University of Groningen

Rent-seeking group contests with one-sided private information

Everhardt, R.J.; Schoonbeek, Lambert

Published in:
Decisions in Economics and Finance

DOI:
[10.1007/s10203-014-0158-1](https://doi.org/10.1007/s10203-014-0158-1)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2015

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Everhardt, R. J., & Schoonbeek, L. (2015). Rent-seeking group contests with one-sided private information. *Decisions in Economics and Finance*, 38(1), 55-73. <https://doi.org/10.1007/s10203-014-0158-1>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Rent-seeking group contests with one-sided private information

Rob J. Everhardt · Lambert Schoonbeek

Received: 11 February 2013 / Accepted: 13 August 2014 / Published online: 28 August 2014
© Springer-Verlag Italia 2014

Abstract We consider a rent-seeking contest where players compete in groups for a prize of given value. One group has private information about its number of members, which can be either small or large. The other groups have possibly different but publicly known sizes. We present an explicit characterization of the groups which are active in the unique equilibrium of the game, and relate the relative magnitude of group efforts to the size of the groups. We compare the decision of each type of the privately informed group to be active in equilibrium to the corresponding decision in a benchmark game with complete information.

Keywords Rent seeking · Group contest · Private information · Group-size paradox

JEL Classification C7 · D7

1 Introduction

We consider a situation where members of groups, with possibly different sizes, compete for a given prize that will be awarded to one of the groups. The prize has private good characteristics. If a group obtains the prize, it is equally shared among its members. The probability that a group wins the prize depends on the (aggregate) effort of this group and the efforts of the other groups. Examples of such group contests are situations where interest groups, like industries, localities or research departments,

We thank an associate editor, an anonymous referee and Allard van der Made for very helpful remarks.

R. J. Everhardt · L. Schoonbeek (✉)

Department of Economics, Econometrics and Finance, Faculty of Economics and Business,
University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands
e-mail: L.Schoonbeek@rug.nl

compete for a fixed budget at the discretion of policy makers, or situations where R&D joint ventures of firms are racing to develop a new product that will create a given economic rent.

Nitzan (1991) has investigated a group contest where the membership of each group is publicly known. Focusing on the case where all groups are active in equilibrium, he demonstrated that larger groups spend less efforts than smaller groups. This result is known as the group-size paradox (Esteban and Ray 2001). It implies that the most likely winner is the group with the smallest number of members. The paradox is caused by two factors: if the size of a group increases, the free-rider incentive within the group becomes more pervasive, and the per-capita reward becomes smaller since each player has to share the prize with more group members. As a result, the magnitude of each individual player's contribution becomes smaller, which dominates the positive effect of the increase of the number of players. It might even occur that large groups become inactive. Ueda (2002) gives an explicit characterization of the groups that are active in the equilibrium of the model of Nitzan (1991).¹ The model of Nitzan generalizes the seminal rent-seeking Tullock (1980) contest between single players to a contest between groups. See Nitzan (1994), Lockard and Tullock (2001), Congleton et al. (2008) and Konrad (2009) for surveys of the rent-seeking literature.

In this paper, we investigate a group contest à la Nitzan (1991). However, we suppose that one group has private information about its number of members, which can be either small (a small-type group) or large (a large-type group). The other groups have publicly known sizes. Our game can be applied to various situations. Take, for example, the competition between groups of researchers for a fixed national research grant that will be awarded to one group. Each member of a group can exert effort to enhance the probability that his group will win the grant (e.g., by contributing to the quality of the joint research proposal). If a group wins the grant, it will be shared among its members. It then might happen that one of the groups has private information about its size. Another example is a market, where joint ventures of firms are involved in an R&D race that will generate a given rent to the winner. We then can have a situation where incumbent joint ventures, which know each other, face entry of a newly formed joint venture, which has private information about the number of firms participating in it.

We will investigate how the equilibrium of our game is affected by the presence of the group with private information. In particular, we address the following questions. Under which conditions is the small type or large type of the privately informed group active in equilibrium? Does activity of the small-type group also depend on the exact size of the large-type group, and vice versa? How is the total effort of the groups without private information affected by the presence of the privately informed group? To what extent does the group-size paradox hold in our game with private information? Finally, for each type of the privately informed group, how does the decision to be (in)active in the equilibrium of the game with private information compare to the

¹ Nitzan (1991) and Ueda (2002) allow for a general sharing-rule to divide the prize within the winning group, i.e., it can be divided equally, or proportionally to each member's effort, or based on a combination of this. For simplicity, and to obtain unambiguous results, we focus on the case where the prize is equally shared among the members of the winning group.

corresponding decision of a related group in the equilibrium of a benchmark game with complete information?

To begin with, we show that our game has a unique (pure-strategy Nash) equilibrium in which members of the same group exert the same effort level. Extending [Ueda \(2002\)](#), we provide an explicit characterization of the groups that are active in this equilibrium, including the two types of the privately informed group. We find the following equilibrium properties. First, in equilibrium either none of the two types of the privately informed group is active, only the small type is active, or both types are active. The decision of the small-type group to be (in)active only depends on its own size, not on the exact size of the large-type group. On the contrary, the (in)activity of the large-type group depends both on its own size and on that of the small-type group. Second, the total effort of all groups without private information is largest in the case where both the small type and large type of the privately informed group are inactive, and smallest if they are both active. This can be understood intuitively since the groups without private information have the largest (smallest) chance of obtaining the prize in the first (last) case, which makes them more (less) eager. Third, in accordance with the group-size paradox, the effort of the large-type privately informed group is smaller than the effort of the small-type group (unless both are inactive). The same holds if we compare the effort levels of two groups without private information with different sizes. However, comparing the small-type privately informed group to the groups without private information, we find cases in which both the size and effort level of the small-type group are smaller than that of a group without private information. Similarly, we have instances where both the size and effort level of the large-type group are larger than those of a group without private information. These cases, which are not in accordance with the group-size paradox, occur since the groups without private information choose their effort levels while taking into account the possible occurrence of both the small-type and large-type group. Related to this, we also find situations where an increase in the size of the small-type privately informed group implies that only the *large*-type group becomes inactive. There is no such indirect effect the other way round. Fourth, for each type of the privately informed group we find that the decision to be (in)active in the game with private information might be different from the corresponding decision of a related group in a benchmark game with complete information. We explain intuitively why these differences can arise.

A small but growing number of studies examine contests with private information regarding the valuation of the prize. The literature started with [Hurley and Shogren \(1998a\)](#), who consider a contest with two players where one player has private information about his valuation, which is the realization of a random variable with a continuous density function defined on a given interval of real numbers. [Hurley and Shogren \(1998b\)](#) also study a contest with two players. First, they consider the case where one player has private information about his valuation, which is the realization of a random variable with a discrete probability distribution with three possible values. Next, they extend the model and numerically investigate the situation with two-sided private information, where the valuation of the other player is drawn from a discrete probability distribution with two possible values. [Malueg and Yates \(2004\)](#) analytically investigate the case with two-sided private information, assuming that the valuation of each player is either high or low, and imposing a simple structure on the probabilities

associated with these valuations (allowing for correlation between the two valuations). Other related papers are [Hurley and Shogren \(1998c\)](#), [Wärneryd \(2003\)](#), [Schoonbeek and Winkel \(2006\)](#), [Fey \(2008\)](#) and [Ryvkin \(2010\)](#). Note, however, that these studies focus on contests between individual players rather than groups. We remark that, formally speaking, our game can be reformulated in terms of a related contest between individual players where one player has private information regarding its valuation of the prize (which can be either high or low), while the other players are allowed to have different valuations of the prize. This game has not been studied in the existing literature.

The paper is organized as follows. Section 2 presents the game. Section 3 gives the analysis of the equilibrium. Section 4 concludes. The ‘Appendix’ contains proofs.

2 The game

Consider a rent-seeking contest where risk-neutral players compete in $m \geq 2$ groups for a single prize of value $V > 0$ by exerting non-refundable efforts. Every player is in exactly one group. Group 1 has n_S players with probability p and n_L players with probability $1 - p$, where $n_L > n_S \geq 1$ and $0 < p < 1$. Hence, we either have a *small-type* or *large-type* group 1. The players of group 1 know the realization of their type, while the players of the other groups only know the distribution. Group $i = 2, \dots, m$ has $n_i \geq 1$ players, which is publicly known. We assume without loss of generality that $n_2 \leq n_3 \leq \dots \leq n_m$.

Let t ($t = S, L$) be the realized type of group 1, n_t the number of players in group 1 of type t , $x_{1jt} \geq 0$ the effort of player j ($j = 1, \dots, n_t$) from group 1 of type t , and $x_{ij} \geq 0$ the effort of player j ($j = 1, \dots, n_i$) from group $i = 2, 3, \dots, m$. The corresponding (aggregate) group efforts are denoted by $x_{1t} = \sum_{j=1}^{n_t} x_{1jt}$ and $x_i = \sum_{j=1}^{n_i} x_{ij}$, $i = 2, \dots, m$, respectively. If group 1 is of type t , the probability that group i wins the prize is given by the Tullock success function

$$q_{it} = \begin{cases} \frac{x_{1t}}{x_{1t} + \sum_{k=2}^m x_k}, & \text{for } i = 1, \\ \frac{x_i}{x_{1t} + \sum_{k=2}^m x_k}, & \text{for } i = 2, 3, \dots, m. \end{cases} \quad (1)$$

If $x_{1t} + \sum_{k=2}^m x_k = 0$, the probability that group $i = 1, 2, \dots, m$ wins is $1/m$.

Each player will choose his effort to maximize his expected payoff. If a group wins, all its players receive an equal share of the prize. Disregarding the case where all group efforts are zero, since that will not occur in equilibrium, the expected payoff of player j ($j = 1, \dots, n_t$) of type t from group 1, π_{1jt} , is

$$\pi_{1jt} = \frac{x_{1t} V}{(x_{1t} + \sum_{k=2}^m x_k) n_t} - x_{1jt}, \quad (2)$$

while the expected payoff π_{ij} for player j ($j = 1, \dots, n_i$) from group $i = 2, 3, \dots, m$ is

$$\pi_{ij} = p \frac{x_i V}{(x_{1S} + \sum_{k=2}^m x_k) n_i} + (1-p) \frac{x_i V}{(x_{1L} + \sum_{k=2}^m x_k) n_i} - x_{ij}. \quad (3)$$

3 The equilibrium

We investigate the equilibrium properties of our game. We focus on equilibria in which players employ pure strategies and all players of the same group choose the same effort level. Section 3.1 gives preliminary results and characterizes which groups are active in equilibrium, if there exists one. Section 3.2 shows that the game has a unique equilibrium. Section 3.3 further discusses whether players of group 1 are active in equilibrium depending on the size of n_S and n_L . Section 3.4 compares our game with private information to the case with complete information.

3.1 Preliminary results

An equilibrium, denoted by $(\hat{x}_{1S}, \hat{x}_{1L}, \hat{x}_2, \dots, \hat{x}_m)$, is characterized by the following first-order conditions (FOCs), with $\hat{x} = \sum_{k=2}^m \hat{x}_k$:

$$\frac{\hat{x} V}{(\hat{x}_{1S} + \hat{x})^2 n_S} \leq 1, \quad (4)$$

$$\frac{\hat{x} V}{(\hat{x}_{1L} + \hat{x})^2 n_L} \leq 1, \quad (5)$$

$$p \frac{(\hat{x}_{1S} + \hat{x} - \hat{x}_i) V}{(\hat{x}_{1S} + \hat{x})^2 n_i} + (1-p) \frac{(\hat{x}_{1L} + \hat{x} - \hat{x}_i) V}{(\hat{x}_{1L} + \hat{x})^2 n_i} \leq 1, \quad \text{if } i = 2, \dots, m, \quad (6)$$

where (4), (5) and (6) hold with equality if $\hat{x}_{1S} > 0$, $\hat{x}_{1L} > 0$ and $\hat{x}_i > 0$, respectively.²

For $i, k \in \{2, 3, \dots, m\}$ with $k > i$, we have either (i) $n_k = n_i$ and $\hat{x}_k = \hat{x}_i$, or (ii) $n_k > n_i$ and either $0 < \hat{x}_k < \hat{x}_i$, or $0 = \hat{x}_k < \hat{x}_i$, or $0 = \hat{x}_k = \hat{x}_i$. Thus, considering the groups with indices 2 to m , we see that a larger group makes a smaller effort than a smaller one (unless both are inactive), which is in accordance with the group-size paradox. Further, group 2 will always exert a positive effort.³

Let $r \in \{1, 2, \dots, m-1\}$ be such that (i) $\hat{x}_{r+1} > 0$ and (ii) either $\hat{x}_{r+2} = 0$ or $r+1 = m$. If $r = m-1$, the $\hat{x}_{r+2} = 0$ condition does not exist. The variable r represents the number of groups that play a positive effort, *excluding* group 1, i.e., the group possessing private information. For example, suppose there are four groups ($m = 4$) of which group 1, 2 and 3 play a positive effort, $\hat{x}_{1S} > 0$, $\hat{x}_{1L} > 0$, $\hat{x}_2 > 0$

² If each group would choose its effort level cooperatively rather than non-cooperatively, the private-information problem would no longer be relevant, because in that case group 1 would choose the same effort level regardless of its membership. In fact, each group would exert the same (total) effort level $(m-1)V/m^2$ in equilibrium.

³ The free-rider effect and per-capita reward size effect give the intuition behind the group-size paradox. For example, take player j of group 1 of type $t = S, L$. The FOC of this player is $\partial \pi_{1jt} / \partial x_{1jt} = \partial q_{1t} / \partial x_{1jt} \times (V/n_t) - 1 = 0$. An increase in n_t decreases V/n_t , which is the per-capita reward size effect. If all members of group 1 exert the same effort, then $\partial q_{1t} / \partial x_{1jt}$ is decreasing in n_t , which denotes the free-rider effect.

and $\hat{x}_3 > 0$, while group 4 does not, $\hat{x}_4 = 0$. In this case, $r = 2$. Note also that $\hat{x}_2 \geq \hat{x}_3$.

Remark 1 Formally speaking, our group contest is isomorphic to a contest between m individual players, where player 1 has private information regarding its valuation of the prize, which equals $V_H = V/n_S$ (high) with probability p and $V_L = V/n_L$ (low) with probability $1 - p$ (with $V_H > V_L$), and where the valuation of the prize by player $i = 2, \dots, m$ is $V_i = V/n_i$. The FOCs of the two games coincide. The link between the games can be understood since maximizing the expected payoff of a member of a group in the equilibrium of the group contest with m groups (in which all players of the same group exert the same effort level and share the prize equally) is equivalent to maximizing the expected payoff of the corresponding player in the equilibrium of the contest between m individuals. $\square$

Proceeding with our group contest, we obtain the following lemma that considers whether $\hat{x}_{1S}$, $\hat{x}_{1L}$ and/or $\hat{x}$ can be positive or zero in equilibrium.

Lemma 1 *In equilibrium we can only have the following three cases: $\hat{x}_{A1S} = 0$, $\hat{x}_{A1L} = 0$, $\hat{x}_A > 0$, $m > 2$ (case A); $\hat{x}_{B1S} > 0$, $\hat{x}_{B1L} = 0$, $\hat{x}_B > 0$ (case B); $\hat{x}_{C1S} > 0$, $\hat{x}_{C1L} > 0$, $\hat{x}_C > 0$ (case C). The first index attached to a symbol denotes the relevant case.*

Using Lemma 1, (4) and (5), we find that the effort of the large-type group 1 is smaller than the effort of the small-type group 1 (unless both are inactive), which again is in accordance with the group-size paradox.

We consider the three cases of Lemma 1 one by one. First, we introduce the following functions for $i \in \{1, \dots, m - 1\}$:

$$\phi_A(i) = \frac{i - 1}{\sum_{j=2}^{i+1} n_j} V, \quad (7)$$

$$\phi_B(i) = \left(\frac{ip\sqrt{Vn_S} + \sqrt{i^2 p^2 Vn_S + 4(pn_S + \sum_{j=2}^{i+1} n_j)(1-p)V(i-1)}}{2(pn_S + \sum_{j=2}^{i+1} n_j)} \right)^2, \quad (8)$$

$$\phi_C(i) = \left(\frac{ip\sqrt{Vn_S} + i(1-p)\sqrt{Vn_L}}{pn_S + (1-p)n_L + \sum_{j=2}^{i+1} n_j} \right)^2. \quad (9)$$

In the ‘Appendix,’ we show for each of the three cases that if the FOCs hold, the solution is unique, and we derive the corresponding values of $\hat{x}$ (i.e., the total effort of groups 2 to m) and r . We find the following.

Case A Assume $m > 2$. If this case holds, we have $\hat{x}_{A1S} = 0$, $\hat{x}_{A1L} = 0$, $\hat{x}_{Ai} > 0$ for $2 \leq i \leq r_A + 1$, and $\hat{x}_{Ai} = 0$ for $r_A + 1 < i \leq m$, with

$$\hat{x}_A = \phi_A(r_A), \quad (10)$$

and

$$r_A = \max\{i \in \{1, 2, \dots, m - 1\} \mid V - \phi_A(i)n_{i+1} > 0\}. \quad (11)$$

Case B If this case holds, we have $\hat{x}_{B1S} > 0$, $\hat{x}_{B1L} = 0$, $\hat{x}_{Bi} > 0$ for $2 \leq i \leq r_B + 1$, and $\hat{x}_{Bi} = 0$ for $r_B + 1 < i \leq m$, with

$$\hat{x}_B = \phi_B(r_B), \quad (12)$$

and

$$r_B = \max\{i \in 1, 2, \dots, m-1 \mid p\sqrt{Vn_S\phi_B(i)} + (1-p)V - \phi_B(i)n_{i+1} > 0\}. \quad (13)$$

Case C If this case holds, we have $\hat{x}_{C1S} > 0$, $\hat{x}_{C1L} > 0$, $\hat{x}_{Ci} > 0$ for $2 \leq i \leq r_C + 1$, and $\hat{x}_{Ci} = 0$ for $r_C + 1 < i \leq m$, with

$$\hat{x}_C = \phi_C(r_C), \quad (14)$$

and

$$r_C = \max \left\{ i \in 1, 2, \dots, m-1 \mid pn_S + (1-p)n_L - in_{i+1} + \sum_{j=2}^{i+1} n_j > 0 \right\}. \quad (15)$$

The next three lemmas give useful properties of $\phi_A(i)$, $\phi_B(i)$ and $\phi_C(i)$.

Lemma 2 For $i \in \{1, \dots, m-1\}$ the following holds:

$$\phi_A(i) \geq \frac{V}{n_S} \Leftrightarrow (i-1)n_S \geq \sum_{j=2}^{i+1} n_j, \quad (16)$$

$$\phi_B(i) < \frac{V}{n_S} \Leftrightarrow (i-1)n_S < \sum_{j=2}^{i+1} n_j, \quad (17)$$

$$\phi_B(i) \geq \frac{V}{n_L} \Leftrightarrow (i-1)(p\sqrt{n_S n_L} + (1-p)n_L) \geq pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{i+1} n_j, \quad (18)$$

$$\phi_C(i) < \frac{V}{n_L} \Leftrightarrow (i-1)(p\sqrt{n_S n_L} + (1-p)n_L) < pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{i+1} n_j. \quad (19)$$

Lemma 3 In case A, we have $\phi_A(i-1) < \phi_A(i)$ for $i = 2, \dots, r_A$ (if $r_A > 2$) and $\phi_A(i) \geq \phi_A(i+1)$ for $i = r_A, \dots, m-2$ (if $r_A < m-1$). Hence, the number of active groups of groups 2 to m , represented by r_A , is such that it maximizes $\phi_A(i)$ for $i = 1, \dots, m-1$. The same holds in case B and case C for, respectively, r_B and $\phi_B(i)$, and r_C and $\phi_C(i)$.

Lemma 4 The game defined by $(m, V, p, n_S, n_L, n_2, \dots, n_m)$ has an equilibrium in

- (i) case A if and only if $\phi_A(r_A) \geq \frac{V}{n_S}$,
- (ii) case B if and only if $\frac{V}{n_S} > \phi_B(r_B) \geq \frac{V}{n_L}$,
- (iii) case C if and only if $\frac{V}{n_L} > \phi_C(r_C)$.

Lemmas 3 and 4 give an explicit characterization of the groups that are active in each of the three possible equilibrium situations A , B or C .

3.2 Main results

Using Lemmas 1–4, we can show that exactly one of the cases A , B and C is true for our game. As a result, we present the next proposition.

Proposition 1 *The game defined by $(m, V, p, n_S, n_L, n_2, \dots, n_m)$ has exactly one (pure-strategy) equilibrium in which players of the same group exert the same effort. It can be expressed as $(\hat{x}_{1S}, \hat{x}_{1L}, \hat{x}_2, \dots, \hat{x}_m)$.*

Lemma 4 implies that the total equilibrium effort of groups 2 to m is larger in case A , when both the small-type and large-type group 1 play a zero effort, than in case B , when there is a chance that group 1 plays, namely if it is the small type. In turn, the total equilibrium effort of groups 2 to m in case B is larger than in case C , when it is certain that group 1 plays, both when it is a small type and large type. Hence, the total effort of the groups without private information is larger if the group with private information is less likely to play. This makes sense, as the former groups have a larger chance of obtaining the prize in that case, making them more eager in the contest.

We give a numerical example of the equilibrium. Consider our game with ten groups ($m = 10$), which all value the prize at $V = 1$. Group 1 has size $n_S = 1$ with probability $p = 0.4$ and size $n_L = 8$ with probability $1 - p = 0.6$. Groups 2–10 have the following group sizes: (3, 4, 6, 9, 9, 10, 10, 14, 16). With the help of (11), (13) and (15), we calculate the number of groups excluding group 1 that play a positive effort in each case: $r_A = 3$, $r_B = 3$ and $r_C = 3$. Furthermore, by (10), (12) and (14), we compute the total effort of those groups for each case: $\hat{x}_A = 0.154$, $\hat{x}_B = 0.121$ and $\hat{x}_C = 0.119$. Because $\hat{x}_A$ is smaller than $V/n_S = 1$ and $\hat{x}_B$ and $\hat{x}_C$ are both smaller than $V/n_L = 1/8$, only case C can yield an equilibrium. It can be verified that in the equilibrium, we have $\hat{x}_{1S} = 0.226$ and $\hat{x}_{1L} = 0.03$. The equilibrium efforts of groups 2, 3 and 4 are $\hat{x}_2 = 0.070$, $\hat{x}_3 = 0.047$ and $\hat{x}_4 = 0.002$. As $r_C = 3$, $\hat{x}_5 = \hat{x}_6 = \dots = \hat{x}_{10} = 0$.

Concluding this section, we examine a game where $m = 4$, $V = 10$, $p = 0.2$, $n_S = 11$, $n_L = 21$, $n_2 = 12$, $n_3 = 13$ and $n_4 = 20$. The equilibrium efforts are then given by $\hat{x}_S = 0.200$, $\hat{x}_L = 0.031$, $\hat{x}_2 = 0.202$, $\hat{x}_3 = 0.181$ and $\hat{x}_4 = 0.029$. Although the small-type group 1 is smaller than group 2 ($n_S < n_2$), we have $\hat{x}_S < \hat{x}_2$, which is not in accordance with the group-size paradox. Since $n_L > n_4$ and $\hat{x}_L > \hat{x}_4$, a similar conclusion holds for the large-type group 1 and group 4. Thus, the group-size paradox does not necessarily hold between the two types of the privately informed group and the groups without private information. The reason is that the latter groups choose their effort levels such that they are optimal ‘on average,’ i.e., while taking into account the possible occurrence of both types of group 1.

3.3 The group size and (in)activity of the types of group 1

We further investigate whether group 1 is active or inactive in equilibrium depending on the size of n_S and n_L . The small-type group 1 is always active in equilibrium if $m = 2$. Hence, if we consider this group in this section, we (implicitly) focus on the case with $m > 2$. We see from (i) of Lemma 4 and (16) that the equilibrium is in case A if and only if

$$(r_A - 1)n_S \geq \sum_{j=2}^{r_A+1} n_j. \quad (20)$$

Since r_A is independent of n_S and n_L , the equilibrium is in case A if n_S is 'large.' Then, both the small-type and large-type group 1 are inactive due to the pervasive free-rider incentive and small per-capita reward in either of these groups (recall that $n_L > n_S$). In fact, the small-type group 1 is inactive if and only if the equilibrium is in case A. Hence, the decision of this group to be (in)active only depends on n_S and not on the exact size of n_L . Next, examine the situation where (20) does not hold, i.e., when n_S is 'small.' Then, we are in case B or C and the small-type group 1 is active in equilibrium. Using (ii) and (iii) of Lemma 4 and the fact that $\phi_B(r_B)$ does not depend on n_L , we see that (given n_S) we have case B if n_L is 'large' and case C if n_L is 'small.' If n_L is 'large,' then the large-type group is inactive because the free-rider incentive is pervasive and the per-capita reward is small. If n_L is 'small,' then this group is active for the opposite reasons. Remark that the decision of the large-type group 1 to be (in)active depends on the values of both n_S and n_L .

Proceeding, take a situation where the game has an equilibrium in case C, i.e., in which both types of group 1 are active. Next, let us increase n_S , while keeping n_L fixed (taking into account that $n_L > n_S$). In other words, we aggravate the free-rider incentive and decrease the per-capita reward in the small-type group 1 only. It then might happen for some values of n_S that the large-type group 1 rather than the small-type one becomes inactive. The reason is that the small-type group 1 decreases its effort if n_S becomes larger. As a result, the total effort of groups 2 to m increases, which in turn is unattractive for the large-type group 1. Hence, we can have an indirect effect of the size of the small-type group 1 on the activity of the large-type group 1. This indirect effect can be illustrated if we increase n_S from 1 to 2 in the first numerical example of Sect. 3.2. In that case, the equilibrium falls in case B with $r_B = 3$, $\hat{x}_{1S} = 0.125$ and $\hat{x}_{1L} = 0$. The total effort of groups 2, 3 and 4 has increased to $\hat{x}_B = 0.132$, with $\hat{x}_2 = 0.077$, $\hat{x}_3 = 0.052$ and $\hat{x}_4 = 0.003$. Reversely, it cannot occur that only the small-type group 1 becomes inactive if n_L increases, while the large-type stays active.

3.4 Comparison to the case with complete information

We will compare the equilibrium behavior of each type of group 1 in our private-information game to that of a corresponding group in a benchmark game with complete information. We have two benchmark games. In benchmark game B_S , we have groups $2, \dots, m$ plus a group S with n_S members. In benchmark game B_L , we have groups

2, $\dots$, m and a group L with n_L members. We can find the equilibrium results of the benchmark games from our previous analysis. That is, take the game with private information and replace n_S and n_L by, respectively, $\tilde{n}_S$ and $\tilde{n}_L$. We obtain game B_S if $\tilde{n}_S = n_S$ and $\tilde{n}_L = n_S$, and game B_L if $\tilde{n}_S = n_L$ and $\tilde{n}_L = n_L$. Hence, in the benchmark games we fix both variables at the same value. Now, take game B_S and replace n_L by n_S in (9) and (15). We denote $\phi_C(i)$ and r_C thus obtained by $\phi_S(i)$ and r_S , respectively. It can be verified, analogously to (iii) of Lemma 4, that group S is active in the equilibrium of game B_S if and only if $V/n_S > \phi_S(r_S)$. Similarly, replacing n_S by n_L in (9) and (15), we define, respectively, $\phi_L(i)$ and r_L corresponding to game B_L (note that $r_L \geq r_C$). Group L is active in the equilibrium of game B_L if and only if $V/n_L > \phi_L(r_L)$. We present the first result.

Proposition 2 *Compare the equilibrium behavior of group S with n_S members in the complete-information game B_S to that of the small-type group 1 in the private-information game. We have the following.*

- (i) *Let $n_S < n_2$. Then, both groups are active.*
- (ii) *Let $n_{k_S} \leq n_S < n_{k_S+1}$, where $k_S \in \{2, \dots, m-1\}$. Then, if $r_A < k_S - 1$, both groups are inactive. If $r_A = k_S - 1$, group S is active if and only if the small-type group 1 is active. If $r_A > k_S - 1$, group S is active if the small-type group 1 is active, but the reverse does not hold in general.*
- (iii) *Let $n_S \geq n_m$. Then, if $r_A = m - 1$, group S is active if and only if the small-type group 1 is active. If $r_A < m - 1$, both groups are inactive.*

The proposition shows that group S and the small-type group 1 are (in)active under the same condition except possibly in the case mentioned in part (ii) for $r_A > k_S - 1$. It then can happen that group S is active, while the small-type group 1 is not active. One can verify that this case can only occur if $r_S < r_A$, which means that a smaller number of the groups 2, $\dots$, m are active in game B_S than in the game with private information. Further, recall that in game B_S the groups 2, $\dots$, m face group S with n_S members with certainty, whereas in the game with private information they face the small-type group 1 with n_S members only with probability $p < 1$. Intuitively speaking, both effects make groups 2, $\dots$, m more aggressive in the game with private information, i.e., their total effort is larger in that case than in game B_S . Apparently, the effects can be so large that the small-type group 1 becomes inactive, while group S remains active.

Next, we present our second result.

Proposition 3 *Compare the equilibrium behavior of group L with n_L members in the complete-information game B_L to that of the large-type group 1 in the private-information game. We have the following.*

- (i) *Let $r_L = 1$. Then, group L is active, while the large-type group 1 may be inactive.*
- (ii) *Let $r_L \geq 2$ and $r_C = 1$. Then, both groups may be active or inactive. (In)activity of one group does not necessarily imply (in)activity of the other group.*
- (iii) *Let $r_L \geq 2$ and $r_C \geq 2$. Then, the large-type group 1 is active if group L is active, but the reverse does not hold in general.*

According to Proposition 3, we have instances where group L and the large-type group 1 are not (in)active at the same time. In order to understand this, first take part (i). Note

that $r_L = 1$ means that in game B_L , only the first one of the groups $2, \dots, m$ is active. As a result, group L must be active as well, since in equilibrium always at least two groups must be active. Next, $r_L = 1$ implies $r_C = 1$. Hence, the large-type group 1 is inactive if and only if $pn_S - p\sqrt{n_S n_L} + n_2 \leq 0$ (use 19), which is true if n_L is large enough, given p , n_S and n_2 . Turning to part (ii), we again can have cases where group L is active while the large-type group 1 is inactive. Now group L is active if and only if n_L is small enough, i.e., $n_L < \sum_{j=2}^{r_L+1} n_j / (r_L - 1)$ (use 19), while the large-type group 1 is inactive if and only if n_L is large enough, given p , n_S and n_2 (analogously to part (i)). There is enough leeway such that both conditions on n_L can hold simultaneously. Further, it can also happen that group L is inactive while the large-type group 1 is active. In order to understand this, let us start with a situation where both group L and the large-type group 1 are inactive. Next, decrease the size of n_S in the game with private information. Then, the total effort of the groups $2, \dots, m$ will become smaller. This decrease might become so large that, in turn, the large-type group 1 becomes active. On the other hand, the changes in n_S have no impact in game B_L ; hence, group L will remain inactive. Finally, in part (iii), it can also happen that group L is inactive while the large-type group 1 is active. This can be explained similarly to part (ii).

Remark 2 One might also consider an alternative benchmark game with complete information, game B_W say, with groups $i = 2, \dots, m$ plus a group W with $n_W \equiv pn_S + (1 - p)n_L$ members. Thus, the membership of group W is a weighted average of n_S and n_L . Game B_W is well defined if and only if n_W is a positive integer, which is not true in general. Hence, this benchmark game does not always exist. If n_W is a positive integer, we can find again the equilibrium results for game B_W from our previous analysis. In particular, we then take the game with private information and replace both n_S and n_L by n_W . We do the same in (9) and (15) and denote $\phi_C(i)$ and r_C , thus obtained by $\phi_W(i)$ and r_W , respectively. We see that group W in game B_W is active in equilibrium if and only if $V/n_W > \phi_W(r_W)$. Using this, it is straightforward to compare the (in)activity of group W in game B_W to the (in)activity of the small-type and large-type group 1 of the game with private information. We leave the details to the interested reader.

As an example, we take a contest with three groups ($m = 3$), where $n_S = 1$, $n_L = 3$, $n_2 = 4$, $n_3 = 4$, $p = 0.5$ and $V = 1$. In the equilibrium of the game with private information, we have $\hat{x}_{1S} = 0.199$, $\hat{x}_{1L} = 0.083$, $\hat{x}_2 = 0.037$ and $\hat{x}_3 = 0.037$. We have calculated the equilibrium efforts of the different benchmark games with complete information. In game B_S , the efforts of groups S , 2 and 3 amount to 0.173, 0.025 and 0.025, respectively. In game B_L , the efforts of groups L , 2 and 3 equal 0.083, 0.050 and 0.050, respectively. In game B_W , the efforts of groups W , 2 and 3 read 0.120, 0.040 and 0.400, respectively. $\square$

4 Conclusion

We have fully characterized the efforts of all groups in the equilibrium of our group contest. In equilibrium, either none of the two types of the privately informed group is active, only the small-type is active, or both types are active. Whether the small-type group is active does not depend on the exact size of the large-type group. The groups

without private information play a larger total effort if the privately informed group is less likely to be active. In accordance with the group-size paradox, the effort of the large-type privately informed group is smaller than the effort of the small-type group. The same holds if we compare a large and small group within the set of groups without private information. However, the result does not necessarily hold if we compare the small-type or large-type privately informed group to the groups without information. This is because the latter groups set their effort level while taking into account the occurrence of both the small-type and large-type group. We have also identified an indirect effect where an increase in the size of the small-type group with private information implies that only the large-type group becomes inactive.

Next, we have compared the game with private information to two benchmark games with complete information. The first benchmark game includes a group with the same membership as the small-type privately informed group. It turns out that if the small-type group is active in the game with private information, the related group in the first benchmark game is active as well. However, the reverse is not true in general. The second benchmark includes a group with the same membership as the large-type privately informed group. For the large-type group, we find that (in)activity in the game with private information does not necessarily imply (in)activity of the related group in the second benchmark game, and vice versa. Finally, we have briefly examined a third benchmark model with complete information that includes a group of which the size is equal to a weighted average of the sizes of the small-type and large-type group of the game with private information. A drawback of this benchmark game is that it is not always well defined.

An interesting topic for future research is to allow for more groups with private information.

Appendix: Proofs⁴

Derivation of (10) and (11)

Considering case A, assume $m > 2$, $\hat{x}_{A1S} = 0$, $\hat{x}_{A1L} = 0$ and $\hat{x}_A > 0$. The FOCs become

$$\frac{V}{\hat{x}_{Ans}} \leq 1, \quad (21)$$

$$\frac{V}{\hat{x}_{AnL}} \leq 1, \quad (22)$$

$$\frac{(\hat{x}_A - \hat{x}_{Ai})V}{(\hat{x}_A)^2 n_i} = 1, \quad \text{if } 2 \leq i \leq r_A + 1, \quad (23)$$

$$\frac{V}{\hat{x}_{Ani}} \leq 1, \quad \text{if } r_A + 1 < i \leq m. \quad (24)$$

⁴ For brevity, the ‘Appendix’ does not give the (straightforward) proof of Lemma 1, the (lengthy) proof of Lemma 3, and details of a number of tedious derivations. They are available upon request from the authors.

From (23), we find for $2 \leq i \leq r_A + 1$ that

$$\hat{x}_{Ai} = \hat{x}_A - \frac{\hat{x}_A^2 n_i}{V}, \quad (25)$$

so $\hat{x}_A = \sum_{j=2}^{r_A+1} \hat{x}_{Aj} = \phi_A(r_A)$, i.e., (10). Using this and the definition of r_A , we find (11). If (21)–(24) hold, there is a unique solution where $\hat{x}_{A1S} = \hat{x}_{A1L} = 0$, $\hat{x}_{Ai}$ can be calculated from (25) for $2 \leq i \leq r_A + 1$, and $\hat{x}_{Ai} = 0$ for $r_A + 1 < i \leq m$.

Derivation of (12) and (13)

Examining case B , assume $\hat{x}_{B1S} > 0$, $\hat{x}_{B1L} = 0$ and $\hat{x}_B > 0$. The FOCs now read

$$\frac{\hat{x}_B V}{(\hat{x}_{B1S} + \hat{x}_B)^2 n_S} = 1, \quad (26)$$

$$\frac{V}{\hat{x}_B n_L} \leq 1, \quad (27)$$

$$p \frac{(\hat{x}_{B1S} + \hat{x}_B - \hat{x}_{Bi}) V}{(\hat{x}_{B1S} + \hat{x}_B)^2 n_i} + (1-p) \frac{(\hat{x}_B - \hat{x}_{Bi}) V}{(\hat{x}_B)^2 n_i} = 1, \quad \text{if } 2 \leq i \leq r_B + 1, \quad (28)$$

$$p \frac{V}{(\hat{x}_{B1S} + \hat{x}_B) n_i} + (1-p) \frac{V}{\hat{x}_B n_i} \leq 1, \quad \text{if } r_B + 1 < i \leq m. \quad (29)$$

Substituting (26) in (28), we find for $2 \leq i \leq r_B + 1$ that

$$\hat{x}_{Bi} = \frac{p \sqrt{\hat{x}_B V n_S} + (1-p) V - \hat{x}_B n_i}{p n_S + (1-p) \frac{V}{\hat{x}_B}}. \quad (30)$$

Using $\hat{x}_B = \sum_{j=2}^{r_B+1} \hat{x}_{Bj}$, finding $\hat{x}_B$ is equivalent to solving the equation

$$\left(p n_S + \sum_{j=2}^{r_B+1} n_j \right) y^2 - \left(r_B p \sqrt{V n_S} \right) y - (1-p) V (r_B - 1) = 0, \quad (31)$$

where $y = \sqrt{\hat{x}_B} > 0$. We find $\hat{x}_B = \phi_B(r_B)$, i.e., (12). Note that $\hat{x}_{Bi} > 0$ if and only if the numerator of (30) is positive, as the denominator is positive. Using this and the definition of r_B , we obtain (13). If (26)–(29) hold, there is a unique solution where $\hat{x}_{B1S}$ is defined by (26), $\hat{x}_{B1L} = 0$, $\hat{x}_{Bi}$ can be derived from (30) for $2 \leq i \leq r_B + 1$, and $\hat{x}_{Bi} = 0$ for $r_B + 1 < i \leq m$.

Derivation of (14) and (15)

Considering case C, assume $\hat{x}_{C1S} > 0$, $\hat{x}_{C1L} > 0$ and $\hat{x}_C > 0$. The FOCs now equal

$$\frac{\hat{x}_C V}{(\hat{x}_{C1S} + \hat{x}_C)^2 n_S} = 1, \quad (32)$$

$$\frac{\hat{x}_C V}{(\hat{x}_{C1L} + \hat{x}_C)^2 n_L} = 1, \quad (33)$$

$$p \frac{(\hat{x}_{C1S} + \hat{x}_C - \hat{x}_{Ci})V}{(\hat{x}_{C1S} + \hat{x}_C)^2 n_i} + (1-p) \frac{(\hat{x}_{C1L} + \hat{x}_C - \hat{x}_{Ci})V}{(\hat{x}_{C1L} + \hat{x}_C)^2 n_i} = 1, \quad \text{if } 2 \leq i \leq r_C + 1, \quad (34)$$

$$p \frac{V}{(\hat{x}_{C1S} + \hat{x}_C)n_i} + (1-p) \frac{V}{(\hat{x}_{C1L} + \hat{x}_C)n_i} \leq 1, \quad \text{if } r_C + 1 < i \leq m. \quad (35)$$

Substituting (32) and (33) in (34), we derive for $2 \leq i \leq r_C + 1$ that

$$\hat{x}_{Ci} = \frac{p\sqrt{\hat{x}_C V n_S} + (1-p)\sqrt{\hat{x}_C V n_L} - n_i \hat{x}_C}{pn_S + (1-p)n_L}. \quad (36)$$

Since $\hat{x}_C = \sum_{j=2}^{r_C+1} \hat{x}_{Cj}$, finding $\hat{x}_C$ is equivalent to solving

$$\left(pn_S + (1-p)n_L + \sum_{j=2}^{r_C+1} n_j \right) y^2 - \left(r_C p \sqrt{V n_S} + r_C (1-p) \sqrt{V n_L} \right) y = 0, \quad (37)$$

where $y = \sqrt{\hat{x}_C} > 0$. It follows that $\hat{x}_C = \phi_C(r_C)$, i.e., (14). Note that $\hat{x}_{Ci} > 0$ if and only if the numerator of (36) is positive, as the denominator is positive. Using this and the definition of r_C , we find

$$r_C = \max \left\{ i \in \{1, 2, \dots, m-1\} \mid p\sqrt{V n_S} + (1-p)\sqrt{V n_L} - n_{i+1}\sqrt{\phi_C(i)} > 0 \right\}, \quad (38)$$

which can be rewritten as (15). If (32)–(35) hold, there is a unique solution where $\hat{x}_{C1S}$ and $\hat{x}_{C1L}$ are found by solving (32) and (33), respectively. The values of $\hat{x}_{Ci}$ for $2 \leq i \leq r_C + 1$ can be derived from (36), while $\hat{x}_{Ci} = 0$ for $r_C + 1 < i \leq m$.

Proof of Lemma 2 Using (7), we directly obtain (16). Next, from (8), $\phi_B(i) < V/n_S$ can be written as

$$\begin{aligned} i^2 p^2 + 4 \left(p + \frac{1}{n_S} \sum_{j=2}^{i+1} n_j \right) (1-p)(i-1) &< (2-i)^2 p^2 \\ + 4(2-i) \frac{p}{n_S} \sum_{j=2}^{i+1} n_j + 4 \left(\frac{1}{n_S} \sum_{j=2}^{i+1} n_j \right)^2. \end{aligned}$$

Straightforward calculations give (17). Using (8), $\phi_B(i) \geq \frac{V}{n_L}$ can be written as

$$\begin{aligned} & i^2 p^2 n_S n_L + 4n_L \left(pn_S + \sum_{j=2}^{i+1} n_j \right) (1-p)(i-1) \\ & \geq \left(2 \left(pn_S + \sum_{j=2}^{i+1} n_j \right) - ip\sqrt{n_S n_L} \right)^2. \end{aligned}$$

Straightforward manipulations give (18). Finally, using (9), we easily find (19). $\square$

Proof of Lemma 4 For each of the cases A , B and C , when the associated FOCs hold, a unique solution is defined. If for a case not all four FOCs hold, the case cannot yield an equilibrium. In each case, the third and fourth FOC—i.e., (23) and (24) for case A , (28) and (29) for case B , and (34) and (35) for case C —always hold, as they are implied by the selection of r_A in (11), r_B in (13), and r_C in (15), respectively. This directly follows for the third FOC of each case. We give a proof of this for the fourth FOC in all three cases.

For case A , note that (11) implies for $i = r_A + 1, \dots, m-1$ that $n_{i+1} \geq V/\phi_A(i)$, which by Lemma 3 gives $n_{i+1} \geq V/\phi_A(r_A)$, which is equivalent to (24). For case B , (13) implies for $i = r_B + 1, \dots, m-1$ that

$$n_{i+1} \geq p\sqrt{\frac{Vn_S}{\phi_B(i)}} + (1-p)\frac{V}{\phi_B(i)}, \quad (39)$$

which by Lemma 3 gives

$$n_{i+1} \geq p\sqrt{\frac{Vn_S}{\phi_B(r_B)}} + (1-p)\frac{V}{\phi_B(r_B)}. \quad (40)$$

Remark that (40) is equivalent to (29) if you substitute in (26). For case C , notice that (38) implies for $i = r_C + 1, \dots, m-1$ that

$$n_{i+1} \geq \frac{p\sqrt{Vn_S} + (1-p)\sqrt{Vn_L}}{\sqrt{\phi_C(i)}}, \quad (41)$$

which by Lemma 3 learns that

$$n_{i+1} \geq p\frac{\sqrt{Vn_S}}{\sqrt{\phi_C(r_C)}} + (1-p)\frac{\sqrt{Vn_L}}{\sqrt{\phi_C(r_C)}}. \quad (42)$$

We see that (42) is equivalent to (35) if you substitute in (32) and (33).

Finally, considering the two FOCs of each case that do not hold for all parameter values—i.e., (21) and (22) for case A , (26) and (27) for case B , and (32) and (33) for case C —we can easily complete the proof of the lemma. $\square$

Proof of Proposition 1 – Step 1: If we have an equilibrium in case A, (16) and part (i) of Lemma 4 give

$$(r_A - 1)n_S \geq \sum_{j=1}^{r_A+1} n_j, \quad (43)$$

which by (17) implies $\phi_B(r_A) \geq V/n_S$. Using Lemma 3, this gives $\phi_B(r_B) \geq V/n_S$, which is inconsistent with an equilibrium for case B. Hence, the game cannot have an equilibrium in both case A and case B.

– Step 2: When we have an equilibrium in case C, we have by Lemma 3 and part (iii) of Lemma 4 that $\phi_C(r_A) < V/n_L$. By (19), and using $n_S < n_L$, we then have

$$\begin{aligned} (r_A - 1)(p\sqrt{n_S n_L} + (1-p)n_L) &< pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{r_A+1} n_j \\ \Rightarrow r_A(pn_S + (1-p)n_L) &< pn_S + (1-p)n_L + \sum_{j=2}^{r_A+1} n_j \\ \Rightarrow (r_A - 1)n_S &< \sum_{j=2}^{r_A+1} n_j. \end{aligned} \quad (44)$$

Note that (44) implies by (16) that $\phi_A(r_A) < V/n_S$, which contradicts with an equilibrium for case A. Hence, the game cannot have an equilibrium in both case A and case C.

– Step 3: When we have an equilibrium in case C, we have by Lemma 3 and part (iii) of Lemma 4 that $\phi_C(r_B) < V/n_L$. Then, (19) yields

$$(r_B - 1)(p\sqrt{n_S n_L} + (1-p)n_L) < pn_S - p\sqrt{n_S n_L} + \sum_{j=2}^{r_B+1} n_j, \quad (45)$$

which by (18) gives $\phi_B(r_B) < V/n_L$, implying there is no equilibrium in case B. Hence, the game cannot have an equilibrium in both case B and case C.

– Step 4: We finally show that the game must have an equilibrium in at least one of the cases A, B and C. Assume case B does not yield an equilibrium. Then, either $\phi_B(r_B) \geq V/n_S$ or $\phi_B(r_B) < V/n_L$. For the case $\phi_B(r_B) \geq V/n_S$, note that (17) implies

$$(r_B - 1)n_S \geq \sum_{j=2}^{r_B+1} n_j. \quad (46)$$

Using (7) and Lemma 3, we find that

$$(r_A - 1)n_S \geq \sum_{j=2}^{r_A+1} n_j. \quad (47)$$

Applying (16) to (47), we know that case A yields an equilibrium.

For the case $\phi_B(r_B) < V/n_L$, Lemma 3 implies that $\phi_B(r_C) < V/n_L$. From (18) and (19), we know that case C yields an equilibrium. $\square$

Proof of Proposition 2 The proof follows in a straightforward way from Lemmas 2, 3, 4, and the facts that the small-type group 1 is active in the game with private information if and only if $V/n_S > \phi_A(r_A)$, and group S is active in game B_S if and only if $V/n_S > \phi_S(r_S)$. $\square$

Proof of Proposition 3 Take game B_L . Group L in this game will be active if and only if (use 19)

$$(r_L - 1)n_L < \sum_{j=2}^{r_L+1} n_j. \quad (48)$$

We derive from (19) when the large-type group 1 is active (case C) in the private-information game. Note that (15) implies $r_L \geq r_C$. So, if $r_L = 1$, then $r_C = 1$ and group L will always be active in game B_L , while the large-type group 1 can be inactive in the game with private information. This gives part (i).

If $r_L \geq 2$ and $r_C = 1$, we compare (48) to (19) with $i = r_C = 1$ and conclude that both group L in game B_L and the large-type group 1 in the private-information game can be active and inactive, without in one game being (in)active necessarily implying (in)activity for the other. This proves part (ii).

Finally, we examine the situation with $r_L \geq 2$ and $r_C \geq 2$. Suppose that group L is active in the equilibrium of game B_L . By (15), we then have

$$\begin{aligned} & \frac{\sum_{j=2}^{r_L+1} n_j}{r_L} + \frac{n_L}{r_L} - n_{r_L+1} > 0 \\ & \Leftrightarrow 1 - \frac{r_L - 1}{\sum_{j=2}^{r_L+1} n_j} n_{r_L+1} - \frac{n_{r_L+1}}{\sum_{j=2}^{r_L+1} n_j} + \frac{n_L}{\sum_{j=2}^{r_L+1} n_j} > 0 \\ & \Rightarrow 1 - \frac{r_L - 1}{\sum_{j=2}^{r_L+1} n_j} n_{r_L+1} - \frac{n_{r_L+1}}{\sum_{j=2}^{r_L+1} n_j} + \frac{1}{r_L - 1} > 0, \end{aligned} \quad (49)$$

where (49) is due to (48). Suppose now that $r_L > r_A$. Then, (11) implies that

$$\frac{n_{r_L+1}}{\sum_{j=2}^{r_L+1} n_j} \geq \frac{1}{r_L - 1}. \quad (50)$$

If we substitute (50) in (49) and compare the result with (11), we obtain $r_A \geq r_L$, which gives a contradiction. Therefore, we have $r_A \geq r_L \geq r_C$. By Lemma 3 and (48), we now know that

$$\frac{1}{n_L} > \frac{r_L - 1}{\sum_{j=2}^{r_L+1} n_j} \Rightarrow \frac{1}{n_L} > \frac{r_C - 1}{\sum_{j=2}^{r_C+1} n_j}. \quad (51)$$

Note from (19) that the large-type group 1 in the private-information game is active if and only if

$$\begin{aligned} \sum_{j=2}^{r_C+1} n_j &> (r_C - 1) (p\sqrt{n_S n_L} + (1 - p)n_L) - pn_S + p\sqrt{n_S n_L} \\ &= pn_L \left(r_C \sqrt{\frac{n_S}{n_L}} - \frac{n_S}{n_L} \right) + (r_C - 1)(1 - p)n_L. \end{aligned} \quad (52)$$

Furthermore, because $0 < n_S < n_L$, we have for $r_C \geq 2$ that

$$\begin{aligned} \left(\sqrt{\frac{n_S}{n_L}} - 1 \right)^2 &> 0 \\ \Leftrightarrow 2 &> \frac{1 - \frac{n_S}{n_L}}{1 - \sqrt{\frac{n_S}{n_L}}} \\ \Leftrightarrow r_C - 1 &> r_C \sqrt{\frac{n_S}{n_L}} - \frac{n_S}{n_L}. \end{aligned} \quad (53)$$

By (53) and (51), we know that if group L in game B_L is active in case $r_C \geq 2$, then

$$\sum_{j=2}^{r_C+1} n_j > (r_C - 1)n_L > pn_L \left(r_C \sqrt{\frac{n_S}{n_L}} - \frac{n_S}{n_L} \right) + (r_C - 1)(1 - p)n_L. \quad (54)$$

Comparing (54) to (52), we conclude that if group L is active in game B_L , the large-type group 1 is also active in the game with private information. Note that the reverse does not hold in general. This proves part (iii). $\square$

References

- Congleton, R.D., Hillman, A.L., Konrad, K.A. (eds.): 40 Years of Research on Rent Seeking, Volumes I and II. Springer, Berlin (2008)
- Esteban, J., Ray, D.: Collective action and the group size paradox. *Am. Polit. Sci. Rev.* **95**(3), 663–672 (2001)
- Fey, M.: Rent-seeking contests with incomplete information. *Public Choice* **135**(3–4), 225–236 (2008)
- Hurley, T., Shogren, J.F.: Effort levels in a Cournot Nash contest with asymmetric information. *J. Public Econ.* **69**, 195–210 (1998a)
- Hurley, T., Shogren, J.F.: Asymmetric information contests. *Eur. J. Polit. Econ.* **14**, 645–665 (1998b)
- Hurley, T., Shogren, J.F.: Strategic behavior in incomplete information contests. In: Baye, M.R. (ed.) *Advances in Applied Microeconomics, Volume 7: Contests*, pp. 87–101. JAI Press Inc., London (1998c)
- Konrad, K.A.: *Strategy and Dynamics in Contests*. Oxford University Press, Oxford (2009)
- Lockard, A.L., Tullock, G. (eds.): *Efficient rent-seeking: chronicle of an intellectual quagmire*. Kluwer Academic, Boston (2001)
- Malueg, D.A., Yates, A.J.: Rent seeking with private values. *Public Choice* **119**(1–2), 161–178 (2004)
- Nitzan, S.: Collective rent dissipation. *Econ. J.* **101**, 1522–1534 (1991)
- Nitzan, S.: Modelling rent-seeking contests. *Eur. J. Polit. Econ.* **10**, 41–60 (1994)

- Ryvkin, D.: Contests with private costs: beyond two players. *Eur. J. Polit. Econ.* **26**, 558–567 (2010)
- Schoonbeek, L., Winkel, B.M.: Activity and inactivity in a rent-seeking contest with private information. *Public Choice* **127**(1), 123–132 (2006)
- Tullock, G.: Efficient rent seeking. In: Buchanan, J.M., Tollison, R.D., Tullock, G. (eds.) *Toward a Theory of the Rent-Seeking Society*, pp. 97–112. Texas A&M University Press, College Station (1980)
- Ueda, K.: Oligopolization in collective rent-seeking. *Soc. Choice Welf.* **19**, 613–626 (2002)
- Wärneryd, K.: Information in conflicts. *J. Econ. Theory* **110**, 121–136 (2003)