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Highlights

- An improved solution of the compression test with integrated barrelling data
- Developed a technique to evaluate a proposed deformation model
- Derived and solved 3 incremental models of BCT profile
- Evaluated and compared the results with those of 2 reference FEA and BCT models

A power-based approach to assess the barrelling test's weak solution

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Abstract:

Physical simulation of forming is an analytical-experimental branch of mechanical science. Carefully formulated models and their solutions are essential to interpret the simulation data meaningfully and to understand its underlying phenomena. Oversimplified models of deformation and their associated non-unique closed-form solutions are widespread due to the complex and path dependence nature of plastic deformation, its multi-layered governing equations and boundary conditions. Therefore, it is vital to critically evaluate these models at the kinematic level to ensure a reliable outcome.

Incremental Profile Modelling (IPM) is proposed as a technique to integrate free surface geometrical data into the solution to address the path dependence issue. IPM also allows quantification of the solution's worthiness without making a reference to the mass or acting forces involved. The technique relies on calculating an *equivalent area* for a given solution. The area is estimated simply based on the deforming body's geometry and is compared with that of a numerical solution as the reference. The comparison provides a quick and quantitative assessment of the closed-form solution.

To demonstrate the technique, the "Barrelling Compression Test" (BCT) is chosen here as a key example. Three kinematic models of the test are considered and their closed-form solutions are obtained and evaluated using IPM. Also, two reference solutions of the test are presented including those of a finite element model and the conventional Cylindrical Profile Model (CPM). The equivalent areas for the first two models and that of the finite element were in good agreement. The third model rendered an "equivalent area singularity" at the sample's centre, and was concluded to be unsuitable to study BCT. The conclusion was not evident purely based on the model's kinematic solution.

Keyword: deformation modelling; kinematically admissible solution; barrelling parameter; incremental equivalent area; weak-form model assessment

1. Introduction

Forming studies are established around measurement, experiment, observation and formulation of models; see for example [1] and [2]. The models and their solutions are essential to interpret the material data, characterize its properties, predict material behaviour under loading and to understand the phenomena associated with the deformation. A model is typically defined as a boundary value problem in which the relationship between the governing equations (including constitutive, kinematic, force equilibrium and mixed boundary conditions) can be visualized best in terms of a Tonti Diagram, e.g. Felippa [3]. Finding an exact closed-form solution of a complex forming problem is often impossible; the exact solution involves simultaneous fulfilment of the multilayered governing equations. When one or some of the relationships are fulfilled in an *integral* (relaxed) rather than a *point by point* manner, then the formulation is classified as weak. To formulate a deformation problem in a weak form, e.g. Alnæs, Logg, Ølgaard, Rognes and Wells [4] and Batra and Zhang [5], is a quick and convenient route to solve it in a closed-form due to the complexity of the boundary value problems, more details can be found in Lubliner [6]. The challenge in developing a representative model is better understood by noting that the weak form model by its nature is non-unique and non-conservative. The latter is due to the presence of plastic deformation which is responsible for a path dependence energy change in the problem. The non-conservativeness increases when friction and mixed boundary conditions are also involved. These pose serious risks in developing a reliable closed-form solution for a forming problem and calls to evaluate the model and its solution to ensure that they represent the experiment reasonably well.

An insight into the root cause of the issue and a technique to avoid it in future developments for the same or similar forming problems such as upsetting and flash butt welding will be presented in the current work.

Key examples of this are the available closed-form solutions for the Barrelling Compression Test (BCT) which have been widely used by researchers and engineers, e.g. [7-10]. Given the complexity of BCT and its mixed boundary conditions, some of the problem's requirements have to be relaxed or simplified before a closed form solution can be obtained; it is not possible to enforce all governing equations of the problem simultaneously unless a sophisticated optimization is performed with several limiting constraints, e.g. Vanderplaats [11]. An example is the inclusion of the incompressibility condition in the energy conservation functional by Lagrange multiplier or by penalty constant, see for example Bonet and Wood [12]. Further, due to the presence of several admissible velocity fields for the BCT, such as that proposed by, Lee and Altan [13] Altinbalik and Çan [14] and that by Khoddam [15], and many combinations of the constraints and their relaxations, there exist several possible weak-forms of the same problem. A closed-form solution to identify the test's factor of friction developed by Ebrahimi and Najafizadeh [16] was found to have serious limitations and very limited applications in finding the friction factor by Solhjoo [17] and Khoddam and Hodgson [18] almost 50 years after its advent by Avitzur [19], Avitzur and Pan [20]. Similarly, the Cylindrical Profile Model (CPM) has been used for many years to interpret BCT's test data, see for example Dieter, Kuhn and Semiatin [21]. As demonstrated by Khoddam and Hodgson [18], the model fails to estimate the friction factor or the distribution of deformation in the test sample and predicts quite unrealistic uniform strain and strain rates in its test sample.

Here we present a technique, to be called incremental profile modelling (IPM), which addresses the modelling and assessment challenges. IPM relies on the fact that the instantaneous geometry of the test sample during the test, particularly its free surface (profile), provides valuable experimental information that can be utilized to perform an incremental and improved solution. This also reduces the gap between weak and strong form solutions. IPM integrates experimental instantaneous profile data into the kinematic model and its closed-form solution in an incremental fashion via a barrelling parameter. This accommodates the path dependency in the solution.

IPM also evaluates the worthiness of a proposed kinematic solution quickly at an early point. The assessment is made here by comparing an “equivalent area” of a given weak formulation with that of a reference solution.

To demonstrate the technique, it is applied to three closed-form solutions of BCT as case studies.

These include:

- 1- Exponential Profile Model (EPM), see Khoddam [15] and Fardi, Ibrahim, Hodgson and Khoddam [22] for details
- 2- Extended Avitzur Model (EA), e.g. Ebrahimi and Najafizadeh [16] and Solhjoo [17]
- 3- Symmetric Exponential Profile Model (SEPM) (a trial kinematic model, explained in section 6)

Two reference models were used: a finite element model and the conventional CPM. The latter represented an extreme zero barrelling model. An incremental expression was derived for each model to describe its free surface radius in terms of an instantaneous barrelling parameter. For each model, the parameter was identified incrementally, employed in the corresponding solution and correlated to its equivalent area for comparison. The contribution of barrelling in the equivalent area was quantified by comparing the finite element’s equivalent area with that of the CPM. Merits and applications of IPM will be discussed.

2. Weak modelling of BCT

2. Methodology

2.1. Weak modelling of BCT

Modelling the Barrelling Compression Test (BCT) is a key step due to the instrumental role of the test in the physical simulation of industrial and scientific forming processes. Poor modelling of the test results can lead to a false interpretation of the test data and misunderstanding of the scientific phenomena underlying its deformation. Therefore, the test is used in this article to illustrate our proposed IPM technique.

A weak modelling approach starts with proposing a kinematically admissible velocity field which contains a parameter as its primary variable. Next, the velocity field is solved by imposing the

kinematic boundary conditions and the incompressibility principle to find the strain rate and strain in the deforming body. We note that neither the weak model nor its closed-form solution is unique. Nevertheless, the solution has to be extensively developed in order to estimate the internal and external forces during the deformation. The development is typically a *non-iterative procedure*; only the parameter can be varied and calibrated via an optimization step. The optimization cost function is constructed as a *functional*, e.g. Chen and Sun [23], which is based on the conservation of the total energy (mechanical and internal) in the deforming body as an isolated system. Upon completion of the step, the optimum parameter minimizes the functional with respect to its primary variable. Ebrahimi and Najafizadeh [16] used the optimum parameter to identify the friction factor in a closed-form solution. This procedure calibrates the solution based on its boundary conditions and improves the weak-form solution; see for example Kobayashi and Thomsen [24] and Avitzur and Pan [20]. However, it does not guarantee an accurate force calculation at the tool-body interface.

The outlined procedure does not account for path dependency in the solution and does not involve an assessment step to replace a poor model with a more suitable one if needed. This is in contrast to a numerical solution of the problem such as finite element applied to the same problem; the true solution is obtained incrementally and iteratively without a need for a velocity field model. Completing the iterations via an optimization routine such as line search, the optimum admissible solution is found; the numerical solution iteratively identifies the (near) “strong-form solution”. Unfortunately, a numerical solution won’t enable material characterization in a forward fashion, see Khoddam, Hodgson and Bahramabadi [25] and Khoddam [15] for details, and therefore is not a replacement for a closed-form solution to the problem. Therefore, methods to assess or improve a proposed weak-form kinematic model and its associated closed-form solution are of vital importance.

2.1.1. A general description of the barrelled profile

Fig. 1 shows the geometry of a half BCT sample described in a cylindrical coordinate system (r, θ, z) .

It includes the sample at its initial configuration ($t = 0$; undeformed, **Fig. 1a**) and the position of a material point p_0 in the undeformed sample using its radial, tangential and axial coordinates, r_0, θ_0 and z_0 , respectively. **Fig. 1b** shows the half sample at an arbitrary time frame ($t = t_n$; deformed,) together

with three velocity components of the material point p_0 within the sample which corresponds to the cylindrical coordinate axes; u , w and v denote the magnitude of the radial, tangential and axial components of displacements, respectively and the accent dot derives the time derivative of each component (velocities). The material point's velocity components ($\dot{u}$, $\dot{w}$ and $\dot{v}$) act at point p but are shifted along the coordinate axes in **Figure 1 b** for the sake of clarity. Radial and tangential axes of the cylindrical coordinate system are located in the sample's plane of symmetry (mid-plane). The coordinate system's centre, O , is fixed at the sample's centre. Sample's top and bottom surfaces move towards the mid-plane with velocities of $-0.5\dot{H}$ and $0.5\dot{H}$, respectively. Symbols D_n and d_n are the largest and shortest sample's diameters (for an arbitrary time step $t = t_n$) at the mid and top-planes, respectively. The barrelled profile can be represented parametrically using its arbitrary profile radius, $R_n(z)$, at a given plane which is parallel to the mid-plane and has an axial distance of z from the mid-plane.

Fig. 1. Schematic of a deforming half sample and illustration of symbols. (a) Initial geometry and coordinates of a material point $p_0(r_0, \theta_0, z_0)$ and (b) an arbitrary deformed geometry ($t = t_n$)- showing an arbitrary $R_n(z)$ (barrel profile's radius), the coordinate system's position and velocity components of a material point.

The modelling of BCT starts with a new kinematically admissible velocity field as input. By solving the model, the strain and strain rate within the sample are found in a closed-form which provides inputs to the problem's other governing layers, such as the constitutive and frictional models.

An arbitrary material point p_0 in the deforming sample has a velocity vector field, $\vec{\varphi}$ of the following form:

$$\vec{\varphi} = \dot{u}\vec{e}_r + \dot{w}\vec{e}_\theta + \dot{v}\vec{e}_z \quad (1)$$

with $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_z$ as the unit vectors and $\dot{u}$, $\dot{w}$ and $\dot{v}$ denote magnitudes of its velocity vector components in radial, tangential and axial directions, respectively.

2.1.2. Incremental profile

In this section, we seek to mathematically describe the “quasi-static motion” of a typical material point located on the test sample's profile after completing a small time step.

For large height reductions in a BCT sample, incremental calculations provide better estimations. This could be better understood by noting that deformation during BCT is a non-conservative process due to the presence of friction and plastic deformation. As a result, the deformation is path dependent and cannot be described merely based on the initial and final configuration of the deforming isolated system. The path-dependent description can be formulated using a backward kinematic approach in which each material point in the sample, specifically a point on the profile, is traced from its undeformed configuration to its current position. Alternatively, this can be done in a forward kinematic approach depending on the availability of the boundary condition or experimental data. To identify the profile incrementally, points located on the sample's free surface have to be traced along deformation streamline starting from their initial position when the sample is undeformed.

Fig. 2 shows seven selected tracing points o, k, q, f, p, l and s at three-time steps; 0 (initial undeformed, a), t_1 (1st time/deformation step, b) and t_2 (2nd time/deformation step, c). The shown tracing points represent how deformation paths for different points move differently.

Fig. 2. Seven tracing points o, k, q, f, p, l and s in a quarter of sample at three consecutive time frames t_0 (a), t_1 (b) and t_2 (c). Subscripts for each point designate their time frame. Inset (d) shows an overlay of the sample in the three-time steps together with trajectories of tracing points l, q and s , respectively during deformation stages $t_0 \leq t \leq t_2$; the trajectory vectors are shown in **Fig. 2d** as green solid line arrows.

One has to distinguish between the radial coordinate of a material point (r or r_i) with the radial coordinate of a material point which belongs to the free surface (e.g. tracing points f_i, l_i and s_i). The later represents the sample's *profile radius* $R(z_i)$ at a plane normal to the z axis as a function the plane's axial distance from the origin, z_i . This is illustrated in **Fig. 2**; a typical tracing material point is p_i ($i = 1, 2$ and 3) where p_i coordinates are r_i and its distance from the mid-plane z_i can be compared with a typical tracing profile point l_i ($i = 1, 2$ and 3) where l_i coordinates are the profile radius $R(z_i)$ and z_i .

A quantitative description of the profile's tracing point for each of the three case studies will be derived later in this article.

Incremental vs. generic notation

Throughout this article, we frequently refer to a deformation parameter with emphasis on its time-step, n , when it is evaluated or measured. Such cases, are presented in this article using an “incremental notation”. Examples of these are $R_n(z)$, D_n and H_n , which represent the instantaneous profile radius, mid-plane diameter and height corresponding to time step n , respectively. Incremental expressions are typically correlating parameters between two consecutive quasi-static time steps namely n and $n - 1$. Alternatively, we may refer to the same parameters using a generic notation when the deformation step

is close to the start time ($n = 0$). In these cases, the subscript for the current time step (n) is dropped for the sake of brevity and to simplify the subsequent derivations. In the expressions represented by generic notations, reference is made to the initial configurations denoted by subscripts 0 (e.g. D_0 and H_0). The subscript in both notations indicates the time associated with the parameter.

2.1.3. A general note on barrelling parameter

The barrelling parameter has been widely used in the literature as a key indicator of shearing deformation in the BCT sample, examples include Ebrahimi and Najafizadeh [16], Fardi, Ibrahim, Hodgson and Khoddam [22], Avitzur [26] and Khoddam [15]. We note that it is a very important geometrical parameter which indicates the non-conservativeness of the deformation process (i.e. friction). It is also a pre-requisite to estimate several deformation parameters, such as the strain and strain rate components.

It is not practical to eliminate friction nor shearing strain during the entire course of BCT for a wide range of the test conditions, particularly hot deformation. A key limitation is a fact that the lubrication conditions cannot be maintained as unchanged during the entire test. Also, a multi-axial deformation test with a reliable closed-form solution is an asset to represent many real scientific and engineering deformation scenarios. Given a deformation stage with a noticeable barrelling, it can be shown that the barrelling parameter is not constant; strictly speaking, it changes with the coordinate of the material point to allow enforcement of volume constancy in the entire sample and to account for foldover. To simplify the derivations, we assume that a constant “representative barrelling parameter” for each deformation step is adequate to describe the deformation. Given variations of strain rate in the sample, for the sake of simplicity and practicality of the derivations, Khoddam [15] defined the sample’s centre as a representative point to estimate strain, strain rate and their associated barrelling parameter in the sample. Inevitably, the adopted concept reduces the accuracy of the EPM’s estimated effective strain and strain rate. This simplification is merely to allow a practical and usable estimate.

IPM, demonstrated using three case studies

2.2. IPM, demonstrated using three case studies

Developing a complete forming theory for BCT or similar processes such as upsetting, flash butt welding etc. is complex and cumbersome. Two major concerns to find a model and its solution are:

1. to account for the path dependency which can reduce the gap between the weak and strong formulation
2. to be able to evaluate the closed-form solution at the kinematic level.

We propose a procedure called IPM which addresses the concerns. The procedure is an efficient incremental and iterative procedure to obtain a model and its closed-form solution; if the kinematic model is evaluated as unworthy of development, it is replaced with another model until a proper model is found and solved. IPM evaluates a weak model and its solution simply by evaluating a geometrical quantity called “equivalent area” which will be explained in section 4 of this article.

Foldover of the sample during deformation is an example of the first concern. It occurs when a part of the free surface of BCT’s sample contacts the die and as suggested by Li, Onodera and Chiba [27], it increases friction significantly. The second concern is addressed in this article by proposing a procedure to evaluate a weak formulation using IPM. The main six steps in this procedure and the required derivations/calculations are shown parametrically in **Fig. 3**.

Fig. 3. The IPM procedure to solve a weak model incrementally and to evaluate its worthiness

We demonstrate and explain the details of the procedure using three nominated weak formulations of BCT represented by the three kinematic models outlined earlier in section 1.

All variables which are commonly used in each model, are shown in the flow chart using the same symbols for easier comparison except for the barrelling parameters, which will be presented with different symbols B , b and $\mathcal{B}$ for models EPM, EA and SEPM, respectively. In **Fig. 3**, the barrelling parameter is shown using b_n which corresponds to time step n .

4. Evaluation of EPM's model

3. Evaluation of five test scenarios

3.1. Evaluation of EPM's model

The Exponential Profile Model (EPM) is the first weak formulation of BCT to be evaluated here. Early results from EPM were introduced by Fardi, Ibrahim, Hodgson and Khoddam [22] and Khoddam [15]. As the first evaluation step (**Fig. 3**), Khoddam [15] proposed EPM's weak form kinematic model with the following radial and axial velocity components:

$$\dot{u}(r, z) = r\dot{H} \left(B + 2 \frac{(1-2BH)}{H^2} z \right) \quad (1)$$

$$\dot{v}(z) = 2\dot{H}z \frac{(BH(2z-H)-z)}{H^2} \quad (0 \leq z \leq 0.5H) \quad (2)$$

where B is EPM's barrelling parameter which has a dimension of L^{-1} . Details on EPM's second evaluation step, can be found in [15] where the following closed-form solution was derived to calculate its effective plastic strain rate:

$$\dot{\epsilon} = \frac{2\dot{H}}{\sqrt{3}H^2} \sqrt{(1-2BH)^2 r^2 + 3(BH(H-4z)+2z)^2} \quad (3)$$

It can be seen in Eqs. 1 and 2 that the barrelling parameter, B , appears in both velocity components $\dot{u}$ and $\dot{v}$ to link them and is an input to estimate $\dot{\epsilon}$.

3.1.1. EPM's profile identification

A rate expression of the sample's profile $R(z)$ for EPM is derived in this section; this is shown as the third step in **Fig. 3**. Given the mathematical description of $\dot{u}$ and the instantaneous geometrical parameters at a time frame t_n , i.e. d, D and H , the sample's profile radius $R(z)$ can be estimated by

imposing the boundary conditions of the deformed profile and solving the resultant differential equation.

A backward solution for the profile can be started noting that $\dot{u}(z, R) = \dot{R}$, so:

$$\dot{R} = R\dot{H} \left(B + 2 \frac{(1-2BH)}{H^2} z \right) \quad (4)$$

Applying the chain rule to Eq. 4:

$$\frac{dR}{dt} = \frac{-dR}{dH} \frac{dH}{dt} \approx \frac{\Delta R}{\Delta H} \dot{H} = R\dot{H} \left(B + 2 \frac{(1-2BH)}{H^2} z \right) \quad (5)$$

where ΔR and ΔH are the sample's profile and height changes, respectively during a time step of Δt duration. Also, derivatives of R and H with respect to t or H represent the changes for an infinitesimally small time or height increments during the step, respectively. Eq. 4 may be considered as a differential equation:

$$-\frac{dR}{dH} - R \left(B + 2 \frac{(1-2BH)}{H^2} z \right) = 0 \quad (6)$$

Solving Eq. 6 for R generically with $R(z, H_0) = 0.5D_0$ as the boundary condition:

$$R = 0.5D_0 \left(\frac{H}{H_0} \right)^{4Bz} e^{B(H_0-H)+2z\left(\frac{1}{H}-\frac{1}{H_0}\right)} \quad (7)$$

Due to the exponential nature of the above profile radius, we refer to its model as Exponential Profile Model (EPM). It should be noted that this profile description is valid only for a small barrelling/reduction. This is due to the path-dependent nature of the deformation process. Also, in solving differential Eq. 6, it is assumed that a constant B can represent the barrelling in the sample for the current deformation step with respect to the undeformed dimensions H_0 and D_0 .

EPM's incremental profile

Alternatively, for medium to large reductions in the sample's height, a "quasi-static" incremental approach can be taken to solve Eq. 6. to express EPM's profile incrementally. Derivations for this approach will be presented next.

To find the incremental motion of a typical profile tracing point l , let us start with the vertical motion of the point from $z_{(n-1)}$ to z_n in a time increment of Δt . Since all tracing points move simultaneously:

$$\Delta t = \frac{R_n - R_{n-1}}{\dot{u}_{n-1}} = \frac{z_n - z_{n-1}}{\dot{v}_{n-1}} = \frac{H_{n-1} - H_n}{\dot{H}} \quad (8)$$

Eq.8 can be manipulated to estimate the vertical position changes for a typical point l located on the profile as:

$$z_n = z_{n-1} + \frac{\dot{v}_{n-1}}{\dot{H}} (H_{n-1} - H_n) \quad (9)$$

An incremental expression for the vertical velocity can be written based on Eq. 2 as:

$$\frac{\dot{v}_{n-1}}{\dot{H}} = 2z_{n-1} \frac{(BH_{n-1}(2z_{n-1} - H_{n-1}) - z_{n-1})}{H_{n-1}^2} \quad (10)$$

Combining Eqs. 9 and 10, the following backward kinematic formulation is obtained that estimates the vertical motion of a profile point during Δt as:

$$z_n = z_{n-1} + 2(H_{n-1} - H_n)z_{n-1} \frac{(BH_{n-1}(2z_{n-1} - H_{n-1}) - z_{n-1})}{H_{n-1}^2} \quad (11)$$

Using Eq. 1 and the chain rule, the radial displacement for a typical on the profile (e.g. the tracing point l) is expressed as: $dR = R_n - R_{n-1}$ which can be related to its radial velocity as:

$$\dot{u}(R, z) = \frac{-du(R, z)}{dH} \times \frac{dH}{dt} = \frac{-dR}{dH} \dot{H} \quad (12)$$

Using Eq. 8, the profile radius can be found using the following integrals:

$$\int_{R_{n-1}}^{R_n} \frac{dR}{R} = - \int_{H_{n-1}}^{H_n} \left(B + 2z \frac{1 - 2BH}{H^2} \right) dH \quad (13)$$

We assume quasi-static incremental deformation, described by Eq. 13, during which B and z don't change with H . Equation 13 can be solved as:

$$R_n = R_{n-1} \left(\frac{H_n}{H_{n-1}} \right)^{4Bz} \exp \left(B(H_{n-1} - H_n) + \frac{2z}{H_n} - \frac{2z}{H_{n-1}} \right) \quad (14)$$

Equation 14 allows to update a typical profile radius based on the vertical positions of the point and the sample's height before and after the time increment, respectively and the average barrelling parameter during the time increment.

Given Eqs. 11 and 14, the incremental motion of an arbitrary profile point and therefore the sample's deformed profile can now be estimated.

From an experimental perspective, we note that H_n and H_{n-1} are typically recorded for a conventional BCT test. To update profile between two consecutive quasi-static deformation steps of n and $n - 1$ using EPM's key relationships 13 and 14, one also requires to measure barrelling parameter B_{n-1} . Methods of measuring the parameter (step 4 in **Fig. 3**) will be presented for each case study separately later in this article.

Comparing equation 13 and 7, we notice that Eq. 7 represents a coarse estimate of the profile by assuming that the time-steps for the quasi-static approach was set to the total deformation time.

Identification of EPM's barrelling parameter

The fourth evaluation step, as shown in **Fig. 3**, is dedicated to identifying the barrelling parameter. The parameter is a key input for all kinematic and dynamic calculations in BCT. Examples of these include velocity components, profile estimation, strain rate, strain, deformation power and friction. Therefore, one has to consider the available avenues to identify the barrelling parameter and to choose the optimum option. To indirectly measure B , one notes that Equations 13 and 14 correlate the parameter to the *measurable* diameter at either the mid-plane, D_n , or top-plane, d_n .

Unfortunately, either of these boundary conditions leads to a different estimation of the parameter. More realistically, one expects to identify the parameter by equating “volume integration” of $R(z)$ in the z direction to the sample's initial volume to account for all profile points to ensure the instantaneous profile can regenerate the exact initial volume. However, this makes the derivations and their solutions too complicated. In this article, estimations of the parameter at the top and mid-planes will be derived and discussed.

Evaluating the parameter at the top-plane is not straightforward due to an uncertainty in calculating the top-plane diameter; a pre-requisite to identify—identifying barrelling parameter at the top-plane (see Eq. 13) is to account for foldover; the fact that a part of the initially free surface comes into contact with the die during compression, e.g. Kobayashi, Oh and Altan [28]. Foldover may be defined as partial annexation of the “sample's top edge free surface” to the “die-sample interface” during deformation. Foldover is a source of error in BCT particularly for medium to large strains and high frictions. To compensate for foldover, one has to modify the profile namely the top-plane diameter and

to use supplementary velocity fields in the vicinity of the samples free surface. Therefore, to approximate B_n at the top-plane, one has to know the contribution of “foldover” in the top-plane diameter first. Currently, the contribution can only be measured experimentally. However, as far as profile estimation is concerned, d can be estimated analytically as will be discussed in the next section.

An approximation for d

We suggest a simple method to approximate the sample’s instantaneous minimum diameter (top diameter), d_n based on the volume constancy principle. The estimated d_n is only valid for sample profile characterization. It is calculated using the following equation derived by Fardi, Ibrahim, Hodgson and Khoddam [22]:

$$d_n = \frac{\sqrt{5} \sqrt{H_n(9D_0^2H_0 - 4D_n^2H_n) - 2D_nH_n}}{3H_n} \quad t_n \neq 0 \quad (15)$$

Equation 15 correlates the largest and smallest diameters, D_n and d_n , respectively (for an arbitrary time step t_n) using initial dimensions of the sample and its current height H_n .

Eq. 15 has a limited scope and should be used carefully. While it provides a reasonably close estimate of the upper plane diameter d and therefore a good estimate of the sample profile $R(z)$, it is not useful to estimate B at the top plane. A closer examination of Eq. 15 shows that to estimate B , one needs to know the incremental variations of d with respect to H . Since derivations of the estimated d with respect to H based on Eq. 15 is different with from that of the real d , therefore their estimates of B at the top plane are likely to be different. Therefore, Eq. 15 is not suitable to calculate the barrelling parameter at the top plane.

However, if experimental data for d_n is available, it can be used to indirectly measure EPM’s barrelling parameter, B_n , using the measured instantaneous top-plane diameter. As a special case of Eq. 14, R_n can be estimated at $z_n = 0.5H_n$, $z_{n-1} = 0.5H_{n-1}$ and $2R_n = d_n$ and eventually to solve it for B ; Eq. 14 can be solved at the top-plane. Using the incremental notation, B can be found between two consecutive steps:

$$B = \frac{1}{H_{n-1} - H_n} \ln \left(\frac{H_{n-1}d_{n-1}}{H_nd_n} \right) \quad (\text{at the top-plane}) \quad (16)$$

where $H \neq H_0$ ($H = H_0 \rightarrow B = 0$). Equation 16 estimates B at the top-plane.

Alternatively, the barrelling parameter can be identified using a special case of Eq. 13 by applying boundary conditions at the mid-plane; $z_n = 0$ and $2R_n = D_n$ to find B as follows:

$$B = \frac{D_n - D_{n-1}}{D_{n-1}H_{n-1} - D_{n-1}H_n} \quad (\text{at the mid-plane}) \quad (17)$$

Equation 17 estimates B at the mid-plane.

Based on several case studies to choose the best representative point for profile, strain and strain rate characterization, it was found by the authors that the calculated parameter at the sample centre; $z = r = 0$ delivered the best agreement between the characterized values by the model and their counterpart reference numerical solutions. This justifies the point as a good candidate for the representative point for strain and strain rate calculations and, therefore, we recommend to evaluate the barrelling parameter at the mid-plane using Eq. 17. The choice of representative point for other applications such as friction force estimations have has to be investigated separately.

As a result, in order to utilize IPM, the BCT rig has to record D_n and d_n concurrently with its conventional instantaneous measurements such as H_n .

EPM's equivalent area

The effective strain rate distribution at a given deformed configuration can be estimated based on the test model's weak formulation (e.g. Eq. 3 for EPM). The estimated strain rate is used here in an upper bound analysis to calculate the equivalent area associated with each model (step 5 in **Fig. 3**). The equivalent area is a simple criterion to evaluate a proposed kinematic model. Using the generic notation, the upper bound deformation power is found as $\dot{W}_i = \int_V \sigma \sqrt{\frac{2}{3} \dot{\epsilon}_{ij} \cdot \dot{\epsilon}_{ij}} dV$, where W_i is the work performed due to deformation. A change of variables as $dV = 2\pi r dr dz$ is required to perform the integration for EPM:

$$\dot{W}_i = \frac{4\pi\sigma\dot{H}}{\sqrt{3}H^2} \int_{z=0}^{0.5H} \int_{r=0}^{\bar{R}} 2r \sqrt{(1 - 2BH)^2 r^2 + 3(BH(H - 4z) + 2z)^2} dr dz \quad (18)$$

the upper limit of integral, $\bar{R}$, in the above equation is an imaginary effective radius, $\bar{R} = 0.5D_0\sqrt{\frac{H_0}{H}}$, defined based on the sample's volume constancy during the test. Choosing EPM's velocity field and

employing $\bar{R}$ has enabled the integration in a closed-form which after simplifying, $\dot{W}_i$ for EPM becomes:

$$\dot{W}_i = \pi \sigma \dot{H} \beta \quad (19)$$

where

$$\beta = \frac{1}{6\sqrt{3}H^2\gamma_2^3} (H(6\Gamma_1 - \Gamma_2 + \Gamma_3) + \Gamma_4) \quad (20)$$

In which:

$$\Gamma_1 = \sqrt{3}BH(B^2H^4)^{3/2} - \sqrt{3}\gamma_1(H^2\gamma_1^2)^{3/2} \quad (21)$$

$$\Gamma_2 = BH \gamma_4(6B^2H^4 + 5\bar{R}^2\gamma_2^2) \quad (22)$$

$$\Gamma_3 = \gamma_1 \gamma_3(6H^2\gamma_1^2 + 5\bar{R}^2\gamma_2^2) \quad (23)$$

$$\Gamma_4 = \sqrt{3}\bar{R}^4(\ln(\sqrt{3}\gamma_4 - 3BH^2) - \ln(\sqrt{3}\gamma_3 - 3H\gamma_1))\gamma_2^4 \quad (24)$$

where $\gamma_1 = 1 - BH$, $\gamma_2 = 1 - 2BH$, $\gamma_3 = \sqrt{3H^2\gamma_1^2 + \gamma_2^2\bar{R}^2}$ and $\gamma_4 = \sqrt{3B^2H^4 + \gamma_2^2\bar{R}^2}$ in Equations 21 to 24 are defined to avoid a lengthy expression for β in Eq. 20 and to allow its calculation in a multi-layer fashion.

For the ease of comparison between different kinematic models, we define EPM's equivalent area, $\hat{A}_{EPM}$, as:

$$\hat{A}_{EPM} = \frac{\dot{W}_i}{\sigma \dot{H}} = \pi \beta \quad (25)$$

In deriving Eq. 25, it is assumed that the deforming material behaves as a “perfectly plastic” material. This reduces the proposed equivalent area to a purely geometrical value which can be easily found without the need to know the sample's constitutive behaviour; the equivalent area is essentially a normalized deformation power whose dimension is L^2 .

The derivations presented in this section enable the application of the IPM procedure to the case of EPM. Similar but more concise derivations will be presented for the two remaining case studies, i.e. EA and SEPM, next before a comparative assessment of the three case studies.

5. EA's evaluation

3.2. EA's evaluation

The Extended-Avitzur (EA) is a popular model in the literature to describe the BCT problem. Different solutions have been proposed to estimate friction using this model. The following is the weak model proposed by Avitzur [19]:

$$\dot{u} = \frac{b\dot{H}re^{-\frac{bz}{H}}}{4H(1-e^{-0.5b})} \quad (26)$$

$$\dot{v} = -0.5\dot{H} \frac{\left(e^{-\frac{bz}{H}} - 1\right)}{e^{-0.5b} - 1} \quad (27)$$

where b is the barrelling parameter of the model. The parameter is deliberately shown with a different symbol to that of the other models while the rest of the variables are denoted using the same symbols.

Using the calculated strain rate components by Avitzur [26], it can be shown that the effective strain rate (second step in **Fig. 3**) for the EA model becomes:

$$\dot{\epsilon} = \frac{\dot{H}be^{\left(\frac{-zb}{H}\right)}}{H^2(e^{-0.5b}-1)} \sqrt{\frac{H^2}{4} + \frac{b^2r^2}{72}} \quad (28)$$

In order to identify the barrelled profile radius, the third step in **Fig. 3**, we use a rate expression of the sample profile $R(z)$ using the generic notation as:

$$\dot{R} = R \frac{b\dot{H}e^{-\frac{bz}{H}}}{4H(1-e^{-0.5b})} \quad (29)$$

Using the chain rule:

$$\frac{dR}{dt} = \frac{-dR}{dH} \frac{dH}{dt} \approx \frac{\Delta R}{\Delta H} \dot{H} = R\dot{H} \frac{be^{-\frac{bz}{H}}}{4H(1-e^{-0.5b})} \quad (30)$$

Profile radius (R) and H change in opposite directions (i.e. $\frac{\Delta R}{\Delta H}$ is negative), therefore:

$$\frac{-(0.5D_0-R)}{H_0-H} \dot{H} = R\dot{H} \frac{be^{-\frac{bz}{H}}}{4H(1-e^{-0.5b})} \quad (31)$$

Simplifying and solving Eq. 31, EA's profile using the generic convention becomes:

$$R(z) = \frac{2D_0(e^{0.5b}-1)H}{4H(e^{0.5b}-1)-b(H_0-H)e^{b\left(0.5-\frac{z}{H}\right)}} \quad (32)$$

Equation 32 is only valid for a negligible barrelling or a small reduction in H . An incremental expression for EA's profile was derived by Solhjoo and Khoddam [29] for the larger deformations as:

$$R_n(z) = R_{n-1}(z) / \left(1 - \frac{(H_{n-1}-H_n)e^{-\frac{bz_n}{H_n}}}{4H_n(1-e^{-0.5b_n})}\right) \quad (33)$$

Given the exponential nature of the Avitzur model, identification of b (fourth step in **Fig. 3**) is not straight forward. In fact, there exist several solutions for the EA barrelling parameter, b , depending on the performed approach. The first derivation of b was done by Avitzur, by expressing b as a function of $\bar{R}$ and H with a constant value of friction factor m :

$$b = \frac{6}{1 + \frac{3\sqrt{3}}{2} \frac{\bar{R}}{mH}} \quad (34)$$

Other estimations of the parameter include those proposed by Ebrahimi and Najafizadeh [16] and later by Solhjoo [17] based on a series expansion as follows:

$$b = \frac{2(D_n - d_n)}{0.5(D_n + d_n)} \left(\frac{2H_n}{H_0 - H_n} - 1 \right) \quad (35)$$

Moreover, Solhjoo [17] performed a number of deformation scenarios for which the friction factors estimated by the EA model agreed poorly with those utilized in a finite element simulation of the same cases.

IPM's fifth step to evaluate the EA model requires the model's equivalent area, as its input. Using the

deformation power defined as $\dot{W}_i = \int_V \sigma \sqrt{\frac{2}{3} \dot{\epsilon}_{ij} \dot{\epsilon}_{ij}} dV$ and Eq. (28), we find the EA's equivalent area

$\hat{A}_{EA}$ as:

$$\hat{A}_{EA} = \frac{\dot{W}_i}{\sigma H} = \frac{-\pi b \bar{R}^3}{3\sqrt{3}H} \left[\left(1 + \frac{12H^2}{\bar{R}^2 b^2} \right)^{1.5} - \left(\frac{12H^2}{\bar{R}^2 b^2} \right)^{1.5} \right] \quad (36)$$

The IPM derivations for the third case study, SEPM, will be presented next.

6. SEPM's evaluation

3.3. SEPM's evaluation

The Symmetric Exponential Profile Model, SEPM, is an extended version of the EPM. The former is a trial kinematic model for BCT. It is proposed by the authors here as an example of a new weak-form model and its closed-form solution is assessed as a demonstration of IPM.

SEPM's axial velocity, in contrast to EPM, is symmetric about the mid-plane and represents the velocity distribution for the entire sample; $-0.5H \leq z \leq 0.5H$. In contrast, the axial velocity for EPM is not symmetric; it is quadratic which is only suitable to describe the axial velocity distribution in the

upper half of the sample. For the lower part, EPM's axial velocity component has to be mirrored and flipped about the mid-plane and the z axis, respectively.

The weak form velocity model for SEPM, first step in **Fig. 3**, is proposed as:

$$\dot{u} = r(Az^2 + B\dot{H}) \quad (-0.5H \leq z \leq 0.5H) \quad (37)$$

where symbol B presents SEPM's barrelling parameter in a similar way that B and b were utilized for EPM and EA, respectively. Undertaking a methodology similar to that employed for EPM and EA, the radial and longitudinal velocity field components, respectively, can be calibrated as:

$$\dot{u}(r, z) = r\dot{H} \left(B - 6 \frac{(1+2BH)}{H^3} z^2 \right) \quad (-0.5H \leq z \leq 0.5H) \quad (38)$$

$$\dot{v}(z) = 2\dot{H} \left(2 \frac{(1+2BH)z^3}{H^3} - Bz \right) \quad (39)$$

Similarly, as SEPM's second evaluation step in **Fig. 3**, SEPM's components of the strain rate can be found and utilized to calculate SEPM's effective strain rate as:

$$\dot{\epsilon} = \frac{2\dot{H}}{H^3} \sqrt{12(1+2BH)^2 r^2 z^2 + (BH^3 - 6(1+2BH)z^2)^2} \quad (40)$$

Also, a generic rate expression of sample's profile $R(z)$ can be derived by considering a special case of Eq. 38 noting that $\dot{u}(z, R) = \dot{R}$:

$$\dot{R} = R\dot{H} \left(B - 6 \frac{(1+2BH)}{H^3} z^2 \right) \quad (41)$$

Equation 41 may be solved as a differential equation using the chain rule:

$$-\frac{dR}{dH} - R \left(B - 6 \frac{(1+2BH)}{H^3} z^2 \right) = 0 \quad R(z, H_0) = 0.5D_0 \quad (42)$$

Solving for $R(z)$ and presenting it using the generic notation, SEPM's profile becomes:

$$R(z) = 0.5D_0 e^{3z^2 \left(\frac{1}{H_0^2} - \frac{1}{H^2} \right) - B \left(H - H_0 + 12z^2 \left(\frac{1}{H} - \frac{1}{H_0} \right) \right)} \quad (43)$$

Following a similar approach to that of EPM, the SEPM profile is found in an incremental fashion:

$$R_n = R_{n-1} \left(1 + (H_{n-1} - H_n) \left(\frac{3(H_{n-1} + H_n + 4B_{n-1}H_{n-1}H_n)z_{n-1}^2}{H_{n-1}^2 H_n^2} - B_{n-1} \right) \right) \quad (44)$$

$$z_n = z_{n-1} \left(1 + (H_{n-1} - H_n) \left(B_{n-1} - \frac{z_{n-1}^2 (H_{n-1} + H_n + 4B_{n-1}H_{n-1}H_n)}{H_{n-1}^2 H_n^2} \right) \right) \quad (45)$$

The SEPM barrelling parameter is identified as a special case of Eq. 44 at the mid-plane, $R(0) = 0.5D$. This results in finding B_{n-1} at the mid-plane as:

$$B_{n-1} = \frac{D_n - D_{n-1}}{D_{n-1}H_{n-1} - D_{n-1}H_n} \quad (\text{at the mid-plane}) \quad (46)$$

Now, the SEPM deformation power, $\dot{W}_i = \int_V \sigma \sqrt{\frac{2}{3} \dot{\epsilon}_{ij} \dot{\epsilon}_{ij}} dV$ is found by the change of variables as $dV = 2\pi r dr dz$ and rearranging the terms:

$$\dot{W}_i = \frac{8\pi\sigma\dot{H}}{H^3} \int_{z=0}^{0.5H} \int_{r=0}^{\bar{R}} r \sqrt{12(1+2BH)^2 r^2 z^2 + (BH^3 - 6(1+2BH)z^2)^2} dr dz \quad (47)$$

Carrying out the integration with respect to r :

$$\dot{W}_i = \frac{2\pi\sigma\dot{H}}{9} \int_{z=0}^{0.5H} \frac{H^6}{z^2 \gamma^2} \left(\left(\frac{12\gamma^2 \bar{R}^2 z^2}{H^6} + \left(\mathcal{B} - \frac{6\gamma z^2}{H^3} \right) \right)^{1.5} - \frac{BH^3 - 6\gamma z^2}{H^3} \right) d\theta \quad (48)$$

where $\gamma = 1 + 2BH$. It can be seen that the integral cannot be solved analytically due to its singularity at $z = 0$.

To skip the singularity, we carry out a numerical integration of the term. Eventually, SEPM's equivalent area $\hat{A}_{SEP}$ can be estimated as:

$$\hat{A}_{SEP} = \frac{\dot{W}_i}{\sigma\dot{H}} = \frac{\pi}{3} H \bar{R}^2 \left(B + \frac{8}{3} \sqrt{g_1 + \frac{3}{16} g_2} + \frac{4}{3} \sqrt{g_1 + \frac{3}{4} g_2} + \frac{2}{3} \sqrt{g_3 + \frac{3}{4} g_2} + \frac{1}{3} \sqrt{g_3 + 3g_2} \right) \quad (49)$$

where $g_1 = \frac{1}{16H^2} \left(\frac{3}{2} - BH \right)^2$, $g_2 = \frac{1}{H^4} (\bar{R} + 2BHR)^2$ and $g_3 = \frac{4}{H^2} \left(\frac{3}{4} + BH \right)^2$ are defined in Eq. 49 to avoid a lengthy expression for $\hat{A}_{SEP}$.

7. A finite element model of BCT

3.4. A finite element model of BCT

A finite element model of BCT was constructed using SFTC-DEFORM Premier software (Scientific Forming Technologies Corp.) and solved for a series of BCT case studies. This allowed a correlation of friction and geometrical parameters of the deformed sample. To make the results comparable with those published in [17], similar test scenarios were considered: $D_0 = 10\text{mm}$, $H_0 = 16\text{mm}$, $H = 15$ to 9mm with 2mm steps and $m_{FEA} = 0.1$ to 1 with 0.1 steps.

For the case solutions presented, the coordinates shown in **Fig. 1** was chosen. The sample was positioned between the moving upper anvil and the symmetry axis (r -axis). The upper die, the primary die, was moved with a velocity of $-0.5\dot{H} = -10\text{ mms}^{-1}$ (along with negative z -axis).

Due to the axisymmetric nature of the problem, only the upper quarter of the test sample and the upper anvil were modelled. A perfectly plastic behaviour with $\sigma = 100\text{MPa}$ was defined for the sample in

the simulations. Also, a rigid anvil was assumed. A quadrilateral element was chosen to solve each deformation case with 1744 nodes and 1624 elements representing the half sample's deformation. The elements employ reduced integration in the implicit solution of the problem.

Constant friction factors of $m = 0.1, 0.2, \dots, 0.9$ and 1 are used in the finite element calculations with the frictional shearing stress (traction) at the tool-sample interface equal to $m\tau$ in which τ is the yield shearing stress of the material.

3.4.1. The equivalent area for the FEM solutions

Having presented the three models and their equivalent areas, the area is derived for the two reference models (i.e. FEM and CPM) for comparison and to assess the worthiness of the weak-form kinematic models. The first reference is the finite element model of BCT. Since there exists no strong form solution for BCT and due to an iterative and optimization based solution for the FEM, this reference is expected to provide the best possible available weak form solution.

To find the equivalent area for the finite element solutions, the following calculations were carried out:

$$\hat{A}_{FEA} = \frac{1}{\dot{H}} \int_V \dot{\epsilon} dV = \frac{2\pi}{\dot{H}} \sum_{i=1}^N \dot{\epsilon}_i A_i \bar{r}_i \quad (50)$$

where $dV_i = 2\pi A_i \bar{r}_i$ and N is the number of elements. The area of the i^{th} element, A_i , and its centroid's radius, $\bar{r}_i$, with vertices (r_1, z_1) , (r_2, z_2) , (r_3, z_3) and (r_4, z_4) , listed counter-clockwise around element's perimeter are given by:

$$A_i = \frac{1}{2} (r_4(z_1 - z_3) + r_2(-z_1 + z_3) + (r_1 - r_3)(z_2 - z_4)) \quad (51)$$

$$\bar{r}_i = 0.25(r_1 + r_2 + r_3 + r_4) \quad (52)$$

The indices in Eq. 51 and 52 are given in a local coordinate system for a quadrilateral element. To convert these to real nodal numbers, the solution's element connectivity matrix was employed. This is important and enables an easier reproduction of the calculations presented here by an interested reader; FEA solutions typically refer to the global node numbers rather the local node numbers (1, 2, 3 and 4 for a quadrilateral element) which could be confusing for a subsequent post process by a user.

8. Cylindrical Profile Model (CPM)

3.5. Cylindrical Profile Model (CPM)

The conventional and mostly used model of BCT, i.e. cylindrical profile model (CPM), is employed here as the second reference model. It neglects the barrelling and has a cylindrical profile in an extreme case with zero barrelling. A detailed derivation of the model could be found elsewhere, see for example [30] and Dieter, Kuhn and Semiatin [21]. We briefly present CPM's upper bound solution and its equivalent area next.

3.5.1. CPM's equivalent area

Based on the upper bound solution of BCT's conventional model (CPM) with an upsetting load P , the non-conservative mechanical power transformation with a sticking friction factor of m can be written as:

$$P\dot{H} = \pi\dot{H}\sigma\bar{R}^2 + \frac{2\pi m\dot{H}\sigma\bar{R}^3}{3\sqrt{3}H} \quad (53)$$

where the first part, $\pi\dot{H}\sigma\bar{R}^2$, is the upper bound deformation power and the second part is friction power during deformation. Thus, CPM's equivalent area, $\hat{A}_{CPM}$, can be calculated from the following simple relationship:

$$\hat{A}_{CPM} = \frac{W_i}{\sigma\dot{H}} = \pi\bar{R}^2 \quad (54)$$

One can interpret the equivalent area in Equation 54 as a sample's cross-section area for which the radius is equal to the effective radius, $\bar{R} = 0.5D_0\sqrt{\frac{H_0}{H}}$.

A close examination of CPM reveals that its estimated equivalent area is independent of friction; the estimate is not sensitive to friction factor (barrelling).

4. The physical interpretation of the equivalent area

One may generalize the abovementioned interpretation of Eq. 54 and conclude that the equivalent area resembles a circular cross-section of an imaginary sample with an *equivalent radius* of $\hat{R}$. This also justifies the choice of the term area for the quantity which has a dimension of L^2 for all three models presented here. For the case of CPM, it is obvious that $\hat{R}_{CPM} = \bar{R}$. Similarly, the equivalent radii for the three other models can be found by:

$$\hat{R}_k = \sqrt{\frac{\hat{A}_k}{\pi}} \quad (k = EPM, EA, SEPM \text{ and } FEM) \quad (55)$$

where an index $k = EPM, EA, SEPM$ and FEM is used in Eq. 55 for the sake of brevity.

To evaluate the models, one notes that the equivalent radius $\hat{R}_k$ (Eq. 55) has less sensitivity compared the equivalent area $\hat{A}_k$. The former is simply introduced to enable a physical visualization of the equivalent area in comparison with the effective radius $\bar{R} = 0.5D_0\sqrt{\frac{H_0}{H}}$ which is a key parameter in the CPM. The radius is not considered an independent criterion to evaluate the presented kinematic models.

These radii ($\hat{R}_{CPM}$, $\hat{R}_{EPM}$, $\hat{R}_{EA}$ and $\hat{R}_{FEA}$) will be used for a tangible evaluation and comparison of the models.

5. Comparison of the results and discussions

Table 1 summarizes the computational conditions for the finite element simulations. The deformed geometry predicted by the finite element simulation was used as the input to the three kinematic models to allow a meaningful benchmarking. These are shown in **Table 1** for the test models and computational conditions. The range of the deformation parameters is relatively wide, a sample's height reduction of ~%14 to %44 and therefore the incremental approach was adopted. Since friction is responsible for barrelling, a full range of barrelling was considered by changing the friction factor, m , between 0.1 and 1.

For each of the finite element solutions, a constant friction value (m_{FEA}) was chosen and the resulting D and d were recorded. The predicted deformed geometries by the FE simulations were chosen as input for EPM, EA and SEPM case studies. For each simulation scenario, the corresponding barrelling parameters (B , b and $\mathcal{B}$) were calculated based on equations 17, 34, and 46, respectively. The equivalent areas $\hat{A}_{EPM}$, $\hat{A}_{EA}$ and $\hat{A}_{SEPM}$ were calculated using the derivations presented earlier in this article and implemented the calculations using a dedicated Fortran program which also enabled calculation of the corresponding equivalent area, $\hat{A}_{FEA}$, for the finite element solutions. The latter calculations were based on strain rate, mesh coordinates and element connectivity matrix, which are available in a typical finite element output file (see Eqs. 50 to 52).

Table 1. Summary of deformed sample details obtained by FEA simulation; D , d , $\bar{R} = R$ and H are given in mm.

		FEM's friction factor; m_{FEA}									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$H = 15, \bar{R} = 5.164$	0.5D	5.174	5.186	5.196	5.201	5.204	5.207	5.209	5.211	5.213	5.214
	0.5d	5.122	5.095	5.076	5.063	5.054	5.0460	5.040	5.034	5.028	5.023
$H = 13, \bar{R} = 5.547$	0.5D	5.549	5.577	5.600	5.619	5.634	5.6480	5.656	5.663	5.668	5.668
	0.5d	5.438	5.373	5.318	5.271	5.230	5.1930	5.160	5.123	5.107	5.090
$H = 11, \bar{R} = 6.030$	0.5D	6.037	6.093	5.960	6.162	6.184	6.2020	6.213	6.221	6.226	6.229
	0.5d	5.832	5.714	5.515	5.554	5.504	5.4720	5.475	5.404	5.395	5.369
$H = 9, \bar{R} = 6.667$	0.5D	6.674	6.757	6.811	6.847	6.875	6.8960	6.909	6.920	6.924	6.927
	0.5d	6.365	6.223	6.126	6.067	5.994	5.9180	5.932	5.930	5.925	5.910

While the barrelling parameter is specific to a given solution/velocity field, the suggested techniques in this work provide different (non-unique) estimates of the same parameter for each model. The profile and the equivalent areas for EPM and SEPM are based on the incremental formulation of the barrelling parameter and its evaluation at the representative point (mid-plane). In the case of EA, the parameter (b) was calculated using Eq. 34. It has to be noted that these estimations do not automatically guarantee the volume constancy.

Fig. 4 compares the predicted sample's profile based on the four models with that of the reference FE and CPM. The predictions are based on the computational conditions of $H = 9$ and $m = 0.3$. Also, mathematical descriptions of the models are given in Eqs. 13 and 14 (EPM), Eq. 33(EA) and Eqs. 44 and 45 (SEPM). The corresponding cylindrical profile for CPM is $R(z) = \bar{R} = 6.030$. The actual profiles for each model (except CPM and FEA) do not satisfy the incompressibility principle and should not be used for volume integration of the upper bound deformation power. This explains the use of $\bar{R} = 0.5D_0\sqrt{\frac{H_0}{H}}$ in the integrations. It can be seen in **Fig. 4** that the hour glass solution has been prevented for all cases due to the incorporation of the barrelling parameter in the models. A closer examination of the profiles indicates that the EA profile is concave while SEPM and FEM are convex. Also, profiles for EPM is very close to an oblique line, while that of CPM is vertical.

Fig. 4 Comparison of the BCT sample's profile using four different models with the FEA's reference simulation. The solid line (no symbol), shows a quarter of deformed sample predicted by the reference FEM model. Insets (a) and (b) show and compare zoom views of the predictions made by different models at the top and mid zone of the profile, respectively. Solutions were obtained for $H = 9$ mm, $D_0 = 10$ mm, $H_0 = 16$ mm and $m = 0.3$. (EPM using B_t)

The estimated profiles based on the investigated kinematic models show a poor agreement with that of the reference FE solution in Fig.4. The main reason for the deviations is that the BCT cannot be simply described with a one-zone model, and a more rigorous and reliable kinematic solution with multi-zone velocity fields are required. This involves division of sample into at least two zones of different velocity fields to account for the foldover. The first (main) zone comprises the entire sample without a zone adjacent to the sample's free surface (vicinity of its profile). The excluded zone is presented with at least one supplementary zone with a different velocity field. This is essential to reliably present the complex deformation and flow localization in the excluded zone (near the profile). A weak kinematic solution of the foldover during BCT with two zones was presented in [1, 4] in which the EA velocity field, discussed in this paper, is used for the first zone.

The profile estimation presented in this work is an essential prerequisite for a “multi zone solution” since its barrelling parameter is a key input for the more comprehensive solution with the supplementary velocity field. The poor agreement of the profiles in Fig. 4 can be explained by noting that they are located in the sample's excluded zone.

The results for the five models' "equivalent area analysis" are compared in **Fig. 5** for easier visualization and evaluation. It shows the equivalent areas of the three case models against that of the reference finite element model and CPM. **Fig. 5a, b, c** and **d** correspond to deformation cases of $H = 15, 13, 11$ and 9 , respectively. In all cases EPM and EA show a reasonably good agreement with the reference FEM estimation of the equivalent area; the largest difference is less than $\sim 9\%$. The agreements for EPM and EA estimations reduce as the friction factor and the height reduction increase. The equivalent area is a conceptual quantity and its increase with deformation is as expected. CPM's constant estimated power is insensitive to friction factor. It has the best agreement for low friction factors but deviates abruptly with an increase in the friction factor and deformation. Given a friction factor, the difference between the equivalent areas found by FEM and CPM is a good quantitative indication of the barrelling contribution to the equivalent area. Contrary to EPM and EA, SEPM's estimated normalized power (shown using the right-hand side vertical axis) is significantly higher than its corresponding reference values ($\sim 150\%$ - 210%). We note that the SEPM equivalent area cannot be calculated analytically by Eq. 48 due to its singularity at $z = 0$. The integration was carried out numerically using Eq. 49. The significantly higher value of $\hat{A}_{SEP}$ compared to $\hat{A}_{FEA}$ can be explained by noting that at least one of the numerical integration points inevitably falls in close vicinity to the singularity point. One has to note that only a limited conclusion can be made from IPM's assessment; a close agreement of the equivalent areas for EPM, EA with the FEA's reference area does not necessarily mean that these models are perfect. The agreement only indicates that the two models are worthy of further developments for more detailed calculations such as the friction factor. Despite a good agreement between $\hat{A}_{EPM}$ and $\hat{A}_{EA}$ and the fact that they are reasonably comparable with $\hat{A}_{FEA}$, further assessment of EPM and EA is necessary. To make a conclusive judgement about either of the EPM or EA models usefulness, they have to be employed in an upper bound solution to compare their results with that of a finite element model.

As far as the SEPM is concerned, the conclusion that can be drawn from the solutions and criterion presented here is that the model is not suitable to study and utilize BCT's data. The trial model

(SEPM) is not worthy of further development despite its ability to predict a reasonably good profile compared to CPM.

Fig. 5. Comparing the normalized powers $\hat{A}_{CPM}$, $\hat{A}_{EPM}$, $\hat{A}_{EA}$ and $\hat{A}_{SEPM}$ calculate by CPM, EPM [15], EA ([16] [17]) and SEPM methods, respectively with their counterparts in FEA simulation, $\hat{A}_{FEM}$; (a) $H = 15mm$ (b) $H = 13mm$, (c) $H = 11mm$ and (d) $H = 9mm$.

Table 2. Computed equivalent radii $\hat{R}_{EPM}$, $\hat{R}_{SEPM}$, $\hat{R}_{EA}$, $\hat{R}_{CPM}$ and $\hat{R}_{FEM}$ for the computational conditions shown in **Fig. 5**, the $\hat{R}$ and H values are in mm.

m		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$\hat{R}_{CPM}$	H=9	6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67
$\hat{R}_{EPM}$		6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67	6.67
$\hat{R}_{SEPM}$		10.17	10.24	10.25	10.29	10.30	10.31	10.31	10.32	10.31	10.33
$\hat{R}_{EA}$		6.67	6.68	6.69	6.71	6.72	6.74	6.76	6.78	6.80	6.82
$\hat{R}_{FEM}$		6.65	6.63	6.61	6.60	6.60	6.57	6.58	6.56	6.59	6.59
$\hat{R}_{CPM}$	H=11	6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03
$\hat{R}_{EPM}$		6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03	6.03

m		0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$\hat{R}_{SEPM}$		8.52	8.61	8.64	8.70	8.72	8.73	8.73	8.73	8.73	8.74
$\hat{R}_{EA}$		6.03	6.04	6.05	6.06	6.08	6.09	6.10	6.12	6.13	6.14
$\hat{R}_{FEM}$		6.02	6.02	6.02	6.01	6.00	6.00	5.99	6.00	6.00	5.99
$\hat{R}_{CPM}$	H=13	5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55
$\hat{R}_{EPM}$		5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55	5.55
$\hat{R}_{SEPM}$		7.38	7.44	7.48	7.54	7.58	7.62	7.65	7.66	7.67	7.67
$\hat{R}_{EA}$		5.55	5.56	5.56	5.57	5.58	5.59	5.60	5.61	5.62	5.63
$\hat{R}_{FEM}$		5.54	5.55	5.55	5.55	5.56	5.56	5.57	5.57	5.57	5.57
$\hat{R}_{CPM}$	H=15	5.16	5.16	5.16	5.16	5.16	5.16	5.16	5.16	5.16	5.16
$\hat{R}_{EPM}$		5.16	5.17	5.17	5.17	5.22	5.17	5.17	5.17	5.17	5.17
$\hat{R}_{SEPM}$		6.75	6.84	6.92	6.96	6.98	7.01	7.02	7.04	7.05	7.06
$\hat{R}_{EA}$		5.17	5.17	5.18	5.19	5.19	5.20	5.21	5.21	5.22	5.23
$\hat{R}_{FEM}$		5.16	5.17	5.17	5.18	5.19	5.19	5.20	5.21	5.21	5.21

Equivalent radii $\hat{R}_{EPM}$, $\hat{R}_{SEPM}$, $\hat{R}_{EA}$, $\hat{R}_{CPM}$ and $\hat{R}_{FEM}$ were calculated for the computational conditions shown in **Fig. 5** using Eq. 55. The calculated values, shown in **Table 2.**, were found to have low sensitivity to the chosen model and the friction factor for a given sample height, H . This excluded the calculated $\hat{R}_{SEPM}$. The radius was 30% and %51 larger than that of other models in average for the sample height of $H = 15\text{mm}$ and $H = 9\text{mm}$, respectively. The low sensitivity can be explained by noting the relatively small differences between the equivalent areas in **Fig. 5** for different models. A comparative study of EPM, EA and FEA by Khoddam [15] revealed a noticeable difference in the distribution of the effective strain and strain rate for these case models. Therefore, the equivalent area is a more useful quantity to compare and evaluate the proposed models. The low sensitivity of the equivalent radii to its model is alarming; one should not rely on visual inspection of the profile only to assess the presence of shearing strain or barrelling in the test sample.

In Figure 5, changes of $\hat{A}_{FE}$ with m for a given H , are less than 10% for all shown cases. $\hat{A}_{FE}$ increases in Figure 5 a and b but decreases in c and d. The term “equivalent area” was simply picked for $\hat{A}_k$ due to its dimension (L^2). However, $\hat{A}_{FE}$ has to be understood carefully to avoid misinterpretation of the evaluation criterion. A closer look at Equation 25 and the definition of $\dot{W}_i$ shows that in general $\hat{A}_k$ is a volume integral, which varies non-linearly with the H and $\bar{R}$. The integrand depends on both r and $\dot{\epsilon}_{ij}$, which are the position of a typical point in the sample and its corresponding von-Mises effective strain

rate, respectively. The latter is distributed heterogeneously in the deforming sample. These suggest that the changes of $\hat{A}_{FE}$ could be quite unpredictable and different with the trend of changes in $\bar{R}$.

~~The author declares that there exists no conflict of interest or relationships that could potentially influence or bias the submitted work.~~

Profile modelling can be seen as a method to augment a weak solution by including new experimental data such as the barrelling parameter which are measured indirectly based on the instantaneous free surface observations. The methodology presented in this work is particularly useful for the forming processes, e.g. plane strain extrusion and indentation, for which a velocity field has already been proposed and the process can be solved incrementally.

6. Conclusion

The Incremental Profile Modelling (IPM) was proposed as a convenient approach to simply enable a non-conservative and path dependent solution for a range of deformation problems. It can also evaluate a boundary value weak formulation and its closed-form solution in a kinematic framework. The technique was demonstrated using the barrelling compression test (BCT) as a key example. Three weak-form kinematic models of BCT were employed to show the implementation of the six typical steps involved in an IPM. The models included Exponential Profile, Extended-Avitzur and Symmetric Exponential Profile Models (EPM, EA and SEPM). IPM applied to BCT was suggested also as an avenue to integrate more test data, i.e. the sample's instantaneous profile data, into the kinematic model to strengthen the weak formulation. The radial and axial velocity components of each model were linked via a barrelling parameter. The techniques to identify the profile's radius and the barrelling parameter for each model were presented and derivations were made for each example using the incremental approach. Using the identified parameters and effective strain rates for each model, their associated equivalent areas were derived and calculated for a number of example test scenarios. The conventional Cylindrical Profile Model was offered as an extreme reference with zero barrelling. A finite element model was constructed and used as the second reference (near strong formulation) solution. Profile estimates and upper bound solutions of the four models were compared with their

finite element counterpart to quantify the contribution of barrelling in the equivalent area. Comparing EPM, EA and SEPM's calculated normalized powers with that of a reference finite element model, it was concluded that the first two models are worthy of further development. The singularity, identified in the third model, was found responsible to impair further use of SEPM to study strain, strain rate, friction and stress based on BCT's data.

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