Eq.(1.23) that the commutator for massless free fields is different from zero either inside and outside the lightcone on a region of size $\sim 1/M_s$ around the lightcone surface $u = 0 = v$.

term the field equation is given by

$$e^{-\gamma(\Box)}(\Box - m^2)\phi(x) = \frac{\partial V(\phi)}{\partial \phi(x)}, \quad (1.24)$$

and in this case the general solution cannot be simply found by solving the local Klein-Gordon equation, but the exponential operator $e^{-\gamma(\Box)}$ will play a crucial role. Hence, solutions of the inhomogeneous field equation will feel the nonlocal modification. The simplest example of inhomogeneous equation is the one with a delta source $\delta^{(4)}(x-y) = \delta(x^0-y^0)\delta^{(3)}(\vec{x}-\vec{y})$, whose solution corresponds to the propagator of the theory. In Minkowski signature, the propagator $\Pi(x-y)$ satisfies the following differential equation:

$$e^{-\gamma(\Box_x)}(\Box_x - m^2)\Pi(x-y) = i\delta^{(4)}(x-y), \quad (1.25)$$

whose solution can be expressed as

$$\Pi(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{\gamma(-k^2)}}{k^2 + m^2 - i\epsilon} e^{ik \cdot (x-y)}, \quad (1.26)$$

where

$$\Pi(k) = -\frac{ie^{\gamma(-k^2)}}{k^2 + m^2 - i\epsilon}, \quad (1.27)$$

is the Fourier transform of the propagator in Minkowski signature. We now wish to explicitly show that the propagator in Eq.(1.26) can not be identified with the time-ordered product of two fields, $\Pi(x-y) \neq \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$. As we have already seen for the Wightman function, the quantity $\Pi(x-y)$ can be expressed in terms of the local one, $\Pi_L(x-y)$, by acting on the latter with the operator $e^{\gamma(\Box_x)}$:

$$\Pi(x-y) = e^{\gamma(\Box_x)} [\theta(x^0-y^0)W_L(x-y) + \theta(y^0-x^0)W_L(y-x)], \quad (1.28)$$

where we have used the fact that the local propagator $\Pi_L(x-y)$ corresponds to the time-ordered product between two fields $\phi(x)$ and $\phi(y)$. Because of the time-derivative component of the d'Alembertian in the exponential function $\gamma(\Box_x)$, it is clear that the propagator cannot maintain the same causal structure of the Feynman propagator of the standard local field theory.

We now want to find the explicit form of the propagator in the coordinate-space, and in order to do so we need to understand how to deal with the differential operators of infinite order. By using the identity ⁸

$$\begin{aligned} \Box_x^n &= (-\partial_{x^0}^2 + \nabla_{\vec{x}}^2)^n \\ &= \sum_{p=0}^n \binom{n}{p} (-\partial_{x^0}^2)^{(p)} (\nabla_{\vec{x}}^2)^{(n-p)}, \end{aligned} \quad (1.29)$$

⁸The identity in Eq.(1.29) holds in flat spacetime as $[\partial_{x^0}^2, \nabla_{\vec{x}}^2] = 0$. In curved spacetime one has to deal with covariant derivatives and $\Box = g_{\mu\nu} \nabla^\mu \nabla^\nu$, so that the simple decomposition in Eq.(1.29) is not possible.

and the generalized Leibniz product-rule,

$$\partial_{x^0}^{(2p)} [g(x^0)h(x^0)] = \sum_{q=0}^{2p} \binom{2p}{q} \partial_{x^0}^{(q)} g(x^0) \partial_{x^0}^{(2p-q)} h(x^0), \quad (1.30)$$

we can manipulate the expression in the last line of Eq.(1.28) and obtain:

$$\begin{aligned} e^{\gamma(\square_x)} [\theta(x^0 - y^0)W_L(x - y)] &= \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \square_x^n [\theta(x^0 - y^0)W_L(x - y)] \\ &= \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \sum_{p=0}^n \binom{n}{p} (-\partial_{x^0}^2)^{(p)} \left[\theta(x^0 - y^0) (\nabla_{\vec{x}}^2)^{(n-p)} W_L(x - y) \right] \\ &= \theta(x^0 - y^0)W(x - y) + \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \sum_{p=0}^n \binom{n}{p} \binom{2p}{q} i^q \theta(2p - q) \partial_{x^0}^{(q-1)} \delta(x^0 - y^0) \\ &\quad \times \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_{\vec{k}}} (-\vec{k}^2)^{n-p} \omega_{\vec{k}}^{2p-q}, \end{aligned} \quad (1.31)$$

where in the last equality we have introduced the step-function $\theta(2p - q)$, so that we can extend the summation over q up to infinity. We can now note that the identity

$$\begin{aligned} \frac{1}{q!} \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} &= \frac{1}{q!} \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \frac{\partial^{(q)} ((k^0)^2 - \vec{k}^2)^n}{\partial k^{0(q)}} \\ &= \frac{1}{q!} \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \sum_{p=0}^n \binom{n}{p} \frac{\partial^{(q)} (k^0)^{2p}}{\partial k^{0(q)}} (-\vec{k}^2)^{n-p} \\ &= \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \sum_{p=0}^n \binom{n}{p} \binom{2p}{q} \theta(2p - q) (k^0)^{2p-q} (-\vec{k}^2)^{n-k}. \end{aligned} \quad (1.32)$$

allows us to rewrite Eq.(1.31) as follows:

$$\begin{aligned} e^{\gamma(\square_x)} [\theta(x^0 - y^0)W_L(x - y)] &= \theta(x^0 - y^0)W(x - y) \\ &\quad + i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial_{x^0}^{(q-1)} \delta(x^0 - y^0) \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_{\vec{k}}} \left. \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} \right|_{k^0=\omega_{\vec{k}}}. \end{aligned} \quad (1.33)$$

Following the same steps for the second term in Eq.(1.28), one has

$$\begin{aligned} e^{\gamma(\square_x)} [\theta(y^0 - x^0)W_L(y - x)] &= \theta(y^0 - x^0)W(y - x) \\ &\quad - i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial_{x^0}^{(q-1)} \delta(x^0 - y^0) \int \frac{d^3 k}{(2\pi)^3} \frac{e^{ik \cdot (y-x)}}{2\omega_{\vec{k}}} \left. \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} \right|_{k^0=\omega_{\vec{k}}}. \end{aligned} \quad (1.34)$$

We can now substitute Eqs.(1.33),(1.34) into Eq.(1.28), and obtain a very interesting expression for the propagator:

$$\begin{aligned} \Pi(x-y) = & \theta(x^0 - y^0)W(x-y) + \theta(y^0 - x^0)W(y-x) \\ & + i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial_{x^0}^{(q-1)} \delta(x^0 - y^0) [W^{(q)}(x-y) - W^{(q)}(y-x)], \end{aligned} \quad (1.35)$$

which is in agreement with the result obtained in Ref.[118] with a different procedure; in the previous equation we have defined

$$\begin{aligned} W^{(q)}(x-y) &:= \left. \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_{\vec{k}}} \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} \right|_{k^0=\omega_{\vec{k}}} \\ &= \int \frac{d^4k}{(2\pi)^3} e^{ik \cdot (x-y)} \theta(k^0) \delta(k^2 + m^2) \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}}. \end{aligned} \quad (1.36)$$

From Eq.(1.35) it is clear that the propagator is not just a time ordered product, but it also has an extra term that breaks the causal structure of the local Feynman propagator: this is a first example of causality violation induced by nonlocal interactions, as already been shown in Ref. [118]. In the standard local quantum field theory, the time-ordered product corresponds to the Feynman causal propagator that is constructed such that particles with positive-energy travels forward in time, while particles with negative energy (anti-particles) travel backwards in time. Such a structure is not preserved in infinite derivative field theory and causality is violated within $1/M_s$. For energies below M_s , the form factor reduces to $e^{-\gamma(\Box)} \rightarrow 1$, and hence reaches the local field theory limit in the IR. In Section 1.2, we will quantify the violation of causality in more detail. We can also define causal and non-causal (or acausal) parts of the propagator in Eq.(1.35), as follows:

$$\Pi_c(x-y) = \theta(x^0 - y^0)W(x-y) + \theta(y^0 - x^0)W(y-x) \quad (1.37)$$

and ⁹

$$\Pi_{nc}(x-y) = i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial_{x^0}^{(q-1)} \delta(x^0 - y^0) [W^{(q)}(x-y) - W^{(q)}(y-x)], \quad (1.38)$$

so that the propagator in Eq.(1.28) can be rewritten as ¹⁰

$$\Pi(x-y) = \Pi_c(x-y) + \Pi_{nc}(x-y). \quad (1.39)$$

⁹Note that in the non-causal term $\Pi_{nc}(x-y)$ there is an infinite number of contact terms that cannot be absorbed through counterterms, thus they will still be there once the theory is renormalized [118].

¹⁰The concept of propagator assumes physical meaning only when we consider propagation between two interaction vertices; thus, such a causality violation does not appear at the level of free-theory within infinite derivative theory, but only when the interaction is switched on.

Since for free-fields $W(x - y) = W_L(x - y)$, one has $\Pi_c(x - y) = \Pi_L(x - y)$.

1.2 Causality

In this Section we will explicitly show that the presence of nonlocal interactions violate causality in a region whose size is given by $\sim 1/M_s$ in coordinate space, and for momenta $k^2 > M_s^2$ in momentum space.

1.2.1 A brief reminder

Let us consider a real scalar field $\phi(x^0, \vec{x})$ that evolves by means of a differential operator $F(\square)$ in presence of a source $j(x^0, x)$, so that it satisfies the following differential equation:

$$F(\square)\phi(x^0, \vec{x}) = -j(x^0, \vec{x}). \quad (1.40)$$

A formal solution to Eq. (1.40) is given by

$$\phi(x^0, \vec{x}) = \phi_o(x^0, \vec{x}) + i \int dy^0 d^3y G(x^0 - y^0, \vec{x} - \vec{y}) j(y^0, \vec{y}), \quad (1.41)$$

where $\phi_o(x^0, \vec{x})$ is the solution of the homogeneous equation, and $G(x^0 - y^0, \vec{x} - \vec{y})$ is the Green function of the differential operator $F(\square)$, defined by

$$F(\square)G(x^0 - y^0, \vec{x} - \vec{y}) = i\delta(x^0 - y^0)\delta^{(3)}(\vec{x} - \vec{y}). \quad (1.42)$$

A system whose evolution is governed by Eq. (1.40) is said to be *causal* if the corresponding Green function $G(x^0, \vec{x})$ can be chosen, such that

$$G(x^0, \vec{x}) = 0, \text{ if } x^0 < 0. \quad (1.43)$$

The statement in Eq. (1.43) means that a physical system cannot respond to an interaction-source before the source was turned on¹¹. The previous definition of causal response holds for both relativistic and non-relativistic systems. A stronger version of the condition in Eq.(1.43) is given by the concept of *sub-luminality* [122], which is a property that has to be satisfied by any relativistic system. A physical system is said to be *sub-luminal*, if the Green function $G(x^0, \vec{x})$ is causal and also vanishes outside the light cone, i.e.

$$G(x^0, \vec{x}) = 0, \text{ if } x^0 < |\vec{x}|. \quad (1.44)$$

By *causality* we will also refer to the concept of sub-luminality, namely a causal system will be characterized by a vanishing Green function for space-like separations. Such a Green function is often indicated with a subscript "R" due

¹¹Note that such a definition of causality in terms of the Green function, not only holds for classical fields, but also for the expectation value of quantum fields in presence of a source, $\langle \phi(x) \rangle_j$.

to its retarded behavior, and we will use the symbol $G_{L,R}$ in the case of the standard local field theory. Another definition of causality is given through the commutator of two fields evaluated in two different space-time points. From a physical-measurement point of view, to preserve causality, we would require that the commutator of the two observables has to vanish outside the lightcone, i.e. for space-like separations. For a real scalar field, such a property can be formulated in the following way ¹²:

$$[\phi(x), \phi(y)] = 0, \text{ if } (x - y)^2 > 0. \quad (1.45)$$

When two observables commute, it means that they can be measured simultaneously, i.e. namely one measurement cannot influence the other. If the condition in Eq.(1.45) is violated, there would be correlations between the two measurements performed at two different spacetime points with space-like separation, implying transmission of information at a speed faster than light, thus violating causality. The property in Eq.(1.45) is called *local commutativity*, or sometime *microcausality*.

Note that the two conditions of causality given in terms of the Green function, see Eq. (1.44), and local commutativity, see Eq. (1.45), are closely related in local field theory. Let us consider a Hamiltonian interaction between a real scalar field $\phi(x)$ and a source $j(x)$, $H_{\text{int}} = \int d^3x j\phi$. Consider an initial configuration with a vacuum state at a time $y^0 = -\infty$ and then switch on the source at a later time. The expectation value of ϕ at a spacetime point $(x^0, \vec{x})$, with $x^0 > y^0$, can be calculated in the interaction picture, and it is given by [122]

$$\begin{aligned} \langle \phi(x) \rangle_j &= \langle 0 | e^{i \int_{-\infty}^{x^0} dy^0 d^3y j(y) \phi(y)} \phi(x) e^{-i \int_{-\infty}^{x^0} dy^0 d^3y j(y) \phi(y)} | 0 \rangle \\ &= \langle \phi(x) \rangle_{j=0} - \int d^4y j(y) i\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle + \dots \end{aligned} \quad (1.46)$$

where the dots stand for higher order contributions in the interaction-source term. By comparing Eq.(1.41) with Eq.(1.46) we can identify $\phi_o(x) = \langle \phi(x) \rangle_{j=0}$, and also note that in local field theory, the expectation value of the commutator between the two fields is related to the retarded Green function through the following relation:

$$G_{L,R}(x - y) = -\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (1.47)$$

Hence, if the commutator vanishes for space-like separations, the interaction-source can only generate non-zero modes inside its future lightcone, and thus the definition of causality given in terms of the Green function is consistent

¹²Let us remind that in the mostly positive metric signature $(x - y)^2 > 0$ stands for space-like separation and $(x - y)^2 < 0$ for time-like separation. In the mostly negative convention we would have had the opposite situation.

with the local commutativity condition. For completeness, we can also write the analog of Eq.(1.47) for the advanced Green function:

$$G_{L,A}(x-y) = \theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle. \quad (1.48)$$

1.2.2 Acausal Green functions

We have already given an example of causality violation in Section 1.1, where we have shown that the propagator is not simply given by a time-ordered product, but it has an extra non-causal term which becomes relevant inside the nonlocal region. We now want to show that the presence of nonlocal interactions leads inevitably to a violation of causality inside the region $\sim 1/M_s$. In particular, we wish to show explicitly that the nonlocal analog of the retarded Green function, $e^{\gamma(\Box)}[G_{L,R}(x^0, \vec{x})]$, is not vanishing outside the light-cone. We will simply indicate the nonlocal analog of the retarded Green function with the symbol G_R , meaning that it is a nonlocal quantity, while in presence of the subscript "L" we would refer to local quantities.

Let us remind that in local quantum field theory the retarded Green function is defined in terms of its Fourier transform as

$$-iG_{L,R}(x-y) = \int_{C_R} \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot (x-y)}}{(k^0)^2 - \vec{k}^2 - m^2}, \quad (1.49)$$

where the integration contour C_R is given by the real axis where both the poles: $\pm\omega_{\vec{k}} = \pm\sqrt{\vec{k}^2 + m^2}$ are avoided from above with two semi-circles. By evaluating the integral in Eq.(1.49) in the massless case, one obtains the retarded Green function in coordinate space:

$$-iG_{L,R}(x-y) = \frac{1}{2\pi} \theta(t) \delta(\rho) = \frac{1}{4\pi r} \delta(t-r), \quad (1.50)$$

where $t = x^0 - y^0$, $\vec{r} = \vec{x} - \vec{y}$ and $\rho = t^2 - r^2$. From Eq.(1.50) it is obvious that the retarded propagator is vanishing outside the light-cone, i.e. in the region $t < r$.

We now want to treat the case of infinite derivative field theory and explicitly see that the retarded Green function shows an acausal behavior due to nonlocal interactions. By following the steps in Eqs.(1.33) and (1.34) together with Eq.(1.47), one can write the nonlocal retarded Green function as follows

$$\begin{aligned} G_R(x-y) &= e^{\gamma(\Box_x)} G_{L,R}(x-y) \\ &= -\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle - \Pi_{nc}(x-y), \end{aligned} \quad (1.51)$$

Note the presence of the acausal (non-causal) term Π_{nc} introduced in Eq.(1.38). In particular, we will consider form factors with polynomial exponents as in Eq.(1.10), and for this specific choice we will see which is the form of Π_{nc} .

First of all, note that such form factors are divergent at infinity along some directions in the complex plane k^0 : for example, it can happen that they diverge at $-\infty$ and $+\infty$ along the real axis making it impossible to compute the integral in Eq.(1.50) in Minkowski signature. These kind of divergences make also impossible to define the usual Wick-rotation; this is one of the mathematical reason why in infinite derivative field theory one has to define all amplitudes in the Euclidean space, and in the end of the calculation go back to Minkowski signature by making an appropriate analytic continuation. Below we will give a more detailed discussion about this last observation.

However, in the case of the exponential choice in Eq.(1.10) with even powers n the nonlocal form factor does not diverge along the real axis at infinity, and we can *still* compute the principal value of the integral in Minkowski signature. Therefore, in this subsection we will consider the following form factors:

$$e^{-\gamma(\Box)} = e^{\left(\frac{\Box}{M_s^2}\right)^{2n}} \quad (1.52)$$

and we will work in the massless case for simplicity, $m = 0$. The aim is to compute the following integral:

$$-iG_R(x-y) = \int_{C_R} \frac{d^4k}{(2\pi)^4} \frac{e^{-\left(\frac{k^2}{M_s^2}\right)^{2n}} e^{ik \cdot (x-y)}}{(k^0)^2 - \vec{k}^2}. \quad (1.53)$$

The integral in Eq.(1.53) can be split into its principal value plus the contribution coming from the two semi-circles that avoid the two poles from above:

$$-iG_R = I_{PV} + I_{2C}, \quad (1.54)$$

where I_{2C} can be calculated by using the residue theorem, and one can easily show that is equal to

$$I_{2C} = \frac{1}{8\pi r} [\delta(t-r) - \delta(t+r)] = \frac{1}{4\pi} \varepsilon(\rho) \delta(\rho). \quad (1.55)$$

As for the principal value, one has

$$I_{PV} = \frac{1}{16i\pi^3} \frac{1}{r} \int_{-\infty}^{\infty} k dk \text{ P.V.} \int_{-\infty}^{\infty} dk^0 \frac{e^{-\left(\frac{-k_0^2 + k^2}{M_s^2}\right)^{2n}}}{k_0^2 - k^2} \left(e^{i(kr - k^0 t)} - e^{-i(kr + k^0 t)} \right), \quad (1.56)$$

where $k \equiv |\vec{k}|$ and $\omega_{\vec{k}} = k$, as we are working in the massless case. Note that all information about nonlocality is contained in the principal value I_{PV} , while I_{2C} is just a local contribution as it is evaluated at the residues, i.e. on-shell.

Figure 1.1: In this plot we have shown the behavior of the nonlocal analog of the retarded Green function as a function of the space-like distance $|\rho|^{1/2} = |x - y|$, $\rho < 0$, for several values of the power in the exponent: $2n = 2$ (continuous thick blue line), $2n = 4$ (dashed orange line), $2n = 8$ (dotted red line) and $2n = 10$ (continuous thin green line). The first two cases can be computed analytically and expressed in terms of the Meijer-G functions (see Eq.(1.58) for the case $2n = 2$), while the last two cases have been obtained numerically. We have set $M_s = 1$ as we are only interested in the qualitative behavior of the functions. It is evident that for small distances nonlocality is relevant and we have an acausal behavior, but as soon as $|\rho|^{1/2}$ increases nonlocality becomes less important and the Green function tends to a zero value recovering the local result, as expected. The oscillation-effects induced by nonlocality increase with the power $2n$.

After some manipulations, one can show that the principal value in Eq.(1.56) can be recast in the following form ¹³:

$$I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \left\{ \varepsilon(\rho) \int_0^\infty \frac{d\zeta}{\zeta} e^{-\frac{\zeta^{4n}}{M_s^{4n} \rho^{2n}}} \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\}, \quad (1.57)$$

where $\varepsilon(\rho)$ is equal to $+1$ if $\rho > 0$ (time-like separation), while it is -1 if $\rho < 0$ (space-like separation); Y_0 and K_0 are Bessel functions of the second kind and the modified Bessel function, respectively.

We can now find an explicit form for the integral in Eq.(1.57), for example we can consider the power $2n = 2$. In such a case the integral can be computed

¹³See Appendix B for all the details of the calculation.

and expressed in terms of the Meijer-G functions [123], so that the acausal Green function in Eq.(1.53),(1.54) reads

$$\begin{aligned}
 -iG_R(\rho) = & \frac{1}{4\pi}\varepsilon(\rho)\delta(\rho) + \frac{1}{2\pi^4}\frac{\varepsilon(\rho)}{\rho} \left\{ G_{2,5}^{4,1} \left(\begin{matrix} 0 \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| \frac{M_s^4 \rho^2}{256} \right) \right. \\
 & \left. + 2\pi^2 G_{3,6}^{4,1} \left(\begin{matrix} 0, -\frac{1}{4}, \frac{1}{4} \\ 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4} \end{matrix} \middle| \frac{M_s^4 \rho^2}{256} \right) \right\}. \quad (1.58)
 \end{aligned}$$

From Eq.(1.58) it follows that the Green function G_R is not vanishing for space-like separation ($\rho < 0$). In Fig. 1.1 we have plotted such a Green function for $\rho < 0$ so that it is very clear that it assumes values different from zero, but for large value of ρ , i.e. for $M_s^2 \rho \rightarrow \infty$, $G_R \rightarrow 0$, as expected. Thus, the violation of causality is restricted to the spacetime region of size approximatively given by $\sim 1/M_s$.

In the limit $M_s^2 \rho \rightarrow \infty$ the integral in Eq. (1.57) reduces to:

$$\lim_{M_s^2 \rho \rightarrow \infty} I_{PV} = \frac{1}{8\pi r} [\delta(t+r) + \delta(t-r)], \quad (1.59)$$

so that the sum of the two contributions $I_{2C} + I_{PV}$ would recover the local result in Eq.(1.50).

It is worth mentioning that for other values of $2n$ the integral in Eq.(1.57) also shows an acausal behavior, for example we have checked that in the case $2n = 4$ the integral can be still expressed as a combination of Meijer-G functions; for larger values of $2n$ one can proceed numerically. In Fig. 1.1 we have also shown the behavior of the acausal Green function for $2n = 4, 8, 10$. Moreover, the same procedure that we have used above can be used to compute the nonlocal analog of the advanced Green function, and it will lead to an opposite situation in which G_A will be non-vanishing for time-like separations.

1.2.3 Acausality for nonlocally interacting fields

We now wish to show that, due to the acausal feature of the Green functions, nonlocal interaction also implies the presence of acausality in the evolution of the fields; in particular we will see that the fields can depend acausally upon the initial data. Let us consider the Lagrangian for a real scalar field $\phi(x)$ with a quartic interaction¹⁴ as an example:

$$\mathcal{L} = \frac{1}{2}\phi(x)(\square - m^2)\phi(x) - \frac{\lambda}{4!} \left(e^{\frac{1}{2}\gamma(\square)} \phi(x) \right)^4, \quad (1.60)$$

¹⁴We could have considered any kind of interaction, but as an example we have chosen ϕ^4 . Moreover, we are working in the case in which the kinetic term is the standard Klein-Gordon operator and the interaction term is modified by the introduction of a form factor; of course the results would be the same if we considered nonlocal kinetic operator and local interaction vertices.

with corresponding field equations given by

$$(-\square + m^2)\phi(x) = -\frac{\lambda}{3!}e^{\frac{1}{2}\gamma(\square)}\left(e^{\frac{1}{2}\gamma(\square)}\phi(x)\right)^3, \quad (1.61)$$

where λ is a dimensionless coupling constant. The field equation in Eq.(1.61) can be solved perturbatively by continuous iterations; the zeroth and first order are given by:

$$\begin{aligned} (-\square + m^2)\phi^{(0)}(x) &= 0, \\ (-\square + m^2)\phi^{(1)}(x) &= -\frac{\lambda}{3!}e^{\frac{1}{2}\gamma(\square)}\left(e^{\frac{1}{2}\gamma(\square)}\phi^{(0)}(x)\right)^3, \end{aligned} \quad (1.62)$$

where the zeroth order is nothing but the homogeneous Klein-Gordon equation, whose local solutions are given by the free-field decomposition in Eq.(1.15), that we rewrite for clarity:

$$\phi^{(0)}(x^0, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-i\omega_{\vec{k}}x^0 + i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^* e^{i\omega_{\vec{k}}x^0 - i\vec{k}\cdot\vec{x}} \right). \quad (1.63)$$

Note that the Fourier-transform of the field $\phi^{(0)}(x^0, \vec{x})$ with respect to the spatial coordinate $\vec{x}$, $\tilde{\phi}^{(0)}(x^0, \vec{k})$, can be expressed in terms of the initial field configuration, $\tilde{\phi}^{(0)}(0, \vec{k})$ and $\dot{\tilde{\phi}}^{(0)}(0, \vec{k})$, as follows ¹⁵

$$\tilde{\phi}^{(0)}(x^0, \vec{k}) = \tilde{\phi}^{(0)}(0, \vec{k})\cos(\omega_{\vec{k}}x^0) + \dot{\tilde{\phi}}^{(0)}(0, \vec{k})\frac{\sin(\omega_{\vec{k}}x^0)}{\omega_{\vec{k}}}, \quad (1.64)$$

so that the free-field in Eq. (1.63) can be rewritten as

$$\phi^{(0)}(x^0, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(\tilde{\phi}^{(0)}(0, \vec{k})\cos(\omega_{\vec{k}}x^0) + \dot{\tilde{\phi}}^{(0)}(0, \vec{k})\frac{\sin(\omega_{\vec{k}}x^0)}{\omega_{\vec{k}}} \right). \quad (1.65)$$

Let us now compute the variation of the free-field with respect to an initial field-configuration $\phi^{(0)}(y)$, with $y \equiv (0, \vec{y})$, such that the distance between x and y is space-like, $(x - y)^2 > 0$ (or, equivalently, $|\vec{x} - \vec{y}| > x^0$):

$$\frac{\delta\phi^{(0)}(x^0, \vec{x})}{\delta\phi^{(0)}(0, \vec{y})} = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(\vec{x}-\vec{y})} \cos(\omega_{\vec{k}}x^0) = -\dot{\Delta}(x^0, \vec{x} - \vec{y}), \quad (1.66)$$

$$\frac{\delta\phi^{(0)}(x^0, \vec{x})}{\delta\dot{\phi}^{(0)}(0, \vec{y})} = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(\vec{x}-\vec{y})} \frac{\sin(\omega_{\vec{k}}x^0)}{\omega_{\vec{k}}} = -\Delta(x^0, \vec{x} - \vec{y}), \quad (1.67)$$

where we have used

$$\tilde{\phi}^{(0)}(0, \vec{k}) = \int d^3x' e^{-i\vec{k}\cdot\vec{x}'} \phi^{(0)}(0, \vec{x}') \quad \text{and} \quad \frac{\delta\phi^{(0)}(0, \vec{x}')}{\delta\phi^{(0)}(0, \vec{y})} = \delta^{(3)}(\vec{x}' - \vec{y}).$$

¹⁵Note that the symbol " $\cdot$ " means derivative with respect to x^0 .

Note that we have obtained the Pauli-Jordan function as a result of the functional differentiation. As we have already emphasized, $\Delta(x^0, \vec{x} - \vec{y})$ has only support inside the lightcone, and the same holds for its time-derivative; thus for space-like separations they vanish and causality is preserved at the level of free-theory.

Let us now consider the first order in perturbation, i.e. the differential equation in the second line of Eq.(1.63). The solution $\phi^{(1)}(x)$ is given by the sum of the homogeneous and the particular solutions, which we can indicate by $\phi_o^{(1)}(x)$ and $\phi_p^{(1)}(x)$, respectively:

$$\phi^{(1)}(x) = \phi_o^{(1)}(x) + \phi_p^{(1)}(x), \quad (1.68)$$

where the particular solution has the physical information about the nonlocal interaction. The homogeneous solution, $\phi_o^{(1)}(x)$, satisfies the same equation of $\phi^{(0)}(x)$, while the particular solution can be formally expressed in terms of the Green function as in Eq.(1.41):

$$\phi_p^{(1)}(x^0, \vec{x}) = \frac{i\lambda}{3!} \int dx'^0 d^3x' e^{\frac{1}{2}\gamma(\Box_{x'})} [G_{L,R}(x^0 - x'^0, \vec{x} - \vec{x}')] \left(\phi^{(0)}(x'^0, \vec{x}') \right)^3, \quad (1.69)$$

where we have used the fact that $e^{\frac{1}{2}\gamma(\Box)}\phi^{(0)}(x) = \phi^{(0)}(x)$, at the zeroth order we have a free-field field propagation - satisfying the homogeneous Klein-Gordon equation, and we have also made use of the kernel representation of the exponential differential operator:

$$e^{\frac{1}{2}\gamma(\Box_x)}g(x) = \int d^4y e^{\frac{1}{2}\gamma(\Box_y)}\delta^{(4)}(x - y)g(y).$$

All information about the presence of nonlocal interactions is contained in the particular solution; thus let us now calculate, as done for the zeroth order in Eqs.(1.66) and (1.67), the variation of the field $\phi_p^{(1)}(x^0, \vec{x})$ with respect an initial field configuration $\phi^{(0)}(0, \vec{y})$, $\dot{\phi}^{(0)}(0, \vec{y})$ ¹⁶:

$$\begin{aligned} \frac{\delta\phi_p^{(1)}(x^0, \vec{x})}{\delta\phi^{(0)}(0, \vec{y})} &= -\frac{i\lambda}{2} \int dx'^0 d^3x' e^{\frac{1}{2}\gamma(\Box_{x'})} [G_{L,R}(x^0 - x'^0, \vec{x} - \vec{x}')] \\ &\quad \times \dot{\Delta}(x'^0, \vec{x}' - \vec{y}) \left(\phi^{(0)}(x'^0, \vec{x}') \right)^2, \end{aligned} \quad (1.70)$$

$$\begin{aligned} \frac{\delta\phi_p^{(1)}(x^0, \vec{x})}{\delta\dot{\phi}^{(0)}(0, \vec{y})} &= -\frac{i\lambda}{2} \int dx'^0 d^3x' e^{\frac{1}{2}\gamma(\Box_{x'})} [G_{L,R}(x^0 - x'^0, \vec{x} - \vec{x}')] \\ &\quad \times \Delta(x'^0, \vec{x}' - \vec{y}) \left(\phi^{(0)}(x'^0, \vec{x}') \right)^2. \end{aligned} \quad (1.71)$$

¹⁶A similar computation was also done, for example, in Ref. [111] in the case of scalar field with cubic interaction, $[e^{\Box/M_s^2}\phi(x)]^3$, that can represent the interaction vertex for a dilaton field in string field theory.

The action of the differential operator on the local Green function in the integrals in Eqs.(1.70) and (1.71) makes the interacting field $\phi_p^{(1)}$ depending acausally upon the initial data: in fact, the integrals in Eq.(1.70) and (1.71) are not vanishing for space-like separations $|\vec{x} - \vec{y}| > x^0$ due to the non-zero contribution coming from the integration-region $x'^0 < |\vec{x}'|$ as the function $e^{\frac{1}{2}\gamma(\Box_{x'})}[G_{L,R}(x'^0, \vec{x}')]]$ exhibits an acausal behavior, i.e. it is non-vanishing for the space-like separations $x'^0 < |\vec{x}'|$, as previously shown for the case $\gamma(\Box) = (-\Box + m^2)^n/M_s^{2n}$. Indeed, more explicitly one has the following situations.

- In the local case, $\gamma(\Box) = 0$, the integrals in Eqs.(1.70),(1.71) get non-vanishing contributions coming from the integration region on which both the retarded Green function and the Pauli-Jordan function are non-zero. Such a region is defined by the following two inequalities: $x^0 - x'^0 \geq |\vec{x} - \vec{x}'|$, $x'^0 \geq |\vec{x}' - \vec{y}|$. Moreover, since the initial time condition is $y^0 = 0$ and we are looking at the future evolution by means the nonlocal analog of the retarded Green function, the following inequality has to hold: $x^0 > x'^0 > 0$. We can now ask if the field $\phi_p^{(1)}(x)$ depends acausally upon the initial spacetime configuration $(0, \vec{y})$. One can easily show that by putting together the above inequalities, we obtain

$$x^0 \geq |\vec{x} - \vec{y}|, \quad (1.72)$$

which implies that for space-like separation, $x^0 < |\vec{x} - \vec{y}|$, the integrals in Eqs. (1.70),(1.71) are vanishing. Thus, in local field theory the field evolution turns out to be causal.

- In the case of nonlocal interactions, $\gamma(\Box) \neq 0$, the Green function G_R shows an acausal behavior, i.e. it is non-vanishing for space-like separations, thus we can not use the first inequality $x^0 - x'^0 \geq |\vec{x} - \vec{x}'|$, $x'^0 \geq |\vec{x}' - \vec{y}|$, as done above for the local case. It follows that for space-like separation $x^0 < |\vec{x} - \vec{y}|$ the functional derivatives in Eqs.(1.70),(1.71) do not vanish and the field can depend acausally on the initial data.

The acausal behavior is confined to a region of size $\sim 1/M_s$, as it would be more explicit once a specific choice for the form factor is made.

1.2.4 Local commutativity violation

We have shown that the presence of nonlocal interaction implies an acausal behavior of the Green functions, which in turn makes the interacting fields depending acausally upon the initial data. However, we have not investigated yet whether also the commutator between interacting fields is modified in such a way that the local commutativity condition is also violated. In Section 1.1.3,

we have shown that the commutator for free-fields is not modified by infinite derivatives, maintaining the same structure of the local theory. We now want to show that local commutativity is violated when nonlocal interaction is switched on.

Let us still consider the nonlocal ϕ^4 -theory as an example, i.e. the Lagrangian in Eq.(1.60), and let us compute the commutator between two interacting fields by using the perturbative field solution $\phi(x) = \phi^{(0)}(x) + \phi^{(1)}(x) + \mathcal{O}(\lambda^2)$ introduced in the previous subsection. The commutator between two interacting fields up to order $\mathcal{O}(\lambda^2)$ is given by

$$\begin{aligned} [\phi(x), \phi(y)] = & [\phi^{(0)}(x), \phi^{(0)}(y)] + [\phi^{(0)}(x), \phi^{(1)}(y)] \\ & + [\phi^{(1)}(x), \phi^{(0)}(y)] + [\phi^{(1)}(x), \phi^{(1)}(y)] + \mathcal{O}(\lambda^3). \end{aligned} \quad (1.73)$$

Note that $[\phi^{(0)}(x), \phi^{(0)}(y)]$ obeys local commutativity as $\phi^{(0)}$ is not affected by nonlocality, see Eqs.(1.62),(1.22). Moreover, from Eq.(1.68) we know that $\phi^{(1)} = \phi_o^{(1)} + \phi_p^{(1)}$, where $\phi_o^{(1)}$ is also not affected by nonlocality being a homogeneous solution; thus all the information about the nonlocal modification of the commutator are taken into account by the terms involving $\phi_p^{(1)}$. The second and third terms contributing to the commutator in Eq.(1.73) are of the following form:

$$[\phi^{(0)}(x), \phi^{(1)}(y)] \sim \frac{i\lambda}{3!} \int d^4y' e^{\frac{1}{2}\gamma(\Box_{y'})} [G_{L,R}(y-y')] \left(\phi^{(0)}(y') \right)^2 [\phi^{(0)}(x), \phi^{(0)}(y')]. \quad (1.74)$$

- In the local case, $\gamma(\Box) = 0$, the integral in Eq.(1.74) gets a non-vanishing contribution when the following two inequalities are satisfied:

$$y^0 - y'^0 \geq |\vec{y} - \vec{y}'|, \quad y'^0 \geq |\vec{y}' - \vec{x}| \quad (1.75)$$

where we have taken $x^0 = 0$ without any loss of generality. We can now notice that, by choosing $y^0 > x^0 = 0$, the last two inequalities together imply

$$y^0 \geq |\vec{y} - \vec{x}|, \quad (1.76)$$

which means that the commutator in Eq.(1.74) gets non-vanishing contributions only for either time-like or null separations, in local field theory.

- In the case of nonlocal interaction, $\gamma(\Box) \neq 0$, the first inequality in Eq.(1.75) can not be used as the Green function is acausal and gives a non-vanishing contribution also for space-like separations. As a result, the commutator in Eq.(1.74) will be non-vanishing for space-like separations.

The same scenario also holds for the fourth term in Eq.(1.73), whose expression is of the following form:

$$[\phi^{(1)}(x), \phi^{(1)}(y)] \sim \left(\frac{i\lambda}{3!}\right)^2 \int d^4x d^4y e^{\frac{1}{2}\gamma(\Box_{x'})} [G_{L,R}(x-x')] e^{\frac{1}{2}\gamma(\Box_{y'})} [G_{L,R}(y-y')] \\ \times \left(\phi^{(0)}(y')\right)^2 \left(\phi^{(0)}(x')\right)^2 [\phi^{(0)}(x'), \phi^{(0)}(y')]. \quad (1.77)$$

- In the local case, $\gamma(\Box) = 0$, the integral in Eq.(1.77) gets a non-vanishing contribution when the following three inequalities are satisfied:

$$x^0 - x'^0 \geq |\vec{x} - \vec{x}'|, \quad y^0 - y'^0 \geq |\vec{y} - \vec{y}'|, \quad x'^0 - y'^0 \geq |\vec{x}' - \vec{y}'|. \quad (1.78)$$

For simplicity, we can take $y^0 > x^0 = 0$, without any loss of generality, and we can notice that all together the inequalities in Eq.(1.78) imply

$$y^0 \geq |\vec{x} - \vec{y}|, \quad (1.79)$$

which means that the commutator in Eq.(1.77) gets non-vanishing contributions only for either time-like or null separations, in local field theory.

- In the case of nonlocal interaction, $\gamma(\Box) \neq 0$, the first two inequalities in Eq.(1.78) can not be used as the Green functions are acausal and give non-vanishing contributions also for space-like separations. As a result, the commutator in Eq.(1.77) will be non-vanishing for space-like separation, meaning a violation of the local commutativity condition.

Note that we have only considered the commutator up to quadratic order in the coupling constant, but it is clear that local commutativity will be also violated at higher order in perturbation theory.

In order to quantify the degree of local commutativity violation we need to perform the computation by specifying an explicit form for the Green function and the Pauli-Jordan function to see how the integrals in Eqs.(1.74,1.77) behaves for space-like separation; but it will be subject of future works.

1.3 Amplitudes

From a physical point of view the presence of acausal effects means that there is no concept of Minkowski spacetime in the nonlocal region, nonlocality is such that we can not define the usual concepts of space and time. We can not define clock and rulers to make any kind of measurements inside $1/M_s$. For this reason, defining physical quantities in Minkowski signature in such a region would not make sense from a physical point of view, but the appropriate way to proceed

would be to define Euclidean amplitudes and Euclidean correlators. Indeed, in Euclidean space we do not have any concept of real time, all Euclidean distances are space-like by definition.

Such a physical argument also has a mathematical counterpart. As we have already briefly mentioned in the previous subsections, in infinite derivative field theory the form factors introduce some ambiguities when performing calculations of integrals in momentum space. For example, the exponential form factors with polynomial exponents introduced in Eq.(1.10) can always appear in loop-integral and amplitudes in the form $e^{-(k^2/M_s)^n}$ where n is a positive integer. For example, for the calculation of either propagator or any other Green functions, one has to deal with integrals of the following type:

$$I(x) = \int_{-\infty}^{\infty} dk^0 e^{\frac{-\left(\frac{-k_0^2 + \vec{k}^2}{M_s^2}\right)^n}{k_0^2 - \omega_{\vec{k}}^2}} e^{-ik^0 x^0 + i\vec{k} \cdot \vec{x}}. \quad (1.80)$$

It is easy to understand that the presence of the form factor gives divergent contributions along certain directions in the complex plane k^0 ; for instance, we can consider as examples $n = 1$ and $n = 2$.

- In the case $n = 1$ one has:

$$e^{\frac{k_0^2 - \vec{k}^2}{M_s^2}} e^{-ik^0 x^0 + i\vec{k} \cdot \vec{x}} \sim e^{\frac{\text{Re}^2(k^0)}{M_s^2}} e^{\frac{\text{Im}(k^0)x^0}{M_s^2}} e^{-\frac{\text{Im}^2(k^0)}{M_s^2}}, \quad (1.81)$$

which diverges at infinity along the directions belonging to the region $|\text{Re}(k^0)| > |\text{Im}(k^0)|$, while it converges to zero along the directions such that $|\text{Re}(k^0)| \leq |\text{Im}(k^0)|$.

- In the case $n = 2$, the relevant contribution at infinity is given by:

$$e^{\frac{-(k_0^2 + \vec{k}^2)^2}{M_s^4}} e^{-ik^0 x^0 + i\vec{k} \cdot \vec{x}} \sim e^{-\frac{\text{Re}^4(k^0)}{M_s^4}} e^{-\frac{\text{Im}^4(k^0)}{M_s^4}} e^{6\frac{\text{Im}^2(k^0)\text{Re}^2(k^0)}{M_s^4}}, \quad (1.82)$$

that only diverges along the directions $\text{Im}(k^0) = \pm \text{Re}(k^0)$, while in the rest of the complex plane it approaches to zero at infinity.

Note that such divergences in most of the cases it makes impossible to calculate integrals in Minkowski signature, for example the usual Feynman contour prescription does not work anymore, because the contribution coming from the semi-circle in either the lower or the upper half of the complex plane receive an infinite contribution at infinity. It implies that the usual Wick-rotation cannot be defined. Therefore, one has to be careful when defying scattering amplitudes. For instance, by starting in Minkowski signature the optical theorem is not satisfied and unitarity seems to be lost [124].

However, one can show that by defying all amplitudes in Euclidean space and then making an appropriate analytical continuation of the external momenta to Minkowski, the theory turns out to be unitary [125, 124, 126, 127]. The important property of such exponential form factors is that they always go to zero along the imaginary axis $\mathcal{I}$ directions, $\text{Im}(k^0) \rightarrow \pm\infty$, so that amplitudes in Euclidean signature are well defined and each loop integral can be legitimately computed on the integration domain $\mathcal{I} \times \mathbb{R}^3$. Subsequently, these amplitudes are analytically continued to real and positive external energies by means of a new prescription such that no Wick rotation is performed by the integration contour is deformed in a finite region of the complex plane; see Refs.[125, 127, 126] for more details.

As mentioned above, in the region of nonlocality $< 1/M_s$, or in momentum space, for momenta $k^2 > M_s^2$, we cannot define any classical concept of spacetime point, but vertices are smeared out such that the external legs and internal lines do *not* join in one point but they overlap in a region of size $1/M_s$. The crucial role is played by the *acausal* term Π_{nc} defined in Eq.(1.38), which implies causality violation in the vertices. In momentum space Π_{nc} reads [118]:

$$\begin{aligned} \Pi_{nc}(k) &= \int d^4x \Pi_{nc}(x) e^{-ik \cdot x} \\ &= i \sum_{q=1}^{\infty} \frac{1}{q!} \frac{1}{2\omega_{\vec{k}}} \left[\left. \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} \right|_{k^0=\omega_{\vec{k}}} (k^0 - \omega_{\vec{k}})^{q-1} - \left. \frac{\partial^{(q)} e^{\gamma(-k^2)}}{\partial k^{0(q)}} \right|_{k^0=-\omega_{\vec{k}}} (k^0 + \omega_{\vec{k}})^{q-1} \right], \end{aligned} \quad (1.83)$$

and it is evident that it has no poles and has no absorptive components, namely it is not made of on-shell intermediate states, but it is purely *off-shell*; indeed, it can be seen as a nonlocal vertex: $\Pi_{nc}(k) = iV(k)$. Thus, in momentum space the nonlocal propagator in Eq.(1.39) reads:

$$\Pi(k) = \Pi_c(k) + iV(k), \quad (1.84)$$

where $\Pi_c(k)$ is causal and $V(k)$ tends to zero for $k^2 \ll M_s^2$, but would be relevant for $k^2 \geq M_s^2$. Note that the nonlocal acausal part of the propagator, $iV(k)$, is made of infinite derivatives and this is the main cause of the smearing of the vertices, which are not point-like anymore. Thus, from Eqs.(1.83,1.84) it is now more clear that in infinite derivative field theories nonlocality and acausality manifest as *off-shell* effects.

In the standard local quantum field theory all internal lines of a Feynman diagram are seen as off-shell, while in infinite derivative quantum field theory the degree of "*off-shellness*" increases as also the vertices become nonlocal. In particular, there is no energy and momentum conservation in one single point, as legs and internal lines overlap on a smeared region of size $1/M_s$, or in terms of momentum, M_s . See Fig. 1.2 for an illustration of local and nonlocal vertices.

Figure 1.2: We have shown a pictorial illustration for local (left side) and nonlocal (right side) vertices. In the latter case the legs do not meet in one single point due to nonlocal effects spread out on a region (black blob) of size $L_s = 1/M_s$.

1.3.1 Euclidean 2-point correlation function

In local quantum field theory one has to deal with infinities which need to be regularized in order to give physical meaning to the theory. There are at least three kind of divergences that one can encounter:

1. UV divergences ($k \rightarrow \infty$);
2. IR divergences ($k \rightarrow 0$);
3. lightcone singularities ($|x - y| \rightarrow 0$).

In principle, one can cure IR and UV divergences but, even after the renormalization procedure has been applied, the lightcone singularity, which corresponds to the singularity at the origin in Euclidean space, still remains uncured. In this section we wish to compute the 2-point correlation function in infinite derivative quantum field theory; as an example we will still consider ϕ^4 -theory. In particular, we want to analyze its behavior on the light-cone surface, or in other words at the Euclidean origin, and see whether nonlocal interactions can regularize the divergence at $(x - y) \rightarrow 0$ from which the local theory suffers. For simplicity, we will focus on the form factor $e^{\gamma(\Box)} = e^{-(\Box - m^2)/M_s^2}$. We will work in Euclidean signature but the results can be reinterpreted in Minkowski space by analytic continuation as briefly explained above. Therefore, let us consider the following Euclidean generating functional:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{-S_E[\phi] + \int d^4x J\phi}, \quad (1.85)$$

where $J(x)$ is the source-term and the Euclidean action is given by:

$$S_E[\phi] = \int d^4x \left(-\frac{1}{2}\phi(x)e^{-(\Box - m^2)/M_s^2}(\Box - m^2)\phi(x) + \frac{\lambda}{4!}\phi^4(x) \right). \quad (1.86)$$

The functional in Eq.(1.86) can be rewritten in the following way:

$$\mathcal{Z}[J] = e^{-\frac{\lambda}{4!} \int d^4x \left[\frac{\delta}{\delta J(x)} \right]^4} \mathcal{Z}_0[J] = e^{-\frac{\lambda}{4!} \int d^4x \left[\frac{\delta}{\delta J(x)} \right]^4} e^{\frac{1}{2} \int d^4x d^4y J(x) \Pi(x-y) J(y)}, \quad (1.87)$$

where $\mathcal{Z}_0[J]$ is the free generating functional and $\Pi(x-y)$ is the propagator in the Euclidean space where, now, $k \equiv (k_4, \vec{k})$ stands for the Euclidean momentum, with $k^4 = -ik^0$, and $x \equiv (x^4, \vec{x})$ for the Euclidean coordinate, with $x^4 = ix^0$. We are interested in computing the 2-point correlation function that is defined as

$$\mathcal{G}(x-y) := \frac{\delta^2 \mathcal{Z}[J]}{\delta J(x) \delta J(y)} \Big|_{J=0}. \quad (1.88)$$

By expanding the exponential in Eq.(1.87), we can compute perturbatively the correlator $\mathcal{G}(x-y)$; for instance up to the first order in λ , we obtain:

$$\mathcal{G}(x-y) = \Pi(x-y) - \frac{\lambda}{2} \Pi(0) \int d^4z \Pi(x-z) \Pi(z-y) + \mathcal{O}(\lambda^2), \quad (1.89)$$

where at zeroth order we have the free-propagator, while at the first order a tadpole contribution.

Let us start by analyzing the zeroth order of the perturbative expansion in Eq.(1.89), i.e. the Euclidean propagator, in both massless and massive case.

- In the massless case the Euclidean 2-point function at zeroth order reads:

$$\Pi(x-y)|_{m=0} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-k^2/M_s^2} e^{ik \cdot (x-y)}}{k^2} = \frac{1 - e^{-\frac{M_s^2(x-y)^2}{4}}}{4\pi^2(x-y)^2}. \quad (1.90)$$

First of all, note that in the limit $M_s^2(x-y)^2 \rightarrow \infty$, we recover the local massless propagator, $\Pi_L(x-y)|_{m=0} = \frac{1}{4\pi^2(x-y)^2}$. More importantly, in the limit in which nonlocality becomes relevant, i.e. $M_s^2(x-y)^2 \rightarrow 0$, unlike in the local case the massless propagator in Eq.(1.90) does not diverge but it tends to a finite constant value:

$$\Pi(0)|_{m=0} = \frac{M_s^2}{16\pi^2}. \quad (1.91)$$

The result in Eq.(1.91) is extremely important for what concerns the UV behavior of the theory. The quantity $\Pi(0)|_{m=0}$ appears as a coefficient of the perturbative series in Eq.(1.89), and in local field theory the renormalization problem arises because of the presence of divergent coefficients. Thus, we have seen a first concrete example of how nonlocal interaction can improve the UV behavior of the theory. In particular, ϕ^4 -theory with nonlocal interaction becomes finite as discussed in Ref.[87, 86].

- As for the massive propagator, by using again the Schwinger parametrization for $1/(k^2 + m^2)$, we can write

$$\Pi(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-(k^2 + m^2)/M_s^2} e^{ik \cdot (x-y)}}{k^2 + m^2} = \frac{m^2}{16\pi^2} \int_{\frac{m^2}{M_s^2}}^{\infty} ds \frac{e^{-s} e^{-\frac{m^2(x-y)^2}{4s}}}{s^2}. \quad (1.92)$$

Although the integral in Eq.(1.92) cannot be solved in terms of elementary functions as in the massless case (Eq.(1.90)), it can be expressed in terms of the so called *cylindrical incomplete function* of Sonine-Schlaefli:

$$\Pi(x - y) = -\frac{1}{4\pi} \frac{m}{|x - y|} S_1 \left(-\frac{m^2}{M_s^2}, -\infty; im|x - y| \right), \quad (1.93)$$

where the Sonine-Schlaefli function is defined as [128]

$$S_\nu(-p, -q; iz) := \frac{e^{-\frac{i\pi\nu}{2}}}{2\pi i} \left(\frac{z}{2} \right)^\nu \int_p^q dt t^{-\nu-1} e^{-t - \frac{z^2}{4t}}.$$

We can study the limit $(x - y) \rightarrow 0$, and note that the massive propagator is non-singular at the Euclidean origin. Indeed, in this limit the integral in Eq.(1.92) gives

$$\Pi(0) = \frac{M_s^2 e^{-\frac{m^2}{M_s^2}}}{16\pi^2} \left[1 + e^{\frac{m^2}{M_s^2}} \text{Ei} \left(-\frac{m^2}{M_s^2} \right) \right], \quad (1.94)$$

where

$$\text{Ei}(x) := - \int_{-x}^{\infty} dt \frac{e^{-t}}{t}$$

is the so called *exponential-integral function*.

First-order correction $\mathcal{O}(\lambda)$

So far we have learnt that at the zeroth order in the perturbative expansion in Eq.(1.89) the 2-point correlation function is regular at the Euclidean origin unlike in the local case where singularities are present. We now want to study the first order correction (tadpole) in Eq.(1.89) and see whether such a regularization property is maintained.

- In the massive case, one can check numerically that the first order correction is non-singular at the Euclidean origin.

- In the massless case, the first order correction to the 2-point Euclidean correlator is singular at the origin, as we will now show with an explicit calculation. However, this can be made non-singular by dressing the propagator; see below.

At the first order in perturbation theory, by going to polar coordinates in 4-dimensions, $d^4k = k^3 \sin\theta \sin^2\alpha dk d\theta d\alpha d\varphi$, the 2-point function for the massless case can be expressed as

$$\begin{aligned} \mathcal{G}^{(1)}(x-y)|_{m=0} &= -\frac{\lambda}{2} \Pi(0)|_{m=0} \int d^4z \Pi(x-z)|_{m=0} \Pi(z-y)|_{m=0} \\ &= -\frac{\lambda}{2} \Pi(0)|_{m=0} \frac{1}{4\pi^2|x-y|} \int_0^\infty dk \frac{e^{-\frac{2k^2}{M_s^2}} J_1(k|x-y|)}{k^2}. \end{aligned} \quad (1.95)$$

First of all, note that the integral in Eq.(1.95) has an IR divergence, as we can see more explicitly by introducing an IR cut-off L :

$$\begin{aligned} \mathcal{G}^{(1)}(x-y)|_{m=0} &= -\frac{\lambda}{2} \Pi(0)|_{m=0} \left\{ -\frac{1}{2\pi^2} \frac{1}{M_s^2(x-y)^2} \left(1 - e^{-\frac{M_s^2(x-y)^2}{8}} \right) \right. \\ &\quad + \frac{1}{16\pi^2} \text{Ei} \left(-\frac{M_s^2(x-y)^2}{8} \right) \\ &\quad \left. + \lim_{L \rightarrow \infty} \left[\frac{1}{4\pi^2} \frac{L}{|x-y|} \left(1 - e^{-\frac{|x-y|}{4L}} \right) - \frac{1}{16\pi^2} \text{Ei} \left(-\frac{|x-y|}{4L} \right) \right] \right\}. \end{aligned} \quad (1.96)$$

It is obvious that in the limit $L \rightarrow \infty$, the IR divergence comes from

$$\lim_{L \rightarrow \infty} \text{Ei} \left(-\frac{|x-y|}{4L} \right) = \infty. \quad (1.97)$$

Note that the IR divergence comes together with the singularity at the Euclidean origin (or lightcone singularity in Minkowski signature) in the massless case - the correlator at the first order in λ in Eq.(1.96) also diverges for $|x-y| \rightarrow 0$.¹⁷

We have seen that the first order correction seems to suggest that nonlocality is not sufficient to regularize the singularity of the 2-point correlation function in the massless case. However, as we will show below, if we consider the full correlator, namely taking into account all the quantum perturbative corrections through the so called dressed propagator, then we will see that the physical 2-point function becomes regular at the origin.

¹⁷The singularity at the Euclidean origin will also appear for any power of the d'Alembertian $\square^n$, with any n ; indeed, it is purely related to the infrared divergence that one has in the massless case. It so happens that infrared divergence and singularity at the Euclidean origin (or lightcone singularity) are mixed.

Figure 1.3: In this plot we have shown the behavior of the dressed 2-point correlation function obtained by solving numerically the integral in Eq.(1.100). The blue-line represent the massless case, while the orange line represents the massive case with $m = 1$. We have set $M_s = 1$ for simplicity, as we are only interested in the qualitative behavior around the origin. We can notice that for $|x - y| \rightarrow 0$ the correlators tend to a finite value that of course differ for massless and massive cases.

Dressed 2-point correlation function

Let us consider the Fourier transform $\mathcal{G}(k)$ of the correlator in Eq.(1.89), in the more general case of massive scalar field and then we will also specialize to the massless case. It is well known that once one takes into account all the perturbative corrections to the 2-point correlation function in momentum space one obtains the so called dressed-propagator that can be expressed in terms of the self-energy $\Sigma(k)$. Thus, we have [87]

$$\begin{aligned}
 \mathcal{G}(k) &= \Pi(k) \sum_{n=0}^{\infty} (-1)^n [\Sigma(k)\Pi(k)]^n \\
 &= \frac{\Pi(k)}{1 + \Sigma(k)\Pi(k)} \\
 &= \frac{e^{-\frac{k^2 + m^2}{M_s^2}}}{k^2 + m^2 + \Sigma(k)e^{-\frac{k^2 + m^2}{M_s^2}}}.
 \end{aligned} \tag{1.98}$$

The self-energy at 1-loop is independent on the external momenta (tad-pole),

and reads:

$$\Sigma = \lambda \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-\frac{(p^2+m^2)}{M_s^2}}}{p^2 + m^2}, \quad (1.99)$$

The integral in Eq.(1.99) turns out to be finite in both massless and massive cases; in fact it has the same expression of the Euclidean propagator evaluated at the origin in Eqs.(1.91) and (1.94) for massless and massive cases, respectively.

We are interested in the coordinate-space dressed-correlator, so we need to consider the following Fourier transform:

$$\begin{aligned} \mathcal{G}(x-y) &= \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\frac{k^2+m^2}{M_s^2}} e^{ik \cdot (x-y)}}{k^2 + m^2 + \Sigma(k) e^{-\frac{k^2+m^2}{M_s^2}}} \\ &= \frac{1}{4\pi^2 |x-y|} \int_0^\infty \frac{k^2 J_1(k|x-y|) dk}{e^{\frac{k^2+m^2}{M_s^2}} (k^2 + m^2) + \Sigma(k)}. \end{aligned} \quad (1.100)$$

The integral in Eq.(1.100) cannot be solved analytically, but we can calculate it numerically, and we note that the full 2-point correlation function is non-singular at the Euclidean origin for both massive and massless cases. In Fig.1.3, we have shown the numerical solutions of the integral in Eq.(1.100) for both massive and massless cases.

1.3.2 s- and t-channels

We now wish to discuss an open problem of infinite derivative field theories which is related to an apparent divergent behavior of scattering amplitudes. We will propose a possible solution to this issue even though deeper investigations are need for a definite resolution.

Some ambiguities may appear when working with amplitudes with the exponentials of the kind $e^{-(p^2/M_s^2)^n}$:

1. $e^{-(p^2/M_s^2)^{2n}}$: tree level amplitudes with *even* power of the exponent would be always exponentially suppressed for both time-like and space-like momentum exchange;
2. $e^{-(p^2/M_s^2)^{2n+1}}$: tree level amplitudes with *odd* power of the exponent would be exponentially suppressed *only* for space-like momentum exchange, but they blow up for time-like exchange, i.e. for $p^2 < 0$.

As a consequence, for both even and odd powers the *tree level* scattering amplitudes turn out to be exponentially suppressed in the case of *t*- and *u*-channels, while the *s*-channel amplitude is exponentially suppressed only for even powers, but it blows up for odd powers as in this case the momentum exchange

is time-like, giving a positive exponent in the exponential which causes the divergence for high energies. Such a divergent behavior appears when the amplitude is made of an internal propagator that connects two cubic vertices (e.g. ϕ^3 -theory).

This unstable behavior should be only apparent and a resolution that goes beyond standard local quantum field theory techniques may be found, indeed here we present an attempt. Note that such divergences only manifest at the level of the *bare propagator*, i.e. at the tree level. In fact, once all quantum corrections are taken into account through the *dressed propagator*, all the scattering amplitudes become exponentially suppressed, thus physically well-defined; see also Ref. [129] for analogue discussions, where the tree level scattering amplitudes were computed, and also the dressed vertices and the dressed propagator.

Let us explicitly show that by correctly dressing the propagator *no* ambiguity would arise. For simplicity, we will consider the case of $\lambda\phi^3$ -theory with a nonlocal kinetic operator, and work in the massless case. A generic tree level scattering amplitude will be given by:

$$\mathcal{M}_n \sim \lambda^2 \frac{e^{-(p^2/M_s^2)^n}}{p^2}. \quad (1.101)$$

Once we dress the propagator, we obtain (see Eq.(1.98))

$$\mathcal{M}_n \sim \lambda^2 \frac{e^{-(p^2/M_s^2)^n}}{p^2 + \Sigma_n(p)e^{-(p^2/M_s^2)^n}}, \quad (1.102)$$

where the self-energy $\Sigma_n(p)$ for $\lambda\phi^3$ -theory, for example at 1-loop, reads

$$\Sigma_n(p) = \lambda^2 \int \frac{dk^4}{(2\pi)^4} \frac{e^{-(k^2/M_s^2)^n} e^{-((k-p)^2/M_s^2)^n}}{k^2(k-p)^2}, \quad (1.103)$$

which for even powers of n turns out to be always exponentially suppressed for any value of p^2 , while for odd powers of n can blow up for time-like momenta, $p^2 < 0$. Note that, generally, integrals of the type in Eq.(1.103) can blow up for some values of the integration variable k ; however, by working in Euclidean signature, $k^0 = ik^4$ and $p^0 = ip^4$, the integrals can be computed, and after the computation is made the momentum p can be analytically continued back to Minkowski signature, $p^4 = -ip^0$; see for example for discussions Refs.[125, 87].

Let us analyze t -channel and s -channel, for both even and odd powers.

Even powers $\square^{2n}$

Note that in the case of even powers, $e^{-(p^2/M_s^2)^{2n}}$, the high energy behavior of the scattering amplitudes is the same for both bare and dressed propagator. Indeed, in the UV regime, $p^2/M_s \rightarrow \infty$ we have the following asymptotic behavior

for the dressed amplitude in Eq.(1.102) with even powers:

$$\mathcal{M}_{2n} \sim e^{-(p^2/M_s^2)^{2n}}, \quad (1.104)$$

as $\Sigma_{2n}(p) e^{-(p^2/M_s^2)^{2n}} \rightarrow 0$ for $p^2/M_s^2 \rightarrow \infty$. Moreover, the asymptotic behavior is also the same for both t- and s-channels, as the even power, $2n$, does not distinguish between space-like and time-like momentum exchange: $(\pm p^2)^{2n} = (p^2)^{2n}$.

Thus, we have shown that in both cases of bare and dressed propagator, and s- and t-channels, the asymptotic behaviors of the scattering amplitudes are the same for even powers ($2n$) of the d'Alembertian.

Odd powers $\square^{2n+1}$

We now want to address the case of odd powers, $e^{-(p^2/M_s^2)^{2n+1}}$. Let us distinguish the cases of t-channel and s-channel.

- **t-channel scattering:** In the case of t-channel scattering, the momentum exchange is space-like, $k^2 > 0$, and the asymptotic behavior for high energies turns out to be the same for both bare and dressed propagator, as it also happens for the case of even power. Indeed, for space-like momentum exchange, for $p^2/M_s^2 \rightarrow \infty$, we have:

$$\mathcal{M}_{2n+1} \sim e^{-(p^2/M_s^2)^{2n+1}}, \quad (1.105)$$

as $\Sigma_{2n+1}(p) e^{-(p^2/M_s^2)^{2n+1}} \rightarrow 0$.

- **s-channel scattering:** In the case of s-channel scattering, the momentum exchange is time-like, $p^2 < 0$, and it is clear that for high energy the propagator blows up. However, as we have already emphasized, nonlocality is inherently off-shell, and as such quantum effects are not negligible and what is physically meaningful is the dressed propagator in the region of nonlocality, $p^2 > M_s^2$.

By dressing the propagator, it so happens that for high energy the amplitude has the same asymptotic behavior as for the t-channel, with the same exponential suppression, as one would expect for consistency. We will show this by making an explicit calculation for the simplest case $n = 1$. In this case the s-channel amplitude with dressed propagator is given by:

$$\mathcal{M}_1 \sim \lambda^2 \frac{e^{-p^2/M_s^2}}{p^2 + \Sigma_1(k) e^{-p^2/M_s^2}}. \quad (1.106)$$

Note that the self-energy $\Sigma_1(k)$ at 1-loop can be explicitly computed by performing the integration in Euclidean space and then analytically

continuing back to Minkowski the momentum p ; indeed by using the Schwinger parameterization we obtain:

$$\begin{aligned}\Sigma_1(p) &= \frac{\lambda^2}{16\pi^2} \int_{\frac{1}{M_s^2}}^{\infty} dt_1 \int_{\frac{1}{M_s^2}}^{\infty} dt_2 \frac{e^{-p^2 \frac{t_1 t_2}{t_1 + t_2}}}{(t_1 + t_2)^2} \\ &= \frac{\lambda^2}{16\pi^2} \left[\frac{2M_s^2}{p^2} \left(e^{-p^2/2M_s^2} - e^{-p^2/M_s^2} \right) + \text{Ei} \left(-\frac{p^2}{2M_s^2} \right) - \text{Ei} \left(-\frac{p^2}{M_s^2} \right) \right],\end{aligned}\tag{1.107}$$

which in the high energy regime goes to zero for space-like momentum exchange, while diverges for time-like exchange. In particular, the asymptotic behavior of the self-energy for $p^2/M_s^2 \rightarrow \infty$ is given by:

$$\Sigma_1(p) \sim e^{-p^2/M_s^2}.\tag{1.108}$$

It is now clear that the interplay between the divergences of the self-energy and the bare propagator, for time-like exchange, is such that the fully dressed s -channel scattering amplitude turns out to be exponentially suppressed in the UV regime, showing the same behavior as in the case of the t -channel. Indeed, the asymptotic behavior of the s -channel amplitude in Eq.(1.106) is given by

$$\mathcal{M}_1 \sim \lambda^2 \frac{e^{-p^2/M_s^2}}{p^2 + \lambda^2 e^{-p^2/M_s^2} \cdot e^{-p^2/M_s^2}} \sim e^{p^2/M_s^2},\tag{1.109}$$

which is exponentially suppressed in the UV regime, $p^2/M_s^2 \rightarrow -\infty$, with $p^2 < 0$.

Therefore, we have shown that all scattering amplitudes in ϕ^3 -theory, for any kind of channel,¹⁸ and for any power n of the d'Alembertian, are physically well-defined and have exactly the same UV behavior, once the propagator is consistently dressed. However, a more general study is needed to include more complicated theories and understand whether the dressing procedure can always help to cure the s -channel divergence for form factors with odd powers of the d'Alembertian.

Note that our procedure and result rely on the fact that we can define the dressed propagator by summing up the geometric series in Eq.(1.98). However, the geometric series converges if and only if $|\Pi(p)\Sigma(p)| < 1$, and such a condition is always satisfied if we define the series in the Euclidean signature where the momenta are space-like, and only after we allow the momenta to be also time-like by performing analytic continuation.

¹⁸Note that the case of u -channel is similar to the t -channel where the momentum exchange is space-like.

Furthermore, in the case of 1-loop or multi-loops amplitudes, we still have finite results, for example, quartic interaction with propagators that are exponentially suppressed was considered in Ref.[87], where scattering amplitudes ($\phi\phi \rightarrow \phi\phi$) and cross sections were computed. One can also consider loop-amplitudes for decay of unstable particles ($\phi \rightarrow \psi\psi$), but in this case, although there is no UV divergence, the amplitudes blow up for some values of the integration variables. However, such divergent integrals can be regularized with appropriate prescriptions and are still physically meaningful [125, 130, 126].

1.3.3 Unitarity

A quantum field theory is unitary if and only if the S-matrix satisfies the unitarity condition (see Appendix A):

$$S^\dagger S = \mathbb{1}, \quad (1.110)$$

which, by introducing the amplitude T such that $S = \mathbb{1} + iT$, can be also expressed as

$$2\text{Im}\{T\} = T^\dagger T > 0. \quad (1.111)$$

The last relation is also known as *optical theorem* and one of its implication is that the imaginary part of any amplitude T has to be always positive. As already mentioned above and discussed in details in Appendix A, higher derivative theories suffer from the presence of unhealthy degrees of freedom which can break the unitarity condition at the tree level. We will now show that, instead, when considering infinite order derivatives this is not the case and unitarity can be preserved.

Let us consider the nonlocal propagator in Eq.(1.27) and the associated amplitude:

$$T = i\Pi(k) = \frac{e^{\gamma(-p^2)}}{p^2 + m^2 - i\epsilon}. \quad (1.112)$$

If we compute the imaginary part of this amplitude we obtain:

$$\begin{aligned} \text{Im}\{T\} &= \frac{e^{\gamma(-p^2)}\epsilon}{(p^2 + m^2)^2 + \epsilon^2} \\ &= \pi\delta^{(4)}(p^2 + m^2) \\ &\equiv \text{Im}\{\text{Res}(i\Pi(p))\}_{p^2=-m^2}, \end{aligned} \quad (1.113)$$

which is positive if and only if the residue of the propagator is positive; to go from the second to the third line we have taken the limit $\epsilon \rightarrow 0$ and the normalization $e^{\gamma(m^2)} = 1$. In the simple case of a scalar field we obtain: $\text{Im}\{\text{Res}(i\Pi(p))\}_{p^2=-m^2} = \pi\delta^{(4)}(p^2 + m^2) > 0$, i.e. the unitarity condition is preserved at the tree level.

In Refs.[125, 130, 126] the optical theorem and the Cutkosky rules [131] were proven at all order in perturbation theory, therefore nonlocal (infinite derivative) field theories respect perturbative unitarity. It is also worthwhile mentioning that the *largest time equation* technique cannot be used in this case to prove unitarity, indeed one of its main assumptions is that the propagator has a time ordered structure and that the vertexes are polynomials of the momenta [132].

1.3.4 Unitarity and dressed propagator

The concept of dressed propagator has been very useful to obtain a singularity-free Euclidean correlator and to make the amplitudes well defined. We now wish to point out that the presence of extra poles in the dressed propagator is harmless, as far as perturbative unitarity is concerned [132]. Indeed, in quantum field theory what is needed to prove perturbative unitarity are bare propagators and bare vertexes, in terms of which higher loop amplitudes are constructed¹⁹. However, it is interesting to understand which is the pole structure of the dressed propagator and whether the presence of infinite massive complex poles may create some instabilities. As an example we will consider a massless scalar field in ϕ^3 - and ϕ^4 -theory, with the simplest choice $\gamma(\Box) = \Box/M_s^2$.

In the case of the dressed propagator, the pole structure is described by the following equation (see Eqs.(1.98,1.106)):

$$p^2 + \Sigma(p)e^{-p^2/M_s^2} = 0. \quad (1.114)$$

Infinite massive complex conjugate poles

As a first example, let us consider the ϕ^4 -theory for which the self-energy at 1-loop is just a constant, so that one has the simple complex equation

$$z = ce^z, \quad (1.115)$$

where $-p^2/M_s^2 =: z = x + iy$ and c is a finite positive constant. We can now study the complex equation in Eq.(1.115) and understand how many and which kind of solutions it has.

- First all note that Eq.(1.115) can have at most two real solutions, when the line $z = x$ intersects the exponential $ce^z = ce^x$. We can check that one of the solution can be associated to the usual shifted mass value, while

¹⁹As a more familiar example we can think of quantum electrodynamics, which is unitary at the perturbative level. At the same time, it is also known that by dressing the propagator an extra ghost-like pole emerges, which is related to the Landau pole. Analogously in GR, when dressing the graviton propagator by taking into account matter contributions, infinite pairs of complex conjugate poles appear [133]. However, such a more complicated pole structure of the dressed propagator does not spoil perturbative unitarity.

the second one comes with a negative residue, whose mass value is larger than the scale of nonlocality, ($x = -p^2/M_s^2 > 1$). Moreover, the smaller is c , which is related to the coupling constant, the larger will be the mass of this ghost-mode beyond the scale of nonlocality²⁰.

- Secondly, we can check whether there are any massive complex poles. By decomposing z in real and imaginary components, we obtain two equations coming from the real and imaginary parts of Eq.(1.115):

$$\begin{aligned} x &= -ce^x \cos y, \\ y &= -ce^x \sin y, \end{aligned} \quad (1.116)$$

which can be rewritten as

$$x = \frac{y}{\tan y}, \quad y = -ce^{y/\tan y} \sin y. \quad (1.117)$$

We note that this system of two equations has an infinite number of solutions, which means that the dressed propagator has an *infinite number of massive complex poles*. Moreover, by studying the system of equations in Eq.(1.117) we can easily understand that if $x + iy$ is a solution, also its complex conjugate, $x - iy$, will be a solution as the equations are unchanged under the transformation $y \rightarrow -y$.

Note that also in the case of ϕ^3 -theory, in which the structure of the 1-loop self-energy is more complicated, see Eq.(1.107), one has a similar scenario with infinite complex conjugate poles.

Unitarity with infinite complex conjugate poles

The amplitude with an internal dressed propagator in Minkowski space reads:

$$T = i\mathcal{G}(p) = \frac{e^{-p^2/M_s^2}}{p^2 - i\Sigma(p)e^{-p^2/M_s^2}}, \quad (1.118)$$

$\Gamma \sim \Sigma(p)e^{-p^2/M_s^2}$ is related to the width of the particle. If we now compute the imaginary part of Eq.(1.118) we obtain:

$$\text{Im} \{ \lambda^2 \mathcal{G}(p) \} = \text{Im} \left\{ \frac{e^{-p^2/M_s^2}}{p^2 - i\Sigma(p)e^{-p^2/M_s^2}} \right\} = \frac{e^{-2p^2/M_s^2} \Sigma(p)}{p^4 + \Sigma^2(p)e^{-2p^2/M_s^2}}. \quad (1.119)$$

Note that the sign in Eq.(1.119) is determined by the sign of $\Sigma(p)$. First of all, we can observe that for ϕ^4 -theory the self-energy at 1-loop is a positive

²⁰See Ref.[134] for a similar real particle spectrum beyond the scale of nonlocality but in the context of Higgs mechanism.

constant, see the subsection 1.3.1, so that the imaginary part of the dressed propagator is positive too. Furthermore, we can also perform the same check for a non-constant self-energy ($\Sigma(p) \neq c$). For instance in the case of ϕ^3 -theory we need to consider the 1-loop expression in Eq.(1.107) which can be checked to be always positive for any value of p .

Therefore, we have shown that also for a dressed propagator the imaginary part can be still positive consistently with the optical theory, despite the presence of extra poles.

1.3.5 Transmutation of nonlocal scale

In this Subsection we will study the behavior of scattering amplitudes involving a very large number of interacting particles [92]. We will consider a simple model with nonlocal kinetic term and ϕ^3 local interaction potential, and work in Euclidean signature. Subsequently, we will also comment on more complicated and realistic Lagrangians.

Let us consider an infinite derivative massless scalar field with cubic interaction and form factor $e^{\gamma(\Box_s)} = e^{(-\Box/M_s^2)^n}$. The corresponding Euclidean action reads

$$S = \int d^4x \left(\frac{1}{2} \phi(x) e^{(-\Box_s/M_s^2)^n} \Box \phi(x) + \frac{\lambda}{3!} \phi^3(x) \right), \quad (1.120)$$

where λ being the coupling constant, and the Euclidean bare propagator is given by

$$\Pi(p) = \frac{e^{-(p^2/M_s^2)^n}}{p^2}, \quad (1.121)$$

which is exponentially suppressed in the UV regime, $p^2 \gg M_s^2$, with $p^2 = (p^4)^2 + \vec{p}^2 > 0$ being the squared of the 4-momentum in an Euclidean signature $p \equiv (ip^4, \vec{p})$. The bare vertex is just a constant:

$$V(k_1, k_2, k_3) = \lambda. \quad (1.122)$$

N -point scattering amplitude

We now want to compute N -point amplitudes, $\mathcal{M}_N$, for the action in Eq.(1.120). Let us consider $n = 1$ to start with, and then we will generalize to generic powers of $\Box$. A generic tree level N -point amplitude for the action in Eq.(1.120) will be made up of N external legs, $N - 2$ vertices and $N - 3$ internal propagators; see Fig. 1.4. The simplest scattering amplitude we can construct is a 4-point diagram:

$$\mathcal{M}_4 = \lambda^2 \frac{e^{-p^2/M_s^2}}{p^2}, \quad (1.123)$$

Figure 1.4: 6-point amplitude for scalar cubic interaction.

with p being the sum of the two ingoing (or, equivalently, outgoing) momenta, $p_1 + p_2 = p_3 + p_4 \equiv p$. Similarly, by increasing the number of external legs to 6, we can consider a 6-point amplitude²¹ as in Fig. 1.4, where our convention is that all p_i with $i = 1, 2, 3, 4$ are ingoing, while p_5 and p_6 are outgoing momenta. From the conservation law of the total 4-momentum, we have:

$$p_1 + p_2 + p_3 + p_4 = p_5 + p_6. \quad (1.124)$$

The 6-point amplitude in the UV regime then reads:

$$\mathcal{M}_6 = \lambda^4 \frac{e^{-\frac{(p_1+p_2)^2}{M_s^2}} e^{-\frac{(p_1+p_2+p_3)^2}{M_s^2}} e^{-\frac{(p_1+p_2+p_3+p_4)^2}{M_s^2}}}{(p_1 + p_2)^2 (p_1 + p_2 + p_3)^2 (p_1 + p_2 + p_3 + p_4)^2}. \quad (1.125)$$

For simplicity, we can make the following choice for the incoming momenta:

$$|p_1 + p_2| = |p_3 + p_4| \equiv |p|, \quad \vec{p}_1 = -\vec{p}_2, \quad \vec{p}_3 = -\vec{p}_4; \quad (1.126)$$

thus, the amplitude in Eq.(1.125) is roughly given by

$$\mathcal{M}_6 \sim \frac{\lambda^4}{(2!)^2} \frac{e^{-(2(1)^2+2^2)p^2/M_s^2}}{p^6} = \frac{\lambda^4}{(2!)^2} \frac{e^{-6p^2/M_s^2}}{p^6}, \quad (1.127)$$

where we have neglected the terms such as $(p_3^4)^2$ and $2p_3^4|p|$ since $2p^2 > (p_3^4)^2$, $2p_3^4|p|$, and this approximation becomes even more justified for a very large number of external legs, i.e. when $N \gg 1$; see below. By adding two extra external legs, p_7 and p_8 , and making similar choices as in Eq.(1.126) and neglecting the cross-terms, one can see that the 8-point scattering amplitude will behave as

$$\mathcal{M}_8 \sim \frac{\lambda^6}{(3!)^2} \frac{e^{-(2(1^2+2^2)+3^2)p^2/M_s^2}}{p^{10}} = \frac{\lambda^6}{(3!)^2} \frac{e^{-19p^2/M_s^2}}{p^{10}}. \quad (1.128)$$

²¹In principle, we can also consider more complicated diagrams, but in the large N limit all the correct features related to the exponential form factors can be captured universally.

By inspecting Eqs. (1.127, 1.128), it is clear that by increasing the number of external legs, the scattering amplitude becomes even more exponentially suppressed. We can now easily find the expression for an N -point scattering amplitude, which will be roughly given by²²

$$\begin{aligned}
 \mathcal{M}_N = & V(p_1, p_2, p_1 + p_2) \prod_{i=2}^{N-2} \Pi(p_1 + \cdots + p_i) \\
 & \times \prod_{j=2}^{N-3} V(p_1 + \cdots + p_j, p_{j+1}, p_1 + \cdots + p_{j+1}) \\
 & \times V(p_1 + \cdots + p_{N-2}, p_{N-1}, p_N) \\
 \sim & \frac{\lambda^{N-2}}{[(N-2)/2]!^2} \frac{e^{-2 \left(\sum_{l=1}^{\frac{N-2}{2}} l^2 - \frac{1}{2} \left(\frac{N-2}{2} \right)^2 \right) p^2 / M_s^2}}{p^{2(N-3)}},
 \end{aligned} \tag{1.129}$$

where now the conservation law of the 4-momenta in Eq.(1.124) and the choice in Eq.(1.126) generalize to

$$p_1 + p_2 + \cdots + p_{N-2} = p_{N-1} + p_N, \tag{1.130}$$

$$|p_i + p_{i+1}| \equiv |p|, \quad \vec{p}_i = -\vec{p}_{i+1}, \quad i = 1, \dots, N-3; \tag{1.131}$$

and we have used the relation

$$2j^2 p^2 \gg (p_{2j+1}^4)^2, \quad 2p_{2j+1}^4 j p, \tag{1.132}$$

to neglect terms like $(p_{2j+1}^4)^2$ and $2p_{2j+1}^4 j p$, as $j \gg 1$. Note that the second set of equalities in Eq.(1.131) corresponds to the choice of the centre of mass frame for $N-2$ incoming particles; indeed, for two incoming particles we would only have $\vec{p}_1 = -\vec{p}_2$ and recover the usual relation between the spatial part of the two incoming momenta in the case of a 4-point scattering amplitude.

The numeric series in Eq.(1.129) can be summed up and in the limit $N \gg 1$ reads:

$$\sum_{l=1}^{\frac{N-2}{2}} l^2 = \frac{N(N-1)(N-2)}{12} \xrightarrow{N \gg 1} N^3; \tag{1.133}$$

therefore, for a large number of interacting particles the N -point amplitude in Eq.(1.129) shows the following behavior:

$$\mathcal{M}_N \sim \lambda' \frac{e^{-N^3 p^2 / M_s^2}}{p^{2(N-3)}} = \lambda' \frac{e^{-P^2 / M_{\text{eff}}^2}}{p^{2(N-3)}}, \tag{1.134}$$

²²The formula in Eq.(1.129) is valid for N even, but the analog formula for N odd can be easily derived. However, in the large N limit the results are the same and do not depend on the parity of the number of legs.

Figure 1.5: The dashed blob represents the region in which the nonlocal interaction takes place. The larger is the number of interacting particles, the larger will be the nonlocal region in coordinate space and time.

where we have defined $\lambda' := \lambda^{N-2}/[(N-2)/2]!^2$, while in the last step we have introduced the total momentum $P = Np$ and the *effective scale*

$$M_{\text{eff}} = \frac{M_s}{\sqrt{N}}. \quad (1.135)$$

Hence, from Eqs.(1.134, 1.135) we have obtained that by increasing the number of external legs, or in other words the number of interacting particles, the scattering amplitude becomes more exponentially suppressed. This feature can be understood as follows: there is a *transmutation* of scale under which the fundamental scale of nonlocality M_s *shifts* towards lower energies, i.e. $M_{\text{eff}} \ll M_s$ when $N \gg 1$. In this process, the nonlocal length and time scales can be made much larger than the original scale of nonlocality, i.e. $L_s = M_s^{-1} \rightarrow \sqrt{N}L_s$, therefore its affect can be felt in the IR. In other words, for larger is the number of interacting particles, the larger will be the spacetime region on which nonlocal interaction takes place; see Fig. 1.5 for an illustration.

The above calculations have been performed for the form factor $e^{-\square/M_s^2}$, i.e. with $n = 1$. We can generalize straightforwardly the previous results to generic powers n of the d'Alembertian, such as $e^{(-\square/M_s^2)^n}$:

$$\mathcal{M}_N \sim \lambda' e^{\frac{-2 \left(\sum_{l=1}^{\frac{N-2}{2}} l^{2n} - \frac{1}{2} \left(\frac{N-2}{2} \right)^{2n} \right) p^{2n}/M_s^{2n}}{p^{2(N-3)}}}. \quad (1.136)$$

The numeric series in Eq.(1.136) can be expressed in terms of the *Faulhaber*

formula which is given by [135]

$$\sum_{l=1}^{\frac{N-2}{2}} l^{2n} = \frac{1}{2n+1} \sum_{i=0}^{2n} (-1)^i \binom{2n+1}{i} B_i \cdot \left(\frac{N}{2} - 1\right)^{2n+1-i}, \quad (1.137)$$

where B_i is the so called Bernoulli number. The expression in Eq.(1.137) seems rather complicated, but fortunately we are only interested in the limit $N \gg 1$, which gives

$$\sum_{l=1}^{\frac{N-2}{2}} l^{2n} \xrightarrow{N \gg 1} N^{2n+1}. \quad (1.138)$$

Hence, the N -point scattering amplitude for generic powers n of the d'Alembertian will behave as

$$\mathcal{M}_N \sim \lambda' \frac{e^{-(N^{2n+1})p^{2n}/M_s^{2n}}}{p^{2(N-3)}} = \lambda' \frac{e^{-p^{2n}/M_{\text{eff}}^{2n}}}{p^{2(N-3)}}, \quad (1.139)$$

where in this more general case the effective nonlocal scale is defined as:

$$M_{\text{eff}} = \frac{M_s}{N^{\frac{2n+1}{2n}}} = \frac{M_s}{N^{1+1/2n}}. \quad (1.140)$$

We can immediately notice that in the case $n = 1$ we consistently recover the case with $e^{-\Box/M_s^2}$ studied above.

Our study has been made for the simple scalar action in Eq.(1.120) but the analysis becomes more complicated for other kind of models. For instance, in the case of gravity we would expect the presence of exponential operators both in the propagator and in the vertex. In this case, one way to make the amplitude well behaved at all energy scales is to dress both propagators and vertexes as it was implemented in Ref.[92]. However, a more general and rigorous investigation has been still work out and will be subject of future works.

1.4 Enlarging (local) symmetries

In the previous Section we have studied a very peculiar property of some non-polynomial Lagrangians according to which the effect of nonlocality becomes more relevant when more particles are involved in a scattering process. In this Section, we will discuss another interesting novel feature related to the possibility to enlarge the class of symmetries under which a Lagrangian is invariant once infinite derivative generalization are introduced. We will show that at least for a simple example it works and leave more general possibilities for future [136].

The symmetry class we wish to generalize through nonlocal operators is the so called *Galilean symmetry* in Minkowski spacetime, which is defined by the following transformation:

$$\phi(x) \rightarrow \phi(x) + b_\mu x^\mu + a, \quad \text{where} \quad \{b_\mu, a\} \quad (1.141)$$

is a family of five constant parameters which generate the whole Galilean symmetry. Lagrangians invariant under the above transformations can be constructed, in terms of a massless scalar field $\phi(x)$, where only specific derivative self-interactions are present. For example, the following Lagrangian exhibits such a symmetry [137]:

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi - \frac{1}{2\Lambda^3}\Box\phi\partial_\mu\phi\partial^\mu\phi + \dots \quad (1.142)$$

where Λ can be, for example, a cut-off energy scale above which an effective field theory description may break down. A more general form of the Lagrangian in Eq. (1.142) can be found in Ref. [137].

We now wish to understand whether we can enlarge the Galilean symmetry while working with nonlocal differential operators. In order to satisfy such a (local) symmetry, we need to construct other nonlocal operators which are different from the ones considered so far. Further note that second order derivative operators, containing terms like the ones in Eq. (1.142), are always invariant under the Galilean transformation. It is easy to understand that if $\Box\phi$ is invariant under the Galilean transformation, then any power $\Box^n\phi$ will be also invariant, which in turn implies that the following nonlocal operator is also invariant under the Galilean transformation:

$$\mathcal{O}\phi \equiv \left(e^{-\Box/M_s^2} - 1\right)\phi = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\Box}{M_s^2}\right)^n \phi, \quad (1.143)$$

which is slightly different from the differential operators in Eq.(1.10).

The question to ask is the following - is there any function $\psi(x)$, such that the transformation $\phi(x) \rightarrow \phi(x) + \psi(x)$, leaves invariant the nonlocal operator in Eq. (1.143), but *not* the local one? Or, in other words, can we find a function $\psi(x)$ such that $\left(e^{-\Box/M_s^2} - 1\right)\psi = 0$, but $\Box\psi \neq 0$? The answer is *yes*.

Let us find solutions to the following nonlocal differential equation:

$$\left(e^{-\Box/M_s^2} - 1\right)\psi(x) = 0 \iff e^{-\Box/M_s^2}\psi(x) = \psi(x), \quad (1.144)$$

which also means to find the eigenfunctions ψ of the operator $e^{-\Box/M_s^2}$ of unit eigenvalue. First of all, note that the following property holds

$$e^{-\Box/M_s^2} = e^{-\Box/M_s^2 + i2\pi k} \equiv e^{\theta_2}, \quad (1.145)$$

where k is an integer number and we have defined the differential operator $\theta_2 := -\square/M_s^2 + i2\pi k$. We can show that the solutions of the differential equation $\theta_2\psi(x) = 0$ are given by [136]:

$$\psi_k(x) = \frac{B}{\sqrt{k}} \sinh\left(\frac{\sqrt{k}}{B} b_\mu x^\mu\right) + a e^{-\frac{\sqrt{k}}{B} b_\mu x^\mu}. \quad (1.146)$$

where the integration constants have been fixed by demanding to recover the Galilean shift symmetry in Eq. (1.141) for $k = 0$, in particular $B = \sqrt{b_\mu b^\mu}/((1+i)\sqrt{\pi}M_s)$. Hence, we have explicitly shown that by working with the nonlocal, infinite derivative operator introduced in Eq. (1.143), we can enlarge the Galilean shift symmetry which now becomes a subclass of a wider class described by

$$\phi(x) \rightarrow \phi(x) + \psi_k(x), \quad \text{where} \quad \{b_\mu, a, k\} \quad (1.147)$$

is a family of six parameters, so that the Galilean transformations correspond to the subfamily $\{b_\mu, a, 0\}$.

1.4.1 A nonlocal Galilean model

The tree level Lagrangians in local Galilean theories are free from any kind of instabilities as the corresponding field equations turn out to be of second order in the derivatives [137]. However, as shown in Ref.[138, 139], quantum corrections can introduce new higher derivative terms like $\phi\square^2\phi$, $\phi\square^3\phi$ and $\phi\square^4\phi$, which cause classical instabilities and unitarity violation, since ghost modes are introduced.

Nonlocal generalizations of Galilean theories can avoid any ghost degrees of freedom, not only at the tree level but also when quantum loop corrections are taken into account, due to their improved UV behavior. As an example, we can consider the following nonlocal model:

$$\begin{aligned} \mathcal{L}_{\text{NL}} = & -\frac{M_s^2}{2} \phi \left(e^{-\square/M_s^2} - 1 \right) \phi \\ & -\lambda \frac{M_s^3}{2} \left(e^{\square/M_s^2} - 1 \right) \phi \frac{\left(e^{\square/M_s^2} - 1 \right)}{\square} \partial_\mu \phi \frac{\left(e^{\square/M_s^2} - 1 \right)}{\square} \partial^\mu \phi, \end{aligned} \quad (1.148)$$

where λ is a dimensionless coupling constant and the corresponding (Euclidean) propagator reads

$$\Pi(p) = \frac{1}{M_s^2 (e^{p^2/M_s^2} - 1)}, \quad (1.149)$$

while the vertex is given by

$$V(k_1, k_2, k_3) = \lambda M_s^3 \left(e^{-k_1^2/M_s^2} - 1 \right) \left(e^{-k_2^2/M_s^2} - 1 \right) \left(e^{-k_3^2/M_s^2} - 1 \right) \\ \times \left(\frac{k_1 \cdot k_2}{k_1^2 k_2^2} + \frac{k_1 \cdot k_3}{k_1^2 k_3^2} + \frac{k_2 \cdot k_3}{k_2^2 k_3^2} \right). \quad (1.150)$$

Very interestingly, in the low energy regime, $\square \ll M_s^2$, the action in Eq.(1.148) reduces to some version of local Galilean actions with the cubic term in Eq.(1.142) plus higher order terms including the 1-loop quantum corrections found in Ref.[138, 139]; for instance, by expanding up to $\mathcal{O}(1/M_s^6)$ we obtain:

$$\mathcal{L}_{\text{NL}} = \frac{1}{2} \phi \square \phi - \frac{1}{4M_s^2} \phi \square^2 \phi - \frac{\lambda}{2M_s^3} \phi \partial_\mu \phi \partial^\mu \phi + \frac{1}{12M_s^4} \phi \square^3 \phi \\ - \frac{\lambda}{M_s^5} \phi \partial_\mu \phi \square \partial^\mu \phi - \frac{1}{24M_s^6} \phi \square^4 \phi + \mathcal{O}(1/M_s^7), \quad (1.151)$$

and if we truncate the series at a certain order in the expansion the nonlocal scale M_s assumes the meaning of a cut-off scale. Therefore, the action in Eq. (1.148) can represent a ghost-free UV completion of the local cubic Galilean action considered in Eq.(1.142); we will now clarify it with a 1-loop computation.

The presence of the exponentials in both propagator and interaction vertex ameliorates the UV behavior of the theory not only at the tree level but also at higher loop orders. The power counting arguments gives the following superficial degree of divergence which turns out to be always negative: $D = -I$, where I is the number of internal propagators. This is a good hint in favor of the finiteness of loop integrals for the model in Eq.(1.148) and we can explicitly show this feature by computing as an example the self-energy $\Sigma(p)$ at 1-loop (bubble diagram),

$$\Sigma^{(1)}(p) = \int \frac{d^4 k}{(2\pi)^4} \Pi(k) \Pi(p-k) V^2(k, p-k, p) \\ = \lambda^2 M_s^2 \left(e^{-p^2/M_s^2} - 1 \right)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\left(e^{-k^2/M_s^2} - 1 \right)^2 \left(e^{-(p-k)^2/M_s^2} - 1 \right)^2}{\left(e^{k^2/M_s^2} - 1 \right) \left(e^{(p-k)^2/M_s^2} - 1 \right)} \\ \times \left(\frac{k \cdot (p-k)}{k^2 (p-k)^2} + \frac{k \cdot p}{k^2 p^2} + \frac{p \cdot (p-k)}{p^2 (p-k)^2} \right)^2, \quad (1.152)$$

whose behavior in the UV regime is given by

$$\Sigma^{(1)}(p) \xrightarrow{UV} \lambda^2 M_s^2 \left(e^{-p^2/M_s^2} - 1 \right)^2 e^{-p^2/M_s^2} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-2k^2/M_s^2}}{k^2}, \quad (1.153)$$

where the factor 2 in the exponent is in agreement with the power counting argument: $D = -2$. From Eq.(1.153) we can notice that the loop integral is

exponentially suppressed at high energy and does not need to be renormalized. Therefore, for the model in Eq.(1.148) we would expect no ghost degrees of freedom to emerge when quantum loop corrections are taken into account, unlike the local case studied in Ref.[138, 139].

Propagator and tree level unitarity

The pole structure of the propagator in Eq.(1.149) is more complicated than the ones seen so far, besides the usual real massless pole at $p^2 = 0$, we now have an infinite number of complex conjugate poles. Indeed, the denominator in Eq.(1.149) vanishes for $p^2 = i2\pi M_s^2 \ell$ where ℓ is an integer number: $\ell = 0$ corresponds to the only real massless pole, while each value of $\ell \neq 0$ is associated with two complex poles, whose conjugates are the ones corresponding to the opposite integer $-\ell$.

The pole structure becomes more explicit by rewriting the momentum space propagator in Eq.(1.149) as follows [136]

$$\Pi(p) = \frac{e^{-p^2/2M_s^2}}{p^2} + e^{-p^2/2M_s^2} \sum_{\ell=1}^{\infty} (-1)^\ell \left(\frac{1}{p^2 + i2\pi M_s^2 \ell} + \frac{1}{p^2 - i2\pi M_s^2 \ell} \right), \quad (1.154)$$

where we have used the following identity

$$\frac{1}{\sin(iz/2)} = \frac{2}{i} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \frac{1}{z + i2\pi \ell}. \quad (1.155)$$

Note that the presence of complex poles, in general, may spoil perturbative unitarity, in such a way that predictability would be lost at the quantum level. However, we can show, at least at tree level, that the model in Eq. (1.148) is healthy. As already discussed above for other operators, the unitarity condition implies that the imaginary part of the propagator has to be positive. If we compute the imaginary part of the propagator in Eq.(1.149) we obtain [136]

$$\begin{aligned} \text{Im}\{\Pi(p)\} &= e^{-p^2/2M_s^2} \text{Im}\left\{ \frac{1}{p^2 - i\epsilon} + \sum_{\ell=1}^{\infty} (-1)^\ell \left(\frac{1}{p^2 + i2\pi M_s^2 \ell} + \frac{1}{p^2 - i2\pi M_s^2 \ell} \right) \right\} \\ &= \frac{e^{-p^2/2M_s^2} \epsilon}{p^4 + \epsilon^2} \\ &= \pi \delta^{(4)}(p^2) > 0, \end{aligned} \quad (1.156)$$

where the infinite complex poles do not contribute to the imaginary part of the propagator due to the fact that they appear in conjugate pairs, and in the last step we have taken the limit $\epsilon \rightarrow 0$. This implies that the propagator in Eqs.(1.149,1.154) satisfies the tree level unitarity condition.

Hence, we have showed that by working with nonlocal actions it is possible to enlarge the Galilean shift symmetry. Indeed, we have found a new class of transformations, Eq. (1.147), under which our nonlocal actions are invariant, and such a wider class includes the Galilean shift symmetry as a subclass. At the same time, we have been also able to enlarge the class of nonlocal operators which make the propagator ghost-free, without restricting ourself to exponential of entire functions, which we introduced in the previous sections. However, further studies aimed to show that for the model in Eq.(1.148) the optical theorem holds at all order in perturbation theory will be subject of future works. Note that in Refs.[31, 32, 30], perturbative unitarity was proved for *Lee-Wick theories* which are local higher derivative theories with a finite number of complex conjugate poles, and these results, or their generalization to infinite pairs of complex conjugate poles, might be useful in our case.

Furthermore, the question on the possibility to enlarge symmetries of local Lagrangian through infinite derivative operators remains still open, indeed we have only showed a very simple example which might be very unique.

Infinite derivative gravity

Einstein's GR turns out to be perturbatively non-renormalizable, while by adding quadratic curvature terms to the Einstein-Hilbert action one obtains a renormalizable theory of gravity which, however, lacks of predictability due to the lost of unitarity caused by the presence of a ghost in the physical spectrum. In this Chapter we study the most general quadratic curvature action which is made up of higher order differential operators, and show that it is possible to choose peculiar non-polynomial operators that make the propagator ghost-free around Minkowski background.

In Section 2.1 we introduce the action and the field equations. By linearizing around Minkowski background, we compute the graviton propagator. In Section 2.2 we find linearized metric solutions in several cases. In Section 2.3 we move towards the non-linear regime and show how nonlocality can help to resolve curvature singularities.

This Chapter is mostly based on **P1**, **P2**, **P4**, **P5**.

2.1 Quadratic curvature action

The most general gravitational action in the curvature, parity-invariant and torsion-free can be written in a compact form as follows [41]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \mathcal{R}_{\mu_1\nu_1\lambda_1\sigma_1} \mathcal{D}_{\mu_2\nu_2\lambda_2\sigma_2}^{\mu_1\nu_1\lambda_1\sigma_1} \mathcal{R}^{\mu_2\nu_2\lambda_2\sigma_2} \right\}, \quad (2.1)$$

where $\kappa = \sqrt{8\pi G}$, $\mathcal{R}_{\dots}$ stands for any curvature tensor, like Ricci scalar, Ricci tensor and Riemann tensor, while $\mathcal{D}$ must be a covariant differential operator containing only covariant derivatives and the metric tensor $g_{\mu\nu}$. In Refs.[41, 62,

63] it was shown that up to linear order in perturbation around any maximally symmetric background, i.e. Minkowski, dS and AdS, the action in Eq.(2.1) reduces to a simpler form:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \frac{1}{2} (\mathcal{R}\mathcal{F}_1(\square)\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{F}_2(\square)\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\square)\mathcal{R}^{\mu\nu\rho\sigma}) \right\}, \quad (2.2)$$

where the form factors $\mathcal{F}_i(\square)$ are analytic operators of the d'Alembertian $\square$ and can be either local (polynomial) or nonlocal (non-polynomial)¹:

$$\mathcal{F}_i(\square) = \sum_{n=0}^N f_{i,n} \square^n, \quad i = 1, 2, 3. \quad (2.3)$$

We will show examples for both cases of positive and negative powers of $\square$, namely $n \geq 0$ or $n < 0$ which correspond to ultraviolet or infrared modifications of Einstein's GR, respectively. Note that, if N is finite ($< \infty$) and $n > 0$ the form factors are polynomial functions and we have a local theory of gravity of order $2N + 4$ in derivatives, whereas if $N = \infty$ the form factors are non-polynomial functions, namely we have a nonlocal theory of gravity.

The action in Eq.(2.2) can be also written in the Weyl basis in which the Riemann tensor is replaced by the Weyl tensor:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \frac{1}{2} (\mathcal{R}F_1(\square)\mathcal{R} + \mathcal{R}_{\mu\nu}F_2(\square)\mathcal{R}^{\mu\nu} + \mathcal{C}_{\mu\nu\rho\sigma}F_3(\square)\mathcal{C}^{\mu\nu\rho\sigma}) \right\}, \quad (2.4)$$

where the new form actors $F_i(\square)$ are related to $\mathcal{F}_i(\square)$ by the following relations:

$$F_1(\square) = \mathcal{F}_1(\square) + \frac{1}{3}\mathcal{F}_3(\square), \quad F_2(\square) = \mathcal{F}_2(\square) + 2\mathcal{F}_3(\square), \quad F_3(\square) = \mathcal{F}_3(\square). \quad (2.5)$$

Since the two actions above are equivalent and related by the simple linear transformations in Eq.(2.5), to derive the EOM we can variate the action in Eq.(2.4) as the Weyl tensor is less cumbersome than the Riemann tensor. The full derivation of the EOM can be found in Refs.[73, 140] where it was shown

¹Each form factor also depends on at least one new physical scale M_s , indeed to be more precise we should write $\mathcal{F}_i(\square/M_s^2)$. However, to make the notation less heavy we just write $\mathcal{F}_i(\square)$.

that:

$$\begin{aligned}
E_\nu^\mu &= (1 + 2\mathcal{F}_1(\square)\mathcal{R}) G_\nu^\mu + \frac{1}{2}\delta_\nu^\mu \mathcal{R}\mathcal{F}_1(\square)\mathcal{R} - 2(\nabla^\mu \partial_\nu - \delta_\nu^\mu \square) \mathcal{F}_1(\square)\mathcal{R} \\
&\quad - \Omega_{1\nu}^\mu + \frac{1}{2}\delta_\nu^\mu (\Omega_{1\sigma}^\sigma + \bar{\Omega}_1) + 2\mathcal{R}^{\mu\sigma} \mathcal{F}_2(\square)\mathcal{R}_{\nu\sigma} \\
&\quad - \frac{1}{2}\delta_\nu^\mu \mathcal{R}_\rho^\sigma \mathcal{F}_2(\square)\mathcal{R}_\sigma^\rho - 2\nabla_\sigma \nabla^\mu \mathcal{F}_2(\square)\mathcal{R}_\nu^\sigma + \square \mathcal{F}_2(\square)\mathcal{R}_\nu^\mu \\
&\quad + \delta_\nu^\mu \nabla_\alpha \nabla_\beta (\mathcal{F}_2(\square)\mathcal{R}^{\alpha\beta}) - \Omega_{2\nu}^\mu + \frac{1}{2}\delta_\nu^\mu (\Omega_{2\sigma}^\sigma + \bar{\Omega}_2) - 2\Delta_{2\nu}^\mu \\
&\quad - \frac{1}{2}\delta_\nu^\mu \mathcal{C}^{\alpha\beta\lambda\sigma} \mathcal{F}_3(\square)W_{\alpha\beta\lambda\sigma} + 2\mathcal{C}_{\alpha\beta\sigma}^\mu \mathcal{F}_3(\square)\mathcal{C}_\nu^{\alpha\beta\sigma} - \Omega_{3\nu}^\mu \\
&\quad - 2(\mathcal{R}_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta)(\mathcal{F}_3(\square_s)\mathcal{C}^\mu{}_\nu{}^{\alpha\beta}) + \frac{1}{2}\delta_\nu^\mu (\Omega_{3\sigma}^\sigma + \bar{\Omega}_3) - 4\Delta_{3\nu}^\mu \\
&= \kappa^2 T_\nu^\mu, \tag{2.6}
\end{aligned}$$

where $G_\nu^\mu = \mathcal{R}_\nu^\mu - \frac{1}{2}\delta_\nu^\mu \mathcal{R}$ is the Einstein tensor, $T_\nu^\mu = -(2/\sqrt{-g})\delta S_m/\delta g_\mu^\nu$ is the stress-energy tensor defined in terms of the matter action S_m and

$$\begin{aligned}
\Omega_{1\nu}^\mu &= \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \partial^\mu \mathcal{R}^{(l)} \partial_\nu \mathcal{R}^{(n-l-1)}, \quad \bar{\Omega}_1 = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \mathcal{R}^{(l)} \mathcal{R}^{(n-l)}, \\
\Omega_{2\nu}^\mu &= \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \mathcal{R}_\beta^{\alpha;\mu(l)} \mathcal{R}_{\alpha;\nu}^{\beta(n-l-1)}, \quad \bar{\Omega}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \mathcal{R}_\beta^{\alpha(l)} \mathcal{R}_\alpha^{\beta(n-l)}, \\
\Delta_{2\nu}^\mu &= \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} [\mathcal{R}_\sigma^{\beta(l)} \mathcal{R}_{;\nu}^{(\mu\sigma(n-l-1))} - \mathcal{R}_{\sigma;\nu}^{\beta(l)} \mathcal{R}^{\mu\sigma(n-l-1)}]_{;\beta}, \\
\Omega_{3\nu}^\mu &= \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} \mathcal{C}_{\beta\lambda\sigma;\nu}^{\alpha(l)} \mathcal{C}_\alpha^{\beta\lambda\sigma;\mu(n-l-1)}, \quad \bar{\Omega}_3 = \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} \mathcal{C}_{\beta\lambda\sigma}^{\alpha(l)} \mathcal{C}_\alpha^{\beta\lambda\sigma(n-l)}, \\
\Delta_{3\nu}^\mu &= \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} [\mathcal{C}_{\sigma\alpha}^{\lambda\beta(l)} \mathcal{C}_{\lambda;\nu}^{\mu\sigma\alpha(n-l-1)} - \mathcal{C}_{\sigma\alpha;\nu}^{\lambda\beta(l)} \mathcal{C}_\lambda^{\mu\sigma\alpha(n-l-1)}]_{;\beta}. \tag{2.7}
\end{aligned}$$

The notation $\mathcal{R}^{(l)} \equiv \square^l \mathcal{R}$ is used for the curvature tensors and their covariant derivatives.

Taking the trace of Eq.(2.6), we obtain

$$\begin{aligned}
E &= -(1 - 6\square\mathcal{F}_1(\square_s)) \mathcal{R} + \square(\mathcal{F}_2(\square_s)\mathcal{R}) + 2\nabla_\mu \nabla_\nu (\mathcal{F}_2(\square_s)\mathcal{R}^{\mu\nu}) \\
&\quad + (\Omega_{1\sigma}^\sigma + 2\bar{\Omega}_1) + (\Omega_{2\sigma}^\sigma + 2\bar{\Omega}_2) + (\Omega_{3\sigma}^\sigma + \bar{\Omega}_3) - 2\Delta_{2\sigma}^\sigma - 4\Delta_{3\sigma}^\sigma \\
&= \kappa^2 T, \tag{2.8}
\end{aligned}$$

where $E \equiv \delta_\mu^\nu E_\nu^\mu$ and $T \equiv \delta_\mu^\nu T_\nu^\mu$.

The EOM above are very cumbersome and finding exact non-linear solutions is very challenging. Indeed, so far exact non-trivial vacuum solutions have been only found in the context of cosmology [40, 58, 59, 60, 101]. We will now work in the linearized regime around Minkowski background analytic solutions can be found.

2.1.1 Linearized action and propagator

Let us consider linear perturbation around the Minkowski spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (2.9)$$

where $h_{\mu\nu}$ is the metric perturbation. By working in the linear regime around Minkowski we can further simplify then in Eq.(2.2), by using the fact that the relevant contribution is of the order $\mathcal{O}(h^2)$, and in such a regime the term $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{F}_3(\square)\mathcal{R}^{\mu\nu\rho\sigma}$ in Eq.(2.2) or, equivalently, the term $\mathcal{C}_{\mu\nu\rho\sigma}F_3(\square)\mathcal{C}^{\mu\nu\rho\sigma}$ in Eq.(2.4), can be neglected. In fact, since the following identity holds true

$$\mathcal{R}_{\mu\nu\rho\sigma}\square^n\mathcal{R}^{\mu\nu\rho\sigma} = 4\mathcal{R}_{\mu\nu}\square^n\mathcal{R}^{\mu\nu} - \mathcal{R}\square^n\mathcal{R} + \mathcal{O}(\mathcal{R}^3) + \text{div}, \quad (2.10)$$

where div stands for total derivative, up to order $\mathcal{O}(h^2)$ we can always replace the Riemann square contributions by combinations of Ricci scalar and Ricci tensor square, indeed $\mathcal{O}(\mathcal{R}^3)$ only contributes at order $\mathcal{O}(h^3)$. Hence, by working in the linearized regime we are allowed to set $\mathcal{F}_3(\square) = 0$ without any loss of generality.

Therefore, the quadratic gravitational action in Eq.(2.2) up to order $\mathcal{O}(h^2)$ reads [41]:

$$S^{(2)} = \frac{1}{4} \int d^4x \left\{ \frac{1}{2} h_{\mu\nu} f(\square) \square h^{\mu\nu} - h_{\mu}^{\sigma} f(\square) \partial_{\sigma} \partial_{\nu} h^{\mu\nu} + h g(\square) \partial_{\mu} \partial_{\nu} h^{\mu\nu} - \frac{1}{2} h g(\square) \square h + \frac{1}{2} h^{\lambda\sigma} \frac{f(\square) - g(\square)}{\square} \partial_{\lambda} \partial_{\sigma} \partial_{\mu} \partial_{\nu} h^{\mu\nu} \right\}, \quad (2.11)$$

where $h \equiv \eta_{\mu\nu} h^{\mu\nu}$ defines the trace and

$$\begin{aligned} f(\square) &\equiv 1 + \frac{1}{2} \mathcal{F}_2(\square) \square, \\ g(\square) &\equiv 1 - 2\mathcal{F}_1(\square) \square - \frac{1}{2} \mathcal{F}_2(\square) \square. \end{aligned} \quad (2.12)$$

Note that, the linearized action in Eq.(1) can be also recast as

$$S^{(2)} = \frac{1}{4} \int d^4x h_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} h_{\rho\sigma}, \quad (2.13)$$

where $\mathcal{O}^{\mu\nu\rho\sigma} = \mathcal{O}^{\nu\mu\rho\sigma} = \mathcal{O}^{\nu\mu\sigma\rho} = \mathcal{O}^{\rho\sigma\mu\nu}$ is the kinetic operator of the graviton field and is defined as [141]

$$\begin{aligned} \mathcal{O}^{\mu\nu\rho\sigma} \equiv & \frac{1}{4} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}) f(\Box) \Box - \frac{1}{2} (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho) f(\Box) \\ & - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} g(\Box) \Box + \frac{1}{2} (\eta^{\mu\nu} \partial^\rho \partial^\sigma + \eta^{\rho\sigma} \partial^\mu \partial^\nu) g(\Box) \\ & + \frac{1}{2} \frac{f(\Box) - g(\Box)}{\Box} \partial^\mu \partial^\nu \partial^\rho \partial^\sigma. \end{aligned} \quad (2.14)$$

By varying the action in Eq.(2.11) with respect to the field $h_{\mu\nu}$ we can obtain the corresponding linearized field equations:

$$\begin{aligned} f(\Box) (\Box h_{\mu\nu} - \partial_\sigma \partial_\nu h_\mu^\sigma - \partial_\sigma \partial_\mu h_\nu^\sigma) + g(\Box) (\eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h) \\ + \frac{f(\Box) - g(\Box)}{\Box} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\rho\sigma} = -2\kappa^2 T_{\mu\nu}, \end{aligned} \quad (2.15)$$

where now

$$T_{\mu\nu} \simeq 2 \frac{\delta S_m}{\delta h^{\mu\nu}}, \quad (2.16)$$

which satisfies the conservation law $\partial^\mu T_{\mu\nu} = 0$ consistently with the Bianchi identity.

Propagator

We can compute the momentum space propagator for the field $h_{\mu\nu}$ by inverting the kinetic operator in Eq.(2.14). Note that, being gravity a gauge theory for a two-rank symmetric tensor, one has to introduce a gauge fixing term in order to make the kinetic operator invertible. One can show that the gauge independent part of the graviton propagator around Minkowski is given by [37, 39, 41, 142, 141]

$$i\Pi_{\mu\nu\rho\sigma}(k) = \frac{i}{f} \Pi_{\text{GR},\mu\nu\rho\sigma}(k) + \frac{3}{2} \frac{f - g}{(f - 3g)k^2} \mathcal{P}_{s,\mu\nu\rho\sigma}^0, \quad (2.17)$$

where

$$i\Pi_{\text{GR},\mu\nu\rho\sigma}(k) = \frac{\mathcal{P}_{\mu\nu\rho\sigma}^2}{k^2} - \frac{\mathcal{P}_{s,\mu\nu\rho\sigma}^0}{2k^2} \quad (2.18)$$

is the GR graviton propagator, and the functions f and g in momentum space depends on k^2 . $\mathcal{P}_{\mu\nu\rho\sigma}^2$ and $\mathcal{P}_{s,\mu\nu\rho\sigma}^0$ are the so called spin-2 and spin-0 projector operators [143, 142, 141], respectively; see Appendix C for their definition and more details. Notice that the graviton propagator in GR has six degrees of freedom off-shell, five coming from the spin-2 and one from the spin-0. The

scalar component has a minus sign in front which means it is a ghost. One can show that it is a good ghost (see Appendix A) and it is needed to cancel the longitudinal component with helicity 0 of the spin-2 when going on-shell. In fact, by imposing the on-shell condition and gauge invariance, we remain only with the ± 2 helicities. However, in higher derivative theories of gravity more physical degrees of freedom can appear.

2.1.2 Several applications

Let us now consider several gravitational theories and see the corresponding form of the propagator, which can be done by making specific choices for the functions $f(-k^2)$ and $g(-k^2)$ in Eq.(2.17). We will suppress the indexes to make the notation less heavy.

General Relativity

The choice

$$\mathcal{F}_1 = \mathcal{F}_2 = 0 \quad \Longrightarrow \quad f = g = 1 \quad (2.19)$$

gives

$$i\Pi(k) = i\Pi_{\text{GR}}(k), \quad (2.20)$$

which recovers the graviton propagator of GR in Eq.(2.18). The only pole of the propagator is $k^2 = 0$ and describes a massless transverse spin-2 particle. Einstein's GR is unitarity but non-renormalizable.

$F(\mathcal{R})$ -theory

The $F(\mathcal{R})$ gravity is actually a family of theories, each one defined by a different function of the Ricci scalar. The general action is

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(\mathcal{R}). \quad (2.21)$$

Since we are interested in computing the propagator around Minkowski, the expansion up to second order in the curvature is sufficient:

$$F(\mathcal{R}) = F(0) + F'(0)\mathcal{R} + \frac{1}{2}F''(0)\mathcal{R}^2 + \dots \quad (2.22)$$

The zero order term identifies with the cosmological constant, and the first order term should reduce to the Einstein-Hilbert term in a healthy theory, $F'(0) = 1$. By neglecting the cosmological constant term, the corresponding choice for the non-polynomial differential operators in Eq.(2.12) is

$$\mathcal{F}_1 = F'' \equiv \alpha, \quad \mathcal{F}_2 = 0 \quad \Longrightarrow \quad f = 1, \quad g = 1 - 2\alpha \square. \quad (2.23)$$

Therefore, gauge independent part of propagator (2.17) reads

$$i\Pi(k) = i\Pi_{\text{GR}} + \frac{1}{2} \frac{\mathcal{P}_s^0}{k^2 + m_0^2}, \quad (2.24)$$

where $m_0 \equiv (3\alpha)^{-1/2}$ is the mass of the spin-0 part, which is an additional degree of freedom besides the massless spin-2. This extra mode is not a ghost and neither a tachyonic as long as $\alpha = F''(0) > 0$. $F(\mathcal{R})$ -theories are unitary but non-renormalizable.

Fourth order gravity

The gravitational action is defined by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \frac{1}{2} (\alpha \mathcal{R}^2 + \beta \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}) \right\}, \quad (2.25)$$

which implies

$$\mathcal{F}_1 = \alpha, \mathcal{F}_2 = \beta \implies f = 1 + \frac{1}{2}\beta\Box, \quad g = 1 - 2\alpha\Box - \frac{1}{2}\beta\Box. \quad (2.26)$$

The propagator is given by

$$i\Pi(k) = i\Pi_{\text{GR}} + \frac{1}{2} \frac{\mathcal{P}_s^0}{k^2 + m_0^2} - \frac{\mathcal{P}^2}{k^2 + m_2^2}, \quad (2.27)$$

where $m_2 = 2(-\beta)^{-1/2}$ and $m_0 = (3\alpha + \beta)^{-1/2}$, with $\beta < 0$ in order to avoid the presence of tachyonic instabilities, are the masses of the healthy spin-0 component and massive spin-2 ghost mode, respectively. The latter cause classical and quantum instabilities; see Appendix A for a proof. The action in Eq.(2.25) describes a renormalizable but non-unitarity theory.

Sixth order gravity

As last example of finite higher order derivatives, we consider an action whose derivative order is six. It is given by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \frac{\alpha}{2} \mathcal{R} \Box \mathcal{R} \right\}, \quad (2.28)$$

so that the functions in Eq.(2.12) read

$$\mathcal{F}_1 = \alpha\Box, \mathcal{F}_2 = 0 \implies f = 1, \quad g = 1 - 2\alpha\Box^2. \quad (2.29)$$

In this case the propagator in Eq.(2.17) gives

$$i\Pi(k) = i\Pi_{\text{GR}} + \frac{3}{2} \frac{1 - 2\alpha k^4}{(1 - 6\alpha k^4)k^2} \mathcal{P}_s^0, \quad (2.30)$$

which possesses a pair of complex poles, $k^2 = \pm(6\alpha)^{-1/2}$ besides the massless spin-2 graviton. Such a gravitational theory belongs to the wide class of Lee-Wick gravity theories which are super-renormalizable and the unitarity can be still preserved by implementing an alternative quantization prescription to the Feynman one [28, 29, 30].

2.1.3 Ghost-free higher derivative gravity

So far we have only seen examples of finite order theories of gravity. We will now show that it is possible to introduce non-polynomial form factors such that the graviton propagator in Eq.(2.17) turns out to be ghost-free. In fact, by imposing

$$f(-k^2) = e^{\gamma_1(-k^2)}, \quad f(-k^2) - 3g(-k^2) = e^{\gamma_2(-k^2)}(k^2 + m_0^2), \quad (2.31)$$

with $\gamma_1(-k^2)$ and $\gamma_2(-k^2)$ being two entire functions, the propagator becomes

$$i\Pi(k) = \frac{i}{e^{\gamma_1(-k^2)}}\Pi_{\text{GR}}(k) + \frac{3}{2} \frac{f - g}{e^{\gamma_2(-k^2)}(k^2 + m_0^2)k^2} \mathcal{P}_s^0, \quad (2.32)$$

which only possesses the poles $k^2 = 0$ and $k^2 = -m_0^2$ and no massive spin-2 ghost degree of freedom. From Eq.(2.31) we can derive

$$f(-k^2) = e^{\gamma_1(-k^2)}, \quad g(-k^2) = \frac{1}{3} \left[e^{\gamma_1(-k^2)} - e^{\gamma_2(-k^2)}(k^2 + m_0^2) \right], \quad (2.33)$$

which implies

$$\begin{aligned} \mathcal{F}_1(\square) &= \frac{2 - [f(\square) + g(\square)]}{\square} = \frac{2 - \{e^{\gamma_1(\square)} + \frac{1}{3} [e^{\gamma_1(\square)} + e^{\gamma_2(\square)}(-\square + m_0^2)]\}}{\square}, \\ \mathcal{F}_2(\square) &= 2 \frac{f(\square) - 1}{\square} = 2 \frac{e^{\gamma_1(\square)} - 1}{\square}. \end{aligned} \quad (2.34)$$

Note that we can also impose a further condition such that only the massless spin-2 graviton propagates. Indeed, by imposing

$$f = g \quad \implies \quad \mathcal{F}_1 = -\frac{1}{2}\mathcal{F}_2, \quad (2.35)$$

the propagators reads

$$i\Pi(k) = \frac{i}{f(-k^2)}\Pi_{\text{GR}}(k) = \frac{i}{e^{\gamma_1(-k^2)}}\Pi_{\text{GR}}(k), \quad (2.36)$$

which only possesses the pole $k^2 = 0$.

As already mentioned above, the key role is played by the functions γ_1 and γ_2 which are entire so that the exponentials do not introduce any new pole besides the standard healthy ones. Therefore, we have seen an explicit example of how nonlocality can help to resolve the ghost problem by means the introduction of non-polynomial differential operators in the action.

Below, when considering physical applications, we will mainly work with the relations in Eq.(2.35) and with the simplest choice for the entire function:

$$e^{\gamma_1(-k^2)} = e^{k^2/M_s^2} \iff e^{\gamma_1(\Box)} = e^{-\Box/M_s^2}, \quad (2.37)$$

which together with Eq.(2.35) corresponds to the following gravitational action:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + G_{\mu\nu} \frac{e^{-\Box/M_s^2} - 1}{\Box} \mathcal{R}^{\mu\nu} \right\}. \quad (2.38)$$

Working in this simpler case will allow us to make exact analytic computations and get physical insights on features of nonlocal gravitational interaction.

Furthermore, this class of non-polynomial ghost-free higher derivative theories of gravity, being nonlocal, are also known as *infinite derivative theories of gravity* (IDG).

2.2 Linearized metric solutions

We are now interested in finding linearized stationary metric solutions for a slowly rotating source in the weak-field regime. We will first present a very general method which applies to any kind of spacetime metric and any kind of higher derivative theory of gravity, and subsequently we will specify to IDG.

Let us consider the following linearized metric:

$$ds^2 = -(1 + 2\Phi)dt^2 + 2\vec{h} \cdot d\vec{r} dt + (1 - 2\Psi)(dr^2 + r^2 d\Omega^2), \quad (2.39)$$

where $\vec{r} \equiv (x, y, z)$ is the isotropic radial coordinate, $r = \sqrt{x^2 + y^2 + z^2}$ and $dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$, while $h_{00} = -2\Phi$, $h_{ij} = -2\Psi\delta_{ij}$ and $h_i = h_{0i}$ are the metric potentials generated by a non-diagonal stress-energy tensor $T_{\mu\nu}$. In particular, we assume that the source is pressureless, $T \equiv \eta^{\mu\nu}T_{\mu\nu} \simeq -T_{00}$, therefore the non-vanishing components are given by $T_{00} = \rho(r)$ and T_{0i} , i.e. the source is modelled as a rotating dust of density $\rho(r)$.

By making the assumption of stationary source, $\Box \simeq \nabla^2$, and using the 00-component and the trace of the linearized field equations in Eq.(2.15), one can show that the metric potentials in Eq.(2.39) are solutions of the following

differential equations:

$$\begin{aligned} \frac{f(\nabla^2)[f(\nabla^2) - 3g(\nabla^2)]}{f(\nabla^2) - 2g(\nabla^2)} \nabla^2 \Phi(r) &= 8\pi G T_{00}(r), \\ \frac{f(\nabla^2)[f(\nabla^2) - 3g(\nabla^2)]}{g(\nabla^2)} \nabla^2 \Psi(r) &= -8\pi G T_{00}(r), \\ f(\nabla^2) \nabla^2 h_{0i}(r) &= -16\pi G T_{0i}(r), \end{aligned} \quad (2.40)$$

where $f \equiv f(\nabla^2)$, $g \equiv g(\nabla^2)$ are now functions of the Laplacian operator. We can notice that the non-diagonal components of the metric are only modified through the function $f(\nabla^2)$, which means that only if $\mathcal{F}_2(\square) \neq 0$ the quadratic part of the action will induce modification on the non-diagonal part of the metric; see the first expression in Eq.(2.12).

General solution

The differential equations in Eq.(2.40) can be formally solved by finding the Green function and using the method of Fourier transform. Indeed, from Eq.(2.40) we obtain:

$$\Phi(r) = -2G \int d^3 r' \mathcal{G}_1(\vec{r} - \vec{r}') T_{00}(\vec{r}'), \quad (2.41)$$

$$\Psi(r) = 2G \int d^3 r' \mathcal{G}_2(\vec{r} - \vec{r}') T_{00}(\vec{r}'), \quad (2.42)$$

$$h_{0i}(r) = 4G \int d^3 r' \mathcal{G}_3(\vec{r} - \vec{r}') T_{0i}(\vec{r}'), \quad (2.43)$$

where the integration region is defined by the volume of the gravitational source, while $\mathcal{G}_\ell(\vec{r} - \vec{r}')$ with $\ell = 1, 2, 3$ are the Green functions which eventually will only depend on the modulus $|\vec{r} - \vec{r}'|$ and are defined by

$$\frac{f(\nabla^2)[f(\nabla^2) - 3g(\nabla^2)]}{f(\nabla^2) - 2g(\nabla^2)} \nabla^2 \mathcal{G}_1(\vec{r} - \vec{r}') = -4\pi \delta^{(3)}(\vec{r} - \vec{r}'), \quad (2.44)$$

$$\frac{f(\nabla^2)[f(\nabla^2) - 3g(\nabla^2)]}{g(\nabla^2)} \nabla^2 \mathcal{G}_2(\vec{r} - \vec{r}') = -4\pi \delta^{(3)}(\vec{r} - \vec{r}'), \quad (2.45)$$

$$f(\nabla^2) \nabla^2 \mathcal{G}_3(\vec{r} - \vec{r}') = -4\pi \delta^{(3)}(\vec{r} - \vec{r}'). \quad (2.46)$$

Since we assume to make the experiment in a weak field regime far outside the gravitational source, we can perform a multipole expansion:

$$\begin{aligned} \mathcal{G}_\ell(|\vec{r} - \vec{r}'|) &= \mathcal{G}_\ell(r) + \partial'_j \mathcal{G}_\ell(|\vec{r} - \vec{r}'|)|_{r'=0} x'^j + \dots \\ &= \mathcal{G}_\ell(r) - \frac{1}{r} \frac{\partial \mathcal{G}_\ell(r)}{\partial r} x'_j x^j + \dots, \end{aligned} \quad (2.47)$$

where "... " stand for higher order multipole contributions and $\mathcal{G}_\ell(r)$ can be computed as (see Eqs. (2.44),(2.45) and (2.46))

$$\mathcal{G}_1(r) = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \frac{f-2g}{f(f-3g)} e^{i\vec{k}\cdot\vec{r}} = \frac{2}{\pi} \frac{1}{r} \int_0^\infty dk \frac{f-2g}{f(f-3g)} \frac{\sin kr}{k}, \quad (2.48)$$

$$\mathcal{G}_2(r) = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \frac{f-2g}{g} e^{i\vec{k}\cdot\vec{r}} = \frac{2}{\pi} \frac{1}{r} \int_0^\infty dk \frac{g}{f(f-3g)} \frac{\sin kr}{k}, \quad (2.49)$$

$$\mathcal{G}_3(r) = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{f k^2} e^{i\vec{k}\cdot\vec{r}} = \frac{2}{\pi} \frac{1}{r} \int_0^\infty dk \frac{1}{f} \frac{\sin kr}{k}. \quad (2.50)$$

Let us first consider the diagonal components Φ and Ψ and subsequently the cross term h_{0i} . By using the multipole expansion in Eq.(2.47), the first non-vanishing contributions for the diagonal components are

$$\Phi(r) = -G \mathcal{G}_1(r) \int d^3r' T_{00}(\vec{r}') = -Gm \mathcal{G}_1(r), \quad (2.51)$$

$$\Psi(r) = G \mathcal{G}_2(r) \int d^3r' T_{00}(\vec{r}') = Gm \mathcal{G}_2(r), \quad (2.52)$$

where we have defined the mass of the system as

$$m = \int d^3r' T_{00}(\vec{r}'). \quad (2.53)$$

As for the non-diagonal components, we can proceed as done for the diagonal part but the first non-vanishing contribution will come from the dipole in the multipole expansion. Indeed, we can rewrite the solution in Eq.(2.43) as

$$\begin{aligned} h_{0i}(r) &= 4G \mathcal{G}_3(r) \int d^3r' T_{0i}(\vec{r}') - \frac{4G}{r} \frac{\partial \mathcal{G}_3(r)}{\partial r} x_j \int d^3r' x'^j T_{0i}(\vec{r}') \\ &= -\frac{2G}{r} \frac{\partial \mathcal{G}_3(r)}{\partial r} (\vec{r} \wedge \vec{J})_i, \end{aligned} \quad (2.54)$$

where we have used the property $\int d^3r' T_{0i}(\vec{r}') = 0^2$ and introduced the angular momentum of the source which is defined as

$$J^i = \int d^3r' \varepsilon_{ijk} T_{0j}(\vec{r}') x'^k \iff \int d^3r' T_{0i}(\vec{r}') x'^j = \frac{1}{2} \varepsilon_{ijk} J^k. \quad (2.55)$$

²Note that this relation is a consequence of the continuity equation $\partial_\mu T^{\mu\nu} = 0$. Indeed, one can easily show that

$$\int d^3r' T^{\mu i}(\vec{r}') = \int d^3r' T^{\mu k}(\vec{r}') \delta_k^i = \int d^3r' T^{\mu k}(\vec{r}') \frac{\partial x'^i}{\partial x'^k} = - \int d^3r' \left(\partial'_k T^{\mu k}(\vec{r}') \right) x'^i = 0.$$

Hence, we have found a formal expression for the cross-term in the metric in Eq.(2.39). By choosing the direction of angular momentum along the z -axis, $\vec{J} = J\hat{z}$, and making the coordinate transformations $x = r \sin\theta \cos\varphi$ and $y = r \sin\theta \sin\varphi$, we can write

$$\begin{aligned}
2\vec{h} \cdot d\vec{r}dt &= -\frac{4J}{r} \frac{\partial \mathcal{G}_3(r)}{\partial r} (\vec{r} \wedge \vec{J})_i dx^i dt \\
&= -\frac{4J}{r} \frac{\partial \mathcal{G}_3(r)}{\partial r} (y dx - x dy) dt \\
&= 4Jr \frac{\partial \mathcal{G}_3(r)}{\partial r} \sin^2\theta d\varphi dt \\
&\equiv 2\chi(r) \sin^2\theta d\varphi dt,
\end{aligned} \tag{2.56}$$

where we have defined

$$\chi(r) \equiv 2Jr \frac{\partial \mathcal{G}_3(r)}{\partial r}. \tag{2.57}$$

Therefore, the spacetime metric in Eq.(2.39) can be recast in the form

$$ds^2 = -(1 + 2\Phi)dt^2 + 2\chi(r) \sin^2\theta d\varphi dt + (1 - 2\Psi)(dr^2 + r^2 d\Omega^2), \tag{2.58}$$

where the metric potentials Φ , Ψ and χ can be found by using the expressions in Eqs.(2.41,2.42,2.57). Note that in Einstein's GR we have $f = g = 1$ which implies $\mathcal{G}_1(r) = -\mathcal{G}_2(r) = \mathcal{G}_3(r) = 1/r$, thus recovering the weak-field limit of the metric potentials for the Kerr metric, i.e the metric in Eq.(2.58) will be Lense-Thirring [144].

2.2.1 Static point-like source

We now specialize the above framework to the case of IDG theories, in particular we will work with the choice of the form factors in Eqs.(2.35,2.37) and find solutions for several types of stress-energy tensors. In this case, the field equations (2.40) become

$$\begin{aligned}
e^{-\nabla^2/M_s^2} \nabla^2 \Phi(r) &= 8\pi G T_{00}(r), \\
e^{-\nabla^2/M_s^2} \nabla^2 \Psi(r) &= -8\pi G T_{00}(r), \\
e^{-\nabla^2/M_s^2} \nabla^2 h_{0i}(r) &= -16\pi G T_{0i}(r).
\end{aligned} \tag{2.59}$$

For a neutral static point-like source, the stress energy tensor is given by

$$T_{\mu\nu} = m\delta_\mu^0 \delta_\nu^0 \delta^{(3)}(\vec{r}), \tag{2.60}$$

so that the modified Poisson equations (2.59) reduce to

$$e^{-\nabla^2/M_s^2} \nabla^2 \Phi = e^{-\nabla^2/M_s^2} \nabla^2 \Psi = 4\pi m \delta^{(3)}(\vec{r}), \tag{2.61}$$

implying that the two metric potentials are equal, $\Phi = \Psi$. Note that we have to deal with a differential equation of infinite order due to the presence of the exponential operator $e^{-\nabla^2/M_s^2}$; however it can be easily solved by going to Fourier space and then anti-transforming back to coordinate space as shown above. Indeed, by following the steps in Eqs.(2.51,2.52) we obtain:

$$\begin{aligned}\Phi(r) = \Psi(r) &= -4\pi Gm \int d^3k \frac{e^{-k^2/M_s^2}}{k^2} e^{i\vec{k}\cdot\vec{r}} \\ &= -\frac{2Gm}{\pi} \frac{1}{r} \int_0^\infty dk \frac{\sin(kr)}{k} e^{-k^2/M_s^2} \\ &= -\frac{Gm}{r} \text{Erf}\left(\frac{M_s r}{2}\right),\end{aligned}\tag{2.62}$$

where

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,\tag{2.63}$$

is the so called error-function.

Very interestingly, the gravitational potential in Eq.(2.62) is non-singular at $r = 0$, indeed it tends to the finite constant value $\Phi(0) = 2GmM_s/\sqrt{\pi}$; while for large distances we recover the $1/r$ behavior of the Newtonian potential, as expected. Note that the linearized regime is valid as long as the inequality

$$\frac{2GmM_s}{\sqrt{\pi}} < 1\tag{2.64}$$

holds true. It is worthwhile mentioning that non-singular time-dependent solutions in a dynamical scenario were obtained in Refs.[71, 70].

From a physical point of view, nonlocality is able to regularize the singularity at the origin, indeed the point-like source at $r = 0$ is smeared out on a region of size $1/M_s$ due to the presence of infinite order derivative. Not only the linearized metric potentials but also all the linearized curvature invariants turn out to be non-singular. For instance, the expression for the Ricci scalar is given by

$$\mathcal{R} = \frac{e^{-\frac{M_s^2 r^2}{4}} Gm M_s^3}{\sqrt{\pi}},\tag{2.65}$$

while the Kretschmann invariant reads

$$\mathcal{K} = \frac{e^{-\frac{M_s^2 r^2}{2}} G^2 m^2}{3\pi r^6} \left\{ 5M_s^6 r^6 + 4 \left[6M_s r + M_s^3 r^3 - 6\sqrt{\pi} e^{\frac{M_s^2 r^2}{4}} \text{Erf}\left(\frac{M_s r}{2}\right) \right]^2 \right\},\tag{2.66}$$

and one can easily show that both curvatures tend to finite values in the limit $r \rightarrow 0$. Furthermore, at $r = 0$ the metric becomes conformally-flat, since all the components of the Weyl tensor vanish at the origin; see Ref.[76] for more details.

2.2.2 Electrically charged static point-like source

Let us now consider the case of a static electric charge as a gravitational source and find the corresponding metric potentials. The stress-energy tensor in this case is given by electro-magnetic one which reads

$$T_{\mu\nu} = \frac{1}{4\pi} \left(\eta_{\rho\nu} F_{\mu\sigma} F^{\rho\sigma} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \quad (2.67)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electro-magnetic field strength defined in terms of the potential-vector A_μ . In the simplest case of an electric charge, only the components related to the electric field are non-vanishing:

$$F_{10} = -F_{01} = E_r \quad \text{and} \quad E_r = \frac{Q}{r^2}, \quad (2.68)$$

with E_r being the radial component of the electric field. Given the stress-energy tensor in Eq.(2.67) with the components in Eq.(2.68), the infinite derivative differential equations in Eq. (2.59) become [79]:

$$\begin{aligned} e^{-\nabla^2/M_s^2} \nabla^2 \Phi &= \frac{GQ^2}{r^4}, \\ e^{-\nabla^2/M_s^2} \nabla^2 \Psi &= \frac{GQ^2}{2r^4}. \end{aligned} \quad (2.69)$$

Also in this case we can solve the modified Poisson equations by using the Fourier transform method. By making the field redefinitions $\tilde{\Phi} := e^{-\nabla^2/M_s^2} \Phi$, $\tilde{\Psi} := e^{-\nabla^2/M_s^2} \Psi$, we obtain the following two solutions from Eq. (2.69):

$$\begin{aligned} \tilde{\Phi}(r) &= -\frac{C_1}{r} + \frac{GQ^2}{2r^2} + C_2 \\ \tilde{\Psi}(r) &= -\frac{C_1}{r} + \frac{GQ^2}{4r^2} + C_2, \end{aligned} \quad (2.70)$$

where we fix $C_1 = Gm$ by requiring that we want to recover the static neutral case in Eq. (2.62) when $Q = 0$, and $C_2 = 0$ since we want asymptotic flatness (for $r \rightarrow \infty$). We can now go back to the fields Φ and Ψ , which are given by

$$\begin{aligned} \Phi(r) &= -Gme^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) + \frac{GQ^2}{2} e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right), \\ \Psi(r) &= -Gme^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) + \frac{GQ^2}{4} e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right). \end{aligned} \quad (2.71)$$

By using the fact that $4\pi/k^2$ is the Fourier transform of $1/r$, we can write

$$\begin{aligned} e^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) &= \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} e^{-k^2/M_s^2} e^{i\vec{k}\cdot\vec{r}} \\ &= \frac{2}{\pi} \int_0^\infty dk \frac{\sin(kr)}{kr} e^{-k^2/M_s^2} \\ &= \frac{1}{r} \text{Erf} \left(\frac{M_s r}{2} \right), \end{aligned} \quad (2.72)$$

which gives the neutral part of the potentials; then, by using the Fourier transform $\frac{2\pi^2}{k} \text{sign}(k)$ of $1/r^2$, we can write

$$\begin{aligned} e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right) &= \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k} \text{sign}(k) e^{-k^2/M_s^2} e^{i\vec{k}\cdot\vec{r}} \\ &= \int_0^\infty dk \frac{\sin(kr)}{r} e^{-k^2/M_s^2} \\ &= \frac{M_s}{r} \text{F} \left(\frac{M_s r}{2} \right), \end{aligned} \quad (2.73)$$

where

$$\text{F}(x) := e^{-x^2} \int_0^x e^{t^2} dt \quad (2.74)$$

is the so called Dawson function. Therefore, the two metric potentials in Eq. (2.71) read [79]

$$\begin{aligned} \Phi(r) &= -\frac{Gm}{r} \text{Erf} \left(\frac{M_s r}{2} \right) + \frac{GQ^2 M_s}{2r} \text{F} \left(\frac{M_s r}{2} \right), \\ \Psi(r) &= -\frac{Gm}{r} \text{Erf} \left(\frac{M_s r}{2} \right) + \frac{GQ^2 M_s}{4r} \text{F} \left(\frac{M_s r}{2} \right). \end{aligned} \quad (2.75)$$

As expected, for large distances $M_s r \gg 1$, we recover the metric potentials of the linearized Reissner-Nordström metric of GR [79], while in the limit $r \rightarrow 0$ the two metric potentials tend to finite values, $\Phi(0) = -GmM_s/\sqrt{\pi} + GQ^2 M_s^2/4$ and $\Psi(0) = -GmM_s/\sqrt{\pi} + GQ^2 M_s^2/8$, as can be easily checked. The same regularized behavior can be shown for all linearized curvature invariants; in particular, the Weyl tensor vanishes implying that the metric is conformally-flat at the origin [79], as it also happens for the case of a neutral source. Therefore, also for a point-like electric charge nonlocality is able to regularize the singularity at $r = 0$.

The linearized regime holds all the way from $r = \infty$ up to $r = 0$, as long as the inequalities $2|\Phi(0)|, 2|\Psi(0)| < 1$ are satisfied, which means $mM_s < M_p^2$ and $|Q|M_s < M_p$, where we have neglected constant factors of order one [79].

2.2.3 Stationary rotating ring

We now aim to determine the spacetime metric generated by a Dirac Delta distribution on a ring and show how nonlocality can regularize Kerr-like ring singularities. It is well known that the Kerr metric suffers from the presence of a ring singularity which in Boyer-Lindquist coordinates is described by the equation $r^2 + a^2 \cos^2 \vartheta = 0$, or in Cartesian coordinates by $z = 0$, $x^2 + y^2 = a^2$, where a is the radius of the ring [3].

To mimic such a ring distribution, we consider a ring of radius a rotating with constant angular velocity ω , described by following stress-energy tensor:

$$T_{00} = m\delta(z)\frac{\delta(x^2 + y^2 - a^2)}{\pi}, \quad T_{0i} = T_{00}v_i, \quad (2.76)$$

where v_i is the tangential velocity and its magnitude is related to the angular velocity through the relation $v = \omega a$; moreover, by assuming that the rotation happens around the z -axis, we can write $v_x = -y\omega$, $v_y = x\omega$ and $v_z = 0$. The stress-energy tensor in Eq.(2.76) will source the following linearized non-diagonal metric in Eq.(2.39), in which the coordinate r is the isotropic radius which should not be confused with the Boyer-Lindquist radial coordinate introduced above. By still following the general method introduced above, we can notice that the metric components in Eq.(2.39) can be found by solving the following set of decoupled infinite order differential equations [80]:

$$\begin{aligned} e^{-\nabla^2/M_s^2} \nabla^2 \Phi(\vec{r}) &= e^{-\nabla^2/M_s^2} \nabla^2 \Psi(\vec{r}) = 4Gm\delta(z)\delta(x^2 + y^2 - a^2), \\ e^{-\nabla^2/M_s^2} \nabla^2 h_{0x}(\vec{r}) &= -16Gm\omega y\delta(z)\delta(x^2 + y^2 - a^2), \\ e^{-\nabla^2/M_s^2} \nabla^2 h_{0y}(\vec{r}) &= 16Gm\omega x\delta(z)\delta(x^2 + y^2 - a^2). \end{aligned} \quad (2.77)$$

Note that, by going to cylindrical coordinates, we can Fourier transform the stress-energy tensor components as follows [80]:

$$\begin{aligned} \mathcal{F}[\delta(z)\delta(x^2 + y^2 - a^2)] &= \int_{-\infty}^{\infty} dz \delta(z) e^{ik_z z} \int_0^{\infty} d\rho \rho \delta(\rho^2 - a^2) \int_0^{2\pi} d\varphi e^{ik_x \rho \cos \varphi} e^{ik_y \rho \sin \varphi} \\ &= \pi \int_0^{\infty} d(\rho^2) \delta(\rho^2 - a^2) I_0 \left(i\rho \sqrt{k_x^2 + k_y^2} \right) \\ &= \pi I_0 \left(ia \sqrt{k_x^2 + k_y^2} \right), \end{aligned} \quad (2.78)$$

$$\begin{aligned}
\mathcal{F}[\delta(z)\delta(x^2+y^2-a^2)] &= \int_{-\infty}^{\infty} dz \delta(z) e^{ik_z z} \int_0^{\infty} d\rho \rho^2 \delta(\rho^2 - a^2) \int_0^{2\pi} d\varphi e^{ik_x \rho \cos\varphi} e^{ik_y \rho \sin\varphi} \cos\varphi \\
&= \pi \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \int_0^{\infty} d(\rho^2) \rho \delta(\rho^2 - a^2) I_1 \left(i\rho \sqrt{k_x^2 + k_y^2} \right) \\
&= \pi a \frac{k_x}{\sqrt{k_x^2 + k_y^2}} I_1 \left(ia \sqrt{k_x^2 + k_y^2} \right),
\end{aligned} \tag{2.79}$$

and

$$\mathcal{F}[y\delta(z)\delta(x^2+y^2-a^2)] = \pi a \frac{k_y}{\sqrt{k_x^2 + k_y^2}} I_1 \left(ia \sqrt{k_x^2 + k_y^2} \right), \tag{2.80}$$

where $\mathcal{F}[\dots]$ stands for the Fourier transform operation, I_0 and I_1 are two Modified Bessel functions. For simplicity, let us only consider the plane $z = 0$ and work with the cylindrical radial coordinate $\rho = \sqrt{x^2 + y^2}$, as we know that in GR the ring singularity lies in the x - y plane. Therefore, by anti-transforming to coordinate space, the metric potentials will be given by [80]

$$\Phi(\rho) = \Psi(\rho) = -Gm \int_0^{\infty} d\zeta I_0(ia\zeta) I_0(i\zeta\rho) \text{Erfc}\left(\frac{\zeta}{M_s}\right), \tag{2.81}$$

$$h_{0x}(x, y) = 4Gm\omega a \frac{y}{\rho} H(\rho), \quad h_{0y}(x, y) = -4Gm\omega a \frac{x}{\rho} H(\rho), \tag{2.82}$$

with

$$H(\rho) := \int_0^{\infty} d\zeta I_1(ia\zeta) I_1(i\zeta\rho) \text{Erfc}\left(\frac{\zeta}{M_s}\right), \tag{2.83}$$

which in the limit $M_s \rightarrow \infty$ reduce to GR case.

The above three integrals cannot be solved analytically but we can do it numerically as shown in Fig. 2.1. We can explicitly see that in GR the metric potentials suffers from the presence of a ring singularity at $\rho = a$, while in IDG such a singularity is regularized. Also in this case, one can show that all the curvature invariants are non-singular in the entire spacetime and that the Weyl tensor vanishes at $r = 0$.

Multipole expansion

We now wish to determine the generic form of the metric in IDG outside the rotating source, without assuming any large distance limit; we will follow all the steps above that brought us to the metric form in Eq.(2.58).

Figure 2.1: In this plot we have shown the behavior of the components $-h_{00}$ and $h_{0i} \sim H$ as functions of the cylindrical radius ρ , in both cases of IDG (orange line) and IDG (blue line). We have set $M_s = 1$, $G = 1$, $m = 0.5$ and $a = 1$. It is clear that, while the metric potentials in GR have a singularity for $\rho = a$, they turn out to be regular in IDG. We have also compared with the multipole expansion result in IDG (dashed red line).

Let us expand in multipole the potential $\text{Erf}(M_s|\vec{r} - \vec{r}'|/2)/|\vec{r} - \vec{r}'|$:

$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} \text{Erf}\left(\frac{M_s|\vec{r} - \vec{r}'|}{2}\right) &= \frac{1}{r} \text{Erf}\left(\frac{M_s r}{2}\right) + \left[\frac{1}{r^3} \text{Erf}\left(\frac{M_s r}{2}\right) \right. \\ &\quad \left. - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M_s^2 r^2}{4}} \right] \sum_{j=1}^3 x^j x'^j + \dots \end{aligned} \quad (2.84)$$

Such a multipole expansion holds true for $r > r' \sim a$, which means outside the source. By using Eq.(2.84) we can now compute the h_{0i} components

$$\begin{aligned} h_{0i}(\vec{r}) &= 4G \int d^3 r' \frac{T_{0i}(\vec{r}')}{|\vec{r} - \vec{r}'|} \text{Erf}\left(\frac{M_s|\vec{r} - \vec{r}'|}{2}\right) \\ &= 2G \left[\frac{1}{r^3} \text{Erf}\left(\frac{M_s r}{2}\right) - \frac{M_s}{\sqrt{\pi} r^2} e^{-\frac{M_s^2 r^2}{4}} \right] (\vec{r} \times \vec{J})_i. \end{aligned} \quad (2.85)$$

We can move from Cartesian to isotropic coordinates, so that the $d\varphi dt$ component of the metric will be given by:

$$2\vec{h} \cdot d\vec{x} dt = -4GJ \left[\frac{1}{r} \text{Erf}\left(\frac{M_s r}{2}\right) - \frac{M_s}{\sqrt{\pi}} e^{-\frac{M_s^2 r^2}{4}} \right] \sin^2 \theta d\varphi dt. \quad (2.86)$$

Moreover, by expressing $J = I\omega = ma^2\omega$ and imposing $|h_{0i}| \sim GmM_s^2\omega a^2 < 1$, we can notice that the slow rotation regime means $\omega < 1/a$.

Therefore, the linearized spacetime metric in Eq.(2.58) outside the source,

$r > a$, in the case of IDG reads:

$$ds^2 = - \left[1 - \frac{2Gm}{r} \text{Erf} \left(\frac{M_s r}{2} \right) \right] dt^2 + \left[1 + \frac{2Gm}{r} \text{Erf} \left(\frac{M_s r}{2} \right) \right] (dr^2 + r^2 d\Omega^2) - 4GJ \left[\frac{1}{r} \text{Erf} \left(\frac{M_s r}{2} \right) - \frac{M_s}{\sqrt{\pi}} e^{-\frac{M_s^2 r^2}{4}} \right] \sin^2 \theta d\varphi dt. \quad (2.87)$$

From Fig. 2.1, it is very clear that the metric constructed by using the multipole expansion is a very good approximation to describe the spacetime outside the source, $r > a$; while in the regime $M_s r \gg 2$, we recover the GR predictions, indeed Eq.(2.87) reduces to the Lense-Thirring metric [144]³.

Note that, the linearized regime holds from $r = \infty$ all the way up to $r = 0$ as long as the inequalities $2|\Phi| < 1$ and $|h_{0i}| < 1$ are satisfied for any r , which, by neglecting constant factors of order one, also means $GmM_s < 1$ and $GmM_s^2 \omega a^2 < 1$, respectively [80].

2.3 Towards non-linear solutions

So far we have found linearized metric solutions for several configurations. In this Section we wish to move towards the full non-linear regime and understand whether nonlocality can resolve singularities for spherical symmetric solutions also when the linearized approximation does not hold. In such a case one should deal with the full set of EOM in Eq.(2.6) which are highly nonlocal due to the presence of infinite order covariant derivatives. Exact solutions have been only obtained in cosmology where non-singular bouncing solutions were shown to exist [40, 58, 145], while *no* non-linear spherical symmetric metric solution has been found so far.

Here, to get some physical insights on the role of nonlocality, we do not follow a brute force approach but we ask whether the Schwarzschild metric of Einstein's GR can be a full solution also in IDG. Before doing so, let us recall some general aspects of the Schwarzschild metric which, usually, are not sufficiently emphasized.

³Recall that the Lense-Thirring metric represents the weak-field and slow-rotation limit of the Kerr metric [144]:

$$ds^2 = - \left(1 - \frac{2Gm}{r} \right) dt^2 - \frac{4GJ}{r} \sin^2 \theta dt d\varphi + \left(1 + \frac{2Gm}{r} \right) (dr^2 + r^2 d\Omega^2). \quad (2.88)$$

2.3.1 A brief remark on Schwarzschild metric

First of all, let us recall the Schwarzschild metric for clarity,

$$ds^2 = -(1 + 2\Phi) dt^2 + (1 + 2\Phi)^{-1} dr^2 + r^2 d\Omega^2, \quad \Phi = -\frac{Gm}{r}. \quad (2.89)$$

It is usually stated that the Schwarzschild metric is a *vacuum* solution ($T_\nu^\mu = 0$ everywhere) of the Einstein EOM, or in other words, that it is a Ricci flat solution, $\mathcal{R}_\nu^\mu$, but this is *not* correct. Indeed, as rigorously shown in Refs.[146, 147], strictly speaking the Schwarzschild metric is vacuum everywhere except at $r = 0$ where a Dirac delta is sitting and plays the role of stress-energy tensor for the such a metric solution.

We can easily understand this feature by looking at components of the Einstein EOM when the metric in Eq.(2.89) is plugged in. For instance, the (22)-component is roughly given by

$$\nabla^2 \Phi(r) \sim \nabla^2 \left(\frac{1}{r} \right) \sim T_{22}, \quad (2.90)$$

where $\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r$ is the radial component of the Laplacian. It is clear that if we consider vacuum everywhere, $T_{22} = 0$, then Eq.(2.90) would not be true at $r = 0$ since the left-hand side diverges while the right-hand side is zero. In fact, Eq.(2.90) would hold for any value of r only when $T_{22} \sim \delta^{(3)}(\vec{r})$, so that at $r = 0$ the left-hand side and the right-hand side assume consistently the same divergent value. Therefore, the true singular Schwarzschild metric is *not* a vacuum solution, but it is a solution of the Einstein EOM sourced by a point-like Dirac delta source at the origin.

The same feature also occurs for the other components of the Einstein EOM. Indeed, by using the theory of distributions one can rigorously show the stress-energy tensor for the Schwarzschild metric is given by [146, 147]

$$T_\nu^\mu = \text{diag} \left(-m\delta^{(3)}(\vec{r}), -m\delta^{(3)}(\vec{r}), \frac{m}{2}\delta^{(3)}(\vec{r}), \frac{m}{2}\delta^{(3)}(\vec{r}) \right). \quad (2.91)$$

It is worth emphasizing that a more complicated stress-energy tensor can be constructed in the case of the Kerr-Newman family [147].

2.3.2 Nonlocality and singularity resolution

The form of the stress-energy tensor in Eq.(2.91) is crucial in order to understand whether Schwarzschild is still a *vacuum* (i.e. except at $r = 0$) in both cases of finite and infinite order higher derivative theories of gravity.

It has been known that a finite number of spatial derivatives acting on a Dirac delta distribution *still* yields a point-like source [9, 148]. Would one repeat

the same argument mathematically for any local theory of gravity, including quadratic curvature Stelle's gravity [8, 9], one would obtain in the right hand side of the EOM, a schematic form of $\delta(r) + \delta'(r)$, which is still a point-like source, but a bit less conventional one. Going further, more but finite number of derivatives will generate a finite series like

$$\delta(r) + \delta'(r) + \delta''(r) + \cdots + \delta^{(N)}(r), \quad (2.92)$$

for some finite N . As long as N is finite we will still have a point-like source and the singular Schwarzschild metric is still a valid solution, even at $r = 0$.

However, if we continue up to the infinite order in derivatives, then one would obtain a series with infinitely many terms. By consulting the theory of distributions in this subtle case one can easily find out (see the book by Vladimirov [149]) that a series of infinitely many terms with derivatives acting on the Dirac delta distribution corresponds to a function with a non-point support. To see this explicitly, let us consider the following distribution $e^{\alpha \partial_x^2} \delta(x)$ on a real axis, where α is a constant. By employing the Fourier transform, we obtain:

$$e^{\alpha \partial_x^2} \delta(x) = \frac{1}{2\pi} e^{\alpha \partial_x^2} \int dk e^{ikx} = \frac{1}{2\pi} \int dk e^{-\alpha k^2} e^{ikx} = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}, \quad (2.93)$$

which is manifestly a regular function with a non-point support. We have seen this argument before in Refs. [69, 40, 41] that an infinite tower of derivatives can smear the singular behavior of a Dirac delta mass distribution⁴. The usual notion of vacuum solution, that applies for the Schwarzschild metric in GR, does not apply in IDG, where nonlocality smears out the point-like source on a region of size $\sim 1/M_s$.

Although the singular Schwarzschild metric in Eq.(2.89) cannot be a full solution of the EOM (2.6) with a point-like source, in principle, there can exist some complicated non-point source which can support it as a full non-linear solution. We argue that this is not the case, namely that nonlocality does *not* admit the $1/r$ metric potential as a full solution valid in the entire spacetime. We will now show an argument in favor of such a statement by working in the linearized regime.

⁴For example the gravitational potential in Eq.(2.62) is a solution of the modified Poisson equation $e^{-\nabla^2/M_s^2} \nabla^2 \Phi(r) = 4\pi G m \delta^{(3)}(r)$, which can be also rewritten as

$$\nabla^2 \Phi(r) = 4\pi G m e^{\nabla^2/M_s^2} \delta^{(3)}(r) = \frac{G m M_s^3}{2\sqrt{\pi}} e^{-\frac{M_s^2 r^2}{4}}.$$

We can notice that the presence of infinite order derivatives has smeared out the Dirac delta source and produced a Gaussian-like distribution which has a non-point support.

Hint from the linearized regime

In the linearized regime around Minkowski the modified Poisson equation for the gravitational potential would be roughly given by:

$$[1 - \mathcal{F}(\nabla^2)\nabla^2]\nabla^2\Phi(r) = 4\pi G\rho(r). \quad (2.94)$$

In fourth order gravity we have a constant form factor $\mathcal{F}(\square) = -\alpha$, with $\alpha > 0$, from which it is clear that there always exists a source which generates the Schwarzschild solution, $\Phi(r) \sim 1/r$, and such a source is given by

$$\rho(r) = \frac{1}{4\pi G}[1 + \alpha\nabla^2]\nabla^2\left(-\frac{Gm}{r}\right) = -\frac{m}{2\pi r}[\delta'(r) + \alpha\delta^{(5)}(r)] \quad (2.95)$$

which is not positive defined [9]. A similar scenario will also hold for any local higher derivative theory of gravity: for any gravitational theory with polynomial form factor $\mathcal{F}$ there exist always a source which generates a the $1/r$ Schwarzschild singularity. For instance, in sixth order gravity, $\mathcal{F}(\square) = -\alpha\square$, the $1/r$ potential is generated by the density $\rho(r) = -m/(2\pi r)[\delta'(r) + \alpha\delta^{(5)}(r)]$.

In ghost-free IDG, the form factor is a non-polynomial function, therefore one has infinite order derivatives acting on $1/r$, or in other words, by looking at the full field equations they will act on the Schwarzschild curvature $\mathcal{R} \sim \delta^{(3)}(\vec{r})$. Unlike the local scenario in which we have a finite number of terms, nonlocal differential operators would generated a complicated expression on the left-hand side involving infinite order derivatives of the Dirac delta, thus yielding a non-point support.

A-priori the Schwarzschild metric could be still a solution but coupled to a source which is not point-like. However, we argue that in IDG there exist *no* source which can couple to the Schwarzschild metric. By looking at the modified Poisson equation in Eq.(2.94) with the function in Eq.(2.37) and using the method of Fourier transform, we obtain:

$$\rho(r) = \frac{1}{4\pi G}e^{-\nabla^2/M_s^2}\left(-\frac{Gm}{r}\right) = \frac{1}{2\pi^2 r}\int_0^\infty dk k \sin(kr)e^{+k^2/M_s^2} = \infty. \quad (2.96)$$

From the last equation, we can notice that, at least in the linear regime, it is impossible to define any source $\rho(r)$ which can generated the Schwarzschild metric with $1/r$ behaviour. In fact, the only option could be $\rho(r) \sim e^{-\nabla^2/M_s^2}\delta(\vec{r})$, but it turns out that it cannot be mathematically defined since $e^{-\nabla^2/M_s^2}\delta(\vec{r}) = \infty$.

Hence, nonlocality through infinite order derivatives does not allow the Schwarzschild metric as an exact full solution, but it can be solution only in some region of the spacetime, for instance in the local regime when $rM_s \gg 1$, which means at large distances far away from the origin [78].

A rigorous and satisfactory treatment for full non-linear spherically symmetric solutions is still lacking. Such a program is challenging but at the same time very attractive, and it will subject of future works.

Phenomenological implications

We wish to study some phenomenological aspects of nonlocal (infinite derivative) field theories, which have been already applied to many several contexts so far. It is worthwhile highlighting that the first application was related to the problem of singularity resolution in cosmology [40], which was also one of the initial motivation to introduce infinite derivative actions. Many efforts were also made to understand how nonlocality affects the inflationary scenario and corrections to the power spectrum were computed [59, 97, 98, 150, 100, 101]. Moreover, interesting contributions were also made in the context of thermal field theory [93, 94, 95].

In this Chapter, we work in the astrophysical context of ultra-compact objects (UCOs) and show which are the main physical implications due to nonlocal interactions. In Section 3.1, we study nonlocal gravitational interaction and show how in this case nonlocality can help to prevent the formation of a horizon and, thus, supporting the existence of horizonless UCOs as metric solutions. Instead, in Section 3.2, as a second application, we study nonlocal field theory in flat space and ask how infinite order derivatives can affect the dynamics of a scalar wave interacting with a double delta potential. The two Dirac deltas can be used to model either the two symmetric photon spheres of a wormhole [151], or to mimic the two potential barriers at the surface and at the photon sphere of an UCO, located at $2Gm(1 + \epsilon)$ and $3Gm$, respectively, with $\epsilon < 0.5$ [152]. We also compute the echoes and compare with the case of local physics in order to extract insights on nonlocal interaction.

These two Sections are based on **P8** and **P9**, respectively.

3.1 Nonlocal star as a black hole mimicker

In Section 2.2.1 we have computed the linearized spacetime metric for a point-like source and highlighted the regularizing feature of nonlocality. Indeed, the nonlocal nature of spacetime is able to smear out the Dirac delta source turning it into an extended object of size $1/M_s$. We have also noticed that the linearized regime can be trusted as long as the inequality in Eq.(2.64) is valid. Such an inequality can be also recast as

$$r_{\text{nl}} > r_{\text{sch}} \frac{2}{\sqrt{\pi}}, \quad r_{\text{nl}} \sim \frac{2}{M_s}, \quad r_{\text{sch}} = 2Gm, \quad (3.1)$$

which tells us that the radius r_{nl} always engulfs the Schwarzschild radius, namely the nonlocal region on which the smearing effects happen extends even beyond the horizon. Indeed, this feature of nonlocality is the key ingredient to avoid horizons.

Classically, M_s (or L_s) is a fundamental parameter (smaller or equal to the Planck mass M_p), thus there always exists a critical value of the mass above which the inequality does not hold. Therefore, the natural question to ask is whether there exists any mechanism such that the inequality (3.1) can remain valid in the entire spacetime and for any large value of the mass m . To find an answer to this question we need to understand how nonlocality behaves when a large number of particles interact nonlocally through gravity. We will construct a *quantum* framework in which such a mechanism can exist.

3.1.1 Horizon avoidance

In Section 1.3.5 we studied a nonlocal scalar field with cubic interaction and shown that when N particles interact, an effective nonlocal scale emerges. The main feature was that, given two bosons the nonlocal interaction will happen on a region of size $L_s = 1/M_s$; while if the number of interacting particles is N ($\gg 1$) then the spacetime region on which the nonlocal interaction manifests is larger and given by

$$L_{\text{eff}} = \sqrt{N} L_s \quad (3.2)$$

or, in terms of energy $M_{\text{eff}} = M_s/\sqrt{N}$; see Eq.(1.135). Thus, the nonlocal scale is *not* fixed but dynamical and depends on the number of interacting particles. Note that a similar scaling behavior was also obtained in Ref.[75] from a different point of view by demanding that the gravitational entropy of a self-gravitating system preserves the *area-law*.

This result has been obtained only for a scalar toy model. We now assume that similar features are also true for a full nonlocal theory of gravity, even though a full rigorous treatment for the graviton field is still missing and will be subject of future investigations.

Hence, we understood the scaling behavior of M_s for a system of N interacting particles. We now wish to find the scaling for the mass m of the gravitational system, thus we will be then able to understand how the inequality (3.1) changes by increasing the mass of the gravitational source.

Let us assume that our gravitational system is made up of N nonlocal constituents which can be seen as building blocks. Each quanta bring an energy (or effective mass), E_g , and so a wave-length

$$\lambda_g = \frac{1}{E_g}. \quad (3.3)$$

Each individual constituent feels nonlocal interaction if and only $\lambda_g \sim L_s$, while for wave-lengths $\lambda_g > L_s$ the self-gravitational interaction is just local. Therefore, each of them is characterized by a wave-length of the order of the fundamental scale of nonlocality:

$$\lambda_g \sim L_s \sim M_s^{-1}, \quad \text{or equivalently,} \quad E_g \sim M_s. \quad (3.4)$$

Moreover, the total mass of the system is given by:

$$m = N E_g. \quad (3.5)$$

Note that, Eq.(3.2) together with Eq.(3.4) tell us that the wave-length of each constituent increases with N showing a collective behavior typical of a condensate. Therefore, the following scaling behavior for each quanta holds:

$$\lambda_g \sim \sqrt{N} L_s \sim L_{\text{eff}}. \quad (3.6)$$

While, in terms of the mass, each quanta becomes *softer*

$$E_g \sim \frac{M_s}{\sqrt{N}}. \quad (3.7)$$

Note that the quantity in Eq.(3.6) corresponds to the size of the system, which is approximatively given by $L_{\text{eff}} = \sqrt{N} L_s$. From Eq.(3.7) we can now understand that the total mass of the graviton condensate in Eq.(3.5) reads

$$m = N E_g \sim N \frac{M_s}{\sqrt{N}} = \sqrt{N} M_s. \quad (3.8)$$

Knowing the expressions of the total mass and the effective nonlocal scale, see Eq.(3.8) and Eq.(1.135) (or, (3.2)), we find that the inequality $2|\Phi| < 1$, always holds true. Indeed, we have:

$$m M_{\text{eff}} \sim \sqrt{N} M_s \frac{M_s}{\sqrt{N}} = M_s^2 < M_p^2. \quad (3.9)$$

Increasing the mass of a gravitational system also means increasing the number of interacting quanta, which in turn means shifting the nonlocal energy scale towards the infrared regime, in such a way that the inequality in Eq.(3.1) is always satisfied for any value of the mass. This should be seen as a *complementary principle* for nonlocal systems. Hence, by relying on such a quantum framework, it is possible to show that horizonless objects can exist in IDG, and they are also known as *nonlocal stars* [84].

It is worthwhile mentioning that in this Thesis work we only consider a static scenario and do not discuss how such a nonlocal star can be formed in a dynamical process, namely whether the collapse can be avoided. We believe that nonlocal interaction can play a crucial role to stop the collapse and prevent the formation of curvature singularities. Indeed, in our quantum nonlocal model the number of gravitons can be very large compared to the number of baryons so that the self-gravitational interaction can be strong enough to win the attractive gravitational force between baryons¹. We can understand this by showing the relation between the number of gravitons N and the number of baryons N_B . The total mass of the system can be written as $m = m_B N_B$, with m_B being the mass of each single baryon, which we assume to have all the same mass for simplicity. Moreover, from Eq.(3.8) we also know that $m = \sqrt{N} M_s$, therefore we obtain the following relation:

$$N = N_B^2 \left(\frac{m_B}{M_s} \right)^2. \quad (3.10)$$

As an example, let us consider a solar mass compact object, $m \sim 10^{38} M_p$, made up of neutrons, $m_B \sim 10^{-19} M_p$, whose number is $N_B \sim 10^{57}$. From Eq.(3.10) we can immediately notice that the number of gravitons is $N \sim 10^{76} (M_p/M_s)^2$ and, since $M_s \leq M_p$ by definition, one has $N \gg N_B$; for instance if $M_s \sim M_p$ we get $N/N_B \sim 10^{19} \gg 1$. However, a deeper and more satisfactory study of gravitational collapse in presence of nonlocality is still missing and will be part of future works.

3.1.2 Compactness

The spacetime metric for a nonlocal star can be modelled as follows. We have seen that the inside of the star is a non-vacuum region in which nonlocal interactions among the constituents take place, in such a way that the overall metric potential $2|\phi|$ remains bounded by one. While, outside we have a vacuum region whose spacetime geometry is well described by the Schwarzschild metric.

¹See Ref.[153] and references therein for discussions on black holes as condensates of soft gravitons.

Therefore, we can construct the metric for a nonlocal star as follows [75, 78]:

$$ds^2 = -(1 + 2\Phi)dt^2 + \frac{dr^2}{1 + 2\Psi(r)} + r^2 d\Omega^2, \quad (3.11)$$

where²

$$\Phi(r) = \begin{cases} -\frac{Gm}{r} \text{Erf}\left(\frac{r}{r_{\text{nl}}}\right), & r \lesssim r_{\text{nl}}, \\ -\frac{Gm}{r}, & r > r_{\text{nl}}, \end{cases} \quad (3.12)$$

and

$$\Psi(r) = \begin{cases} -\frac{Gm}{r} \text{Erf}\left(\frac{r}{r_{\text{nl}}}\right) + \frac{2Gm e^{-r^2/r_{\text{nl}}^2}}{\sqrt{\pi} r_{\text{nl}}}, & r \lesssim r_{\text{nl}}, \\ -\frac{Gm}{r}, & r > r_{\text{nl}}, \end{cases} \quad (3.13)$$

where the radius of the nonlocal star is approximatively given by $r_{\text{nl}} \sim 2L_{\text{eff}}$. For such a metric all curvature invariants are non-singular, and it approaches conformal-flatness in the limit $r \rightarrow 0$ [76, 78]. Moreover, for the inside metric the Birkhoff theorem is violated, $\Phi \neq \Psi$. Note that, the metric potential Φ is non-vanishing at $r = 0$ meaning a net distinction between the nonlocal star and gravastar [154, 155] metrics, indeed the latter has a de Sitter core.

Note that, we can now recast in inequality in Eq.(3.1) as follows:

$$r_{\text{nl}} \sim 2L_{\text{eff}} = r_{\text{sch}}(1 + \epsilon) \gtrsim 2Gm \frac{2}{\sqrt{\pi}}. \quad (3.14)$$

where

$$\epsilon \gtrsim 0.128, \quad (3.15)$$

which is saturated in the most compact scenario, i.e. $\epsilon \simeq 0.128$. From the inequality in Eq.(3.14), we can understand that the radius of a nonlocal star always engulfs the Schwarzschild radius, which implies that there is no horizon for any value of the mass parameter. A Similar situation arises in the fuzz-ball scenario, which has been constructed in a stringy scenario; for a review see Ref. [156, 157].

Let us define the following function

$$\mu := 1 - \frac{r_{\text{sch}}}{r_{\text{nl}}} = \frac{\epsilon}{1 + \epsilon}, \quad (3.16)$$

²Note that the metric potentials in Eqs.(3.12,3.13) are discontinuous at $r = r_{\text{nl}}$. However, there is no sharp boundary between nonlocal and local regimes, but the metric has to be smooth all the way from $r = \infty$ to $r = 0$; therefore, a more rigorous description should be able to take into account this issue in such a way that there exist two metric potentials which continuously interpolate between the inner and outer regions.

which measures the compactness of a nonlocal star. For a black hole we have $\epsilon = 0$, which implies $\mu_{\text{bh}} = 0$. In the case of the most compact nonlocal star, i.e. $\epsilon \simeq 0.128$, the compactness parameter is equal to $\mu_{\text{nl}} \simeq 0.11$, or in other words $r_{\text{sch}}/r_{\text{nl}} \simeq \sqrt{\pi}/2 \simeq 0.886\dots$. A very intriguing fact is that the compactness of a nonlocal star is of the same order of the Buchdahl limit [158] which is $8/9 \simeq 0.888\dots$ and was derived by Buchdahl by assuming a constant density interior. Indeed, for a nonlocal star when we take the limit $r \rightarrow 0$, the metric approaches conformal flatness with constant metric potentials or, equivalently, the effective Gaussian density source becomes constant in good approximation close to the origin.

Let us distinguish several possibilities:

- **Nonlocal star as an UCO:** For a sufficiently compact nonlocal star we can have

$$r_{\text{nl}} = 2Gm(1 + \epsilon) < 3Gm, \quad 0.128 \lesssim \epsilon \lesssim 0.5, \quad (3.17)$$

which means that it can possess a photon sphere, and can be seen as a new kind of UCO. In this case, the properties of a nonlocal star can be probed by studying the ringdown phase. Indeed, after the merging process of two sufficiently compact objects there is a fraction of waves which will be able to cross the photon sphere, but there will also be a fraction of them which will interact with the photon sphere and travel back towards the central object. Since there is no horizon the waves can interact with the surface and travel back. This periodic behavior of a small fraction of gravitational waves would produce *echoes* in the wave-form signal [152, 159, 160, 161].

- **Echoes after ringdown phase:** In Ref.[159], an interesting distinction was made in the class of ultracompact objects, by distinguishing the so called *ClePhOs* (Clean Photon sphere) from the non-ClePhOs. The main difference between these two kind of objects is that for the former echoes are produced at later times, and can be more easily distinguished by the rest of the wave-form by future LIGO/VIRGO observatories. ClePhOs have a radius

$$R = r_{\text{sch}}(1 + \epsilon), \quad \epsilon \lesssim 0.0165.$$

From Eqs.(3.14,3.15), we can now understand that the nonlocal star is a particular class of ultracompact object which does not belong to the class of ClePhOs, since in its most compact case we have $\epsilon = 0.128 > 0.01665$.

- **Nonlocal star as a dark giant star:** When

$$\epsilon > 0.5,$$

the radius of the nonlocal star is such that it engulfs the photon sphere, i.e. $r_{\text{nl}} > 3Gm$, so that no light ring would be present. Such a gravitationally bound system would be very similar to a dark giant star and they

will hardly radiate, which will become evident when discussing the life time of a nonlocal star and the number of quantum states such an object possesses, see below. In terms of compactness, they will be very similar to a neutron [162], or a boson star [163].

3.1.3 Absorption coefficient

We now wish to obtain an estimation of the absorption coefficient of a nonlocal star in the most compact case, $\epsilon \simeq 0.128$. We can do it by studying the dynamics of the system composed by the compact object and the accretion disk, see [164], and using the experimental bounds on the exchanged fluxes of energies $\dot{M}_{\text{disk}}$ and $\dot{E}$. The former describes the amount of infalling matter coming from the accretion disk and going inside the central object per unit time, while the latter corresponds to the emitted energy from the compact object per unit of time. We can immediately notice that a classical black hole is characterized by $\dot{E}_{\text{bh}} = 0$, as nothing can escape outside beyond the horizon.

First, note that one can compute which is the solid angle $\Delta\Omega$ under which particles coming out of the compact object can escape at infinity, and it is proportional to the parameter ϵ , see [159]:

$$\frac{\Delta\Omega}{2\pi} = \frac{27}{8}\epsilon + \mathcal{O}(\epsilon^2). \quad (3.18)$$

The last equation tells us that the more compact the object is, the more deflected the geodesics are, i.e. the more black the central object appears. This result is crucial in order to study the scenario in which a compact object is surrounded by an accretion disk.

Let us now introduce the following fundamental quantities for a nonlocal star [160]:

- κ is the *absorption coefficient* and measures the amount of energy that is lost inside the nonlocal star.
- γ is the *elastic reflection coefficient* which measures the fraction of energy that reaches and interact elastically with the surface of the nonlocal star, and then is reflected back.
- $\tilde{\gamma}$ is the *inelastic reflection coefficient* which measures the fraction of energy which is reflected after inelastic interaction with the surface, i.e. the portion of energy that goes inside the nonlocal star and subsequently is re-emitted.

Note that the following relation holds: $\tilde{\gamma} = 1 - \kappa - \gamma$. For a classical black hole we have: $\kappa_{\text{bh}} = 1$, $\gamma_{\text{bh}} = \tilde{\gamma}_{\text{bh}} = 0$. See also Ref.[160], for the definition of other fundamental parameters characterizing compact objects.

In the case of a compact object the flux of emitted energy can be non-vanishing, indeed the following relation holds true [160]:

$$\frac{\dot{E}}{\dot{M}_{\text{disk}}} \simeq \frac{(1 - \kappa - \gamma)(1 - \gamma)\Delta\Omega/2\pi}{\kappa + (1 - \kappa - \gamma)\Delta\Omega/2\pi}. \quad (3.19)$$

Very interestingly, the quantity in Eq.(3.19) can be constrained through astronomical observations; indeed, one can put the following upper bound [160]:

$$\frac{(1 - \kappa - \gamma)(1 - \gamma)\Delta\Omega/2\pi}{\kappa + (1 - \kappa - \gamma)\Delta\Omega/2\pi} \lesssim \mathcal{O}(10^{-2}). \quad (3.20)$$

We are mainly interested in obtaining a direct bound on the absorption coefficient, but the way Eq.(3.20) is written would only allow us to constrain the combination of the two parameters κ and γ . However, we will still have to use one of the main features due to nonlocality, namely that it does *not* have a hard surface due to nonlocality which weakens the interactions, thus the corresponding elastic reflection coefficient is very close to zero, $\gamma \approx 0$. However, there is still a non-vanishing inelastic reflection coefficient which measures the fraction of infalling quanta which can come out again and escape from the nonlocal star, but we expect it to be very small by virtue of our previous results.

Therefore, for a nonlocal star Eq.(3.20) becomes

$$\frac{(1 - \kappa)\Delta\Omega/2\pi}{\kappa + (1 - \kappa)\Delta\Omega/2\pi} \lesssim \mathcal{O}(10^{-2}), \quad (3.21)$$

from which, by using the formula Eq.(3.18) and working in the most compact case, $\epsilon \simeq 0.128$, we get a direct bound on the absorption coefficient:

$$0.977 \lesssim \kappa \lesssim 1. \quad (3.22)$$

This result is consistent with the discussions on life time, entropy and number of states made above, indeed as expected the value of the absorption coefficient of a nonlocal star is very close to unity. As a consequence, since $\gamma \approx 0$, we also obtain an upper bound on the inelastic reflection coefficient:

$$0 \lesssim \tilde{\gamma} \lesssim 0.023, \quad (3.23)$$

which turns out to be very small, consistently with our expectations.

Let us summarize the main results of this Section. We have provided a quantum mechanical framework in which non-singular, horizonless compact objects can exist. The gravitational interaction weakens due to the presence of nonlocal form factors made up of infinite order covariant derivatives. For the simplest choice of an entire function, which appears in the graviton propagator, it is possible to avoid any curvature singularity and event horizon, provided that the

complementarity relation holds, $mM_{\text{eff}} \sim M_s^2 < M_p^2$. We wish to emphasize that we have only worked with a scalar toy model for which the complementarity relation can be shown to be true; see Subsection 1.3.5. In fact, one should study the full gravitational action and compute scattering amplitudes between gravitons. This is very challenging and further investigations will be subject of future works.

3.2 Nonlocality as an amplifier of echoes

In this Section we wish to provide a neat smoking gun signature for nonlocal interactions, which is independent of the nonlocal star picture introduced above. We study the propagation of waves for a nonlocal massless scalar field coupled to two Dirac delta potential barriers in one spatial dimension. We work out the quasi-normal modes (QNMs) and the echoes of an initial incoming pulse and make the comparison with the local case. Our analysis allows to make a very powerful analogy in astrophysics. Despite the simplicity of the model, we are able to capture the main features of nonlocality which turn out to be universal and we believe also to hold in a more realistic astrophysical scenarios.

3.2.1 Nonlocal scalar field with double delta potential

Let us consider a nonlocal massless scalar field Φ interacting with a potential V in $1+1$ dimensions with a nonlocal operator given by an exponential of an *entire* function

$$[F(\square)\square - V]\Phi = 0, \quad F(\square) = e^{-L_s^2 \square}; \quad (3.24)$$

we will work with the simplest kinetic operator above but other choices are also possible [39, 42, 74, 81]. In Eq.(3.24), L_s is the fundamental length scale of nonlocality below which new physics should manifest and $\square = -\partial_t^2 + \partial_x^2$ is the flat d'Alembertian operator. By separating the time and spatial variables with the ansatz $\Phi(t, x) = e^{-i\omega t}\psi(x)$, Eq.(3.24) becomes [106]

$$\left[e^{-L_s^2(\partial_x^2 + \omega^2)}(\partial_x^2 + \omega^2) - V(x) \right] \psi(x) = 0, \quad (3.25)$$

which is a nonlocal version of the Schrödinger equation. For general potential barriers, $V(x)$ is assumed to be positive and to satisfy $V(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, so that the general solution of $\psi(x)$ has the form

$$\psi(x) = \begin{cases} Ae^{i\omega x} + Be^{-i\omega x}, & x \rightarrow -\infty, \\ Ce^{i\omega x}, & x \rightarrow +\infty. \end{cases} \quad (3.26)$$

The reflection and transmission coefficients, $\mathcal{R}$ and $\mathcal{T}$ respectively, are defined as

$$\mathcal{R} = \frac{B}{A}, \quad \mathcal{T} = \frac{C}{A}, \quad |\mathcal{R}|^2 + |\mathcal{T}|^2 = 1. \quad (3.27)$$

For a general potential Eq.(3.25) may not be easily solvable, but there exist analytic solution in some cases. Here, we assume a symmetric double Dirac delta potentials

$$V(x) = \lambda\delta(x+a) + \lambda\delta(x-a). \quad (3.28)$$

Note that a single Dirac delta potential in nonlocal theory was first studied in [106]. The nonlocal Schrödinger equation (3.25) with the potential (3.28) can be solved analytically by using the Lippmann-Schwinger method [165]

$$\psi(x) = \psi_0(x) - \int_{-\infty}^{\infty} dx' G(x, x') V(x') \psi(x') \quad (3.29)$$

where $G(x, x')$ is the Green function defined through the equation $e^{-L_s^2(\partial_x^2 + \omega^2)}(\partial_x^2 + \omega^2)G(x, x') = -\delta(x - x')$, and reads [106]

$$\begin{aligned} \mathcal{G}(z) &= \frac{i}{4\omega} [e^{i\omega z} \mathbf{Y}(z) + e^{-i\omega z} \mathbf{Y}(-z)] \\ \text{with } \mathbf{Y}(z) &= 1 + \text{erf}\left(i\omega L_s + \frac{z}{2L_s}\right), \end{aligned} \quad (3.30)$$

which in the limit $L_s \rightarrow 0$ reduces to the local Green function $\mathcal{G}_0(z) = \frac{i}{2\omega} e^{i\omega|z|}$, as expected. The Green function turns out to be symmetric, $G(x, x') = G(x', x)$, and depends only on the variable $z = x - x'$: $G(x, x') \equiv \mathcal{G}(z)$, $\mathcal{G}(z) = \mathcal{G}(-z)$. Hence, by using the Lippmann-Schwinger equation (3.29), the solution for the symmetric double delta in (3.28) can be shown to be

$$\psi(x) = e^{i\omega x} - \lambda \sum_{j,k=1}^2 \mathcal{G}(x - a_k) \Phi_{kj}^{-1} e^{i\omega a_j}, \quad (3.31)$$

where

$$\Phi_{kj} = \begin{cases} 1 + \frac{i\lambda_j}{2\omega} [1 + \text{erf}(i\omega L_s)], & k = j; \\ \frac{i\lambda_j}{4\omega} [e^{i2\omega a} \mathbf{Y}(2a) + e^{-i2\omega a} \mathbf{Y}(-2a)], & k \neq j, \end{cases} \quad (3.32)$$

and we have defined $a_1 = a$, $a_2 = -a$. By assuming that the incoming wave comes from the left, we can find the reflection and transmission coefficients for the solution in Eq.(3.32) by identifying the coefficients in front of $e^{-i\omega x}$ when $x \ll a$ and of $e^{i\omega x}$ when $x \gg a$, respectively; they are given by

$$\begin{aligned} \mathcal{R} &= -\frac{2[-2\frac{\omega}{\lambda}i+1+\text{erf}(i\omega L_s)]\cos(2\omega a)-[1+\text{erf}(i\omega L_s+\frac{a}{L_s})]e^{2i\omega a}-[1+\text{erf}(i\omega L_s-\frac{a}{L_s})]e^{-2i\omega a}}{[-2\frac{\omega}{\lambda}i+1+\text{erf}(i\omega L_s)]^2-\frac{1}{4}\{[1+\text{erf}(i\omega L_s+\frac{a}{L_s})]e^{2i\omega a}+[1+\text{erf}(i\omega L_s-\frac{a}{L_s})]e^{-2i\omega a}\}^2}, \\ \mathcal{T} &= 1 - \frac{2[-2\frac{\omega}{\lambda}i+1+\text{erf}(i\omega L_s)]-\{[1+\text{erf}(i\omega L_s+\frac{a}{L_s})]e^{2i\omega a}+[1+\text{erf}(i\omega L_s-\frac{a}{L_s})]e^{-2i\omega a}\}\cos(2\omega a)}{[-2\frac{\omega}{\lambda}i+1+\text{erf}(i\omega L_s)]^2-\frac{1}{4}\{[1+\text{erf}(i\omega L_s+\frac{a}{L_s})]e^{2i\omega a}+[1+\text{erf}(i\omega L_s-\frac{a}{L_s})]e^{-2i\omega a}\}^2}, \end{aligned} \quad (3.33)$$

and one can easily check that $|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$.

Figure 3.1: We have shown the behaviour of the real part of the QNFs $\mathcal{W}_n$ as a function of the negative imaginary part ω'_n for both local (black line) and nonlocal (orange line) cases; the latter was obtained by solving numerically Eq.(3.35). In the left panel we have set $a = 1$, $L_s = 0.01$ and $\lambda = 10$, while in the right panel $a = 1$, $L_s = 0.01$ and $\lambda = 50$.

3.2.2 Quasi-normal modes

The poles of $\mathcal{R}$ and $\mathcal{T}$ describe the QNMs of the system, which satisfy the boundary condition such that there is no incoming wave at infinity, i.e. $\psi \rightarrow e^{-i\omega x}$ as $x \rightarrow -\infty$ and $\psi \rightarrow e^{i\omega x}$ as $x \rightarrow +\infty$ [166]. In a realistic astrophysical framework, the quantities in Eq.(3.33) are crucial for understanding the effect of nonlocality in the UCOs. We can check that the poles of the reflection and transmission coefficients are given by the equations

$$2[-2\frac{\omega}{\lambda}i + 1 + \text{erf}(i\omega L_s)] = \pm \left\{ [1 + \text{erf}(i\omega L_s + \frac{a}{L_s})]e^{2i\omega a} + [1 + \text{erf}(i\omega L_s - \frac{a}{L_s})]e^{-2i\omega a} \right\}, \quad (3.34)$$

whose roots are of the type $\omega_n = \mathcal{W}_n - i\omega'_n$ and are called quasi-normal frequencies (QNFs). Note that for ω with a negative imaginary part, the time dependent piece $e^{-i\omega t}$ behaves like a damped oscillation: the real part gives the energy carried by the wave and the imaginary part (inverse) gives the time scale over which the wave decays after interacting with the potential barrier. It is worthwhile mentioning that in the limit, $a = 0$, we recover the quasi-normal condition for a single Dirac delta potential with strength 2λ , i.e. $1 - i\omega/\lambda + \text{erf}(i\omega L_s) = 0$ [106]; while in the limit $L_s = 0$, Eq.(3.34) reduces to the well known local case, $1 - i2\omega/\lambda = \pm e^{2i\omega a}$.

Generally, Eq.(3.34) cannot be solved, but we can simplify it and obtain an analytic solution by working in a specific regime which turns out to be the most interesting one from a physical point of view. Since we expect the fundamental scale of nonlocality L_s to be always smaller than the distance between the surface and the photon sphere of a UCO, see [152, 84], it is sensible to require

$a \gg L_s$. Moreover, when studying the QNMs and the echoes produced by an initial pulse (see also below) the low frequencies are the most relevant ones as they correspond to longer damping times, therefore we can also impose the inequality $|\omega|L_s < a/L_s$. In such limits, $\text{erf}(i\omega L_s + a/L_s) \rightarrow 1$ and $\text{erf}(i\omega L_s - a/L_s) \rightarrow -1$, so that Eq.(3.34) reduces to

$$1 - i2\frac{\omega}{\lambda} + \text{erf}(i\omega L_s) = \pm e^{2i\omega a}, \quad (3.35)$$

which in the limit $|\omega|L_s \ll 1$ can be further simplified since $\text{erf}(i\omega L_s) \approx \frac{2}{\sqrt{\pi}}i\omega L_s$. Indeed, in this regime we can find an analytic expression for the QNMs in terms of the Lambert W function:

$$\omega_n = i \left[\frac{W_n(\pm \alpha a e^{\alpha a})}{2a} - \frac{\alpha}{2} \right], \quad \frac{1}{\alpha} = \frac{1}{\lambda} - \frac{L_s}{\sqrt{\pi}}. \quad (3.36)$$

Furthermore, we can solve numerically Eq.(3.35) and study the behaviour of the real part of the QNMs, $\text{Re}(\omega) = \mathcal{W}_n$, as a function of the negative imaginary part $-\text{Im}(\omega) = \omega'_n$. From Fig. 3.1, we can see that nonlocal effects make the imaginary part of the QNFs smaller, i.e. the damping time is longer. The curve corresponding to the nonlocal case is always below the local one in the low frequency regime as shown in the Fig. 3.1. The closer the ring-down energy $\mathcal{W}_n$ is to the energy scale $1/L_s \equiv M_s$, the more relevant nonlocality is. This means the more obvious becomes the difference between the imaginary parts between local and nonlocal. Only for $L_s \rightarrow 0$ the two curves coincide. Moreover, when the strength of the delta potential $\lambda \gtrsim 1/L_s \equiv M_s$, the damping is smaller. Therefore, a crucial implication due to nonlocality is that low frequency waves survive a *longer* life-time as compared to the local case. This feature was expected since it is known that nonlocality weakens the interaction [87, 88]. Furthermore, when $\mathcal{W}_n L_s \sim \mathcal{O}(1)$, we can also notice the presence of a turning point after which the damping time decreases. This property is mathematically related to the structure of the error function with complex argument but it is not relevant from a physical point of view since it will only describe short-lived waves with energies $\mathcal{W}_n L_s \gg 1$.

3.2.3 Echoes

So far we have learned that nonlocal effects weaken the interaction between wave and potential barriers, and as a consequence the time scales over which a transmitted wave decays turn out to be longer than the local case. We now wish to use the spectral features of the QNMs and study the consequences induced by nonlocality on the echoes production. In order to do so we need to work in the time domain where the form of the wave can be cast as an infinite linear

combination of QNMs [151, 167]:

$$\Psi(t) = \sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}, \quad (3.37)$$

which in a more realistic scenario can represent the signal produced at the photon sphere after relaxation. Because the oscillation period of the real parts of the QNMs is $\text{Re}(\omega_{n+1} - \omega_n) \simeq \pi/2a$, which is the same also in the local case, and since $\text{Im}(\omega_n) \ll \text{Re}(\omega_n)$, the function $\Psi(t)$ is in good approximation periodic, with period $T = 4a$. This means that the primary signal is reproduced periodically, as it can be reflected by the second potential barrier (UCO's surface), thus giving rise to echoes.

This feature allows us to express the coefficients c_n as a Fourier transform of the first echo, $\Psi_{1^{\text{st}}\text{echo}}^{(0)}(t)$,

$$c_n = \frac{1}{4a} \int_0^{4a} dt \Psi_{1^{\text{st}}\text{echo}}^{(0)}(t) e^{i\omega_n t}. \quad (3.38)$$

We can choose the first echo as a Gaussian wave packet of the form

$$\Psi_{1^{\text{st}}\text{echo}}^{(0)}(t) = e^{-i\omega_0(t-t_0)} e^{-(t-t_0)^2/2\tau^2}, \quad (3.39)$$

where ω_0 is the leading frequency of the primary signal. Assuming that $\tau \ll 2a$ we can obtain an approximation for the coefficients c_n of the subsequent echoes by computing the integral in Eq.(3.38) [151]

$$c_n \simeq \sqrt{\frac{\pi}{2}} \frac{\tau}{2a} \exp \left[i\omega_n t_0 - \frac{\tau^2}{2} (\omega_n - \omega_0)^2 \right]. \quad (3.40)$$

Hence, we can now use Eq.(3.37) with the coefficients given in Eq.(3.40) to plot the time dependent profile of the signal, see Fig. 3.2 Note that, because only the long-lived QNMs (QNFs with small negative imaginary part) led to the echoes in the late time [168], thus it is also sufficient to analyze these results using Eq.(3.36).

Let us capture the salient features of our analysis. In the regime $\lambda \ll 1/L_s \equiv M_s$, we have already seen that the QNMs are very similar to the local case, therefore the same applies for the echoes. It means that when the height of the potential barriers is below the energy scale $1/L_s \equiv M_s$, nonlocality does not play a relevant role, i.e. the echoes are almost the same as the local case. However, when the height of the potential barrier is either comparable or larger than the nonlocal energy scale, i.e. $\lambda \gtrsim M_s$, the damping time of the nonlocal case is much longer at low frequency. So a very distinct difference arises between local and nonlocal cases. As shown in Fig. 3.2, the amplitudes of the late time echoes are amplified with respect to the local ones, meaning that nonlocality

Figure 3.2: We have plotted the behaviour of the signal in Eq.(3.37), with coefficients (3.40), which describes the evolution of the echoes produced at a fixed spatial point, for both local (black line) and nonlocal (orange line) cases. We have chosen $t_0 = 1$, $\tau = 0.25$, and set $a = 1$, $\lambda = 50$, $L_s = 0.01$.

acts as an *amplifier*. Actually, we can adjust the degree of amplification by adjusting the height of the potential λ . Such a behavior is consistent with the fact that the nonlocal interaction makes the life-times of the waves longer [84], which means they are less damped in time, therefore the echo signal is more amplified. The presence of a smaller damping scale also implies that in the process of transmission less energy is dissipated when the interaction between the wave and the potential barrier is nonlocal, therefore the echoes are louder. Therefore, nonlocal effects can be amplified in presence of double delta potential barriers, which can be experimentally probed.

This analysis has been made in the simple case of flat background but future studies are needed to take into account the presence of a curvature and of a more realistic effective potential, but we believe that the essence will be very similar to what we have presented here. The key smoking-gun signature will be the presence of a smaller damping scale and the amplification of echoes, which is a distinctive feature of nonlocality and is surely absent in a local theory.

It is worthwhile mentioning that this kind of nonlocal effects could be captured in models of condensate matter system, in which the reflection and transmission properties of quantum mechanical barriers can be potentially tested in a table-top experiment. For a different application, see also Ref.[169] where nonlocal effects on quantum mechanical oscillators were studied.

Conclusions and Outlook

In this Thesis we have studied classical and quantum aspects of Lorentz invariant nonlocal (infinite derivative) field theories. Let us briefly summarize and highlight the obtained results and the analyzed relevant properties.

In Chapter 1, we showed that the action can be made nonlocal by introducing Lorentz invariant analytic form factors either in the kinetic operator and/or in the interaction vertex. We demonstrated that in order to not introduce any *ghost-like* degree of freedom, we need to demand the form factors to be *exponentials of entire functions*; in particular, we considered exponentials of polynomials of $\square$; see Eq.(1.10). One first feature we noticed is that the nonlocal propagator is not simply defined in terms of a time-ordered product but it is made of an acausal contribution, unlike in local theory. Moreover, the nonlocal analog of the retarded Green function assumes an acausal behavior, indeed it is non-vanishing for space-like separations. As a consequence, also the local commutativity condition is violated in presence of nonlocal interaction. Therefore, in the nonlocal region, we cannot define any concept of space and time due to the presence of acausal effects. Such a statement is also mathematically justified by the fact that amplitudes are ill-defined with the Minkowski signature due to the presence of the exponential operators, which can diverge along some direction in the complex plane, making it impossible to define the usual Wick-rotation. For this reasons, the recipe to follow is to define the theory in the Euclidean space, where all the amplitudes can be well-defined, and after having performed the computations, we can analytically continue back the external momenta to the Minkowski signature with the prescription described in Ref.[125, 126, 127]. We have also emphasized that nonlocality and acausality manifest as off-shell phenomena.

Subsequently, we showed that the optical theorem, and so the unitarity

condition, are preserved in nonlocal scalar field theory. We also considered N ($\gg 1$) particles interacting in a scattering process and for a simple scalar model with cubic interaction we showed that the scale of nonlocality M_s shifts towards the IR regime by increasing the number of interacting particles. It means that the larger N is, the bigger the size of the region on which the nonlocal interaction takes place is. Moreover, we computed the Euclidean 2-point correlation function, and showed that it is non-singular at the Euclidean origin.

Another intriguing aspect we have discussed is the possibility to enlarge the class of symmetries under which a local Lagrangian is invariant by introducing non-polynomial differential operators. We managed to do it in the case of the Galilean shift symmetry but a more general investigation is needed in future.

In Chapter 2, we built a ghost-free higher derivative theory of gravity by using exponentials of entire functions as done for the simpler case of a scalar field. We computed the propagator which can be made ghost-free, and found several linearized spacetime metric solutions around Minkowski background. One of the physical implication of nonlocal gravitational interaction is the resolution of curvature singularities through the action of infinite order derivatives. Indeed, for the cases of neutral and electrically charged point-like sources we noticed that no singularity appears at $r = 0$, and for a stationary rotating ring that no Kerr-like singularity is present. Subsequently, we moved towards the non-linear regime and showed that also in this case nonlocality can play a crucial role, indeed in IDG the Schwarzschild metric cannot be a full non-linear solution, but can be only valid in the large distance regime, $M_s r \gg 1$ and has to interpolate with a non-singular core in the regime $M_s \ll 1$.

In Chapter 3, we have discussed two phenomenological applications of non-local (infinite derivative) field theories. The first refers to nonlocal gravitational interaction, in particular to the possibility to form horizonless compact objects in IDG. Indeed, we built a quantum framework in which the mass and the nonlocal scale of a system made up of N quanta scale in such a way that the inequality $2|\Phi| < 1$ is always satisfied in the entire spacetime and for any value of the mass, thus preventing the formation of any horizon. We also studied some physical properties like compactness and absorption coefficient. As a second application, we have considered a scalar field interacting nonlocally with a double delta potential in flat spacetime. We computed transmission and reflection coefficients, and afterwards we worked out the QNMs and the echoes. Since nonlocality weakens the interaction, it was possible to show that the damping time of the wave through the barrier is larger and the echoes turn out to be amplified as compared to the local case. Despite its simplicity, this model can be very useful in an astrophysical context to mimic the interaction of gravitational waves with the two photon spheres of a wormhole, or the two potential barriers located at the surface and at the photon sphere of a UCO. Therefore, nonlocality works

as an amplifier and this feature might represent a smoking-gun signature of nonlocal interaction in future experiments.

Open issues

Although nonlocal (infinite derivative) field theories have been showing many interesting features, there are still several open questions which still need to be properly addressed.

- Systematic methods to proof macrocausality¹ at the level of the S-matrix (Bogolyubov's causality condition [120]) have not been developed yet. In local quantum field theory, it is well known that the causality can be also proven by using the *largest time equation* [132]. Such an approach strongly relies on two crucial hypothesis: (i) the propagator has a time-ordered structure, (ii) vertices are local. It is clear that when the principle of locality is given up at the level of interaction, and infinite derivative are introduced, the largest time equation cannot be consistently used, as the propagator is not simply a time-ordered product and the interaction vertices become nonlocal.
- All the theorems on renormalizability of local field theories rely on the crucial assumptions of polynomial counter-terms, which follows from the locality of the Lagrangian [170]. It is clear that similar theorems do not exist in nonlocal field theory since the basic assumption of locality is now missing. For instance, the usual power-counting argument is *not* sufficient in this case due to the presence of exponentials in the propagators and/or in the vertices. Therefore, new smart ideas are needed.
- While in nonlocal scalar field theory all amplitudes can be well defined, in the case of gravity some ambiguities can appear at the tree level. Indeed, in IDG the exponentials of entire functions appear both in the kinetic and in the interaction terms, which means that in a scattering amplitude there can be both suppressed and enhanced nonlocal contributions. By doing a simple power-counting, one can show that the superficial degree of divergence of a tree level diagram in Euclidean space is given by the difference of number of vertices and number of internal propagators: $D = V - I$. This formula means that anytime $V > I$ the amplitude diverges at high energies. A new attempt to solve this problem was worked out in Ref.[86] in which both bare propagators and bare vertices were replaced by the dressed ones in order to make all loop integrals convergent.

¹Macrocausality is a generalization of the concept of causality in which one can have the presence of acausal effects at microscopic scales ($t, r \leq 1/M_s$), but physics is still causal on macroscopic scales ($t, r \gg 1/M_s$).

- As already discussed in this Thesis, nonlocality can help to regularize singularities from which the local theory suffers. This has been shown in the linearized regime but it is still not clear whether such a feature holds universally and, so, if it applies to the non-linear level too. No exact spherically symmetric solution has been found so far in the non-linear regime, therefore a complete satisfactory picture is still lacking.
- Another serious issue of infinite derivative field theories is the huge arbitrariness in the choice of the entire function $\gamma(\Box)$. In principle, any kind of entire function is suitable, since all of them would give a ghost-free propagator. At the moment, it is still not clear how to choose and/or discriminate among this infinite class of entire functions. However, it is worthwhile emphasizing that at least in string theory only one type is recurrent, namely $\gamma(\Box) = \alpha' \Box$. Indeed, by identifying the string functional integral over the worldsheet metrics with a sum over all lattices and using the large- N expansion to define surfaces on these lattices, one can define a continuum limit [114] yielding a string field theory action with Gaussian behaviors ($e^{-\alpha' k^2}$) either in the propagator or in the vertex [115, 116].

To conclude, it is still not very clear how to tackle these problems and find definite answers to these questions. However, the very interesting results obtained so far motivate a better understanding of how to address these issues. Such a program is extremely challenging but equally attractive, therefore infinite derivative field theories will deserve and surely undergo further and deeper future investigations.

Unitarity and ghosts

In this Appendix we give the concept of unitarity and study the conditions under which it is preserved in a generic (either local or nonlocal) quantum field theory. We will also give the definition of ghost degree of freedom and distinguish between *good* and *bad* ghosts.

A.1 Unitarity and optical theorem

In a physical quantum theory, we expect that the sum of all probabilities is equal to 1. This means that the norm of a state $|\alpha\rangle$ at time $t = 0$ should be the same at a later time t :

$$\langle\alpha, t=0|\alpha, t=0\rangle = \langle\alpha, t|\alpha, t\rangle , \quad (\text{A.1})$$

which also implies that the Hamiltonian should be Hermitian, $H^\dagger = H$, since

$$|\alpha, t\rangle = e^{iHt} |\alpha, t=0\rangle . \quad (\text{A.2})$$

The S -matrix is defined as

$$S := e^{iHt} , \quad (\text{A.3})$$

and from Eqs.(A.1,A.2) it follows that the operator S has to be unitary:

$$S^\dagger S = 1. \quad (\text{A.4})$$

Therefore, the unitarity of the S -matrix is equivalent to conservation of probability.

Let us write the S -matrix as

$$S = \mathbb{1} + iT, \quad (\text{A.5})$$

where T is called transfer matrix and its elements are defined as

$$\langle \alpha | T | \beta \rangle = (2\pi)^4 \delta^4(p_\alpha - p_\beta) \mathcal{M}(\alpha \rightarrow \beta), \quad (\text{A.6})$$

with $\mathcal{M}(\alpha \rightarrow \beta)$ being the scattering amplitude. The matrix T is not hermitian, in fact from Eq.(A.5) we have

$$\mathbb{1} = S^\dagger S = (\mathbb{1} - iT^\dagger) (\mathbb{1} + iT) \quad (\text{A.7})$$

$$\implies 2\text{Im}\{T\} = T^\dagger T, \quad (\text{A.8})$$

that is an equivalent to Eq.(A.4) and is well known as *optical theorem*.

Let us now suppose we can construct a Fock space W around Minkowski spacetime and consider the states $|\alpha\rangle, |\beta\rangle \in W$ as in- and out-states, respectively. Therefore, by using the completeness relation

$$\mathbb{1} = \sum_{|n\rangle \in W} |n\rangle \langle n|, \quad (\text{A.9})$$

we can recast Eq.(A.8) as follows:

$$\langle \alpha | (-i)T | \beta \rangle + \langle \alpha | iT^\dagger | \beta \rangle = \sum_{|n\rangle \in W} \langle \alpha | T^\dagger | n \rangle \langle n | T | \beta \rangle. \quad (\text{A.10})$$

Note that the relation in Eq.(A.10) must hold order by order in perturbation theory. While the left hand side is matrix elements, the right hand side is matrix elements squared: this means that at order λ^2 in some coupling the left hand side must be a loop to match a tree level calculation on the right hand side. Thus, the imaginary parts of loop amplitudes can be fully determined once we know the tree level amplitudes; in particular, without loops unitarity would be violated.

A.2 Ghost fields

In this Section we give the definition of ghost field and study its implications both at the classical level and at the quantum level. We will only consider a scalar field for simplicity but the results also hold for vector and tensor field.

Let us consider the following Lagrangian:

$$\mathcal{L} = -\frac{a}{2} \partial_\mu \phi \partial^\mu \phi - \frac{b}{2} m^2 \phi^2, \quad (\text{A.11})$$

where $a = \pm 1$ and $b = \pm$. The momentum conjugate to ϕ is defined by

$$\pi := \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = a \partial_0 \phi \equiv a \dot{\phi}, \quad (\text{A.12})$$

We can obtain the Hamiltonian density by performing the following Legendre transformation

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} \\ &= \pi\dot{\phi} - \frac{a}{2}\partial_\mu\phi\partial^\mu\phi + \frac{b}{2}m^2\phi^2 \\ &= \frac{a}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2\right) + \frac{b}{2}m^2\phi^2\end{aligned}\tag{A.13}$$

while the Hamiltonian reads

$$H = \int d^3x \left[\frac{a}{2} \left(\dot{\phi}^2 + (\nabla\phi)^2 \right) + \frac{b}{2} m^2 \phi^2 \right]. \tag{A.14}$$

We can distinguish the following scenarios [171]:

- $a = b = +1$: the Hamiltonian is positive semi-definite and therefore bounded from below;
- $a = b = -1$: the Hamiltonian is negative semi-definite and therefore bounded from above;
- $a = -b$: the Hamiltonian is indefinite and so it is not bounded either from below or from above.

If $a = b = -1$, the field ϕ is called *ghost field*; if $a = +1$ and $b = -1$ one has a *tachyon field*; while, if $a = -1$ and $b = +1$ it is called *tachyonic ghost field*. More generally one gives the following definition:

“A ghost field is defined as a field which has negative kinetic energy”.

In what follows we shall see what happens if ghosts are present in the Lagrangian. At the classical level the presence of ghost could cause instabilities of the vacuum since the energy is not bounded from below, while at the quantum level they correspond to states with negative norm and could violate the unitarity condition. However, the presence of ghost does not always violate fundamental principles. In fact, we have to distinguish *good* ghosts from *bad* ghosts. A good ghost does not violate any fundamental principles since they never appear as observable physical state. An example of good ghost appears in Electrodynamics where the time component of the four-vector A_μ has a negative kinetic term. The associated states do not belong to the physical spectrum but they are very important because are necessary to cancel out the longitudinal component of the vector field. Another example is given by the Faddeev-Popov ghost in gauge theories which were introduced to maintain the consistency between gauge invariance and path integral formulation and, as Feynman noticed, they were necessary to preserve unitarity. It is remarkable, doubtless of profound

significance, that good ghosts solve, simultaneously, the problem of unitarity and gauge invariance [172]. Moreover, we have also seen an example of good ghost in GR graviton propagator, where we have a scalar graviton component that behave as a good ghost. Instead, a bad ghost does violate fundamental principles because it is associated with physical particles. Examples are the Pauli-Villars ghost that was introduced to regularize loop-integrals in quantum field theory, and the Weyl ghost in fourth order gravity which we have deeply studied in this Thesis.

A.2.1 Ghosts at the classical level

If a Hamiltonian is unbounded from below (like in the cases $a = b = -1$ and $a = -b$) instabilities can emerge in the system. However, if a ghost field ϕ is free, namely if interactions are absent, one can easily see that the system will be still stable as the energy is conserved, independently of its sign. In fact, any constant that multiplies a (classical) Lagrangian does not change the physics, since it does not appear in the equations of motion. Thus, the choices $a = b = +1$ and $a = b = -1$ are completely equivalent at the free level. Translating the previous words in formula, the field equation for the Lagrangian (A.4) is given by

$$(-a\Box + bm^2)\phi(x) = 0. \quad (\text{A.15})$$

Thus, in the cases $a = b = +1$ and $a = b = -1$ the field equations coincide. In momentum space the last equation becomes

$$(ak^2 + bm^2)\phi(k) = 0 \implies a(k^0)^2 - a\vec{k}^2 = bm^2. \quad (\text{A.16})$$

Unlike ghosts, tachyons can cause troubles even when there is no interaction. First, we can easily verify that for $a = -b$ we have a tachyonic solution. In fact, Eq.(A.16)) gives $(k^0)^2 - \vec{k}^2 = -m^2 > 0$ that implies $m^2 < 0$, i.e. complex masses. If we now consider the following Fourier decomposition

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}, \quad (\text{A.17})$$

we obtain that, for $a = b = \pm 1$, every mode $\phi_{\vec{k}}(t)$ evolves independently from the others and satisfies the equation

$$\ddot{\phi}_{\vec{k}}(t) + (m^2 + \vec{k}^2)\phi_{\vec{k}}(t) = 0, \quad (\text{A.18})$$

which exhibits oscillatory solutions of frequency given by $\omega(\vec{k}) = \sqrt{m^2 + \vec{k}^2}$. A small perturbation at $t = t_0$ from the configuration $\phi = 0$ is described by small Fourier coefficients $\phi_{\vec{k}}(t_0)$, and the oscillatory behavior ensures that there is no exponential enhancement, i.e. the perturbation remains small for $t > t_0$.

Instead, if $a = -b$, the frequency $\omega(\vec{k}) = \sqrt{\vec{k}^2 - m^2}$ turns out to be imaginary when $\vec{k}^2 < m^2$ and so the Fourier modes suffer from an exponential growth, implying the presence of a tachyonic instability in the theory.

Let us now switch the interaction on and consider the following Lagrangian:

$$\mathcal{L} = \frac{a}{2} \partial_\mu \phi \partial^\mu \phi - \frac{a}{2} m_\phi^2 \phi^2 + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m_\psi^2 \psi^2 - V_{\text{int}}(\phi, \psi), \quad (\text{A.19})$$

where for simplicity the potential V_{int} is assumed to be independent of field derivative, but only depends on the two fields ϕ, ψ , and admits the solution $\phi = \psi = 0$ as a local minimum. By performing the Legendre transformation we can also obtain the corresponding Hamiltonian:

$$\mathcal{H} = \frac{a}{2} \left(\dot{\phi}^2 + (\nabla \phi)^2 \right) + \frac{a}{2} m^2 \phi^2 + \frac{1}{2} \left(\dot{\phi}^2 + (\nabla \phi)^2 \right) + \frac{1}{2} m^2 \phi^2 + V_{\text{int}}(\phi, \psi). \quad (\text{A.20})$$

Note that, if $V_{\text{int}}(\phi, \psi) = 0$, the configuration $\phi = \psi = 0$ is still stable independently of the sign of a . The stability is preserved as the energy conservation law can be applied separately for the two non-interacting fields ϕ and ψ . However, we have to point out that already at this level there is a difference between the cases $a = +1$ and $a = -1$. Although the system is stable, the choice $a = -1$ corresponds to an infinite number of different states with zero energy which cannot be associated to small perturbations of the vacuum (minimum) $\phi = \psi = 0$. In the case $V_{\text{int}}(\phi, \psi) \neq 0$, the minimum configuration is still a solution of the field equation and one can show that the Hamiltonian can be bounded from below for constant values of the dynamical fields. Since now the interaction term is non-vanishing, the configurations cannot have zero energy anymore. However, by perturbing the vacuum configuration, one can construct states with energy values very close to zero. Therefore, if $a = -1$, the available volume of the momentum space turns out to be infinite with an infinite number of excited states. Thus, since for entropy reasons the total energy is redistributed into the largest possible class of states, the decay towards these excited states is extremely favoured, and this can be summarized saying that the system is unstable for small oscillations [171].

As we will see below a Lagrangian of the type in Eq.(A.19) with $a = -1$ is equivalent to a Lagrangian containing higher order derivatives. In 1850, Ostrogradsky demonstrated a theorem in which he states that the Hamiltonian of a non-degenerate higher derivative theory is unbounded from below, and also from above, so instabilities are present [10, 173]. For this reason, the classical instability due to the presence of a negative kinetic term is also called *Ostrogradsky instability*.

A.2.2 Ghosts at the quantum level

Let us compute the propagator for the Lagrangian in Eq.(A.4) with the ghost choice for the coefficients, $a = b = -1$:

$$\Pi(k) = \frac{i}{k^2 + m^2 - i\epsilon}, \quad (\text{A.21})$$

which turns out to have a minus sign of difference with respect to an ordinary (healthy) propagator; see Eq.(1.27).

We now want to show the physical implications due to the presence of a ghost at the quantum level.

Optical theorem violation

We will now show that the presence of ghost degrees of freedom violates the optical theorem. From Eq.(A.8) it follows that the imaginary parts of amplitudes in a generic quantum field theory have to be positive, but this is not the case for a ghost. Indeed, by defining the tree level amplitude $T = i\Pi(k)$ we obtain

$$\begin{aligned} \text{Im}\{T\} &= \frac{\epsilon}{(k^2 + m^2)^2 + \epsilon^2} \\ &= -\pi\delta^{(4)}(k^2 + m^2) \\ &\equiv \text{Im}\{\text{Res}(i\Pi(k))\}_{k^2=-m^2}, \end{aligned} \quad (\text{A.22})$$

which means that the residue is negative, $\text{Im}\{\text{Res}(\Pi(k))\}_{k^2=-m^2} < 0$, therefore it is in contradiction with the optical theorem¹.

One could think that a possible way out is to restrict the Fock space only to the in- and out-states $|\alpha\rangle, |\beta\rangle$ which are physical. However, this strategy would not help since in Eq.(A.10) the completeness relation would introduce anyway the full set of states through $|n\rangle$. Therefore, in presence of ghost states one cannot prove the identity $S^\dagger S = \mathbb{1}$, but only establish a different relation which is called *pseudo-unitarity equation* and reads:

$$\langle\alpha|(-i)T|\beta\rangle + \langle\alpha|iT^\dagger|\beta\rangle = \sum_{|n\rangle\in W} \langle\alpha|T^\dagger|n\rangle(-1)^{\sigma_n}\langle n|T|\beta\rangle, \quad (\text{A.23})$$

¹Let us remark that our conclusions rely on the fact that we are using the Feynman prescription to quantize the theory, indeed we are shifting the denominator of the propagator by $-i\epsilon$. We would like to emphasize that alternative quantization prescriptions may give different results. For instance, if we consider $i\epsilon$ (Dyson prescription) instead of $-i\epsilon$, the would be ghost becomes healthy while the would be standard degree of freedom becomes unhealthy. However, generally the latter prescription violates macro-causality and does not define a physical theory. We would also like to mention that recently a new quantization prescription has been introduced in order to solve the problem of the Weyl ghost, and in general of ghosts in any higher derivative theory, where the ghost is converted in a *fake* degree of freedom; see Refs.[31, 32, 33, 30, 34, 35, 36] for more details.

where $\sigma_n = 1$ for physical states and $\sigma_n = -1$ for unphysical states. Therefore, in presence of ghost states we can prove the pseudo-unitarity equation which does *not* imply the optical theorem.

Unbounded probabilities

We will now show another implications due to the presence of ghost fields, namely of negative residues in the propagator. From the path integral formulation of the quantum field theory we know that the vacuum-vacuum transition amplitude in the presence of a source J corresponds to the generating function $Z_0[J]$, where the subscript 0 means that we are dealing with the free theory [172]:

$$\begin{aligned} Z_0[J] &= \langle 0, \infty | 0, -\infty \rangle^J \\ &= \exp \left\{ i \int d^4x \int d^4y J(x) \Pi(x-y) J(y) \right\}, \end{aligned} \quad (\text{A.24})$$

with the normalization choice $Z_0[0] = 1$.

Now, we want to recast the integrals in Eq.(A.24) in terms of its Fourier transform. Indeed, by using the relations

$$\begin{aligned} \Pi(x-y) &= \int \frac{d^4k}{(2\pi)^4} \Pi(k) e^{ik \cdot (x-y)} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{ik \cdot (x-y)}, \\ J(x) &= \int \frac{d^4k_1}{(2\pi)^4} J(k_1) e^{ik_1 \cdot x}, \\ J(y) &= \int \frac{d^4k_2}{(2\pi)^4} J(k_2) e^{ik_2 \cdot y}; \end{aligned} \quad (\text{A.25})$$

and the momentum space representation the Dirac delta in four dimensions,

$$\delta^{(4)}(k+k_1) = \int \frac{d^4x}{(2\pi)^4} e^{-i(k+k_1) \cdot x}, \quad \delta^{(4)}(k-k_2) = \int \frac{d^4y}{(2\pi)^4} e^{-i(k-k_2) \cdot x}, \quad (\text{A.26})$$

we obtain (up to 2π factors)

$$\int d^4x \int d^4y J(x) \Pi(x-y) J(y) \sim \int d^4k J(-k) \Pi(k) J(k). \quad (\text{A.27})$$

Thus, the vacuum-vacuum amplitude in the presence of a source J becomes

$$\langle 0, \infty | 0, -\infty \rangle^J \sim \exp \left\{ i \int d^4k J(-k) \Pi(k) J(k) \right\}. \quad (\text{A.28})$$

We can write the integral over k as

$$\int d^4k J(-k) \Pi(k) J(k) = \int d^3k \left[\int dk_0 J(-k) \Pi(k) J(k) \right] \quad (\text{A.29})$$

and calculate first the one dimensional integral over k_0 by using Cauchy's residue theorem:

$$\int d^4k J(-k)\Pi(k)J(k) = \int d^3k [2\pi i \text{Res} \{J(-k)\Pi(k)J(k)\}_{k^2=-m^2}]. \quad (\text{A.30})$$

Hence, Eq.(A.28)) can be recast as

$$\langle 0, \infty | 0, -\infty \rangle^J \sim \exp \left\{ i \int d^3k [2\pi \text{Res} \{J(-k)i\Pi(k)J(k)\}_{k^2=-m^2}] \right\}. \quad (\text{A.31})$$

The integrand represents the current-current amplitude in momentum space: $\mathcal{A}(k) = J(-k)i\Pi(k)J(k)$. We can immediately notice that the sign of the imaginary part of the residue is crucial ²: if it is positive we obtain a negative exponent in Eq.(A.31), while if it is negative the exponential will be positive allowing the vacuum-vacuum amplitude to be greater than 1.

The quantity $\langle 0, \infty | 0, -\infty \rangle^J$ is the transition amplitude to go from the initial $|0, -\infty\rangle$ to the final state $|0, \infty\rangle$. The probability to find the system in the initial state is given by P , which has to be less than 1, ($P < 1$); while the probability to transit to the final state is given by $1 - P$ and it has to be less than one too. Therefore, if the imaginary part of the residue in Eq.(A.31) is negative, then we obtain $1 - P > 1 \implies P < 0$, i.e. as a result we get negative probabilities which also implies unitarity violation.

A.3 Ghosts in higher derivative theories

We have already mentioned that higher derivative field theories suffer from the presence of ghost degrees of freedom. We now wish to make a more systematic study.

Let us consider a scalar field with fourth derivative Lagrangian given by

$$\mathcal{L} = \frac{1}{2} \phi (\square - m_1^2) (\square - m_2^2) \phi, \quad (\text{A.32})$$

with masses m_1 and m_2 ; let us assume $m_2 > m_1$. By defining two new fields [174]

$$\psi_1 = \frac{(\square - m_2^2) \phi}{\sqrt{(m_2^2 - m_1^2)}}, \quad \psi_2 = \frac{(\square - m_1^2) \phi}{\sqrt{(m_2^2 - m_1^2)}}, \quad (\text{A.33})$$

the Lagrangian in Eq.(A.32) can be recast as

$$\mathcal{L} = \frac{1}{2} \psi_1 (\square - m_1^2) \psi_1 - \frac{1}{2} \psi_2 (\square - m_2^2) \psi_2, \quad (\text{A.34})$$

²Note that if the integrand $2\pi i \text{Res} \{J(-k)\Pi(k)J(k)\}_{k^2=-m^2}$ is positive (negative), the integral $\int d^3k [2\pi i \text{Res} \{iJ(-k)\Pi(k)J(k)\}_{k^2=-m^2}]$ will be positive (negative) too.

where the term corresponding to the field ψ_2 has the wrong sign, i.e. it is a ghost field. Thus, we have two degrees of freedom among which one is unhealthy. We are considering a theory without interaction but, naturally, the study can be extended to Lagrangians with interaction potential $V_{\text{int}}(\phi) \equiv V_{\text{int}}(\phi_1, \phi_2)$.

We can also compute the momentum space propagator which reads:

$$\begin{aligned}\Pi(k) &= \frac{-i}{(k^2 + m_1^2)(k^2 + m_2^2)} \\ &= \frac{-i}{(m_2^2 - m_1^2)} \left(\frac{1}{k^2 + m_1^2 - i\epsilon} - \frac{1}{k^2 + m_2^2 - i\epsilon'} \right),\end{aligned}\tag{A.35}$$

where we have introduced two infinitesimal shifts ϵ and ϵ' as the Feynman prescription has to be applied to both components of the propagator. Therefore, it is clear that the higher derivative propagator can be equivalently recast as the difference of the two propagators for ψ_1 and ψ_2 , among which the latter corresponds to a negative residue, i.e. it is responsible for Hamiltonian instabilities³ and violation of unitarity. Indeed, its imaginary part is not positive definite:

$$\text{Im} \{i\Pi(k)\} = \frac{\pi}{(m_2^2 - m_1^2)} \left[\delta^{(4)}(k^2 + m_1^2) - \delta^{(4)}(k^2 + m_2^2) \right].\tag{A.36}$$

We can also consider the case in which one has only one massive field with mass m , while the other is massless. Moreover, an interesting case is when the two masses are equals $m_1 = m_2 = m$. In this case the propagator as a double pole:

$$\Pi(k) = \frac{-i}{(k^2 + m^2 - i\epsilon)^2}.\tag{A.37}$$

Also in this case we have one healthy degree of freedom and one ghost, indeed the double pole can be represented as the convergence of two simple poles with opposite residues. We can introduce a small positive parameter ε , which has to be sent to zero at the end, and define the two masses $m_1 = m$ and $m_2 = m + \varepsilon$. Thus, we can now follow the same steps as in Eqs. (A.35, A.36) and reach similar conclusions.

A.4 Fourth order gravity

In this Section we wish to consider the case of fourth order gravity and explicitly show that the presence of the massive spin-2 Weyl ghost cause instabilities and break unitarity. To do so we will compute the residue of a tree level amplitude and show that it is not positive definite.

³See also Refs.[175, 176], in which the authors show a method to avoid Hamiltonian instabilities in higher derivative field theories by introducing new variables.

The action for fourth order gravity is

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ \mathcal{R} + \frac{1}{2} (\alpha \mathcal{R}^2 + \beta \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}) \right\}. \quad (\text{A.38})$$

By linearizing around Minkowski we can compute the propagator, whose gauge independent part reads

$$i\Pi(k) = \frac{\mathcal{P}^2}{[1 + \frac{1}{4}\beta k^2] k^2} + \frac{\mathcal{P}_s^0}{[-2 - (3\alpha + \beta) k^2] k^2}, \quad (\text{A.39})$$

which, by playing with the fractions, can be also written as

$$i\Pi(k) = \frac{1}{k^2} \left(\mathcal{P}^2 - \frac{\mathcal{P}_s^0}{2} \right) + \frac{1}{2} \frac{\mathcal{P}_s^0}{k^2 + m_0^2} - \frac{\mathcal{P}^2}{k^2 + m_2^2} \quad (\text{A.40})$$

where $m_2 = 2(-\beta)^{-1/2}$ and $m_0 = (3\alpha + \beta)^{-1/2}$. The first piece,

$$i\Pi_{\text{GR}}(k) = \frac{1}{k^2} \left(\mathcal{P}^2 - \frac{\mathcal{P}_s^0}{2} \right), \quad (\text{A.41})$$

is the GR graviton propagator corresponding to the Hilbert-Einstein linearized action, while the second and the third terms in correspond to a healthy massive scalar degree of freedom with mass m_0 and to a massive spin-2 ghost with mass m_2 , respectively.

Let us consider the amplitude

$$\mathcal{A} = T^{*\mu\nu}(k) \Pi_{\mu\nu\rho\sigma}(k) T^{\rho\sigma}(k) = \mathcal{A}_{\text{GR}} + \mathcal{A}_2 + \mathcal{A}_0, \quad (\text{A.42})$$

where

$$\begin{aligned} \mathcal{A}_{\text{GR}} &= T^{*\mu\nu} \Pi_{\text{GR}}(k)_{\mu\nu\rho\sigma}(k) T^{\rho\sigma}(k), \\ \mathcal{A}_2 &= -T^{*\mu\nu}(k) \frac{(-i)\mathcal{P}_{\mu\nu\rho\sigma}^2}{k^2 + m_2^2} T^{\rho\sigma}(k), \\ \mathcal{A}_0 &= T^{*\mu\nu}(k) \frac{1}{2} \frac{(-i)\mathcal{P}_{s,\mu\nu\rho\sigma}^0}{k^2 + m_0^2} T^{\rho\sigma}(k). \end{aligned} \quad (\text{A.43})$$

We need to calculate the imaginary part of the residue of the full amplitude in (A.42), that corresponds to the sum of the residue of the three amplitudes in (A.43). By using the definition of the spin projector operators $\mathcal{P}^2$ and $\mathcal{P}_s^0$ one can show that

$$\begin{aligned} \text{Im} \{ \text{Res}_{k^2=0} \{ \mathcal{A}_{\text{GR}} \} \} &= |T^{\mu\nu}(0)|^2 - \frac{1}{2} |T(0)|^2, \\ \text{Im} \{ \text{Res}_{k^2=-m_2^2} \{ \mathcal{A}_2 \} \} &= - \left(|T^{\mu\nu}(m_2)|^2 - \frac{1}{3} |T(m_2)|^2 \right), \\ \text{Im} \{ \text{Res}_{k^2=-m_0^2} \{ \mathcal{A}_0 \} \} &= \frac{1}{6} |T(m_0)|^2. \end{aligned} \quad (\text{A.44})$$

Let us now expand the source $T(k)$ as [177, 141]

$$\begin{aligned} T_{\mu\nu}(k) = & a(k)k_\mu k_\nu + b(k)k_{(\mu}\tilde{k}_{\nu)} + c_i(k)k_{(\mu}\varepsilon_{\nu)}^i \\ & + d(k)\tilde{k}_\mu\tilde{k}_\nu + e_i(k)\tilde{k}_{(\mu}\varepsilon_{\nu)}^i + f_{ij}(k)\varepsilon_{(\mu}^i\varepsilon_{\nu)}^j, \end{aligned} \quad (\text{A.45})$$

where the expansion basis is $\{k^\mu, \tilde{k}^\mu, \varepsilon_1^\mu, \varepsilon_2^\mu\}$, such that

$$\begin{aligned} k^\mu &\equiv (k^0, \vec{k}), \quad \tilde{k}^\mu \equiv (\tilde{k}^0, -\vec{k}), \quad \varepsilon_i^\mu \equiv (0, \vec{\varepsilon}_i), \\ k^\mu \varepsilon_{i,\mu} &= 0 = \tilde{k}^\mu \varepsilon_{i,\mu}, \quad \varepsilon_i^\mu \varepsilon_{j,\mu} = -\vec{\varepsilon}_i \cdot \vec{\varepsilon}_j = -\delta_{ij}, \end{aligned} \quad i = 1, 2. \quad (\text{A.46})$$

Hence, we are now able to show that

$$\begin{aligned} \text{Im} \{ \text{Res}_{k^2=0} \{ \mathcal{A}_{\text{GR}} \} \} &= |f_{ij}(0)|^2 - \frac{1}{2}|f(0)|^2 \\ &= \frac{1}{2}|f_{11}(0) - f_{22}(0)|^2 + 2|f_{12}(0)|^2 > 0, \\ \text{Im} \{ \text{Res}_{k^2=-m_2^2} \{ \mathcal{A}_2 \} \} &= - \left[\frac{2}{3} (a(m_2) - d(m_2))^2 m_2^4 \right. \\ &\quad + \frac{m_2^2}{2} (|c_i(m_2)|^2 - |e_i(m_2)|^2) \\ &\quad + |f_{ij}(m_2)|^2 - \frac{1}{2}|f(m_2)|^2 \\ &\quad \left. - \frac{2}{3} (a(m_2) - d(m_2)) m_2^2 f_{ii}(m_2) \right], \\ \text{Im} \{ \text{Res}_{k^2=-m_0^2} \{ \mathcal{A}_0 \} \} &= \frac{1}{6} \left[(a(m_0) - d(m_0))^2 m_0^4 + |f_{ii}(m_0)|^2 \right. \\ &\quad \left. - 2(d(m_0) - a(m_0)) m_0^2 f_{ii}(m_0) \right] > 0, \end{aligned} \quad (\text{A.47})$$

where we have defined $f = f_{ii}$, $|f|^2 = f_{ii}^* f_{jj}$ and $|f_{ij}|^2 = \sum_{ij} f_{ij}^* f_{ij}$. From a physical point of view we consider $T > 0$, which implies $f_{ii} < 0$ [141]. Since the source $T_{\mu\nu}(k)$ is arbitrary, its Fourier modes (i.e. the coefficients $a(k)$, $b(k), \dots$) can be freely chosen and we can make choices such that only one of the three residues contributes at a time. Then, we notice that the massless and the massive scalar poles (first and second lines in (A.47)) are well defined physical state. While, if we choose the source $T_{\mu\nu}(k)$ such that only the pole m_2^2 contributes one can see that the second line in Eq.(A.47) is not positive defined. For example, if $|c_i(m_2)|^2 - |e_i(m_2)|^2 > 0$ and $a(m_2) - d(m_2) > 0$ we get

$$\text{Im} \{ \text{Res}_{k^2=-m_2^2} \{ \mathcal{A}_2 \} \} < 0 \quad (\text{A.48})$$

which violates the unitarity condition at tree level. We could imagine to make special choices for the coefficients in the source expansion (A.45) to obtain a

positive value for the sum of the three residues and so a ghost-free theory⁴. However, in this way we would restrict $T_{\mu\nu}(k)$ by hand to get a ghost-free sum and it does not mean that the theory is healthy because interactions can always generate the $T_{\mu\nu}(k)$ -configurations that have not been considered.

Hence, we have shown that the Weyl ghost in fourth order gravity is a bad ghost, indeed it does not satisfy the optical theorem.

⁴Keep in mind that with the nomenclature "ghost-free" we refer to a theory free from "bad" ghosts.

Acausal Green function computation

We now want to show the computation that leads to the expression in Eq.(1.58) for the acausal Green function in infinite derivative field theory. In particular, we want to compute the principal-value integral in Eq. (1.56) that we recall for clarity:

$$I_{PV} = \frac{1}{16i\pi^3} \frac{1}{r} \int_{-\infty}^{\infty} k dk \text{ P.V.} \int_{-\infty}^{\infty} dk^0 \frac{e^{-\left(\frac{-k_0^2 + k^2}{M_s^2}\right)^{2n}}}{k_0^2 - k^2} \left(e^{i(kr - k^0 t)} - e^{-i(kr + k^0 t)} \right), \quad (\text{B.1})$$

where, let us remind that $k \equiv |\vec{k}|$ and $\omega_{\vec{k}} = k$, as we are working with the massless case.

Since we are interested in the modification of the local retarded Green function we will consider the case $t > 0$ ¹. To compute I_{PV} we need to consider several cases corresponding to different regions of the planes t - r and k^0 - k . As for the plane t - r we have to distinguish ²:

1. $t > 0, t^2 > r^2 \iff (x - y)^2 < 0$ (time-like separation):

$$t = \rho^{1/2} \cosh^2 \alpha, \quad r = \rho^{1/2} \sinh^2 \alpha, \quad t^2 - r^2 = \rho > 0;$$

2. $t > 0, t^2 < r^2 \iff (x - y)^2 > 0$ (space-like separation):

$$t = \rho^{1/2} \cosh^2 \alpha, \quad r = \rho^{1/2} \sinh^2 \alpha, \quad t^2 - r^2 = \rho < 0.$$

¹If we considered the case $t < 0$ we would study the modification of the advanced Green function.

²In Ref.[47] the authors consider the same calculation for the case $2n = 2$.

Instead, as for the plane k^0 - k we will split the double integral in Eq. (B.1) in the following two regions:

(i) $k_0^2 > k^2$:

$$k = R \sinh \beta, \quad k^0 = R \cosh \beta, \quad R^2 = k_0^2 - k^2 > 0, \quad -\infty < \beta, R < \infty;$$

(ii) $k_0^2 < k^2$:

$$k = R \cosh \beta, \quad k^0 = R \sinh \beta, \quad -R^2 = k_0^2 - k^2 < 0, \quad -\infty < \beta, R < \infty.$$

By moving to the new integration variables R, β we get a Jacobian factor $|R|$ so that the integral in Eq.(B.1) in the case 1. ($\rho > 0$) reads

$$\begin{aligned} I_{PV} &= \frac{i}{16\pi^3 r} \int_{-\infty}^{\infty} dR d\beta e^{-R^{4n}/M_s^{4n}} \frac{|R|}{R} \left\{ \sinh \beta \left[e^{-iR\rho^{1/2} \cosh(\beta-\alpha)} - e^{-iR\rho^{1/2} \cosh(\beta+\alpha)} \right] \right. \\ &\quad \left. - \cosh \beta \left[e^{iR\rho^{1/2} \sinh(\beta-\alpha)} - e^{-iR\rho^{1/2} \sinh(\beta+\alpha)} \right] \right\} \\ &= -\frac{1}{8\pi^3 r} \int_{-\infty}^{\infty} dR d\beta e^{-R^{4n}/M_s^{4n}} \frac{|R|}{R} \left\{ \sinh \beta \sin \left(R\rho^{1/2} \cosh(\beta+\alpha) \right) \right. \\ &\quad \left. - \cosh \beta \sin \left(R\rho^{1/2} \sinh(\beta+\alpha) \right) \right\}. \end{aligned} \tag{B.2}$$

Defining the new integration variable $\theta = \beta + \alpha$, the integral in Eq.(B.2) becomes:

$$\begin{aligned} I_{PV} &= \frac{1}{8\pi^3 \rho^{1/2}} \int_{-\infty}^{\infty} dR d\theta e^{-R^{4n}/M_s^{4n}} \frac{|R|}{R} \left\{ \cosh \theta \sin \left(R\rho^{1/2} \cosh \theta \right) \right. \\ &\quad \left. - \sinh \theta \sin \left(R\rho^{1/2} \sinh \theta \right) \right\} \\ &= -\frac{1}{\pi^3} \frac{\partial}{\partial \rho} \int_0^{\infty} \frac{dR}{R} e^{-R^{4n}/M_s^{4n}} \int_0^{\infty} d\theta \left\{ \cos \left(R\rho^{1/2} \cosh \theta \right) - \cos \left(R\rho^{1/2} \sinh \theta \right) \right\}. \end{aligned} \tag{B.3}$$

The integrals in θ -variable can be expressed in terms of Bessel functions [178]:

$$\int_0^{\infty} d\theta \cos \left(R\rho^{1/2} \sinh \theta \right) = K_0(R\rho^{1/2}), \quad \int_0^{\infty} d\theta \cos \left(R\rho^{1/2} \cosh \theta \right) = -\frac{\pi}{2} Y_0(R\rho^{1/2}); \tag{B.4}$$

then, by introducing the dimensionless variable $\zeta = R\rho^{1/2}$, the principal-value integral in Eq.(B.3) becomes

$$I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \int_0^\infty \frac{d\zeta}{\zeta} e^{-\frac{\zeta^{4n}}{M_s^{4n} \rho^{2n}}} \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right]. \quad (\text{B.5})$$

The last result holds for the case 1. when $\rho > 0$, but we can also take into account the case 2., when $\rho < 0$, by considering the following expression:

$$I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \left\{ \varepsilon(\rho) \int_0^\infty \frac{d\zeta}{\zeta} e^{-\frac{\zeta^{4n}}{M_s^{4n} \rho^{2n}}} \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\}, \quad (\text{B.6})$$

where the function $\varepsilon(\rho)$ is equal to +1 if $\rho > 0$ (time-like separation), while it is -1 if $\rho < 0$ (space-like separation). The result in Eq.(B.6) corresponds to the integral in Eq.(1.57).

The integral in Eq.(B.6) can be computed analytically for $2n = 2$ and can be expressed in terms of the Meijer-G functions [123]; indeed for space-like separation ($\rho < 0$) one has

$$\begin{aligned} I_{PV} &= \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \left\{ \int_0^\infty \frac{d\zeta}{\zeta} e^{-\frac{\zeta^4}{M_s^4 \rho^2}} \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\} \\ &= \frac{2}{\pi^3 M_s^4 \rho^3} \left\{ \int_0^\infty d\zeta e^{-\frac{\zeta^4}{M_s^4 \rho^2}} \zeta^3 \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\} \\ &= \frac{1}{2\pi^4} \frac{1}{\rho} \left\{ G_{2,5}^{4,1} \left(\begin{matrix} 0 \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| \frac{M_s^4 \rho^2}{256} \right) \right. \\ &\quad \left. + 2\pi^2 G_{3,6}^{4,1} \left(\begin{matrix} 0, -\frac{1}{4}, \frac{1}{4} \\ 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4} \end{matrix} \middle| \frac{M_s^4 \rho^2}{256} \right) \right\}, \end{aligned} \quad (\text{B.7})$$

which explains the expression in Eq.(1.58) for the acausal retarded Green function G_R .

Spin projector operators decomposition

C.1 Tensor decomposition

In this appendix we introduce the formalism of the *spin projector operators* which we used in Chapter 2 to compute the graviton propagator around the Minkowski background. First, we will study the tensor representations of the Lorentz Group, especially the irreducible tensor representations under $SO(3)$; then we will construct the projector operators through which we can decompose four-vector and two-rank tensors. As for the tensor representation we will closely follow the treatment in Ref.[179], while for the spin projector operators Refs.[143, 142] and especially Ref.[141].

C.1.1 Lorentz tensor representation

Let us consider a two-rank tensor $\varphi^{\mu\nu}$ with two contravariant indices in Minkowski space. By definition $\varphi^{\mu\nu}$ is an object that under Lorentz transformations transforms as

$$\varphi'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \varphi^{\rho\sigma}. \quad (\text{C.1})$$

Tensors are examples of representations of the Lorentz group. For instance a generic two-rank tensor $\varphi^{\mu\nu}$ has 16 components and (C.1) shows that these components transform among themselves, i.e. they form a basis for a 16-dimensional representation of the Lorentz group.

In Group Theory the irreducible representations of any group turn out to be very important; for example, they are very useful when we want to decompose a tensor object in its several spin components. We can notice that the 16-dimensional representation, we have just introduced, is reducible in different

irreducible parts. First of all we easily understand that if φ is symmetric (antisymmetric) then also φ' will be symmetric (antisymmetric), so the symmetric and antisymmetric parts of a tensor $\varphi^{\mu\nu}$ do not mix, and the 16-dimensional representation is for sure reducible into a 6-dimensional antisymmetric representation $\psi^{\mu\nu}$ and a 10-dimensional symmetric representation $h^{\mu\nu}$. One can explicitly see this decomposition in symmetric and antisymmetric parts in the following way:

$$\varphi^{\mu\nu} = h^{\mu\nu} + \psi^{\mu\nu}, \quad \begin{cases} h^{\mu\nu} := \frac{1}{2}(\varphi^{\mu\nu} + \varphi^{\nu\mu}) \\ \psi^{\mu\nu} := \frac{1}{2}(\varphi^{\mu\nu} - \varphi^{\nu\mu}) \end{cases}. \quad (\text{C.2})$$

Furthermore, the trace of a symmetric tensor can be also isolated. Indeed, it is invariant under Lorentz transformation:

$$h' = \eta_{\mu\nu} h'^{\mu\nu} = \eta_{\mu\nu} \Lambda_\rho^\mu \Lambda_\sigma^\nu h^{\rho\sigma} = \eta_{\rho\sigma} h^{\rho\sigma} = h; \quad (\text{C.3})$$

so a traceless tensor remains traceless after a Lorentz transformation, and thus the 10-dimensional symmetric representation decomposes into a 9-dimensional irreducible symmetric traceless representation and a 1-dimensional scalar representation. In formula this means that

$$h^{T\mu\nu} := h^{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} h, \quad h = \eta_{\mu\nu} h^{\mu\nu}, \quad (\text{C.4})$$

where the apex "T" means "traceless", in fact $\eta_{\mu\nu} h^{T\mu\nu} = h - \frac{4}{4}h = 0$.

In representation theory the following notation is commonly used: an irreducible representation is denoted by its dimensionality (the number of components), written in boldface. Thus the scalar representation is denoted as **1**, the four-vector representation as **4**, the antisymmetric representation as **6** and the traceless symmetric representation as **9**.

The tensor representation (C.1) sees the action of two Lorentz matrices. It means that the representation (C.1) is a tensor product of two four-vector representations, namely each of two contravariant indices of $\varphi^{\mu\nu}$ transforms separately as a four-vector index. The tensor product of two representation is denoted by the symbol $\otimes$. Since we have found that a two-rank tensor can be decomposed into the direct sum of three irreducible representations, denoting the direct sum by $\oplus$, we can express the irreducible representation in terms of the dimensionality introduced above:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}. \quad (\text{C.5})$$

Analogously one can obtain the tensor decomposition into irreducible parts when more than two indices are present. The most general irreducible representation of the Lorentz group are found starting from a generic tensor with an

arbitrary number of indices, removing first all traces, and then symmetrizing or antisymmetrizing over all pairs of indices. Note also that, by raising and lowering the indices with the Minkowski metric tensor $\eta_{\mu\nu}$, we can always restrict to contravariant tensors. Thus, for instance, V^μ and V_μ are equivalent representations.

All tensor representations are in a sense derived from the four-vector representation, since the transformation law of a tensor is obtained by applying separately on each Lorentz index the matrix Λ_ν^μ that defines the transformation law of a four-vector. This means that tensor representations are tensor products of four-vector representations and for this reason, the four-vectors play a fundamental role.

C.1.2 Decomposition of Lorentz tensors under $SO(3)$

We know how a tensor behaves under a generic Lorentz transformation. Now, we are going to focus particularly on the transformation properties of a tensor under the $SO(3)$ rotation subgroup, and we can therefore ask what is the angular momentum j of the various tensor representations. We will be able to decompose a generic two-rank tensor in terms of its spin components.

Let us recall that the representations of $SO(3)$ are labeled by an index j which assumes integer values $j = 0, 1, 2, \dots$; while the dimension of the representation, labeled by j , is defined by $2j + 1$. Then within each representation, there are $2j + 1$ states labeled by $j_z = -j, \dots, j$. Note that for $SO(3)$ it is more common to denote the representation as $\mathbf{j}$, i.e. to label it with the associated angular momentum rather than with the dimension of the representation, $2j + 1$. Hence in this notation, $\mathbf{0}$ is the scalar (singlet, spin-0), $\mathbf{1}$ is a triplet (spin-1) with components $j_z = -1, 0, 1$, while $\mathbf{2}$ is a representation of dimension 5 (spin-2), and so on with higher dimensionality.

A Lorentz scalar is of course also scalar under rotations, so it has $j = 0$. A four-vector $V^\mu = (V_0, \vec{V})$ is an irreducible representation of the Lorentz group, since a generic Lorentz transformation mixes all four components, but under $SO(3)$ it is reducible. Indeed, spatial rotations do not mix V^0 with $\vec{V}$: V^0 is invariant under spatial rotations, so it has $j = 0$, while the three spatial components V^i form an irreducible 3-dimensional representations of $SO(3)$, with $j = 1$.

By adopting the above convention according to which the representations are indicated by the associated angular momentum, the decomposition of a four-vector into the direct sum of a scalar and a spin-1 representation under $SO(3)$ can be written as

$$V^\mu \in \mathbf{0} \oplus \mathbf{1}. \quad (\text{C.6})$$

While in terms of their dimensions we should write

$$V^\mu \in \mathbf{4} = \mathbf{1} \oplus \mathbf{3}. \quad (\text{C.7})$$

Now we would like to understand which are the spin components of a two-rank tensor $\varphi^{\mu\nu}$, i.e. what angular momenta appear. By definition we know that $\varphi^{\mu\nu}$ transforms as the tensor product of two four-vector representations. Since, from a point of view of $SO(3)$, a four-vector decomposes as $\mathbf{0} \oplus \mathbf{1}$, a generic two-rank tensor has the following decomposition in angular momenta

$$\begin{aligned}\varphi^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}).\end{aligned}\quad (\text{C.8})$$

In (C.8) we have used the usual rule to compose angular momenta, according to which the composition of two angular momentum j_1 and j_2 is given by all angular momentum between $|j_1 - j_2|$ and $j_1 + j_2$:

$$\mathbf{0} \otimes \mathbf{0} = \mathbf{0}, \quad \mathbf{0} \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{0} = \mathbf{1}, \quad \mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}. \quad (\text{C.9})$$

Thus, under the rotation group $SO(3)$, $\varphi^{\mu\nu}$ decomposes as two spin-0 representations, three spin-1 representations and one spin-2 representation. It would be interesting to see how these representations are shared between the symmetric traceless, the trace and the antisymmetric part of the tensor $\varphi^{\mu\nu}$, since these are irreducible Lorentz representations. So, let us see how these two different irreducible decompositions match to each other. The trace is a Lorentz scalar, so it is in particular scalar under $SO(3)$ and therefore is a $\mathbf{0}$ representation. An antisymmetric tensor $\psi^{\mu\nu}$ has six components, which can be written as the direct sum of the two three-vectors ψ^{0i} and $\frac{1}{2}\epsilon^{ijk}\psi^{jk}$. These are two spatial vectors (two triplets) and so¹

$$\psi^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}. \quad (\text{C.10})$$

Since we have identified the trace h with $\mathbf{0}$ and $\psi^{\mu\nu}$ with $\mathbf{1} \oplus \mathbf{1}$, by comparing (C.5) and (C.8) we can see that the nine components of a symmetric traceless tensor $h^{T\mu\nu}$ decompose, under the subgroup $SO(3)$, as

$$h^{T\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}. \quad (\text{C.11})$$

We have seen that a generic two-rank tensor can be written as a tensor product of two four-vectors. So let us observe that when we write $\varphi^{\mu\nu}$ as $(\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1})$, the first $\mathbf{0}$ corresponds to taking the index $\mu = 0$, the first $\mathbf{1}$ corresponds to taking the index $\mu = i$, and similarly for the second factor $(\mathbf{0} \oplus \mathbf{1})$ and the index ν . Therefore in equation (C.8) we have the following correspondence:

$$\mathbf{0} \otimes \mathbf{0} \rightarrow \varphi^{00}, \quad \mathbf{0} \otimes \mathbf{1} \rightarrow \varphi^{0i}, \quad \mathbf{1} \otimes \mathbf{0} \rightarrow \varphi^{i0}, \quad \mathbf{1} \otimes \mathbf{1} \rightarrow \varphi^{ij}.$$

¹In Electrodynamics one has an important example of antisymmetric tensor, i.e the tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this case the two vectors are $E^i = -F^{0i}$ and $B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}$, i.e. the electric and magnetic fields.

It is clear that, under spatial rotations $SO(3)$, φ^{00} behaves like a scalar, while φ^{0i} and φ^{i0} like spatial vectors. As for the spatial components φ^{ij} , its antisymmetric part $\psi^{ij} = \frac{1}{2}(\varphi^{ij} - \varphi^{ji})$ has only three independent component and so it turns out to be a spatial vector, giving a 3-dimensional representation **1**; while its symmetric part can be separated into its trace, which gives the second **0** representation, and the traceless symmetric part, which must have² $j = 2$.

In general, a symmetric tensor with N indices contains angular momentum up to $j = N$. In four dimensions, higher antisymmetric tensors with four indices, $\varphi^{\mu\nu\rho\sigma}$, has only one independent component φ^{0123} , so it must be a Lorentz scalar. An antisymmetric tensor with three indices, $\varphi^{\mu\nu\rho}$, has $\frac{4 \times 3 \times 2}{3!} = 4$ components and it has the same transformation properties of a four-vector.

C.2 Spin projector operators

So far we have seen how to decompose a generic two-rank tensor (or more generally a N -rank tensor) into scalar, vector and tensor components. At this point one question that we can ask could be: can we define a complete set of projector operators through which the tensor $\varphi^{\mu\nu}$ can be projected along its scalar, vector and tensor components? The answer is *yes* as we will now show.

Furthermore, we are going to introduce a basis in the space of four-rank tensor operators $\mathcal{O}^{\mu\nu\rho\sigma}$ appearing in a given parity-invariant Lagrangian

$$\mathcal{L} = \frac{1}{2} \varphi_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma}, \quad (\text{C.12})$$

whose field equations in presence of a source $J^{\mu\nu}$ read

$$\mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma} = \lambda J^{\mu\nu}, \quad (\text{C.13})$$

where λ a coupling constant.

Note that in the case of the quadratic theories of gravity studied in the Chapter 2 we have deal with the symmetric tensor field $\varphi_{\mu\nu} \rightarrow h_{\mu\nu}$ and the symmetric stress-energy tensor $J_{\mu\nu} \rightarrow T_{\mu\nu}$, while the coupling constant is given by $\lambda \rightarrow \kappa$ and the operator $\mathcal{O}^{\mu\nu\rho\sigma}$ is symmetric in all its four indexes $\mathcal{O}^{\mu\nu\rho\sigma} = \mathcal{O}^{\nu\mu\rho\sigma} = \mathcal{O}^{\mu\nu\sigma\rho} = \mathcal{O}^{\nu\mu\sigma\rho}$.

Let us start by discussing how decompose a four-vector and subsequently we will generalize to two-rank tensors.

²In General Relativity an important example is given by the physical graviton. It can be described by a traceless symmetric spatial tensor (transverse to the propagation direction) corresponding to spin-2.

C.2.1 Four-vector decomposition

A generic four-vector V^μ can be decomposed along its transverse and longitudinal components, under the $SO(3)$ representation of the Lorentz group $V^\mu \in \mathbf{0} \oplus \mathbf{1}$, in terms of a set of two projectors, $\{\theta, \omega\}$:

$$V_\mu = \theta_{\mu\nu} V^\nu + \omega_{\mu\nu} V^\nu, \quad (\text{C.14})$$

where the operators $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are defined as

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square}, \quad \omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\square} \quad (\text{C.15})$$

or in momentum space

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \quad (\text{C.16})$$

It is easy to show that the following properties hold³

$$\theta_{\mu\nu} + \omega_{\mu\nu} = \eta_{\mu\nu}, \quad \theta_{\mu\nu} \theta_\rho^\nu = \theta_{\mu\rho}, \quad \omega_{\mu\nu} \omega_\rho^\nu = \omega_{\mu\rho}, \quad \theta_{\mu\nu} \omega_\rho^\nu = 0, \quad (\text{C.17})$$

which means that the set $\{\theta, \omega\}$ is complete and orthogonal. Note also that θ and ω project along the transverse and longitudinal components, respectively:

$$k^\mu \theta_{\mu\nu} = 0, \quad k^\mu \omega_{\mu\nu} = k_\nu. \quad (\text{C.18})$$

Furthermore, we notice that the transverse component has spin-1 and the longitudinal one spin-0 by calculating the trace of the two projectors:

$$\begin{aligned} \eta^{\mu\nu} \theta_{\mu\nu} &= 3 = 2(1) + 1 && (\text{spin-1}), \\ \eta^{\mu\nu} \omega_{\mu\nu} &= 1 = 2(0) + 1 && (\text{spin-0}). \end{aligned} \quad (\text{C.19})$$

C.2.2 Two-rank tensor decomposition

Let us now to implement an analogue decomposition in the case of a two-rank tensor. We know that a two-rank tensor can be seen as a tensor product of two four-vectors, so we can find the projector operators for $\varphi^{\mu\nu}$ by decomposing each index separately:

$$\varphi^{\mu\nu} \equiv V^\mu \otimes U^\nu. \quad (\text{C.20})$$

Moreover, we know that we can decompose $\varphi^{\mu\nu}$ in its symmetric and antisymmetric parts as done in (C.2), so one can study the two sectors separately.

³As we have already said more times, we shall often write the projectors suppressing the indices.

Symmetric decomposition

Let us start with the symmetric part: $h_{\mu\nu} \in \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$. By treating $h_{\mu\nu}$ as a symmetric tensor product of two four-vectors, the decomposition can be performed as follow:

$$\begin{aligned}
 h_{\mu\nu} &= (\theta_{\mu\rho} + \omega_{\mu\rho}) (\theta_{\nu\sigma} + \omega_{\nu\sigma}) h^{\rho\sigma} \\
 &= (\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\rho}\omega_{\nu\sigma} + \omega_{\mu\rho}\theta_{\nu\sigma} + \omega_{\mu\rho}\omega_{\nu\sigma}) h^{\rho\sigma} \\
 &= \frac{1}{2} (\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) h^{\rho\sigma} - \frac{1}{3} \theta_{\mu\nu}\theta_{\rho\sigma} h^{\rho\sigma} \\
 &\quad + \frac{1}{3} \theta_{\mu\nu}\theta_{\rho\sigma} h^{\rho\sigma} + \omega_{\mu\nu}\omega_{\rho\sigma} h^{\rho\sigma} \\
 &\quad + \frac{1}{2} (\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}) h^{\rho\sigma}.
 \end{aligned} \tag{C.21}$$

We can now define the following projectors:⁴

$$\begin{aligned}
 \mathcal{P}_{\mu\nu\rho\sigma}^2 &= \frac{1}{2} (\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3} \theta_{\mu\nu}\theta_{\rho\sigma}, \\
 \mathcal{P}_{m,\mu\nu\rho\sigma}^1 &= \frac{1}{2} (\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \\
 \mathcal{P}_{s,\mu\nu\rho\sigma}^0 &= \frac{1}{3} \theta_{\mu\nu}\theta_{\rho\sigma}, \quad \mathcal{P}_{w,\mu\nu\rho\sigma}^0 = \omega_{\mu\nu}\omega_{\rho\sigma}.
 \end{aligned} \tag{C.22}$$

The set

$$\mathcal{O}^i \equiv \{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0\}, \quad i = 1, 2, 3, 4, \tag{C.23}$$

forms a complete set of spin projector operators in terms of which a symmetric two-rank tensor can be decomposed. In fact, one can easily verify that

$$\mathcal{O}^i \mathcal{O}^j = \delta_{ij} \mathcal{O}^i, \quad \mathcal{O}^1 + \mathcal{O}^2 + \mathcal{O}^3 + \mathcal{O}^4 = \mathbb{1}, \tag{C.24}$$

or in terms of $\mathcal{P}$ s

$$\mathcal{P}_a^i \mathcal{P}_b^j = \delta_{ij} \delta_{ab} \mathcal{P}_a^i, \quad \mathcal{P}^2 + \mathcal{P}_m^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0 = \mathbb{1}. \tag{C.25}$$

Note that we are suppressing the indexes for simplicity, but we mean we should write $\mathcal{O}_{\mu\nu\alpha\beta}^i \mathcal{O}_{\alpha\beta\rho\sigma}^j = \delta_{ij} \mathcal{O}_{\mu\nu\rho\sigma}^i$ or $\mathcal{P}_{a,\mu\nu\alpha\beta}^i \mathcal{P}_{b,\alpha\beta\rho\sigma}^j = \delta_{ij} \delta_{ab} \mathcal{P}_{a,\mu\nu\rho\sigma}^i$. The second property of (C.24) (or (C.25)) has been already showed when we constructed

⁴We are labeling the spin-1 projector operator also with the apex m to distinguish it from the spin-1 of the anti-symmetric sector. However, when working only with the symmetric part no ambiguity can arise and we can just use $\mathcal{P}^1$, as it has been done in Chapter 2.

and defined the set of operators in (C.21), but we can also show it explicitly:

$$\begin{aligned}
(\mathcal{P}^2 + \mathcal{P}_m^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0)_{\mu\nu\rho\sigma} &= \frac{1}{2} (\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) + \omega_{\mu\nu}\omega_{\rho\sigma} \\
&\quad + \frac{1}{2} (\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}) \\
&= \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}) + \frac{1}{2} \eta_{\nu\rho}\omega_{\mu\sigma} + \frac{1}{2} \eta_{\nu\sigma}\omega_{\mu\rho} \\
&\quad - \frac{1}{2} \eta_{\nu\sigma}\omega_{\mu\rho} - \frac{1}{2} \eta_{\nu\rho}\omega_{\mu\sigma} \\
&= \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}) = \mathbb{1}_{\mu\nu\rho\sigma}.
\end{aligned} \tag{C.26}$$

Hence, we have found a complete set of projector operators to decompose $h^{\mu\nu}$:

$$\begin{aligned}
h_{\mu\nu} &= \mathcal{P}_{\mu\nu\rho\sigma}^2 h^{\rho\sigma} + \mathcal{P}_{m,\mu\nu\rho\sigma}^1 h^{\rho\sigma} + \mathcal{P}_{s,\mu\nu\rho\sigma}^0 h^{\rho\sigma} + \mathcal{P}_{w,\mu\nu\rho\sigma}^0 h^{\rho\sigma} \\
&= (\mathcal{P}^2 + \mathcal{P}_m^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0)_{\mu\nu\rho\sigma} h^{\rho\sigma}.
\end{aligned} \tag{C.27}$$

Note that in order to form a basis in the space of symmetric four-rank tensors, the four operators we have introduced in Eq.(C.23) are not enough. In fact, the full basis has to contain five independent elements so that one can expand any of the following five independent operators:

$$\begin{aligned}
&k_\mu k_\nu k_\rho k_\sigma, \quad k_\mu k_\nu \eta_{\rho\sigma}, \quad k_\mu k_\rho \eta_{\nu\sigma} \\
&\eta_{\mu\nu} \eta_{\rho\sigma}, \quad \eta_{\nu\sigma} \eta_{\mu\sigma}.
\end{aligned} \tag{C.28}$$

Therefore, we need to introduce one more operator which is defined as:

$$\mathcal{P}_{\mu\nu\rho\sigma}^\times = \mathcal{P}_{sw,\mu\nu\rho\sigma}^0 + \mathcal{P}_{ws,\mu\nu\rho\sigma}^0, \tag{C.29}$$

where

$$\mathcal{P}_{sw,\mu\nu\rho\sigma}^0 = \frac{1}{\sqrt{3}} \theta_{\mu\nu} \omega_{\rho\sigma}, \quad \mathcal{P}_{ws,\mu\nu\rho\sigma}^0 = \frac{1}{\sqrt{3}} \omega_{\mu\nu} \theta_{\rho\sigma}. \tag{C.30}$$

are two operators that mix the two scalar components s and w ⁵.

The orthogonality relations in (C.25) can be extended to the operators $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{ws}^0$, so that we obtain (when $a \neq b$ and $c \neq d$)

$$\begin{aligned}
\mathcal{P}_a^i \mathcal{P}_b^j &= \delta_{ij} \delta_{ab} \mathcal{P}_a^j, & \mathcal{P}_{ab}^0 \mathcal{P}_c^i &= \delta_{i0} \delta_{bc} \mathcal{P}_{ab}^i, \\
\mathcal{P}_c^i \mathcal{P}_{ab}^0 &= \delta_{i0} \delta_{ac} \mathcal{P}_{ab}^0, & \mathcal{P}_{ab}^0 \mathcal{P}_{cd}^0 &= \delta_{ad} \delta_{bc} \mathcal{P}_{ab}^0,
\end{aligned} \tag{C.31}$$

⁵Note that $\mathcal{P}_{sw}^0$ and $\mathcal{P}_{sd}^0$ are not projectors, as it becomes very clear by looking at the orthogonality relations in Eq.(C.31). Often we will make an abuse of nomenclature by referring to them as projectors.

where $i, j = 2, 1, 0$ and $a, b, c, d = m, s, w$, absent⁶. Hence, the set

$$\{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0, \mathcal{P}_{sw}^0, \mathcal{P}_{ws}^0\} \quad (\text{C.32})$$

forms a basis of symmetric four-rank tensors. We can also verify that $\mathcal{P}^2$ is traceless and transverse, in fact:

$$\begin{aligned} \eta^{\mu\nu} \mathcal{P}_{\mu\nu\rho\sigma}^2 h^{\rho\sigma} &= \left[\frac{1}{2} (\eta^{\mu\nu} \theta_{\mu\rho} \theta_{\nu\sigma} + \eta^{\mu\nu} \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{3} \eta^{\mu\nu} \theta_{\mu\nu} \theta_{\rho\sigma} \right] h^{\rho\sigma} \\ &= \left[\frac{1}{2} (\theta_\rho^\nu \theta_{\nu\sigma} + \theta_\sigma^\nu \theta_{\nu\rho}) - \frac{1}{3} (4-1) \theta_{\rho\sigma} \right] h^{\rho\sigma} \\ &= \left[\frac{1}{2} (\theta_{\rho\sigma} + \theta_{\sigma\rho}) - \theta_{\rho\sigma} \right] h^{\rho\sigma} = 0 \end{aligned} \quad (\text{C.33})$$

Moreover,

$$k^\mu \mathcal{P}_{\mu\nu\rho\sigma}^2 h^{\rho\sigma} = \left[\frac{1}{2} (k^\mu \theta_{\mu\rho} \theta_{\nu\sigma} + k^\mu \theta_{\mu\sigma} \theta_{\nu\rho}) - \frac{1}{3} k^\mu \theta_{\mu\nu} \theta_{\rho\sigma} \right] h^{\rho\sigma} = 0, \quad (\text{C.34})$$

where we have used $k^\mu \theta_{\mu\rho} = k^\mu \theta_{\mu\sigma} = k^\mu \theta_{\mu\nu} = 0$.

Antisymmetric decomposition

Let us now work on the antisymmetric part $\psi^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$. By proceeding as done for the symmetric part, we get:

$$\begin{aligned} \psi_{\mu\nu} &= (\theta_{\mu\rho} + \omega_{\mu\rho}) (\theta_{\nu\sigma} + \omega_{\nu\sigma}) \psi^{\rho\sigma} \\ &= (\theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\rho} \omega_{\nu\sigma} + \omega_{\mu\rho} \theta_{\nu\sigma} + \omega_{\mu\rho} \omega_{\nu\sigma}) \psi^{\rho\sigma} \\ &= \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} - \theta_{\mu\sigma} \theta_{\nu\rho}) \psi^{\rho\sigma} \\ &\quad + \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}) \psi^{\rho\sigma} \end{aligned} \quad (\text{C.35})$$

We can define the spin projector operators for the antisymmetric part as follow:

$$\begin{aligned} \mathcal{P}_{b,\mu\nu\rho\sigma}^1 &= \frac{1}{2} (\theta_{\mu\rho} \theta_{\nu\sigma} - \theta_{\mu\sigma} \theta_{\nu\rho}), \\ \mathcal{P}_{e,\mu\nu\rho\sigma}^1 &= \frac{1}{2} (\theta_{\mu\rho} \omega_{\nu\sigma} - \theta_{\mu\sigma} \omega_{\nu\rho} - \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho}). \end{aligned} \quad (\text{C.36})$$

Thus, we obtain

$$\begin{aligned} \psi_{\mu\nu} &= \mathcal{P}_{b,\mu\nu\rho\sigma}^1 \psi^{\rho\sigma} + \mathcal{P}_{e,\mu\nu\rho\sigma}^1 \psi^{\rho\sigma} \\ &= (\mathcal{P}_b^1 + \mathcal{P}_e^1)_{\mu\nu\rho\sigma} \psi^{\rho\sigma} \end{aligned} \quad (\text{C.37})$$

⁶Note that the projector $\mathcal{P}^2$ does not have any lower index, so it can happen that a, b, c, d are absent.

The set $\{\mathcal{P}_b^1, \mathcal{P}_e^1\}$ is complete and allow us to project every antisymmetric tensor along its two vector components. Observe that the letters b and e refer to *magnetic* spin-1 and *electric* spin-1, respectively, in analogy with electrodynamics where the same happens with the antisymmetric tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Note that, in the antisymmetric case, the completeness relation is given by

$$(\mathcal{P}_b^1 + \mathcal{P}_e^1)_{\mu\nu\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}). \quad (\text{C.38})$$

Full decomposition

We are now able to decompose any two-rank tensor $\varphi^{\mu\nu}$ along the spin components corresponding to the irreducible representations of the group $SO(3)$,

$$\varphi^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}, \quad (\text{C.39})$$

in terms of the spin projector operators. Indeed, we can extend the symmetric set $\{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0\}$ by including the antisymmetric part $\{\mathcal{P}_b^1, \mathcal{P}_e^1\}$:

$$\mathcal{O}^i \equiv \{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0, \mathcal{P}_b^1, \mathcal{P}_e^1\}, \quad i = 1, 2, 3, 4, 5, 6, \quad (\text{C.40})$$

so that any two-rank tensor can be decomposed as

$$\begin{aligned} \varphi^{\mu\nu} &= \mathcal{P}_{\mu\nu\rho\sigma}^2 \varphi^{\rho\sigma} + \mathcal{P}_{m,\mu\nu\rho\sigma}^1 \varphi^{\rho\sigma} + \mathcal{P}_{s,\mu\nu\rho\sigma}^0 \varphi^{\rho\sigma} \\ &\quad + \mathcal{P}_{w,\mu\nu\rho\sigma}^0 \varphi^{\rho\sigma} + \mathcal{P}_{b,\mu\nu\rho\sigma}^1 \varphi^{\rho\sigma} + \mathcal{P}_{e,\mu\nu\rho\sigma}^1 \varphi^{\rho\sigma} \\ &= (\mathcal{P}^2 + \mathcal{P}_m^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0 + \mathcal{P}_b^1 + \mathcal{P}_e^1)_{\mu\nu\rho\sigma} \varphi^{\rho\sigma}. \end{aligned} \quad (\text{C.41})$$

To form a basis in terms of which we can expand any four-rank tensor we need to introduce other two operators that mix the spin-1 components:

$$\begin{aligned} \mathcal{P}_{me,\mu\nu\rho\sigma}^1 &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} - \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} - \theta_{\nu\sigma}\omega_{\mu\rho}), \\ \mathcal{P}_{em,\mu\nu\rho\sigma}^1 &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} - \theta_{\nu\rho}\omega_{\mu\sigma} - \theta_{\nu\sigma}\omega_{\mu\rho}), \end{aligned} \quad (\text{C.42})$$

which finally complete the basis:

$$\{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0, \mathcal{P}_e^1, \mathcal{P}_b^1, \mathcal{P}_{sw}^0, \mathcal{P}_{ws}^0, \mathcal{P}_{em}^1, \mathcal{P}_{me}^1\}. \quad (\text{C.43})$$

It easy to show that the following orthogonal relations hold (when $a \neq b$ and $c \neq d$)

$$\begin{aligned} \mathcal{P}_a^i \mathcal{P}_b^j &= \delta_{ij} \delta_{ab} \mathcal{P}_a^j, & \mathcal{P}_{ab}^0 \mathcal{P}_c^i &= \delta_{i0} \delta_{bc} \mathcal{P}_{ab}^i, \\ \mathcal{P}_c^i \mathcal{P}_{ab}^0 &= \delta_{i0} \delta_{ac} \mathcal{P}_{ab}^0, & \mathcal{P}_{ab}^0 \mathcal{P}_{cd}^0 &= \delta_{ad} \delta_{bc} \mathcal{P}_{ab}^0, \end{aligned} \quad (\text{C.44})$$

where $i, j = 2, 1, 0$ and $a, b, c, d = m, s, w, b, e$, absent. Hence, the introduction of the four operators⁷ $\mathcal{P}_{ws}^0, \mathcal{P}_{sw}^0, \mathcal{P}_{em}^1, \mathcal{P}_{me}^1$ is important to satisfy the relations (C.44) that define the algebra of the operators. Note that for the full decomposition the completeness relations takes into account both symmetric and antisymmetric part, and it is given by

$$\begin{aligned} (\mathcal{P}^2 + \mathcal{P}_m^1 + \mathcal{P}_s^0 + \mathcal{P}_w^0 + \mathcal{P}_b^1 + \mathcal{P}_e^1)_{\mu\nu\rho\sigma} &= \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \\ &+ \frac{1}{2} (\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \quad (\text{C.46}) \\ &= \eta_{\mu\rho}\eta_{\nu\sigma}. \end{aligned}$$

We can also find the spin value of each spin projector operator by contracting with the identity matrix $\mathbb{1}_{\mu\nu\rho\sigma}$ which in the symmetric case is given by $\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$, while in the antisymmetric case by $\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$. Indeed, the following relation hold:

$$\mathbb{1}^{\mu\nu\rho\sigma} \mathcal{P}_{\mu\nu\rho\sigma}^j = 2(j) + 1, \quad (\text{C.47})$$

where j is the spin associated to the spin projector operator $\mathcal{P}^j$. Hence we can easily verify that:

$$\begin{aligned} \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{\mu\nu\rho\sigma}^2 &= 5 = 2(2) + 1 \quad (\text{spin-2}), \\ \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{m, \mu\nu\rho\sigma}^1 &= 3 = 2(1) + 1 \quad (\text{spin-1}), \\ \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{s, \mu\nu\rho\sigma}^0 &= 1 = 2(0) + 1 \quad (\text{spin-0}), \\ \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{w, \mu\nu\rho\sigma}^0 &= 1 = 2(0) + 1 \quad (\text{spin-0}), \\ \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{b, \mu\nu\rho\sigma}^1 &= 3 = 2(1) + 1 \quad (\text{spin-1}), \\ \eta^{\mu\rho}\eta^{\nu\sigma} \mathcal{P}_{e, \mu\nu\rho\sigma}^1 &= 3 = 2(1) + 1 \quad (\text{spin-1}). \end{aligned} \quad (\text{C.48})$$

Note that the relations in Eq.(C.48) do not hold for the operators $\mathcal{P}_{sw}^0, \mathcal{P}_{ws}^0, \mathcal{P}_{me}^1, \mathcal{P}_{em}^1$ because they are not projectors as we have already pointed out. Moreover,

⁷No operators which connect electric and magnetic spin-1 spaces ($\mathcal{P}_{eb}^1$ and $\mathcal{P}_{be}^1$), nor operators which connect the third pair of spin-1 spaces ($\mathcal{P}_{bm}^1$ and $\mathcal{P}_{mb}^1$) exist; let us understand why. The four-rank tensor operators we are dealing with can be part of a Lagrangian and field equations, for instance,

$$\mathcal{L} = \frac{1}{2} \varphi_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma}. \quad (\text{C.45})$$

If the Lagrangian is invariant under parity transformations the presence of such transition operators is excluded. While, in a parity-violating scenario terms like $\epsilon_{\mu\nu\rho\sigma} \varphi^{\mu\nu} \varphi^{\rho\sigma}$ or $\epsilon_{\mu\nu\rho\sigma} \varphi^{\mu\nu} \partial^\rho \psi^\sigma$ would be present in the Lagrangian and so the operators $\mathcal{P}_{eb}^1, \mathcal{P}_{be}^1, \mathcal{P}_{mb}^1$ and $\mathcal{P}_{bm}^1$ would appear. See Ref.[143] for more details.

they tell us that (C.27) corresponds to the decomposition of a symmetric two-rank tensor in terms of one spin-2, one spin-1 and two spin-0 components under the $SO(3)$ representation of the Lorentz group, i.e. $h^{\mu\nu} \in \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$; while (C.37) corresponds to the decomposition of an antisymmetric tensor in terms of two spin-1 components, i.e. $h^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1}$.

The basis (C.43) is important, for instance, when we want to determine the propagator associated to any Lagrangian. We have used only the symmetric space in this Thesis, but generally we can have Lagrangians that require the use of the complete set of operators [143]:

$$\mathcal{O}^i \equiv \{\mathcal{P}^2, \mathcal{P}_m^1, \mathcal{P}_s^0, \mathcal{P}_w^0, \mathcal{P}_b^1, \mathcal{P}_e^1, \mathcal{P}_{ws}^0, \mathcal{P}_{sw}^0, \mathcal{P}_{em}^1, \mathcal{P}_{me}^1\}, \quad i = 1, \dots, 10; \quad (\text{C.49})$$

which allows us to write

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \varphi_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma} \\ &= \frac{1}{2} \varphi_{\mu\nu} \left(\sum_{i=1}^{10} C_i \mathcal{O}^{i, \mu\nu\rho\sigma} \right) \varphi_{\rho\sigma}, \end{aligned} \quad (\text{C.50})$$

or, in other words, we can say that the operator space of the field equations can be spanned as

$$\begin{aligned} \mathcal{O}^{\mu\nu\rho\sigma} \varphi_{\rho\sigma} &= \lambda J^{\mu\nu} \\ \iff \left(\sum_{i=1}^{10} C_i \mathcal{O}^{i, \mu\nu\rho\sigma} \right) \varphi_{\rho\sigma} &= \lambda \left(\sum_{i=1}^6 C_i \mathcal{O}^{i, \mu\nu\rho\sigma} \right) J_{\rho\sigma}, \end{aligned} \quad (\text{C.51})$$

where the coefficients C_i depends the specific Lagrangian theory we are considering.

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