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Dynamic replenishment from two sources with different yields, costs, and leadtimes

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ABSTRACT

We study a single-item periodic-review inventory system with two sources of supply: in-house production and outsourcing. The two sources differ in yields, costs, and leadtimes. Demands in consecutive periods are independent random variables and stockout is fully backlogged. Our problem is to characterize the optimal dynamic policy that simultaneously determines the in-house production quantity and the outsourcing quantity for each period to minimize the total discounted cost. We show that the optimal policy for in-house production can be characterized by a threshold and the production quantity is decreasing in the starting inventory. The optimal outsourcing policy, however, is more complex, and a threshold-type policy is only optimal under a more restrictive assumption about the yields. Comparative statics about the effects of yield uncertainty, outsourcing on the optimal policy, and cost performance are discussed.

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1. Introduction

In this paper, we address the problem of simultaneously determining inventory replenishment strategies from two sources with different leadtimes, yields and costs. The importance of the sourcing strategy has been emphasized by Erickson et al. (1995) in The Purchasing Handbook: “Supplier selection is the most important milestone in the purchasing process. The ultimate success of a new product, the profitability of a product line, and the timeliness of delivery to the marketplace may depend on this decision”. As Johnson (2006) also points out, it is quite common for a manufacturer or distributor to have different supply sources that have different grades of reliability in terms of the quantity and quality of orders delivered. When faced with this situation, it is imperative that replenishment decisions take into account supply uncertainty and related cost implications. Particularly, a company may need to protect itself from the uncertainty by having diversified suppliers. Indeed, after the fiasco of the UPS strike in the United States two years ago, most companies vowed that they would never ever use a single supplier again. Waart (2006) reports a case of Hewlett–Packard (HP). In 2000, HP was unable to obtain sufficient volumes of flash memory cards. Consequently, the company was unable to ship about 2,500,000 printers. Subsequently, HP took a dual-sourcing strategy, which resulted in annual procurement savings of more than 100 million by better managing supply and demand risk. A more recent example is Apple

Computer, which uses Motorola as a unique chip supply source for its very popular G4 model. Currently, Apple is suffering losses as a result of Motorola's inability to supply enough chips. By implementing dual-sourcing, GM has reduced its vehicle development cycle time from four years to 18 months (Gutmann, 2003). Airbus, reported by Wiesmann (2007), has ceded design and production of parts of its planned A350 aircraft to Russia's United Aircraft Corporation as flag-carrier Aeroflot firmed up an order for 22 of the wide-body jets. In other words, the aircraft maker's outsource strategy can secure orders despite fierce competition from US rival Boeing.

The above situations indicate that dual sourcing offers a solution for reducing the risk in the procurement process and closely matching supply with demand. However, how do companies find an effective way of deciding how to source, when and from whom? That is the key issue that our paper addresses. To the best of our knowledge, our paper is the first to consider the combined impacts of leadtimes, yields and costs for achieving an effective dual sourcing strategy. We assume that an original equipment manufacturer can order a critical material or component from two sources: in-house production and outsourcing. Such sourcing practices are popular in various North American industries, such as electronic appliances, automobile and toy.

We assume that in-house production has no leadtime while outsourcing has an one-period leadtime. The reason is that due to the geographical distance, the leadtime of outsourcing is usually quite long compared with the leadtime of in-house production. Moreover, this assumption is commonly used in the literature of dual sourcing, such as Anupindi and Akella (1993), in order to achieve the analytical tractability. Moreover, the variable costs of

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these two sources may be different. Next, we assume that the replenishment yield from in-house production is uncertain while the replenishment from outsourcing is certain. When the firm produces the material in his or her own factory, the production yield is uncertain due to the in-house operations, so that the quantity received from a replenishment is also uncertain (see [Henig and Gerchak, 1990](#)). Another case is that the domestic supplier has a finite capacity and receives multiple orders in each period (the firm's order is one of these orders and the number of orders may change from time to time). As a result, the firm's order from the domestic supplier can be modelled as an uncertain portion of the total amount produced by the domestic supplier in each period. The firm can apply a selective sourcing strategy to safeguard the quality of the manufacturing process at the overseas supplier's factory. For example, when products are being manufactured, a selective sourcing quality engineer is present to insure that the product is being made to the proper standards and quality level (see [McNeill et al., 2005](#)). Another suitable case is that the firm can select the overseas supplier through a third party, such as ONet (<http://www.outsourcingnetwork.net>), who will guarantee the quality of the product on behalf of the firm. Therefore, it is reasonable to assume that the replenishment from the overseas supplier is with certain yield.

The objective of this paper is to characterize the optimal dynamic policy that simultaneously determines the order quantities from two sources to minimize the total discounted cost. We show that the optimal replenishment policy from in-house production is of a threshold type, i.e., it is optimal to order if and only if the starting inventory in a period is below a threshold value. The optimal order quantity from in-house production decreases in the starting inventory. The optimal policy from outsourcing is more complex, and a threshold-type policy is optimal under a more restrictive assumption about the yields. To understand the implication of dual sourcing and random yields, we compare our model with two existing models in the literature: one includes dual sourcing but not uncertain yield; the other includes uncertain yields but only has one supply source. Comparing to the first model, we prove that the threshold of replenishment from in-house production with uncertain yield is higher than that with certain yield. In addition, we demonstrate that the suppliers' variable costs cannot dominate the decision of the choice of sourcing when the yield is uncertain; the optimal order quantity may not be monotone in yield variability; the yield variability deteriorates the cost performance. Next, by comparison with the second model, we find that although the dual-sourcing strategy can achieve a significant cost saving under both supply and demand uncertainties, the percentage cost saving becomes small when either supply or demand uncertainty becomes high. The contribution of this study is twofold. First, we analytically characterize the structure of optimal replenishment policies for the dual-sourcing model with different yields, costs, and leadtimes. Second, we develop fundamental insights about the impact of dual sourcing and yield uncertainty by comparison of the previous literature.

Our paper is closely related to two streams of research: lot sizing with random yields and replenishment from multiple suppliers or multiple delivery modes.

Research on lot sizing with random yields has been initialized by [Karlin \(1958\)](#) and received a lot of attention in operations research. Much work has been done on periodic-review and/or multistage models in dynamic settings, including [Henig and Gerchak \(1990\)](#), [Parlar and Wang \(1993\)](#), [Wang and Gerchak \(1996\)](#), and [Chen et al. \(2001\)](#). The focus of these papers is on the structure of the optimal policy. [Yano and Lee \(1995\)](#) provide a comprehensive review of research before 1995 in this area. The issues recently addressed related to uncertain yields are quite diverse, including for example, dynamic pricing under uncertain yield ([Li and Zheng, 2006](#)), applications in agricultural and food businesses ([Kazaz, 2004](#)), diversification

([Tomlin and Wang, 2005](#); [Dada et al., 2007](#); [Chopra et al., 2007](#)), and the impact of yield improvement on expected profits ([Gupta and Cooper, 2005](#)). Recently, [Xu \(2010\)](#) analyzes a decentralised supply chain comprised of a supplier and a manufacturer with random yield. The author finds that both the supplier and the manufacturer could gain a high expected profit under an optional contract. [Tang et al. \(2012\)](#) study a dynamic pricing strategy for the newsvendor problem with random yield. The authors characterize the optimal price and ordering decisions and compare the profit performances with dynamic and fixed pricing policies. They find that such a dynamic policy can result in a high benefit when demand uncertainty is low. [Wang et al. \(2014\)](#) considers a periodic-review model with random yield, disruption, and limited inventory capacity. Under the assumptions of an additive random yield and a linear production cost, the authors find that an order-up-to policy is optimal.

There is also an extensive literature related to either multiple suppliers or multiple delivery modes. [Fukuda \(1964\)](#) is the first to consider an inventory model with two delivery modes. The lead times of the two modes vary by exactly one period and the faster one is more expensive. The optimal policy is shown to have the following form. There are two base-stock levels. If the initial stock level is below the smaller base stock level, then an order with the faster mode is placed to bring the stock level back up to that base-stock level. No order with the faster mode is placed otherwise. The system stock level is then brought up to the larger base stock level by ordering with the slower mode. [Whittemore and Saunders \(1977\)](#) first analyze a two-supply-option model where the difference of delivery time can be more than one period. [Anupindi and Akella \(1993\)](#) study three different models with two uncertain suppliers. Different from our model, the authors do not simultaneously integrate the impact of leadtimes and yields into one model. [Güllü et al. \(1999\)](#) consider a single supplier with supply uncertainty. The supply availability is three categories: unavailable, partially available, and fully available. The authors find that an order-up-to policy is optimal. [Mohebbi \(2004\)](#) examines a continuous review of a single supplier with random lead time and disruption. [Feng et al. \(2006\)](#) show that when the number of modes is more than two, only the fastest two modes have optimal base stocks. [Veeraraghavan and Scheller-Wolf \(2008\)](#) extend the literature by allowing the difference between the lead times of the two modes to be more than one. [Wang et al. \(2010\)](#) investigate whether a buying firm should do multiple sourcing and/or improve supplier reliability. [Li et al. \(2009\)](#) and [Zhu \(2012\)](#) consider the use of expediting orders as an additional source in response to demand uncertainties. Note that none of the above mentioned papers consider the impact of uncertain yield. [Giri \(2011\)](#) studies a single-period dual-source model under yield uncertainty and risk aversion. Recently, [Kouvelis and Li \(2013\)](#) study how to use an emergency sourcing as contingent action after the yield uncertainty of the regular cycle order is realized. The authors explore the impact of yield and demand uncertainty on the cycle order size, the emergency order size, and the way to split the available good units between the fast and slow shipping modes. [Ahiska et al. \(2013\)](#) investigate a dual-source model with one reliable supplier and one unreliable supplier. The unreliable supplier status is controlled by a two-state Markov process. But, there is no lead time difference between two suppliers. Through numerical experimentation, the authors determine the structure of the optimal ordering policy. [Riezebos and Zhu \(2015\)](#) generalize theory on material requirements planning ordering by including the occurrence of dynamic lead-time variation and order crossovers in a multiple-supplier environment.

We end this section by describing the organization of the paper. In [Section 2](#), we introduce the notation, our assumptions, and some properties of the one-period profit function. In [Section 3](#), we formulate the finite-period problem with Markov decision programming and explore various structural properties of the optimal policies.

We investigate the operational effects of yield uncertainty and dual sourcing in Sections 4 and 5, respectively. We conclude the paper with some discussion of the results and suggestions for possible future research in Section 6. All proofs can be found in the appendix.

2. Model description

We consider a single-item, periodic-review inventory model with two sources of supply: in-house production and outsourcing. In period n , the unit variable cost of in-house production is c_n and that of purchasing from outside is c_n^o . In-house production is instantaneous but outsourcing takes a leadtime of one period for order processing, transportation, etc. We assume that the outside market is large enough and quality inspection is done at the supplier sites. Therefore, there is no capacity constraints on how much the firm can buy and each order is guaranteed to be delivered with satisfactory quality. For the in-house production, the quantity received from a replenishment is a random variable whose distribution depends on the production quantity. We assume a stochastically proportional yield model, i.e., in period n , a production run of q units will yield $\epsilon_n q$ usable units, where ϵ_n is a nonnegative random variable with mean $\bar{\epsilon}_n$ and support $[0, 1]$. This model is widely used in the literature (e.g., Li and Zheng, 2006). It applies to circumstances such as in electronic fabrication and assembly, where yield is sensitive to random environmental changes or variations in materials. In these circumstances, the yield rate could be relatively predictable under a specific environmental condition, but the condition itself is random. Hence, the yield rate distribution reflects the likelihood of various environmental conditions (Yano and Lee, 1995). The assumption that ϵ_n is upper bounded by 1 is made without loss of generality. For an arbitrary finite upper bound, rescaling of the production quantity will lead to the same qualitative results. The demand in period n is given by a random variable D_n with distribution function Φ_n . Demands in consecutive periods are independent. Stockout is fully backlogged.

The sequence of events is as follows. At the beginning of each period, the buyer first observes the current inventory level and simultaneously decides the quantities of in-house production and outsourcing. Then, the actual demand occurs and is fulfilled by the stock available. The unsatisfied demand is backordered. Finally, inventory costs (such as holding cost and backordering cost) are counted based on the inventory level at the end of the period.

Let $R_n(z)$ denote the holding and the backlog cost in period n when the ending inventory is z . We assume that $R_n(z)$ is convex in z , and $R_n(z) \rightarrow \infty$ as $|z| \rightarrow \infty$. In period n , if the initial inventory is x , production quantity is q and the order quantity from outsourcing is v , then the one-period expected cost is

$$L_n(x, q, v) = \mathbf{E}[R_n(x + \epsilon_n q - D_n)] + c_n q + c_n^o v. \quad (1)$$

The formulation in (1) assumes that the firm pays for all the quantity produced. This is the case for in-house production and the firm must pay for the cost regardless of the yield outcome. If random yield arises from an external supplier, however, it is more appropriate that costs are calculated based on the quantity received. In this case, the total expected purchasing cost is $\bar{c}_n q$, where $\bar{c}_n = c_n / \bar{\epsilon}_n$ represents the cost of getting a good unit of product. By rescaling c_n , one should get the same qualitative results. For outsourcing, we assume that payment is made when orders are placed, instead of when shipments arrive. However, this assumption is also made without affecting the qualitative results. We conclude this section by presenting a useful property of $L_n(x, q, v)$.

Lemma 1. $L_n(x, q, v)$ is continuous and jointly convex in (x, q, v) .

3. Structure of optimal replenishment policy

Suppose that the length of the planning horizon is N periods. Let $V_n(x)$ denote the optimal discounted cost from period n to the end of the planning horizon when the initial inventory at the start of period n is x . Then, from the standard theory of Markov decision processes (MDP), we have, for $n = 1, \dots, N$,

$$V_n(x) = \min_{0 \leq q, 0 \leq v} \tilde{J}_n(x, q, v), \quad (2)$$

where $\tilde{J}_n(x, q, v) = L_n(x, q, v) + \alpha \mathbf{E}V_{n+1}(x + v + \epsilon_n q - D_n)$.

For the terminal condition, we define $V_{N+1}(x) = 0$ for $x \in \mathcal{R}$. Here, α is the discount factor, with $0 \leq \alpha \leq 1$.

Lemma 2. For $n = 1, 2, \dots, N$, $\tilde{J}_n(x, q, v)$ are jointly convex in (x, q, v) and $V_n(x)$ is convex in x .

Because $\tilde{J}_n(x, q, v)$ is jointly convex in (x, q, v) , we can find the global optimum by the sequential optimization; i.e., we can rewrite (2) as

$$V_n(x) = \min_{0 \leq q} J_n(x, q), \quad (3)$$

where we define

$$J_n(x, q) = \mathbf{E}[R_n(x + \epsilon_n q - D_n)] + c_n q - c_n^o x + G_n(x, q), \quad (4)$$

$$G_n(x, q) = \min_{y \geq x} \{c_n^o y + \alpha \mathbf{E}V_{n+1}(y + \epsilon_n q - D_n)\}, \quad (5)$$

and $y = x + v$.

For the rest of the paper, we will focus our analysis on (3). Before we discuss the properties of (3), we first present the following technical lemma.

Lemma 3. If $h(z, y)$ is jointly convex and supermodular in (z, y) , then, for $a \leq b$,

$$f(x, y) = \min_{x+a \leq z \leq x+b} h(z, y).$$

is jointly convex and supermodular in (x, y) .

Lemma 4. For $n = 1, \dots, N$,

- (i) $G_n(x, q)$ is jointly convex and supermodular in (x, q) ;
- (ii) $J_n(x, q)$ is jointly convex and supermodular in (x, q) .

Denote $q_n^*(x) = \arg\min_{0 \leq q} J_n(x, q)$, $S_n(x) = \arg\min_y \{c_n^o y + \alpha \mathbf{E}V_{n+1}(y + \epsilon_n q_n^*(x) - D_n)\}$, and $v_n^*(x) = [S_n(x) - x]^+$. Note that $S_n(x)$ could be treated as the desirable stock level from outsourcing. In case of multiple solutions, $(q_n^*(x), v_n^*(x))$ is defined as the smallest minimizer. Throughout this paper, we shall use “decreasing” and “increasing” in the nonstrict sense to mean nonincreasing and nondecreasing, respectively. Note that both $q_n^*(x)$ and $v_n^*(x)$ are continuous in x given that both $J_n(x, q)$ and $G_n(x, q)$ are continuous in (x, q) .

Lemma 5. For $n = 1, \dots, N$, $S_n(x)$ is increasing in x and $q_n^*(x)$ is decreasing in x .

Lemma 5 indicates that when the initial stock is sufficient, the firm intends to reduce the order quantity from in-house production. Moreover, the firm prefers to set a higher stock level from outsourcing due to the reliability. However, a higher stock level does not always indicate a large order quantity as shown later.

Define

$$I_n = \inf\{x : q_n^*(x) = 0\}, \quad (6)$$

where I_n is the threshold value.

The following theorems characterize the structure of the optimal replenishment policies.

Theorem 1. For in-house production, the optimal policy is a threshold type: $q_n^*(x) = 0$ if $x \geq I_n$; $0 < q_n^*(x)$ if $x < I_n$.

Theorem 2. Denote $\hat{S}_n = S_n(I_n)$.

- (i) For $x \geq I_n$, $S_n(x) = \hat{S}_n$ and $v_n^*(x)$ is decreasing in x .
- (ii) If $I_n \geq \hat{S}_n$, then $v_n^*(x) = 0$ for $x \geq \hat{S}_n$.
- (iii) If $I_n < \hat{S}_n$, then $v_n^*(x) = \hat{S}_n - x > 0$ for $x \in [I_n, \hat{S}_n)$ and $v_n^*(x) = 0$ for $x \geq \hat{S}_n$.

Theorem 1 provides a full characterization of the optimal policy for in-house production: there exists a threshold such that an order is placed if and only if the initial inventory is below the threshold. The characterization of the ordering decision from outsourcing by Theorem 2, however, is only partial; there is a region in both (ii) and (iii) where the behavior of v_n^* is unclear. One may easily hypothesize that $v_n^*(x)$ may also be decreasing and hence ordering from outsourcing also follows a threshold-type policy. However, the following example shows that this does not hold in general.

In this example, we consider two periods and all parameters are stationary. Demand is deterministic and given by 50 per period. The one-period inventory cost is $R_n(z) = 0.1(z)^+ + 13(-z)^+$, where 0.1 represents the unit holding cost and 13 represents the unit backorder cost. The random yield is modelled by $\epsilon_n = 0.8$ with probability 0.6 and $\epsilon_n = 0.4$ with probability 0.4. Other parameters are $\alpha = 0.7$, $c = 2$, and $c^o = 2.2$. For this example, after solving the two-period model as a special case of the MDP defined by (2), optimal policy is given by

$$v_1^*(x) = \begin{cases} 0, & x < 0; \\ x, & 0 \leq x \leq 50; \\ 100 - x, & 50 < x < 100; \\ 0, & x \geq 100, \end{cases}$$

and

$$q_1^*(x) = \begin{cases} \frac{50-x}{0.4}, & x \leq 50; \\ 0, & x > 50. \end{cases}$$

The optimal policy is shown in Fig. 1. As we can see from the figure, $v_1^*(x)$ is not decreasing in x . In particular, it is strictly increasing in $[0, 50]$. To understand why this happens, let us examine the optimal policy when $x \in [0, 50]$. Note that in this example, the penalty cost is much larger than the holding cost and hence preventing stockout is the primary concern. The order q_1^* can result in either high or low yields and it is chosen such that

Fig. 1. The optimal policy in period 1.

there is no stockout even when the yield is low. If the yield is low, another order from in-house production should be placed again in period 2. If the yield is high, then no order from either source should be placed in period 2; instead, the unused inventory carried over from period 1 and the order from outsourcing, which is placed in period 1 but arrives in period 2, will be exactly enough to fill the demand in period 2. When x increases, q_1^* decreases, which means that the unused inventory carried over to period 2 is decreasing in x . Therefore, to fill the same demand in period 2, a larger $v_1^*(x)$ needs to be placed.

Lemma 6.

$$V'_n(x) \begin{cases} \leq -\bar{c}_n & \text{if } x < I_n; \\ \geq -\bar{c}_n & \text{if } x \geq I_n. \end{cases}$$

By Lemma 6, when $x < I_n$, the marginal benefit of having more initial inventory is greater than the average cost of producing one (good) unit; when $x \geq I_n$, the marginal benefit of having more initial inventory is smaller than the same average cost. These are consistent with our conclusions that it is optimal to get supply from local source when $x < I_n$, and not to order when $x \geq I_n$. Similar results have appeared in Henig and Gerchak (1990) and Li and Zheng (2006) under different settings.

Now, let us consider a special case of the stochastic proportional yield model. Suppose the yield of the in-house production is given by

$$\epsilon_n = \begin{cases} 1 & \text{w.p. } \beta_n; \\ 0 & \text{w.p. } \bar{\beta}_n = 1 - \beta_n. \end{cases} \quad (7)$$

This is known as all-or-nothing yield model. All-or-nothing yields may arise due to batch failures, acceptance sampling, supply chain disruptions, or supplier delays if the delivery occurs too late to serve demand and the model has been frequently used in the literature (e.g., Anupindi and Akella, 1993; Tomlin and Wang, 2005; Dada et al., 2007).

Under the all-or-nothing yield model, we have

$$V_n(x) = \min_{0 \leq q, 0 \leq v} \mathbf{E}[\{\bar{\beta}_n R_n(x - D_n) + \beta_n R_n(x + q - D_n)\} + c_n q + c_n^o v + \alpha[\bar{\beta}_n V_{n+1}(x + q - D_n) + \beta_n V_{n+1}(x + q + v - D_n)]].$$

Theorem 3. Under the all-or-nothing yield model

- (i) $v_n^*(x)$ is decreasing in x .
- (ii) Let $I_n^o = \inf\{x : v_n^*(x) = 0\}$. Then, I_n^o is the threshold value for replenishment from outsourcing: order $v_n^*(x) > 0$ if $x < I_n^o$, and do not order if $x \geq I_n^o$.

By Theorems 1 and 3, for the all-or-nothing yield model, the optimal policy for both the in-house production and outsourcing is of a threshold type.

In what follows, we suppose that the yield distribution is stationary and hence the subscript of β_n can be dropped. We examine the effects of β on the optimal policy and the optimal cost function. Note that a change in β affects not only the yield variability but also the cost of obtaining a good unit of product. To focus on the effects of variability, we assume that $c_n = \beta \bar{c}_n$, where $\bar{c}_n$ is independent of the yield distribution.

Theorem 4. If the yield distribution is stationary and $c_n = \beta \bar{c}_n$, then

- (i) Both I_n and I_n^o are decreasing in β .
- (ii) $V'_n(x)$ is increasing in β and $V_n(x)$ is decreasing in β .

Note that $(1-\beta)/\beta$ represents the squared coefficient of variation of the yield distribution and it is decreasing in β . Therefore, Theorem 4 implies that the higher the yield variability, the higher the cost, and the more frequently the firm orders to hedge against the risk caused by uncertain yield.

Next, to study the impact of the unit production and purchasing costs on the thresholds and the optimal solution, we denote $c_n^o = \gamma_n c_n$ without loss of generality, where γ_n reflects the cost difference between the unit production cost and the unit purchasing cost in period n .

Theorem 5. Suppose that the unit production cost and the unit purchasing cost are stationary, i.e., $c_n = c$, $c_n^o = c^o$, $c^o = \gamma c$. We have

- (i) I_n and $q_n^*(x)$ are increasing in γ .
- (ii) I_n^o and $v_n^*(x)$ are decreasing in γ .
- (iii) There exists a unique $\underline{\gamma}_n$ and $\bar{\gamma}_n$ such that $q_n^*(x) = 0$ for $\gamma \leq \underline{\gamma}_n$ and $v_n^*(x) = 0$ for $\gamma \geq \bar{\gamma}_n$.
- (iv) $V_n'(x)$ is decreasing in γ and $V_n(x)$ are increasing in γ .

Theorem 5 indicates that when the cost difference between the unit production cost and the unit purchasing cost is significant (i.e., $\gamma \leq \underline{\gamma}_n$), the in-house production is not used with $q_n^*(x) = 0$. Similarly, when the cost difference between the unit production cost and the unit purchasing cost is marginal (i.e., $\gamma \geq \bar{\gamma}_n$), the outsourcing replenishment is not used with $v_n^*(x) = 0$.

4. Effects of yield uncertainty

One of the important features of our model is its consideration of uncertain yield for in-house production. A natural question to ask is: what are the potential benefit of removing this uncertainty that could result from quality improvement efforts or learning over time? In this session, we compare our model with the dual-sourcing certain-yield model. We use the superscript c to signify the notation for the model with certain yield. The optimality equation for the model under certain yield is

$$V_n^c(x) = \min_{0 \leq q, 0 \leq v} \{E[R_n(x+q-D_n) + \bar{c}_n q + c_n^o v + \alpha V_{n+1}^c(x+v+q-D_n)]\}. \quad (8)$$

For (8), we can show that the optimal policy for both the in-house production and outsourcing is a base-stock type. Define

$$I_n^c = \inf\{x : q_n^c(x) = 0\}, \quad (9)$$

where I_n^c represents a base-stock level for in-house production.

By following the similar argument of Theorem 1 by Fukuda (1964), we can show that the optimal replenishment decisions are given by

$$q_n^c(x) = \begin{cases} I_n^c - x, & x < I_n^c; \\ 0, & x \geq I_n^c, \end{cases} \quad (10)$$

and

$$v_n^c(x) = \begin{cases} (\hat{S}_n^c - I_n^c)^+, & x < I_n^c; \\ (\hat{S}_n^c - x)^+, & x \geq I_n^c, \end{cases} \quad (11)$$

where

$$\hat{S}_n^c = \arg \min_y \{c_n^o y + \alpha E V_{n+1}^c(y - D_n)\},$$

which represents the base-stock level for outsourcing. Both $q_n^c(x)$ and $v_n^c(x)$ are decreasing in x .

Lemma 7.

$$V_n^{c'}(x) \begin{cases} = -\bar{c}_n, & x < I_n^c; \\ \geq -\bar{c}_n, & x \geq I_n^c. \end{cases}$$

The following theorem examines the effects of yield uncertainty on the optimal policy and the cost. As a change in yield rate also affects the marginal cost of obtaining a (good) unit, we assume $c_n = \bar{c}_n \bar{c}_n$ in order to focus on the impact of yield uncertainty.

Theorem 6. If $E R_n(x - D_n)$ is continuous and differentiable almost everywhere, and $c_n = \bar{c}_n \bar{c}_n$: (i) $\hat{S}_n \geq \hat{S}_n^c$; (ii) $I_n \geq I_n^c$; (iii) $V_n'(x) \leq V_n^{c'}(x)$; (iv) $V_n(x) \geq V_n^c(x)$; (v) $q_n^c(x) \leq q_n^*(x)$.

Part (i) of Theorem 6 states that if in-house production is not used in the current period, the firm would raise the inventory to a higher level when the production yield is uncertain than otherwise, simply because supply is not reliable in the future. Part (ii) compares the thresholds for in-house production. A higher threshold will allow diversification over time when yield is uncertain. The value of having one more unit in starting inventory is higher under uncertain yield because supply in future periods is not guaranteed, and yield uncertainty leads to a higher total expected cost, as stated in parts (iii) and (iv), respectively. Finally, (v) indicates that the yield uncertainty causes the increase of the in-house production quantity.

Note that under certain yield, if $c_n^o \geq \alpha \bar{c}_n$, the outsourcing is never used because in-house production is more beneficial in both lead time and cost. However, when yield is uncertain, we find that the firm may still outsource even when $c_n^o > \alpha \bar{c}_n$ (see Fig. 1). The presence of yield uncertainty significantly complicates the sourcing decisions.

In the previous analysis, we have been able to compare the optimal policies under uncertain and certain yields. Next, we intend to investigate the impact of the variation of the yield distribution on the firm's replenishment strategy and performance. Let us consider the following mean-preserving transformation: $\epsilon = \theta \xi + (1-\theta)/2$, where $0 \leq \theta \leq 1$, and ξ follows a uniform distribution on $[0, 1]$. Note that ϵ follows a uniform distribution on $[\frac{1-\theta}{2}, \frac{1+\theta}{2}]$. The variance and the coefficient of variation of ϵ are increasing in θ . We use a two-period stationary model and assume $c=2.4$, $c^o=2$, $\alpha=0.6$, and zero initial inventory. The demand per period is constant at d and $R_n(x) = h(x)^+ + b(-x)^+$, where h is the unit holding cost and b is the unit backlog cost. For a benchmark, $h=1$ and $b=12$. For the experiments, we change one parameter at one time. Numerical studies about how the optimal order quantities in period 1 and the minimal total expected cost change when θ is increased are reported in Tables 1 and 2.

From Tables 1 and 2, we observe that when the unit holding cost is relatively high ($h=1.5$) or the unit backlog cost is relatively low ($b=6$), q is decreasing in θ and v is increasing in θ . The reason is that when the yield variability becomes larger, the buyer intends to order less from the uncertain source and order more from the reliable source to fulfill the demand. What might appear to be surprising is that the order quantities from in-house production and outsourcing are not always monotone in yield variability under the condition that b is high or h is low. Note that the supply of in-house production consists of two parts, $\theta \xi$ and $(1-\theta)/2$. When θ is small, the second part dominates the first. Because ensuring supply to avoid shortages is the primary concern under this cost structure, the quantity from relatively reliable in-house production is higher than that from outsourcing. When θ is increased, $(1-\theta)/2$ is reduced or the portion of certain supply is reduced, and hence the firm will respond by increasing the order quantity from in-house production and reducing it from outsourcing. However, when θ is large, the situation becomes

different. For this case, since in-house production is very unreliable, the firm prefers to fulfill the current demand in the next period by ordering from outsourcing with lower cost instead of meeting the current demand by ordering from the very unreliable source. Further, the expected cost become higher when the yield variability increases, which is consistent with an observation by Gupta and Cooper (2005).

5. Benefits of outsourcing

Does the firm necessarily reduce the quantity from in-house production when it buys from outsourcing? How much benefit can outsourcing bring? This section is devoted to answering these questions. The model without outsourcing is formulated as (the superscript s representing sole sourcing)

$$V_n^s(x) = \min_{0 \leq q^s} \{R_n(x + \epsilon_n q^s - D_n) + c_n q^s + \alpha V_{n+1}^s(x + \epsilon_n q^s - D_n)\}.$$

Let $I_n^s = \inf\{x : q_n^s(x) = 0\}$.

Next, we can show that $I_n \leq I_n^s$ and $q_n^*(x) \leq q_n^s(x)$, which indicates that the firm will order smaller amount and less frequently from the in-house production when an alternative source exists.

We perform the sensitivity analysis on the impact of dual sourcing with respect to unit holding cost (h), unit shortage cost (b), demand variability, and yield variability on the total cost. We consider a 10-period stationary model. As a benchmark, cost parameters are set as $h=0.5$, $b=10$, $c=4$, $c^o=2$. Demand is assumed to follow the normal distribution with mean equal to 100 and coefficient of variation equal to 0.1. Yield is also assumed to follow the normal distribution with mean equal to 0.5 and coefficient of variation equal to 0.1. $\alpha=1$ and the initial inventory is set at 0. For the experiments, we only change one parameter at one time. We compare the total expected cost of the sole-source model (V^s) with that of the dual-source model (V) under difference settings. The cost-saving percentage is computed by $(V^s - V)/V^s$.

From Tables 3 to 6, we find that (i) the dual-sourcing strategy results in a considerable amount of cost saving (25–45%) under all the considered scenarios, consistent with the findings of Wang et al. (2010); (ii) the dual-sourcing strategy may achieve a higher cost saving when the initial inventory is low; (iii) the variabilities of demand and yield have a more significant impact on cost saving

than the cost parameters do; (iv) the percentage cost saving becomes small when demand or yield uncertainty increases, which indicates that either demand or yield variability results in a higher increase of the cost performance of the dual-sourcing model than that of the sole-sourcing module. Therefore, facing the demand and yield uncertainty, it is imperative for firms to carefully implement multiple-sourcing strategy in order to hedge the supply risk and maintain the cost efficiency.

6. Extension

In this section, we discuss one extension of the current model with uncertain outsourcing. When an order is outsourced to an overseas supplier, normally either the whole order arrives on time or the whole order is cancelled due to the potential disruption. Thus, similar to (7), we assume that the uncertainty of outsourcing is modelled by the all-or-nothing model, i.e., the outsourcing order can be received with probability γ_n in period n . We can rewrite (2) as

$$V_n(x) = \min_{0 \leq q, 0 \leq v} \{L_n(x, q, v) + \alpha[\gamma_n \mathbf{E}V_{n+1}(x + v + \epsilon_n q - D_n) + (1 - \gamma_n) \mathbf{E}V_{n+1}(x + \epsilon_n q - D_n)]\}.$$

By following the similar analysis in Section 3, we can prove that all the main results (Theorems 1–4) still hold for this model. However, when the uncertainty of outsourcing is also modelled by a stochastic yield model, we find that the in-house production may not follow the threshold-type policy and both the in-house production quantity and outsourcing quantity may not be decreasing in the initial inventory level. Namely, the characterization of the optimal policy could be a direction of future research.

7. Concluding remarks

Our work addresses inventory replenishment strategies from two sources with different leadtimes, yields and costs. We first formulate this problem as a stochastic dynamic programming model and then characterize the structure of the optimal policies for both in-house production and outsourcing. Further, we investigate operational effects of dual sourcing and yield uncertainty on the optimal policies and cost function by comparison of two common models: the certain-yield

Table 1
Effects of increasing yield variability with respect to h .

Parameter	$h=0.5$			$h=1$			$h=1.5$		
	q	v	Cost	q	v	Cost	q	v	Cost
0.2	104.29	44.80	376.11	100.17	46.43	380.81	96.88	47.81	385.99
0.4	104.85	43.99	419.25	100.24	46.30	425.98	93.75	48.11	432.61
0.6	99.08	48.25	465.91	96.90	50.53	473.73	86.50	52.21	481.09
0.8	89.44	55.30	510.98	82.57	56.91	518.94	77.48	58.06	526.28
1	78.92	62.83	551.22	72.98	63.58	558.86	68.56	64.07	565.85

Table 2
Effects of increasing yield variability with respect to b .

Parameter	$b=6$			$b=12$			$b=18$		
	q	v	Cost	q	v	Cost	q	v	Cost
0.2	87.83	52.14	367.2	100.24	46.43	380.81	106.39	43.6	386.56
0.4	79.61	53.54	388.27	98.62	46.3	425.98	110.37	41.8	445.86
0.6	70.88	56.18	407.04	91.9	50.53	473.73	107.04	46.33	515.85
0.8	62.78	59.19	422.9	82.57	56.91	518.94	97.93	54.84	587.97
1	55.74	62.09	436.07	72.98	63.58	558.86	86.65	64.23	654.87

Table 3Effect of holding cost h on the cost efficiency.

Parameter	Init. inv.=0			Init. inv.=30			Init. inv.=50		
	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$
0.50	1656.94	916.12	44.71	1413.77	855.80	39.47	1251.72	832.14	33.52
1.50	1674.23	931.71	44.35	1430.77	870.01	39.19	1268.49	845.83	33.32
2.50	1687.48	945.16	43.99	1443.72	882.11	38.90	1281.23	857.27	33.09
3.50	1698.58	956.98	43.66	1454.52	892.64	38.63	1291.83	867.08	32.88%

Table 4Effect of shortage cost b on the cost efficiency.

Parameter	Init. inv.=0			Init. inv.=30			Init. inv.=50		
	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$
10	1656.94	916.12	44.71	1413.77	855.80	39.47	1251.72	832.14	33.52
20	1724.76	973.45	43.56	1478.59	907.77	38.61	1314.68	881.49	32.95
30	1760.00	1006.19	42.83	1510.19	935.25	38.07	1345.29	915.60	31.94
40	1771.88	1020.96	42.38	1522.68	951.22	37.53	1356.36	924.90	31.81

Table 5

Effect of demand uncertainty on the cost efficiency.

Parameter	Init. inv.=0			Init. inv.=30			Init. inv.=50		
	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$
CV									
0.1	1656.94	916.16	44.71	1413.77	855.80	39.47	1251.72	832.12	33.52
0.3	1733.41	1121.86	35.28	1489.28	1020.90	31.45	1327.93	905.83	31.79
0.6	1965.78	1453.34	26.07	1635.67	1298.13	20.64	1520.85	1070.45	29.62
0.9	2312.99	1833.64	20.72	1911.45	1602.01	16.19	1818.43	1335.22	26.57

Table 6

Effect of yield uncertainty on the cost efficiency.

Parameter	Init. inv.=0			Init. inv.=30			Init. inv.=50		
	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$	V^s	V	$\frac{V^s - V}{V^s} (\%)$
CV									
0.1	1656.94	916.16	44.71	1413.77	855.8	39.47	1251.72	832.12	33.52
0.2	1694.11	976.45	42.36	1438.7	881.44	38.73	1270.43	858.17	32.45
0.3	1744.48	1040.37	40.36	1466.27	916.99	37.46	1286.19	877.36	31.79
0.4	1804.36	1068.70	40.77	1496.88	943.77	36.95	1301.29	885.56	31.95

model with dual sourcing and the single-source model with uncertain yield. Our key findings are summarized as follows:

- The optimal replenishment policy for in-house production is a threshold type.
- The optimal replenishment policy for outsourcing is much complicated in general. For the all-or-nothing yield model, it is a threshold type.
- The outsourcing decision is determined by both the supplier's reliability and variable costs.
- The replenishment threshold from the source with uncertain yield is higher than that with certain yield. Yield uncertainty always leads to a higher cost.
- Under a mean-preserving model, the expected cost of the firm increases when the variation of the yield becomes large.
- Cost saving achieved by the dual-sourcing strategy decreases in the variability of demand and yield.

Besides the extension discussed in Section 6, there are several more possible directions for future study. First, we may explore the optimal policy under a general delay structure. Assume that the lead-time of the in-house production is L periods while that of outsourcing is $L+m$. For $m=1$, by redefining the state variable x as inventory position instead of inventory level, we can prove that the results in Section 3 still hold. However, if the lead-time difference m is more than 1, then the dimension of state space is increased to m dimensions. For example, for $m=2$, (2) is rewritten as

$$V_n(x, x_1) = \min_{0 \leq q, 0 \leq v} \{L_n(x, q, v) + \alpha EV_{n+1}(x + x_1 + \epsilon_n q - D_n, v)\},$$

where x_1 is the order quantity received at the beginning of period $n+1$. As Whittemore and Saunders (1977) point out, for $m > 1$, the optimal policy becomes very complicated, which requires multi-dimensional dynamic programming to find the optimal solution.

Instead, it may be useful to develop a heuristic based on [Veeraraghavan and Scheller-Wolf \(2008\)](#), which could be another direction for future research.

Second, we can extend the current model to the infinite horizon. We may prove that [Theorems 1 and 2](#) still hold. And we can consider the price-sensitive demand for the current model and figure out the joint pricing and replenishment strategy under a dual-sourcing scenario.

Third, since [Wang and Gerchak \(1996\)](#) point out that there are many factors which cause production capacity to be variable, such as unexpected breakdowns and unplanned maintenance, one could also integrate the variable capacity into our model and analyze the impact of the capacity uncertainty on the replenishment policy and the cost-efficiency of the firm.

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Appendix A

Proof of Lemma 1. The continuity is obvious, because all the three terms at the right-hand side of (1) are continuous. To show the convexity, note that the second term and the third at the right-hand side of (1) are linear. It is then sufficient to show the convexity of $\mathbf{E}[R_n(x + \epsilon_n q - D_n)]$. This is true because $R_n(\cdot)$ is convex and $x + \epsilon_n q - D_n$ is linear in x and q . $\square$

Proof of Lemma 2. Clearly, it is true for $n = N + 1$. Suppose that $V_{n+1}(x)$ is convex in x . Then, $\mathbf{E}[V_{n+1}(x + v + \epsilon_n q - D_n)]$ is jointly convex in (x, q, v) because $x + v + \epsilon_n q - D_n$ is linear in x, q , and v . By [Lemma 1](#), $\hat{J}_n(x, q, v)$ is jointly convex in (x, q, v) , which further implies that $V_n(x)$ is convex in x because minimization preserves convexity. $\square$

Proof of Lemma 3. The proof is similar to that of [Lemma 3.2](#) in [Chao et al. \(2009\)](#) and hence omitted. $\square$

Proof of Lemma 4. Denote $\tilde{G}_n(y, q) = c_n^0 y + \alpha \mathbf{E}V_{n+1}(y + \epsilon_n q - D_n)$. Since V_{n+1} is convex, $\tilde{G}_n(y, q)$ is jointly convex in (y, q) . By [Lemma 2.6.2](#) in [Topkis \(1998\)](#), $\tilde{G}_n(y, q)$ is supermodular in (y, q) . Then, part (i) is simply an application of [Lemma 3](#). For part (ii), since $R_n(x + \epsilon_n q - D_n)$ is jointly convex and supermodular in (x, q) for each sample of ϵ_n , by part (i), $J_n(x, q)$ is also jointly convex and supermodular in (x, q) . $\square$

Proof of Lemma 5. The proof follows directly from [Lemma 4](#).

Proof of Theorem 1. The proof immediately follows from [Lemma 5](#).

Proof of Theorem 2. (i) For $x \geq I_n$, by [Theorem 1](#), we have $q_n^*(x) = 0$ and $S_n(x) = S_n(I_n)$ follows from the definition of $S_n(x)$. Consequently, $v_n^*(x) = [S_n(I_n) - x]^+$ for $x \geq I_n$. Thus, $v_n^*(x)$ is decreasing in x . (ii) When $x \geq I_n$, we have $v_n^*(x) = [\hat{S}_n - x]^+ = 0$ because of $I_n \geq \hat{S}_n$. When $x \in [\hat{S}_n, I_n]$, $v_n^*(x) = [S_n(x) - x]^+ \leq [S_n(I_n) - x]^+ = [\hat{S}_n - x]^+ = 0$, where the inequality is true because $S_n(x)$ is increasing in x . The proof for (iii) is similar and thus omitted. $\square$

Proof of Lemma 6. Let us consider the following two cases: $x < I_n$ and $x \geq I_n$.

Case 1: $x < I_n$. The proof is by contradiction. First, suppose that $V'_n(x) > -\bar{c}_n$ for some $x < I_n$. Then, from the convexity of $V_n(x)$, we have $V'_n(y) \geq V'_n(x) > -\bar{c}_n$ for any $y > x$, which implies $V_n(x + \epsilon_n q_n^*) > V_n(x) - \bar{c}_n \epsilon_n q_n^*$ for any sample of ϵ_n , and hence,

$$\mathbf{E}V_n(x + \epsilon_n q_n^*) > V_n(x) - c_n q_n^*. \quad (12)$$

Second, we have

$$V_n(x) = \mathbf{E}R_n(x + \epsilon_n q_n^* - D_n) + c_n q_n^* + G_n(x, q_n^*) - c_n^0 x. \quad (13)$$

For each sample of ϵ_n , we have

$$V_n(x + \epsilon_n q_n^*) \leq J_n(x + \epsilon_n q_n^*, 0), \quad (14)$$

because $q_n = 0$ is feasible but not necessarily optimal. Taking expectation of both sides of (14) and replacing $J_n(x + \epsilon_n q_n^*, 0)$ by (4), we have

$$\begin{aligned} \mathbf{E}V_n(x + \epsilon_n q_n^*) &\leq \mathbf{E}[R_n(x + \epsilon_n q_n^* - D_n) + G_n(x + \epsilon_n q_n^*, 0) - c_n^0(x + \epsilon_n q_n^*)] \\ &= V_n(x) - c_n q_n^* + \mathbf{E}[G_n(x + \epsilon_n q_n^*, 0) - c_n^0 \epsilon_n q_n^*] - G_n(x, q_n^*), \end{aligned} \quad (15)$$

where the equality follows from (13).

In what follows, we shall show that $\mathbf{E}[G_n(x + \epsilon_n q_n^*, 0) - c_n^0 \epsilon_n q_n^*] \leq G_n(x, q_n^*)$. By (5), for each sample of ϵ_n , we have

$$G_n(x + \epsilon_n q_n^*, 0) \leq c_n^0(y_n^* + \epsilon_n q_n^*) + \mathbf{E}V_{n+1}(y_n^* + \epsilon_n q_n^* - D_n),$$

where the inequality is true since $y_n^* + \epsilon_n q_n^* \geq x + \epsilon_n q_n^*$, where $y_n^* = \arg\min_{y \geq x} \{c_n^0 y + \alpha \mathbf{E}V_{n+1}(y + \epsilon_n q_n^* - D_n)\}$. Hence

$$\begin{aligned} \mathbf{E}G_n(x + \epsilon_n q_n^*, 0) &\leq c_n^0(y_n^* + \bar{c}_n q_n^*) + \mathbf{E}V_{n+1}(y_n^* + \epsilon_n q_n^* - D_n) \\ &= G_n(x, q_n^*) + c_n^0 \bar{c}_n q_n^*. \end{aligned}$$

Thus, we have shown that $\mathbf{E}[G_n(x + \epsilon_n q_n^*, 0) - c_n^0 \epsilon_n q_n^*] \leq G_n(x, q_n^*)$. This, together with (15), implies $\mathbf{E}V_n(x + \epsilon_n q_n^*) \leq V_n(x) - c_n q_n^*$, which contradicts (12). Hence, $V'_n(x) \leq -\bar{c}_n$ for $x < I_n$.

Case 2: $x \geq I_n$. we have $q_n^* = 0$. Hence, by (2), we have

$$V_n(x) = \min_{y \geq x} \{\mathbf{E}R_n(y - D_n) + c_n^0(y - x) + \alpha \mathbf{E}V_{n+1}(y - D_n)\},$$

$$V'_n(x) = \begin{cases} \mathbf{E}[R'_n(x - D_n) + \alpha V'_{n+1}(x - D_n)], & x \geq \hat{S}_n; \\ \mathbf{E}[R'_n(x - D_n)] - c_n^0, & x < \hat{S}_n. \end{cases} \quad (16)$$

For $I_n \geq \hat{S}_n$, by the definition of I_n , we have

$$\mathbf{E}[R'_n(I_n - D_n) + \alpha V'_{n+1}(I_n - D_n)] + \bar{c}_n \geq 0.$$

Thus, by (16), $V'_n(I_n) \geq -\bar{c}_n$.

For $I_n < \hat{S}_n$, by the definition of I_n , we have

$$\mathbf{E}R'_n(I_n - D_n) + \bar{c}_n - c_n^0 \geq 0.$$

By (16), we also have $V'_n(I_n) \geq -\bar{c}_n$.

Because of the convexity of $V_n(x)$, $V'_n(x) \geq V'_n(I_n) \geq -\bar{c}_n$. We hence can conclude that $V'_n(x) \geq -\bar{c}_n$ for $x \geq I_n$. $\square$

Proof of Theorem 3. For part (i), let us first denote

$$\hat{J}_n(x, v) = c_n^0 v + \alpha(1 - \beta_n) \mathbf{E}V_{n+1}(x + v - D_n) - c_n x + g_n(x, v), \quad (17)$$

where $g_n(x, v)$ is defined by

$$g_n(x, v) = \min_{z \leq z} \{\beta_n R_n(z - D_n) + c_n z + \alpha \beta_n V_{n+1}(z + v - D_n)\},$$

where $z = x + q$.

Because V_{n+1} is convex and minimization preserves convexity, $g_n(x, v)$ is jointly convex in (x, v) . By [Lemma 2.6.2](#) in [Topkis \(1998, p. 50\)](#) and [Lemma 3](#), $g_n(x, v)$ is supermodular in (x, v) . Because of the convexity of $V_{n+1}(x)$ and the property of $g_n(x, v)$, $\hat{J}_n(x, v)$ is jointly

convex and supermodular in (x, v) . Thus, Part (i) holds true because $\hat{J}_n(x, v)$ is supermodular in (x, v) .

To show part (ii), note from Theorem 2 that $v_n^*(x) = 0$ for all $x \geq \hat{S}_n$ and $\hat{S}_n$ is finite. So by part (i) of this theorem, we know that I_n^0 is well defined and an order is placed if and only if the initial inventory is strictly below I_n^0 . $\square$

Proof of Theorem 4. The proof is by induction.

For period N ,

$$V_N(x) = (1 - \beta)ER_N(x - D_N) + \beta ER_N(x + q_N^*(x) - D_N) + \beta \bar{c}_N q_N^*(x),$$

$$\frac{dV_N(x)}{d\beta} = -ER_N(x - D_N) + ER_N(x + q_N^*(x) - D_N) + \bar{c}_N q_N^*(x).$$

Since $q_N^*(x)$ minimizes $ER_N(x + q - D_N) + \bar{c}_N q$, we know that $dV_N(x)/d\beta$ is negative, and hence $V_N(x)$ is decreasing in β .

Furthermore, we have

$$V'_N(x) = \begin{cases} (1 - \beta)ER'_N(x - D_N) - \beta \bar{c}_N, & x < I_N; \\ ER'_N(x - D_N), & x \geq I_N. \end{cases}$$

For $x < I_N$, we have $dV'_N(x)/d\beta = -ER'_N(x - D_N) - \bar{c}_N$, which is positive because I_N is defined as $I_N = \inf\{x : ER'_N(x - D_N) + \bar{c}_N = 0\}$ and R_N is a convex function. Therefore, $V'_N(x)$ is increasing in β . For $x \geq I_N$, $V'_N(x)$ is independent of β , and hence increasing in β . For the other results, it is easy to see that v_N^* , I_N^0 , q_N^* , and I_N are all independent of β . We hence have shown the result for period N .

Suppose that the conclusions hold for period $n + 1$. We shall show that they also hold for period n .

For (i), the definition of I_n implies that $\beta ER'_n(I_n - D_n) + \beta \bar{c}_n + \frac{\partial G_n(I_n, q)}{\partial q}|_{q=0} = 0$. By (5), we have

$$\frac{\partial G_n(x, q)}{\partial q}\bigg|_{q=0} = \begin{cases} \alpha \beta EV'_{n+1}(x - D_n), & x \geq \hat{S}_n; \\ -\beta c_n^0, & \text{Otherwise.} \end{cases}$$

For $x \geq \hat{S}_n$, I_n is decreasing in β since $V'_{n+1}(x)$ is increasing in β . For $x < \hat{S}_n$, I_n is independent of β . Thus, I_n is decreasing in β .

Then, we show that I_n^0 is decreasing in β . By (17), I_n^0 is given by

$$c_n^0 + \alpha(1 - \beta)EV'_{n+1}(x - D_n)|_{x=I_n^0} + \frac{\partial g_n(x, v)}{\partial v}\bigg|_{x=I_n^0, v=0} = 0. \quad (18)$$

Denote

$$\bar{z}_n(v) = \arg\min_z \{\beta R_n(z - D_n) + c_n z + \alpha \beta V_{n+1}(z + v - D_n)\}.$$

By (17), we need to consider three cases: $\bar{z}_n(v) \leq x$, $x < \bar{z}_n(v)$.

Case 1: $\bar{z}_n(v) \leq x$. We have $\frac{\partial g_n(x, v)}{\partial v}\big|_{x=I_n^0, v=0} = \alpha \beta EV'_{n+1}(I_n^0 - D_n)$. Substituting it into (18), we have $c_n^0 + \alpha \beta EV'_{n+1}(I_n^0 - D_n) = 0$. Since $V'_{n+1}(x)$ is increasing in β , I_n^0 is decreasing in β .

Case 2: $x < \bar{z}_n(v)$. We have $\frac{\partial g_n(x, v)}{\partial v}\big|_{x=I_n^0, v=0} = \alpha \beta EV'_{n+1}(\bar{z}_n(0) - D_n)$. Substituting it into (18), we have $c_n^0 + \alpha(1 - \beta)EV'_{n+1}(I_n^0 - D_n) + \alpha \beta EV'_{n+1}(\bar{z}_n(0) - D_n) = 0$. Since $V'_{n+1}(x)$ is increasing in both β and x , I_n^0 is decreasing in β .

To prove (ii), we need to show that $V'_n(x)$ is increasing in β and $V_n(x)$ is decreasing in β . Then, we need to consider the following two cases.

Case 1: $I_n^0 \leq I_n$. For $x \geq I_n$, we have

$$V_n(x) = E[R_n(x - D_n) + \alpha V_{n+1}(x - D_n)].$$

By induction hypothesis, we see that $V_n(x)$ is decreasing in β and $V'_n(x)$ is increasing in β for this case.

For $I_n^0 \leq x < I_n$, we have

$$V'_n(x) = (1 - \beta)ER'_n(x - D_n) + \beta ER'_n(x + q_n^*(x) - D_n) + \alpha E[(1 - \beta)V'_{n+1}(x - D_n) + \beta V'_{n+1}(x + q_n^*(x) - D_n)]. \quad (19)$$

Noting that $\beta ER'_n(x + q_n^*(x) - D_n) + \beta \bar{c}_n + \alpha \beta EV'_{n+1}(x + q_n^*(x) - D_n) = 0$. We can rewrite (19) as

$$V'_n(x) = (1 - \beta)ER'_n(x - D_n) + \alpha(1 - \beta)EV'_{n+1}(x - D_n) - \beta \bar{c}_n \\ = (1 - \beta)[ER'_n(x - D_n) + \alpha EV'_{n+1}(x - D_n) + \bar{c}_n] - \bar{c}_n,$$

where $ER'_n(x - D_n) + \bar{c}_n + \alpha EV'_{n+1}(x - D_n) \leq 0$. Thus, $V'_n(x)$ is increasing in β .

For $x < I_n^0$, we have

$$V'_n(x) = (1 - \beta)ER'_n(x - D_n) + \beta ER'_n(x + q_n^*(x) - D_n) - c_n^0. \quad (20)$$

Since $q_n^*(x)$ is increasing in β for $x \leq I_n^0$, $V'_n(x)$ is increasing in β .

Case 2: $I_n < I_n^0$. For $x \geq I_n^0$, $V'_n(x)$ satisfies (19). Thus, (ii) is true for period n .

For $I_n < x < I_n^0$, we have

$$V'_n(x) = ER'_n(x - D_n) - c_n^0.$$

Clearly, (ii) is true for this situation.

For $x \leq I_n$, $V'_n(x)$ satisfies (20). Thus, (iii) is also true for this situation. $v_n^*(x)$ is determined by $c_n^0 + \alpha EV'_{n+1}(x + v_n^*(x) + q_n^*(x) - D_n) = 0$. By induction, it is obvious that $v_n^*(x)$ is also decreasing in β .

By following the similar argument, we can also show that $V_n(x)$ is decreasing in β . $\square$

Proof of Theorem 5. The proof is similar to that of Theorem 4 and hence omitted. $\square$

Proof of Lemma 7. Since I_n^0 is the base-stock level, it is clear that $V_n^{c'}(x) = -\bar{c}_n$ for $x < I_n^c$. The rest follows from the convexity of $V_n(x)$. $\square$

Proof of Theorem 6. We prove (i), (ii), and (iii) together by induction. Under the assumption that $ER_n(x - D_n)$ is continuous and differentiable almost everywhere, by induction, we can show that $V'_n(x)$ is continuous in x , which guarantees the existence of the first-order condition.

For period N , since $v_N^*(x) = v_N^c(x) = 0$, there exists only one decision variable q . We obtain $\hat{S}_N = \hat{S}_N^c$. It is shown by Henig and Gerchak (1990) that the threshold value is independent of yield variability for the single-period model. Thus, $I_N \geq I_N^c$.

For the certain-yield model, we have

$$V_N^{c'}(x) = \begin{cases} -\bar{c}_N, & x \leq I_N^c; \\ ER'_N(x - D_N), & \text{Otherwise.} \end{cases}$$

For the uncertain-yield model, we have

$$V'_N(x) = \begin{cases} ER'_N(x + \epsilon_N q_N^*(x) - D_N), & x \leq I_N; \\ ER'_N(x - D_N), & \text{Otherwise.} \end{cases}$$

Because of $I_N = I_N^c$, we have $V'_N(x) \leq V_N^{c'}(x)$.

Suppose that the conclusions are true for period $n + 1$. We have to show that they still hold for period n . By induction hypothesis, i.e., $V'_{n+1}(x) \leq V_{n+1}^{c'}(x)$, we have $\hat{S}_n \geq \hat{S}_n^c$.

Secondly, we show that $I_n \geq I_n^c$ by contradiction. Suppose $I_n < I_n^c$.

By (6) and (9), I_n and I_n^c are uniquely determined by the following two first-order conditions, respectively, i.e.,

$$I_n = \inf \left\{ x : \bar{c}_n ER'_n(x - D_n) + c_n + \frac{\partial G_n(x, q)}{\partial q}\bigg|_{q=0} = 0 \right\},$$

$$I_n^c = \inf \left\{ x : ER'_n(x - D_n) + \bar{c}_n + \frac{\partial G_n^c(x, q)}{\partial q}\bigg|_{q=0} = 0 \right\},$$

where

$$\frac{\partial G_n(x, q)}{\partial q}\bigg|_{q=0} = \alpha \bar{c}_n EV'_{n+1}(\max\{\hat{S}_n, x\} - D_n),$$

$$\left. \frac{\partial G_n^c(x, q)}{\partial q} \right|_{q=0} = \alpha \mathbf{E} V_{n+1}^{c'} (\max\{\hat{S}_n^c, x\} - D_n).$$

Let us consider the following two cases.

Case 1: $I_n > \hat{S}_n^c$. We have

$$\bar{c}_n \mathbf{E} R'_n(I_n - D_n) + c_n + \alpha \bar{c}_n \mathbf{E} V'_{n+1}(I_n - D_n) = 0. \quad (21)$$

Since $\hat{S}_n^c \leq \hat{S}_n$ and $I_n < I_n^c$, we have $I_n^c > \hat{S}_n^c$. Then, we have

$$\mathbf{E} R'_n(I_n^c - D_n) + \bar{c}_n + \alpha \mathbf{E} V'_{n+1}(I_n^c - D_n) = 0. \quad (22)$$

Comparing (21) with (22), since $V'_{n+1}(x) \leq V_{n+1}^{c'}(x)$, we have $I_n \geq I_n^c$, which yields a contradiction.

Case 2: $I_n \leq \hat{S}_n$. We have

$$\bar{c}_n \mathbf{E} R'_n(I_n - D_n) + c_n + \alpha \bar{c}_n \mathbf{E} V'_{n+1}(\hat{S}_n - D_n) = 0. \quad (23)$$

If $I_n^c \leq \hat{S}_n^c$, we have

$$\mathbf{E} R'_n(I_n^c - D_n) + \bar{c}_n + \alpha \mathbf{E} V'_{n+1}(\hat{S}_n^c - D_n) = 0. \quad (24)$$

Since $c_n^0 + \alpha \bar{c}_n \mathbf{E} V'_{n+1}(\hat{S}_n - D_n) = 0$ and $c_n^0 + \alpha \mathbf{E} V'_{n+1}(\hat{S}_n^c - D_n) = 0$, by comparing (23) and (24), we have $I_n = I_n^c$, which yields a contradiction.

If $I_n^c > \hat{S}_n^c$, we have

$$\mathbf{E} R'_n(I_n^c - D_n) + \bar{c}_n + \alpha \mathbf{E} V'_{n+1}(I_n^c - D_n) = 0. \quad (25)$$

Since $\alpha \mathbf{E} V'_{n+1}(I_n^c - D_n) \geq -c_n^0$, by comparing (23) and (25), we have $I_n \geq I_n^c$, which yields a contradiction. Thus, we have shown that $I_n \geq I_n^c$.

Thirdly, we intend to show that (iii) is true. By Lemmas 6 and 7, since $I_n \geq I_n^c$, we obtain that $V_n^{c'}(x) \geq V'_n(x)$ for $x \leq I_n$. Then, we just need to show that it is also true for $x > I_n$. For $x > I_n$, we have

$$V'_n(x) = \begin{cases} \mathbf{E} R'_n(x - D_n) + \alpha \mathbf{E} V'_{n+1}(x - D_n), & x \geq \hat{S}_n; \\ \mathbf{E} R'_n(x - D_n) - c_n^0, & \text{Otherwise.} \end{cases}$$

Let us consider the following three cases.

Case 1: $x \geq \hat{S}_n$. We have

$$V_n^{c'}(x) = \mathbf{E} R'_n(x - D_n) + \alpha \mathbf{E} V'_{n+1}(x - D_n).$$

Since $V'_{n+1}(x) \leq V_{n+1}^{c'}(x)$, we have $V'_n(x) \leq V_n^{c'}(x)$ for this case.

Case 2: $\hat{S}_n^c < x < \hat{S}_n$. Since $\alpha \mathbf{E} V'_{n+1}(x - D_n) > \alpha \mathbf{E} V'_{n+1}(\hat{S}_n^c - D_n) \geq -c_n^0$, we have $V'_n(x) \leq V_n^{c'}(x)$ for this case.

Case 3: $x \leq \hat{S}_n^c$. We have $V_n^{c'}(x) = \mathbf{E} R'_n(x - D_n) - c_n^0$. Therefore, we have $V'_n(x) = V_n^{c'}(x)$ for this case.

In summary, we have shown that $V'_n(x) \leq V_n^{c'}(x)$.

Next, we prove (iv) by induction. It is obviously true for $n = N + 1$ because $V_{N+1}(x) = V_{N+1}^c(x) = 0$.

Suppose that $V_{n+1}(x) \geq V_{n+1}^c(x)$. By invoking Jensen's inequality, we have

$$\begin{aligned} J_n(x, q, v) &= \mathbf{E}\{R_n(x + \epsilon_n q - D_n) + c_n q + c_n^0 v + \alpha V_{n+1}(x + \epsilon_n q + v - D_n)\} \\ &\geq \mathbf{E}\{R_n(x + \bar{c}_n q - D_n) + c_n q + c_n^0 v + \alpha V_{n+1}(x + \bar{c}_n q + v - D_n)\}, \end{aligned}$$

where the inequality is due to the convexity of $R_n(\cdot)$ and $V_{n+1}(\cdot)$. Therefore,

$$\begin{aligned} V_n(x) &= \min_{0 \leq q, 0 \leq v} J_n(x, q, v) \\ &\geq \min_{0 \leq q, 0 \leq v} \mathbf{E}\{R_n(x + \bar{c}_n q - D_n) + c_n q + c_n^0 v + \alpha V_{n+1}(x + \bar{c}_n q + v - D_n)\} \\ &\geq \min_{0 \leq q, 0 \leq v} \mathbf{E}\{R_n(x + \bar{c}_n q - D_n) + c_n q + c_n^0 v + \alpha V_{n+1}^c(x + \bar{c}_n q + v - D_n)\} \\ &= \min_{0 \leq q, 0 \leq v} \mathbf{E}\{R_n(x + \bar{c}_n q - D_n) + \bar{c}_n q + c_n^0 v + \alpha V_{n+1}^c(x + \bar{c}_n q + v - D_n)\} \end{aligned}$$

$$\begin{aligned} &= \min_{0 \leq q, 0 \leq v} \mathbf{E}\{R_n(x + q - D_n) + \bar{c}_n q + c_n^0 v + \alpha V_{n+1}^c(x + q + v - D_n)\} \\ &\geq \min_{0 \leq q, 0 \leq v} \mathbf{E}\{R_n(x + q - D_n) + \bar{c}_n q + c_n^0 v + \alpha V_{n+1}^c(x + q + v - D_n)\} \\ &= V_n^c(x), \end{aligned}$$

and the result follows.

Finally, for (v), noting that $I_n \geq I_n^c$, let us consider three cases. Case I: $x \geq I_n$. It is obvious that $q_n^*(x) = q_n^c(x) = 0$. Case II: $I_n^c \leq x < I_n$, we have $q_n^*(x) \geq q_n^c(x) = 0$. Case III: $x < I_n^c$. Because of $dq_n^*(x)/dx \leq -1$, it implies that $q_n^*(x) \geq I_n - x$. Since I_n^c can be treated as the base-stock level, we have $q_n^c(x) = I_n^c - x$. Since $I_n \geq I_n^c$, we have $q_n^*(x) \geq q_n^c(x)$. $\square$

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