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Discretizing continuous-time continuous-state deterioration processes, with an application to condition-based maintenance optimization

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Abstract

We present an approach for discretizing stationary continuous-time continuous-state non-decreasing deterioration processes. This results in a discrete-time Markov chain that is represented by its transition probability matrix. Based on this approach, the easier specification of the more realistic continuous-time continuous-state models can be combined with the better analytical tractability of the discrete-time discrete-state models. We consider the gamma deterioration process as a special case and use the discretization approach combined with matrix algebra to optimize condition-based maintenance for a continuously monitored single-unit system.

Keywords: Deterioration process, Gamma process, Discretization, Markov chain, Condition-based maintenance

1. Introduction

Because of the ongoing developments in the fields of monitoring, storing and analyzing conditions of technical equipment, condition-based maintenance policies are gaining popularity. A prerequisite for analyzing condition-based maintenance policies is the modeling of deterioration processes. When equipment with a single condition parameter is considered, there are continuous-time continuous-state deterioration processes on one side of the spectrum, and deterioration processes with a finite state space in a discrete-time setting on the other side. The continuous-time continuous-state deterioration processes are most detailed and most realistic. Furthermore, they are often fully characterized by a few parameter values; specifying such processes is therefore relatively easy. The main advantage of discrete-time discrete-state deterioration processes is their analytical tractability: analyzing and optimizing maintenance policies with such deterioration processes is often much easier. However, these processes are more complicated to specify. The main aim of this study is to present a method for discretizing stationary continuous-time continuous-state deterioration processes so that the more straightforward specification of the continuous-time continuous-state models can be combined with the better analytical tractability of the discrete-time discrete-state models.

An example of a continuous-time continuous-state deterioration processes is the gamma process. It is a rather flexible process that is applicable to model a wide variety of deterioration processes, and it is therefore often a suitable choice. The gamma process was first introduced in the area of reliability by Abdel-Hameed (1975). Van Noortwijk (2009) lists some studies that present examples in which the gamma process fits well to deterioration data. Recent studies that

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adopt a gamma deterioration process include Bouvard et al. (2011), Caballé et al. (2015), Castro (2013), De Jonge et al. (2017), Do et al. (2015), Hong et al. (2014), Huynh et al. (2011), Mercier and Castro (2013), Mercier and Pham (2012), Peng and Van Houtum (2016), Van Horenbeek and Pintelon (2013), Shafiee et al. (2015), Wang et al. (2016), Zhang et al. (2014), and Zhang and Zeng (2017).

A special case of a discrete-time discrete-state deterioration process, on the other hand, is the discrete-time Markov chain. Such a process is characterized by specifying its transition probability matrix. The main reason for using a discrete-time Markov chain for modeling deterioration is its analytical tractability. For instance, it is well known that optimizing maintenance policies for multi-unit systems is much more difficult than for single-unit systems. By modeling the deterioration of each component by a discrete-time Markov chain, we obtain a finite state space for the entire system, which enables the use of dynamic programming to determine optimal policies. Examples of recent studies that model deterioration by a discrete-time Markov chain include Icten et al. (2013), Kurt and Kharoufeh (2010a,b), Olde Keizer et al. (2016, 2017), Van Oosterom et al. (2017), and Zhou et al. (2013, 2016).

There are various situations in which the discretization approach that we present is particularly useful. If a discrete-time Markov chain with a large number of deterioration states is used, it is tedious and complicated to manually choose a suitable transition probability matrix (see also Section 3). Instead, the described approach can be used to base this matrix on a practically realistic stochastic process, such as the gamma process. Another example is that of a production system with a deterioration process that depends on the production mode. In such a setting, each production mode has its own transition probability matrix, and these can easily be chosen based on e.g. gamma processes with different parameter values. A similar approach can be used for systems that operate in a stochastic environment, and for which the deterioration process is influenced by the environment (Fouladirad et al., 2008; Kurt and Kharoufeh, 2010a; Xiang et al., 2012), for heterogeneous populations of components that deteriorate according to different deterioration processes (Scarf and Cavalcante, 2012; Van Oosterom et al., 2017; Zhang et al., 2014), and for situations in which maintenance actions change the deterioration process of equipment (Kurt and Kharoufeh, 2010b; Nicolai et al., 2009; Zhang et al., 2015).

The remainder of this paper is organized as follows. We will present our discretization approach in Section 2. We start this section with general stationary continuous-time continuous-state non-decreasing deterioration processes, after which we consider the stationary gamma process as a special case. In Section 3, we use the discretization approach to analyze and optimize condition-based maintenance for a continuously monitored single-unit system that deteriorates according to a gamma process. We end with some conclusions in Section 4.

2. Discretization

In this section we will introduce our approach for discretizing stationary continuous-time continuous-state non-decreasing deterioration processes $\{X(t) : t \in \mathbb{R}_+\}$ with state space $\mathbb{R}_+$. We will refer to such a process as a continuous-time continuous-state process. The result of the discretization is a discrete-time discrete-state non-decreasing deterioration processes with stationary increments $\{Y(t) : t \in \mathbb{N}_0\}$ with state space $\mathbb{N}$, which we will refer to as a discrete-

time Markov chain (DTMC).

We let $F_{\Delta t}$ denote the (cumulative) distribution function of the additional amount of deterioration during a time period with length Δt in the continuous-time continuous-state process. Furthermore, we assume that the deterioration level at time $t = 0$ equals 0, i.e., $X(0) = 0$. We subdivide the state space into equally sized intervals with length Δx . The k th interval, denoted by x_k , is thus given by $x_k = [(k-1)\Delta x, k\Delta x]$, $k = 1, 2, \dots$. This k th interval will correspond to state k in the DTMC. We will also discretize time and denote the length of the time steps by Δt . Time $i\Delta t$ in the continuous-time continuous-state process corresponds to time i in the DTMC, $i = 0, 1, \dots$.

The next step is to define the transition probabilities of the DTMC. The probability of a one-step transition from state k to state $k+i$ in the DTMC coincides with the probability of moving from a deterioration level within the interval x_k at a time t to a deterioration level within the interval x_{k+i} at time $t + \Delta t$ in the continuous-time continuous-state process. However, the latter probability obviously depends on the exact deterioration level within the interval x_k at time t , whereas we need a single transition probability in the DTMC. The approximation approach that we will use is to assume that the deterioration level is uniformly distributed on the interval x_k when it is within this interval at an arbitrary moment in time. We note that the deterioration level at a fixed point in time is gamma distributed, but we are now interested in the distribution of the deterioration level given that it is within a certain interval at an arbitrary moment in time. Because a deterioration level within an interval x_k can be reached after any arbitrary number of time steps, and because we can come from any deterioration level at the previous time period, it is most reasonable to adopt a uniform distribution. Only for the first intervals this approximation is less accurate. As an illustration, the deterioration process always starts at the left endpoint 0 of the first interval x_1 . Thus, given that the process is in this interval, it is more likely that it is at deterioration level 0 than at any other level. However, if a small number of relatively large intervals is chosen, the process is expected to remain in the interval x_1 for some time, making the uniform distribution more reasonable. If, on the other hand, a large number of relatively small intervals is chosen, the exact distribution of the deterioration levels within the intervals becomes less important. The latter will also be seen in the example that we consider in Section 3.1. Furthermore, we also assume the uniform distribution for each interval because we aim to obtain a DTMC with stationary increments. Choosing different distributions for the intervals would result in a DTMC with non-stationary increments.

We let q_i , $i = 0, 1, 2, \dots$, denote the probability of not having exceeded interval x_{k+i} at time $t + \Delta t$ when the deterioration level at time t is uniformly distributed on the interval x_k , for arbitrary t and k . This probability equals

$$\begin{aligned} q_i &= P(X(t + \Delta t) \leq (k+i)\Delta x \mid X(t) \sim \text{Unif}(x_k)) \\ &= P(X(t + \Delta t) \leq (i+1)\Delta x \mid X(t) \sim \text{Unif}(x_1)) \\ &= \frac{1}{\Delta x} \int_0^{\Delta x} F_{\Delta t}((i+1)\Delta x - x) dx. \end{aligned}$$

In the DTMC, q_i will be the probability that a one-step transition brings us at most i states ahead. Then, also in the DTMC, we let p_i , $i = 0, 1, \dots$, denote the probability of moving from

a state k to a state $k + i$. We have that

$$\begin{aligned} p_0 &= q_0, \\ p_i &= q_i - q_{i-1}, \quad i = 1, 2, \dots \end{aligned}$$

The last step concerns the number of states in the DTMC. Studies on reliability and maintenance generally assume that failure occurs when a certain deterioration level L is exceeded. We subdivide the deterioration levels between 0 and L into m equally sized intervals with length $\Delta x = L/m$. All deterioration levels above L will be combined into a single $(m + 1)$ th interval $[L, \infty)$ that represents the failed state.

The discrete-time Markov chain is now described by an $(m+1) \times (m+1)$ transition probability matrix P . This matrix is upper diagonal, i.e., all entries P_{ij} , $j < i$, below the main diagonal are zero. Each entry P_{ij} with $1 \leq i \leq m, i \leq j \leq m$ equals probability p_{j-i} . Each entry $P_{i,m+1}$ in the final column equals the ‘remaining’ probability, i.e., $P_{i,m+1} = \sum_{k=m+2-i}^{\infty} p_k = 1 - \sum_{k=1}^{m+1-i} p_k$, $i \leq m$. Finally, $P_{m+1,m+1} = 1$, i.e., failed units will always remain in the failed state. Summarizing, we get

$$P_{i,j} = \begin{cases} p_{j-i}, & \text{if } i \leq j \leq m, \\ 0, & \text{if } j < i, \\ 1 - \sum_{k=1}^{m+1-i} p_k, & \text{if } i \leq m \text{ and } j = m+1, \\ 1, & \text{if } i = j = m+1. \end{cases}$$

2.1. Gamma process

We will continue to illustrate the discretization approach for a stationary gamma process, which is an example of a stationary continuous-time continuous-state process. We use the following definition for the density function f of the gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$:

$$f_{\alpha,\beta}(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} e^{-\frac{t}{\beta}}, \quad t > 0,$$

in which $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ denotes the gamma function. We let $F_{\alpha,\beta}(t)$ denote the corresponding (cumulative) distribution function. Because no closed-form expression exists for $F_{\alpha,\beta}$, it has to be evaluated numerically. The stationary gamma process has a shape function at with shape parameter $a > 0$ and a scale parameter $b > 0$. It is a stochastic process $\{X(t) : t \in \mathbb{R}_+\}$ with the following properties:

1. $X(0) = 0$ with probability 1;
2. $X(\tau) - X(t) \sim f_{a(\tau-t),b}$ for $\tau > t \geq 0$; and
3. $X(t)$ has independent increments.

We note that, although the gamma process is a continuous-time continuous-state process, the paths of the gamma process are not continuous. Within any time interval of arbitrary length, the gamma process makes an infinite number of jumps.

After specifying the discretization parameters Δt and Δx , and the parameters a and b of the gamma process, the probabilities q_i can be calculated as

$$q_i = \frac{1}{\Delta x} \int_0^{\Delta x} F_{a\Delta t, b}((i+1)\Delta x - x) dx, \quad i = 0, 1, 2, \dots$$

As an example, we will use our approach to discretize the gamma deterioration process with parameter values $a = 1$ and $b = 1$, and with failure deterioration threshold $L = 10$. If we set the number of deterioration states before failure equal to $m = 4$ (implying that $\Delta x = L/m = 2.5$) and the lengths of the time periods equal to $\Delta t = 1$, we get the following transition probability matrix P for the discrete-time Markov chain:

$$P = \begin{pmatrix} 0.632834 & 0.337027 & 0.027664 & 0.002270 & 0.000203 \\ 0.000000 & 0.632834 & 0.337027 & 0.027664 & 0.002473 \\ 0.000000 & 0.000000 & 0.632834 & 0.337027 & 0.030138 \\ 0.000000 & 0.000000 & 0.000000 & 0.632833 & 0.367166 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{pmatrix}.$$

Appendix A contains the implementation of a function that returns the transition probability matrix as a function of the input parameters a , b , L , m , and Δt in the scripting language R (R Core Team, 2013).

3. Condition-based maintenance optimization

In this section we will consider condition-based maintenance optimization for a single-unit system with a stationary continuous-time continuous-state non-decreasing deterioration process. We will use the discretization approach presented in the previous section to obtain a discrete-time Markov chain, and we will explain how the optimal condition-based maintenance policy can be determined by using matrix algebra. In Section 3.1 we will consider an example and compare the results of our approach to the outcomes of another study that is based on simulation, and that therefore requires long calculation times.

We will model the deterioration of the unit by a stationary gamma process with parameters a and b (see Section 2.1). The unit fails when deterioration level L is exceeded, after which corrective maintenance is required. Preventive maintenance can be carried out as long as failure has not occurred. The cost of corrective maintenance is c_{cm} , and the cost of preventive maintenance is c_{pm} , with $c_{pm} < c_{cm}$. Both maintenance types are assumed to require a negligible amount of time and to make the unit as-good-as-new, i.e., they set the deterioration level back to 0. The deterioration process of the unit is monitored continuously, and the preventive maintenance policy that we use is the control-limit policy, i.e., preventive maintenance is carried out when deterioration level M is reached. This preventive maintenance threshold M is the decision variable of the control-limit policy, and the aim is to determine the value for M that minimizes the long-run cost rate.

Based on the discretization approach presented in the previous section we transform the gamma deterioration process into a discrete-time Markov chain. After specifying the parameters a and b of the gamma process, the failure deterioration level L , the number of deterioration states m , and the length Δt of the time steps, we end up with a Markov chain with an $(m+1) \times (m+1)$

transition probability matrix P . Because failed units will remain in the failed state as long as no maintenance is carried out, the Markov chain is an absorbing Markov chain in which state $m + 1$ is the absorbing state. All other states are transient states. The transition probability matrix P can be written as

$$P = \begin{pmatrix} Q & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix},$$

with Q an $m \times m$ matrix, r a column vector of length m , and $\mathbf{0}$ a row vector with zeros of length m .

The probability of going from deterioration state $i \in \{1, \dots, m\}$ to deterioration state $j \in \{1, \dots, m\}$ in exactly k steps is equal to entry (i, j) of the matrix Q^k . We note that entries (i, j) with $j < i$ are 0 because we consider non-decreasing deterioration processes. By summing the matrix Q^k over $k = 0, 1, \dots$ we get the so-called fundamental matrix R . Entry (i, j) of this matrix indicates the expected number of visits to transient state j starting from transient state i , before being absorbed. It can be shown that

$$R = \sum_{k=0}^{\infty} Q^k = (I_m - Q)^{-1},$$

in which I_m is the $m \times m$ identity matrix. Entry (i, j) of the matrix R equals the expected number of time periods that the process is in deterioration state j before it reaches the failed state $m + 1$, given that it started in deterioration state i . When a unit is in the new state, it follows that the expected number of time periods until reaching the failed state equals the sum of all entries in the first row of the matrix R : $\sum_j R_{1j}$.

In the discrete-time model we let state $\bar{M} \in \{1, 2, \dots, m\}$ denote the preventive maintenance threshold, and we let $\bar{\eta}(\bar{M})$ denote the (long-run) cost rate as a function of this threshold. Because the unit is as-good-as-new after each maintenance action, standard renewal theory can be applied to calculate the cost rate (Resnick, 2013). We call the time between two consecutive maintenance actions a cycle, and we calculate the cost rate $\bar{\eta}(\bar{M})$ as the mean cost per cycle, denoted by $\bar{C}(\bar{M})$, divided by the mean cycle length, denoted by $\bar{D}(\bar{M})$. I.e., $\bar{\eta}(\bar{M}) = \bar{C}(\bar{M})/\bar{D}(\bar{M})$.

Because the deterioration process is non-decreasing, entry R_{1j} , $j < \bar{M}$, is not only the expected number of times that deterioration state j is visited before failure occurs, but also the expected number of times it is visited before reaching or exceeding deterioration state $\bar{M}$. Thus, under the control-limit policy, R_{1j} , $j < \bar{M}$, is the mean number of times deterioration state j is visited before maintenance (either preventive or corrective) is carried out. It follows that the mean cycle length equals $\bar{D}(\bar{M}) = \sum_{j < \bar{M}} R_{1j}$.

Every time that the deterioration level j is smaller than the preventive maintenance threshold $\bar{M}$, the probability that the next transition leads to a failure equals $P_{j,m+1}$. It follows that the probability that a cycle ends with a failure equals $\sum_{j < \bar{M}} R_{1j} P_{j,m+1}$, and thus that the probability that a cycle ends with preventive maintenance equals $1 - \sum_{j < \bar{M}} R_{1j} P_{j,m+1}$. This implies that

the mean cost per cycle can be written as

$$\begin{aligned}\bar{C}(\bar{M}) &= c_{\text{cm}} \sum_{j < \bar{M}} R_{1j} P_{j,m+1} + c_{\text{pm}} \left(1 - \sum_{j < \bar{M}} R_{1j} P_{j,m+1} \right) \\ &= c_{\text{pm}} + (c_{\text{cm}} - c_{\text{pm}}) \sum_{j < \bar{M}} R_{1j} P_{j,m+1}.\end{aligned}$$

We now have that the cost rate $\bar{\eta}(\bar{M})$, as a function of the preventive maintenance threshold $\bar{M}$, equals

$$\bar{\eta}(\bar{M}) = \frac{\bar{C}(\bar{M})}{\bar{D}(\bar{M})} = \frac{c_{\text{pm}} + (c_{\text{cm}} - c_{\text{pm}}) \sum_{j < \bar{M}} R_{1j} P_{j,m+1}}{\sum_{j < \bar{M}} R_{1j}}.$$

The deterioration threshold $\bar{M}$ that minimizes this cost rate is the optimal preventive maintenance threshold $\bar{M}^*$. The corresponding optimal cost rate equals $\bar{\eta}(\bar{M}^*)$.

Finally, we go back to the original continuous-time continuous-state model. We approximate the optimal preventive maintenance threshold M^* by the midpoint of the $\bar{M}^*$ -th deterioration interval of the discrete-time model:

$$M^* = \left(\bar{M}^* - \frac{1}{2} \right) L/m.$$

Furthermore, because time steps have length Δt in the continuous-time model instead of length 1 in the discrete-time model, the corresponding optimal cost rate $\eta(M^*)$ is approximated by

$$\eta(M^*) = \frac{\bar{\eta}(\bar{M}^*)}{\Delta t}.$$

3.1. Example

The evaluation of maintenance policies for units that deteriorate according to gamma processes involves some integrals that are complicated and burdensome to compute numerically (Mercier and Castro, 2013). Therefore, several studies have used simulation to optimize maintenance policies for units with a gamma deterioration process. However, simulation is often seen as the method of last resort. When using simulation, a large number of iterations are required to obtain stable results, and the system needs to be simulated for many values of the decision variable(s) to be able to approximate the optimal policy. As a consequence, simulation requires long computation times.

De Jonge et al. (2017) have used simulation to optimize the control-limit policy for the single-unit system that we consider in this section. They consider a base case with a stationary gamma deterioration process with parameter values $a = 5$ and $b = 0.2246$, and with a failure deterioration level of $L = 1$. This results in a mean time to failure of exactly 1. The cost of preventive maintenance is set equal to $c_{\text{pm}} = 0.2$, and that of corrective maintenance to $c_{\text{cm}} = 1$. They evaluate the cost rate $\eta(M)$ as a function of the preventive maintenance threshold M . The result is shown in Figure 1 (a), this figure is taken from De Jonge et al. (2017). However, in order to make this graph, the system needs to be simulated for a large number of values for M , and for each value of M a large number of iterations is needed to obtain an accurate approximation of the corresponding cost rate $\eta(M)$. The exact calculation time depends on the

number of values for M that are considered, the number of iterations, and the computer that is used; the generation of Figure 1 (a) required approximately 15 minutes calculation time.

We will continue to determine the optimal maintenance policy by using the discretization approach described in Section 2 and the matrix algebra described above. We use time periods with length $\Delta t = 0.01$, and we subdivide the deterioration levels between 0 and $L = 1$ into $m = 100$ intervals with length $\Delta x = 0.01$. The result is a discrete-time Markov chain with an 101×101 transition probability matrix P . The first row of the corresponding fundamental matrix R can be used to evaluate the cost rate $\bar{\eta}(\bar{M})$ for all values of $\bar{M} \in \{1, 2, \dots, m\}$. As a consequence, it requires approximately 0.33 seconds to calculate all these cost rates. We find that the cost rate $\bar{\eta}(\bar{M})$ in the discrete-time model is minimized for $\bar{M}^* = 65$ with corresponding optimal cost rate $\bar{\eta}(\bar{M}^*) = 0.0038$. This corresponds to an optimal preventive maintenance threshold of $M^* = 0.645$ in the original continuous-time model. The corresponding optimal cost rate is $\eta(M^*) = 0.3802$.

Figure 1 (b) shows the cost rate $\eta(M)$ as a function of the preventive maintenance threshold for the continuous-time model based on the discretization approach and matrix algebra. We observe that these results are virtually the same as those obtained by using simulation, and this is also the case for other problem instances. The main advantage of our suggested approach is its efficiency; long simulation times are avoided. Furthermore, based on the discretization approach, it is much easier to determine the optimal policy. By using simulation, the cost rate is always fluctuating somewhat around its exact value, making it difficult to identify the exact minimum. Our discretization approach always results in a unimodal function for the cost rate with a clear minimum. If desired, the accuracy of the outcome of the discretization approach can be improved by increasing the number of deterioration states m . If we increase m from 100 to 1,000, the calculation time increases to approximately 0.77 seconds, and we obtain a more precise optimal preventive maintenance threshold of $M^* = 0.6435$.

Figure 1: The cost rate $\eta(M)$ as a function of the preventive maintenance threshold M based on simulation (a) and based on the presented discretization approach (b).

4. Conclusion

We have presented an approach for discretizing stationary continuous-time continuous-state non-decreasing deterioration processes into discrete-time Markov chains with stationary incre-

ments. Furthermore, we have shown how matrix algebra can be used to optimize condition-based maintenance for a single-unit system that deteriorates according to such a discrete-time Markov chain, and we have pointed out the advantages of this approach as opposed to simulation.

Our approach can be used to specify transition probability matrices of discrete-time Markov chains that model deterioration, and is especially useful if a large number of deterioration states is considered. Furthermore, the approach is applicable in settings where the characteristics of the deterioration process change due to e.g. stochastic environmental factors, changing production modes, or maintenance activities that influence the deterioration rate of equipment.

The use of matrix algebra to optimize maintenance policies also encourages future research. The simple setting that we have considered could be extended with, e.g., a lead time between planning and initiating maintenance, imprecise condition monitoring, or randomness in the deterioration level at which failure occurs. Furthermore, we have restricted ourselves to stationary non-decreasing deterioration processes with a single condition parameter. Future research could consider non-stationary processes, processes that could also decrease, non-Markovian processes, or processes consisting of multiple condition parameters.

A. Implementation discretization stationary gamma process in R

Below is the implementation of a function in the scripting language R (R Core Team, 2013) for discretizing stationary gamma deterioration processes as described in Section 2.1. The inputs are the parameters a and b of the stationary gamma process, the failure deterioration threshold L , the number of deterioration states before failure m , and the lengths of the time periods Δt . The function returns the $(m + 1) \times (m + 1)$ transition probability matrix of the resulting discrete-time Markov chain.

```
TPM_gammaprocess = function(a,b,L,m,dt) {

  P = matrix(0,nrow=m+1,ncol=m+1)
  dx = L/m

  prob = function(i,dx,dt,a,b) {
    (1/dx)*integrate(pgamma,(i-1)*dx,i*dx,shape=a*dt,scale=b)$value
  }
  prob = Vectorize(prob,"i")

  q = prob(1:m,dx,dt,a,b)
  p = diff(c(0,q))

  for (i in 1:m) {
    P[i,i:m] = p[1:(m+1-i)]
    P[i,m+1] = 1-sum(P[i,1:m])
  }
  P[m+1,m+1] = 1

  P[P<0] = 0
  return(P)

}
```

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