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Reach control problem for affine multi-agent systems on simplices [★]

Yuhu Wu ^{a,b}, Weiguo Xia ^{a,b}, Ming Cao ^c, Xi-Ming Sun ^{a,b}

^aKey Laboratory of Intelligent Control and Optimization for Industrial Equipment of Ministry of Education, Dalian University of Technology, China

^bSchool of Control Science and Engineering, Dalian University of Technology, China

^cFaculty of Science and Engineering, ENTEG, University of Groningen, the Netherlands

Abstract

This paper studies the reach control problem for a coupled affine multi-agent system, which aims to find an affine feedback control for the trajectories of the agents to reach and exit a particular facet of a given simplex in the state space in finite time. The interactions between agents characterized by diffusive coupling prevent the effective construction of controller using the well developed techniques to study similar problems for affine single-agent systems. In fact, the affine feedback control designed for a single affine system may not work for the multi-agent case anymore as some agent can be driven to exit the simplex through a restricted facet under the influence from its coupled peers. A sufficient condition is developed to guarantee that all the agents move continuously in a cone containing the simplex and exit through the exit facets in finite time under an affine feedback control. A numerical example is given to verify the effectiveness of our derived result.

Key words: Reach control problem, affine control, multi-agent systems, simplex, exit facets.

1 Introduction

The reach control problem concerns steering the state of a system, which starts from a point within an n -dimensional simplex or polytope, to reach a specific facet of this simplex or polytope in finite time without exiting from other facets first. The study of this problem is related to the reachability problem of piecewise linear hybrid systems (Habets and van Schuppen [2004]) and has received much attention in the last decade (Habets et al. [2006], Roszak and Broucke [2006], Broucke [2010], Habets et al. [2012], Helwa et al. [2016]). Fruitful results have been reported regarding affine systems (Habets and van Schuppen [2004], Habets et al. [2006], Roszak and Broucke [2006], Broucke [2010], Habets et al.

[2012]) and discontinuous dynamical systems (Wu and Shen [2016]) with different feedbacks including affine state feedback control and discontinuous state feedback control (Broucke and Ganness [2014], Semsar-Kazerooni and Broucke [2014]). For affine systems, two sets of conditions, the invariance conditions and flow conditions, have been proposed to guarantee that all trajectories exit via the desired facet using an affine state feedback control. Continuous state feedback and discontinuous feedback have been discussed in (Broucke [2010], Broucke and Ganness [2014]). The well-posedness and structural stability of the reach control problem for the affine systems on a simplex or polytope has been considered in (Broucke and Semsar-Kazerooni [2012]). A recent paper (Ornik and Broucke [2018]) investigates the case when a trajectory exits a simplex but does not cross into an outer half-space, i.e., it chatters, and identifies the classes of feedback controls that do not allow chattering.

In the above literature, the reach control problem centers on controlling a single system characterized by an affine differential equation. As networked systems become prevailing in recent years, a system can be composed of a set of subsystems interacting with each other. Typical examples range from physical to natural dynamical sys-

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Email addresses: wuyuhu@dlut.edu.cn (Yuhu Wu), wgxiaseu@dlut.edu.cn (Weiguo Xia), m.cao@rug.nl (Ming Cao), sunxm@dlut.edu.cn (Xi-Ming Sun).

tems, such as artificial neural networks (Bishop [1995]), complex ecosystems (May [2001]), and coupled systems of nonlinear oscillators (Dorfler and Bullo [2014]). Emerging collective behaviors, such as synchronization, flocking, and swarming (Yu et al. [2008], Xia and Cao [2011], Olfati-Saber [2006], Gazi and Passino [2003]), arise when a group of agents are interacting with each other, which have been extensively studied in the past several decades. The study of the reach control problem of multi-agent systems will be useful for controller design with specific motion-trajectory objectives of multi-agent systems. For example, when a group of mobile robots in one area are desired to move to another and rendezvous at some point there (Bullo et al. [2009]). The allowable moving space can be partitioned into simplices and the corresponding controllers are designed for each simplex so that the robots move through the simplices in sequence and finally reach the desired area (Ornik and Broucke [2018]). The reach control problem of such a multi-agent system would require that the state of each agent exits from a given facet. The controllers should be redesigned for the multi-agent system case as the direct application of the proposed controller for a single system may not work for such an interconnected system. For example, some agent can be driven to exit the simplex through a restricted facet under the influence from its coupled peers that are leaving the simplex through the specific facet. An example (Example 2.1) will be given to illustrate this possibility. How to devise new controllers to achieve the reach control objective for a coupled system remains unknown and it is our goal in this paper to deal with this challenging problem.

In this paper, we investigate the reach control problem of a group of coupled affine subsystems. This integrated system is characterized by a set of diffusively coupled differential equations and its reach control problem is reduced to the classical one considered in the literature when only one subsystem is concerned. The invariance and flow conditions have been proposed in the literature for a single affine system. We have to redesign the controller for the multi-agent case to guarantee that all the states of the system exit from the particular facet without leaving the simplex from other facets. An example is provided to illustrate that the controllers should be carefully designed to avoid this scenario by taking into account the interactions among the subsystems. To solve the reach control problem of a group of coupled affine subsystems, (i) a cone that contains the simplex as a subset is introduced and the restricted facets are part of the boundary of the cone; new invariance conditions are proposed so that the states of the system stay within the cone and therefore they will not exit the simplex through the restricted facets; (ii) new flow conditions are proposed so that after a sufficient long time, the states of the system will exit the simplex through the particular facet.

The rest of the paper is organized as follows. Section 2

introduces some preliminaries, formulates the reach control problem of a coupled system consisting of N affine subsystems, and provides an example illustrating that the controller that works for the reach control problem of a single system does not work for a coupled system. Section 3 proposes a set of conditions that solves the reach control problem. Section 4 revisits the example and verifies our derived results. Section 5 concludes the paper.

2 Preliminaries and problem statement

We first introduce some notations used throughout the paper. For a positive integer k , $[k] \triangleq \{1, \dots, k\}$ and $[\bar{k}] \triangleq \{0, 1, \dots, k\}$. Consider an n -dimensional simplex S with vertices $v_0, \dots, v_n$ and its facets $\mathcal{F}_0, \dots, \mathcal{F}_n$, where each facet $\mathcal{F}_i$ is the convex hull of $\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$. Throughout this paper, we always assume that the simplex S is full dimensional in $\mathbb{R}^n$, which means that the set $V := \{v_0, \dots, v_n\}$ of vertices of S are affinely independent points.

Let $\mathcal{I}$ be a given subset of $[\bar{n}]$. The facets $\mathcal{F}_i, i \notin \mathcal{I}$, of S are called admissible exit facets (Habets et al. [2006]), and the facets $\mathcal{F}_i, i \in \mathcal{I}$ of S are called restricted facets. Without loss of generality, in this paper we consider the special case of one admissible exit facet, and assume that it is $\mathcal{F}_0$, which implies that $\mathcal{I} = \{1, \dots, n\}$. This assumption is imposed for the ease of presentation while the results can be extended to the general case of more than one admissible exit facet.

Denote the boundary of a subset Ω of $\mathbb{R}^n$ by $\partial\Omega$, the closure of Ω by $\bar{\Omega}$, and the interior of Ω by $\mathring{\Omega}$, respectively. Then $\partial S = \bigcup_{i=0}^n \mathcal{F}_i$. Additionally, for each $x \in V$ let

$$\mathcal{I}_x = \{k | k \in \mathcal{I}, x \in \mathcal{F}_k\}. \quad (1)$$

Definition 2.1 Let $h \in \mathbb{R}^N$ be a nonzero vector, and let $c \in \mathbb{R}$ be a constant. The hyperplane $H(h, c)$ and the closed half-space $L(h, c)$ of $\mathbb{R}^N$ are defined respectively as

$$H(h, c) = \{x \in \mathbb{R}^N : h^\top x = c\}, \quad (2)$$

$$L(h, c) = \{x \in \mathbb{R}^N : h^\top x \leq c\}. \quad (3)$$

Let $h_i (i \in [\bar{n}])$ be the unit normal vector associated with each facet $\mathcal{F}_i$ pointing out of S . The following equivalent description of simplex S can be obtained.

Remark 2.1 The simplex S can be described by the intersection of $n + 1$ closed half spaces (Habets and van Schuppen [2004]), that is, there exist $n + 1$ scalars $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that

$$S = \bigcap_{k=0}^n L(h_k, c_k). \quad (4)$$

Actually, here the scalars $c_0, c_1, \dots, c_n \in \mathbb{R}$ satisfy

$$c_k = h_k^\top v_j, \forall k, j \in [\bar{n}], k \neq j. \quad (5)$$

Furthermore, each facet $\mathcal{F}_k$ is the intersection of S with one of its supporting hyperplane, $H(h_k, c_k)$, i.e., $\mathcal{F}_k = S \cap H(h_k, c_k)$. $\square$

Now we introduce the reach control problem of a single affine system. The dynamics of the system are described by

$$\dot{x} = Ax + Bu + a, \quad (6)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $a \in \mathbb{R}^n$. The affine feedback is given by

$$u(x) = Kx + b, \quad (7)$$

with $K \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The reach control problem of the affine system (6) is defined as follows.

Problem 1. For a given simplex S , with a set of restricted facets $\mathcal{F}_j, j \in \mathcal{I}$, construct a affine feedback (7) such that for each initial condition $x(0) = x_0 \in S$ of the affine system (6), there exist a time $t_0 \geq 0$ and an $\varepsilon > 0$ such that:

- (i) $x(t) \in S, \forall t \in [0, t_0]$;
- (ii) $x(t_0) \in \mathcal{F}_k$, for some $k \notin \mathcal{I}$;
- (iii) $x(t) \notin S, \forall t \in (t_0, t_0 + \varepsilon)$.

This problem has been studied in the literature such as (Habets and van Schuppen [2004], Habets et al. [2006], Roszak and Broucke [2006], Broucke [2010], Habets et al. [2012]) and the main result on the reach control problem of system (6) is restated as follows.

Theorem 2.1 (Habets et al. [2006], Roszak and Broucke [2006]) *Consider the affine system (6) on simplex S . Then, the reach control problem for the affine system (6) is solvable if and only if there exist a set of inputs $u_0, \dots, u_n \in \mathbb{R}^m$ and a vector $\omega \in \mathbb{R}^n$ such that the following hold:*

(1) *Invariance conditions:*

$$h_l^\top \eta_k \leq 0, k \in [\bar{n}], l \in \mathcal{I}_{v_k}. \quad (8)$$

(2) *Flow conditions:*

$$\omega^\top \eta_k < 0, k \in [\bar{n}], \quad (9)$$

where

$$\eta_k = Av_k + Bu_k + a, k \in [\bar{n}]. \quad (10)$$

Once the control input u_i for each vertex $v_i, i \in [\bar{N}]$ is obtained based on the necessary and sufficient condition

in Theorem 2.1, the affine control (7) can be calculated in view of the following proposition, adapted from Lemma 5 of (Roszak and Broucke [2006]).

Proposition 2.1 *Consider two sets of points $\{v_0, \dots, v_n\}, v_i \in \mathbb{R}^n$ and $\{u_0, \dots, u_n\}, u_i \in \mathbb{R}^m$. Suppose the v_i 's are affinely independent. Then there exists a unique matrix $K \in \mathbb{R}^{m \times n}$ and a unique vector $b \in \mathbb{R}^m$ such that for each $v_i, u_i = Kv_i + b$, where K, b are calculated by*

$$\begin{bmatrix} K^\top \\ b^\top \end{bmatrix} = \begin{bmatrix} v_0^\top & 1 \\ \vdots & \vdots \\ v_n^\top & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_0^\top \\ \vdots \\ u_n^\top \end{bmatrix}.$$

As discussed in Section 1, a networked system is composed of multiple subsystems that interact with each other. Instead of considering the reach control problem of a single affine system, this paper concerns the reach control problem of a coupled system consisting of N subsystems. The dynamics of each subsystem, or an agent, are given by

$$\dot{x}_i = Ax_i + \sum_{j \in [N]} g_{ij} \Gamma(x_j - x_i) + Bu_i + a, \quad i \in [N], \quad (11)$$

where $x_i \in \mathbb{R}^n$. $\sum_{j \in [N]} g_{ij} \Gamma(x_j - x_i)$ describes the diffusive coupling between agents, where $g_{ij} \geq 0$ is a nonnegative constant representing the coupling strength between i and j for every $i, j \in [N]$, and $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$ with $\gamma_k \neq 0$, for all $k \in [n]$. Let $G = (g_{ij})_{N \times N}$. The coupling term $\sum_{j \in [N]} g_{ij} \Gamma(x_j - x_i)$ in system (11) in essence acts as virtual forces that drive the agents to reduce the difference between their states and agree on their states. Systems with similar dynamics arise in several control problems of multi-agent systems like rendezvous and formation control (Bullo et al. [2009]). The study of the reach control problem can be useful for the controller design in other multi-agent control problems. The model (11) has also been used to describe a single-species dynamical system which is composed of several patches connected by discrete diffusion (Lu and Takeuchi [1993]) or coupled systems on networks (Li and Shuai [2010]).

We similarly construct the affine feedback

$$u(x_i) = Kx_i + b. \quad (12)$$

The reach control problem of the multi-agent system (11) is defined as follows.

Problem 2. Construct the affine feedback control law (12) such that for each initial condition $\{x_i(0)\}_{i=1}^N \subset S$ of the multi-agent system (11), there exist $t_i \geq 0, i \in [N]$, and $\varepsilon > 0$ such that for each $i \in [N]$:

- (i) $x_i(t) \in S, \forall t \in [0, t_i]$;
- (ii) $x_i(t_i) \in \mathcal{F}_k$, for some $k \notin \mathcal{I}$;
- (iii) $x_i(t) \notin S, \forall t \in (t_i, t_i + \varepsilon)$.

Note that due to the existence of the coupling term in system (11) compared to system (6), Theorem 2.1 derived for a single agent system does not directly apply to the multi-agent system (11). We give the following example to illustrate and we will identify the sufficient conditions for the reach control problem of system (11) in the next section.

Example 2.1 Let S_2 be the triangle in $\mathbb{R}^2$ with vertices $v_0 = [0, 0]^\top, v_1 = [2.5, 0]^\top, v_2 = [2, 1]^\top$ shown in Fig. 1. The corresponding outer normal vectors on the three facets $\mathcal{F}_0, \mathcal{F}_1$, and $\mathcal{F}_2$ of S_2 are $h_0 = \sqrt{5}/5[2, 1]^\top, h_1 = \sqrt{5}/5[-1, 2]^\top$, and $h_2 = [0, -1]^\top$. Assume that the exit facet is $\mathcal{F}_0$, and so $\mathcal{I} = \{1, 2\}$.

On the simplex S_2 , consider the multi-agent affine system (11) with $N = 6$,

$$A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \Gamma = \begin{bmatrix} 0.1 & 0 \\ 0 & 2 \end{bmatrix}. \quad (13)$$

The coupling matrix $G = [g_{ij}]_{6 \times 6}$ is given by

$$G = \begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0 & 1 & 1 & 1 & 1 \\ 0.25 & 1 & 0 & 1 & 1 & 1 \\ 0.25 & 1 & 1 & 0 & 1 & 1 \\ 0.25 & 1 & 1 & 1 & 0 & 1 \\ 0.25 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

If the 6 agents are not coupled, i.e., each system is described by (6), the solvability of the reach control problem is given by Theorem 2.1. Choose $u_0 = [2, 0]^\top, u_1 = [2, 1]^\top, u_2 = [2, 1.25]^\top$, and it's easy to verify that the conditions (8) and (9) are satisfied. Then one can compute K and b in (12) from Proposition 2.1 as

$$K = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \quad (14)$$

Let the initial conditions of $x_i(t)_{i=1}^N$ be $x_1(0) = [0.5, 0.1]^\top, x_2(0) = [2, 0.8]^\top, x_3(0) = [1.2, 0.2]^\top, x_4(0) = [1.8, 0.8]^\top, x_5(0) = [1.6, 0.6]^\top, x_6(0) = [2, 0.3]^\top$. The trajectory of each agent is illustrated in Fig. 1. However, the same affine feedback control cannot solve the

Fig. 1. Trajectories of the six agents described by system (6) under the control (12) with (14).

reach control problem of the coupled system (11). The trajectories of the six agents are depicted in Fig. 2. It can be seen that agent 1 exists the simplex S_2 from the restricted facet $\mathcal{F}_1$ which violates the requirement of Problem 2. New conditions should be established to find an appropriate affine feedback control.

Fig. 2. Trajectories of the six agents described by system (11) under the control (12) with (14).

3 Main results

In this section, a sufficient condition is established for the reach control problem of the coupled system (11). To solve the reach control problem of system (11), in view of its difference with system (6), the key issue is how to deal with the coupling term $\sum_{j \in [N]} g_{ij} \Gamma(x_j - x_i)$. Our idea is to provide an estimate on this coupling term so that an appropriate feedback controller can be identified. The main result is summarized as follows.

Theorem 3.1 Consider the multi-agent system (11)

with a given simplex S . If there exists a set of inputs $u_0, \dots, u_n \in \mathbb{R}^n$, and a constant $\xi \in (0, 1)$ such that the following hold:

(1) Strong invariance conditions

$$h_l^\top \eta_k + \mu_l(v_k) < \xi h_l^\top \eta_0, \forall k \in [\bar{n}], l \in (I_{v_k} \cup \{0\}) \setminus \{k\}; \quad (15)$$

(2) Strong flow conditions

$$\hat{h}_0^\top \eta_k > 0, \quad \forall k \in [\bar{n}], \quad (16)$$

where η_k is given in (10) of Theorem 2.1, and

$$\mu_l(x) \triangleq \|G\|_\infty \max_{v \in V} h_l^\top \Gamma(v - x), \quad \forall x \in \mathbb{R}^n, \quad (17)$$

$$\hat{h}_0 \triangleq \Gamma^{-1} h_0, \quad (18)$$

and $\|G\|_\infty = \max_i \sum_{j \in [N]} |g_{ij}|$. Then, the reach control problem for the multi-agent system (11) is solvable with feedback control $u(x) = Kx + b$, where K and b are uniquely determined by inputs $u_0, \dots, u_n$ at vertices as stated in Proposition 2.1.

Compared with the standard invariance condition in Theorem 2.1, there is an additional term $\mu_l(v_i)$ in the strong version (15). Note that $\mu_l(v_i)$ estimates the impact of the coupling term $\sum_{j \in [N]} g_{ij} \Gamma(x_j - x_i)$ on the trajectories of the agents by working with the upper bound of $\sum_{j \in [N]} g_{ij} h_l^\top \Gamma(x_j - x_i)$. Compared with the standard flow condition in Theorem 2.1, the specific vector h_0 is picked and $\hat{h}_0$ is defined correspondingly to stimulating the condition (16). It may be possible to state this condition using a vector ω as in Theorem 2.1, but it will bring difficulties in the proof.

Before proving the theorem, we introduce two more notions that will play a key role in the development of the proof.

Definition 3.1 For a given simplex S in $\mathbb{R}^n$, we define

$$S_{v_0}(\delta) \triangleq \left\{ x \in \mathbb{R}^n \mid x = v_0 + \sum_{k=1}^n \lambda_k (v_k - v_0), \sum_{k=1}^n \lambda_k \leq \delta, \lambda_k \geq 0, k \in \mathcal{I} \right\} \quad (19)$$

$$C_{S, v_0} \triangleq \left\{ x \in \mathbb{R}^n \mid x = v_0 + \sum_{k=1}^n \lambda_k (v_k - v_0), \lambda_k \geq 0, k \in \mathcal{I} \right\}. \quad (20)$$

We call $S_{v_0}(\delta)$, with $\delta \geq 1$, the convex extension of S with regard to vertex v_0 , and call C_{S, v_0} the cone extension of S with regard to vertex v_0 , respectively.

For simplex S in a two-dimensional space, its convex extension $S_{v_0}(\delta)$, and its cone extension C_{S, v_0} with regard to vertex v_0 are depicted in Fig. 3 to clarify the above definition. Some geometric relationships of S , its convex extension $S_{v_0}(\delta)$, and cone extension C_{S, v_0} are given as follows without proof.

Fig. 3. A simplex S in grey in the two-dimensional space. Its convex extension $S_{v_0}(\delta)$ with $\delta = 1.5$ is the dashed area, and its cone extension C_{S, v_0} with regard to vertex v_0 is a cone.

Proposition 3.1 Simplex S , $S_{v_0}(\delta)$, C_{S, v_0} defined by (19) and (20) satisfy the following formulas

$$C_{S, v_0} = \bigcap_{k \in \mathcal{I}} L(h_k, c_k), \quad (21)$$

$$S_{v_0}(\delta) = C_{S, v_0} \cap L(h_0, (1 - \delta)h_0^\top v_0 + \delta c_0), \quad (22)$$

$$C_{S, v_0} = \bigcup_{\delta \in [0, +\infty)} S_{v_0}(\delta), \quad (23)$$

$$S_{v_0}(1) = S. \quad (24)$$

For each δ , it is clear that the vertex set of $S_{v_0}(\delta)$ is $V(\delta) = \{v_0(\delta), v_1(\delta), \dots, v_n(\delta)\}$, with $v_i(\delta) \triangleq (1 - \delta)v_0 + \delta v_i$, that is

$$S_{v_0}(\delta) = \text{Co}\{V(\delta)\}. \quad (25)$$

The proof of Theorem 3.1 relies on several lemmas that we start to develop now. The main idea of the proof is the following: Lemma 3.1 provides an intermediate result used in the proof of Lemma 3.2 which shows that under the condition (15) the states of all the agents will stay in the cone extension C_{S, v_0} of S for all time $t \geq 0$; Based on the result of Lemma 3.2, the conditions (15) and (16) guarantee that the tangent vector of each agent's trajectory always has an acute angle with the direction $\hat{h}_0$ and all the agents exit the simplex through the desired facet, which is proved in Lemma 3.3.

The following proposition establishes an upper bound of the term $\sum_{j \in [N]} g_{ij} h_l^\top \Gamma(x_j - x_i)$ on simplex S .

Proposition 3.2 If $x_i \in S$ for all $i \in [N]$, then

$$\sum_{j \in [N]} g_{ij} h_l^\top \Gamma(x_j - x_i) \leq \mu_l(x_i), \quad \forall l \in I, i \in [N]. \quad (26)$$

Proof: Since S is a full n -dimensional simplex in $\mathbb{R}^n$ with vertices $v_0, \dots, v_n$, for any $y \in S$, there exists $\lambda_k^y \geq 0, k \in [\bar{n}]$ with $\sum_{k=0}^n \lambda_k^y = 1$ such that $y = \sum_{k=0}^n \lambda_k^y v_k$.

Hence, for any $y \in S$, we have

$$h_l^\top \Gamma(y - x) = \sum_{k=0}^n \lambda_k^y h_l^\top \Gamma(v_k - x) \leq \max_{v \in V} h_l^\top \Gamma(v - x). \quad (27)$$

Since $x_i \in S$ for all $i \in [N]$, we have

$$\begin{aligned} \sum_{j \in [N]} g_{ij} h_l^\top \Gamma(x_j - x_i) &\leq \sum_{j \in [N]} g_{ij} \max_{v \in V} h_l^\top \Gamma(v - x_i) \\ &\leq \|G\|_\infty \max_{v \in V} h_l^\top \Gamma(v - x_i) = \mu_l(x_i), \end{aligned} \quad (28)$$

recalling definition (17) of μ_l and that $\|G\|_\infty$ is the maximum absolute row sum of G . ■

With the help of Proposition 3.2, the next lemma claims that at each time instant, if all the agents lie in the convex extension $S_{v_0}(\delta_0)$, then an agent on the boundary of $S_{v_0}(\delta_0)$ will move towards the interior of $S_{v_0}(\delta_0)$.

Lemma 3.1 *Assume that the strong invariance conditions (15) hold for the multi-agent system (11). At time $t > 0$, assume*

$$x_i(t) \in S_{v_0}(\delta_0), \text{ for all } i \in [N], \quad (29)$$

with $\delta_0 = \frac{1}{1-\xi}$.

(I) if for some $i_0 \in [N]$, and some $l \in \mathcal{I}$,

$$x_{i_0}(t) \in H(h_l, c_l), \quad (30)$$

then,

$$h_l^\top \dot{x}_{i_0}(t) < 0; \quad (31)$$

(II) if for some $i_0 \in [N]$,

$$x_{i_0}(t) \in H(h_0, c'_0), \quad (32)$$

with $c'_0 = (1 - \delta_0)h_0^\top v_0 + \delta_0 c_0$, then,

$$h_0^\top \dot{x}_{i_0}(t) < 0. \quad (33)$$

Proof: For Case (I), let

$$y_i(t) \triangleq \xi v_0 + (1 - \xi) x_i(t), \quad (34)$$

for all $i \in [N]$. Since $x_{i_0}(t) \in H(h_l, c_l) \cap S_{v_0}(\delta_0)$, and $\delta_0 = \frac{1}{1-\xi}$, we have $y_{i_0}(t) \in \mathcal{F}_l$. So $y_{i_0}(t)$ can be written as a convex combination of $V \setminus \{v_l\}$, i.e., there exist $\bar{\lambda}_k(t) \geq 0, k \in [\bar{n}]$, such that $y_{i_0}(t) = \sum_{k=0, k \neq l}^n \bar{\lambda}_k(t) v_k$, and $\sum_{k=0, k \neq l}^n \bar{\lambda}_k(t) = 1$. Then, we get

$$\begin{aligned} &h_l^\top [A y_{i_0}(t) + B u(y_{i_0}(t)) + a] + \mu_l(y_{i_0}(t)) \\ &= \sum_{k=0, k \neq l}^n \bar{\lambda}_k(t) (h_l^\top \eta_k + \mu_l(v_k)). \end{aligned} \quad (35)$$

Furthermore, for $\|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_l^\top \Gamma(z - x_{i_0}(t))$, we have

$$\begin{aligned} &\|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_l^\top \Gamma(z - x_{i_0}(t)) \\ &= \|G\|_\infty \max_{z \in V(\delta_0)} h_l^\top \Gamma(z - x_{i_0}(t)) \\ &= (1 - \delta_0) h_l^\top \Gamma v_0 + \delta_0 \|G\|_\infty \max_{z \in V} h_l^\top \Gamma z - h_l^\top \Gamma x_{i_0}(t) \\ &= \delta_0 \|G\|_\infty \left(\max_{z \in V} h_l^\top \Gamma(z - y_{i_0}(t)) \right) = \delta_0 \mu_l(y_{i_0}(t)), \end{aligned} \quad (36)$$

noticing that $x_{i_0}(t) = (1 - \delta_0)v_0 + \delta_0 y_{i_0}(t)$ by (34). So, using condition (29) and (36), we have

$$\begin{aligned} &h_l^\top \dot{x}_{i_0}(t) \\ &= h_l^\top [A x_{i_0}(t) + B u(x_{i_0}(t)) + a] \\ &\quad + \sum_{j \in [N]} g_{i_0 j} h_l^\top \Gamma(x_j(t) - x_{i_0}(t)) \\ &\leq h_l^\top [A x_{i_0}(t) + B u(x_{i_0}(t)) + a] \\ &\quad + \|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_l^\top \Gamma(z - x_{i_0}(t)) \\ &= (1 - \delta_0) h_l^\top \eta_0 \\ &\quad + \delta_0 (h_l^\top [A y_{i_0}(t) + B u(y_{i_0}(t)) + a] + \mu_l(y_{i_0}(t))). \end{aligned} \quad (37)$$

Now combining (35) and the above inequality, we get

$$h_l^\top \dot{x}_{i_0}(t) \leq (1 - \delta_0) h_l^\top \eta_0 + \delta_0 \sum_{k=0, k \neq l}^n \bar{\lambda}_k(t) (h_l^\top \eta_k + \mu_l(v_k)). \quad (38)$$

Since, when $l \in \mathcal{I}$ and $k \neq l$, we have $h_l^\top \eta_k + \mu_l(v_k) < \xi h_l^\top \eta_k$, by the condition (15). Then, $\delta_0 \sum_{k=0, k \neq l}^n \bar{\lambda}_k(t) (h_l^\top \eta_k + \mu_l(v_k)) < \delta_0 \xi h_l^\top \eta_0 = (\delta_0 - 1) h_l^\top \eta_0$, by recalling $\delta_0 = \frac{1}{1-\xi}$. Hence, inequality (38) implies $h_l^\top \dot{x}_{i_0}(t) < 0$.

For case (II), since $x_{i_0}(t) \in H(h_0, c'_0) \cap S_{v_0}(\delta_0)$, with $c'_0 = (1 - \delta_0)h_0^\top v_0 + \delta_0 c_0$, we get that $x_{i_0}(t)$ can be written as a convex combination of $\{v_k(\delta_0), k \in [n]\}$, i.e., there exist $\hat{\lambda}_k(t) \geq 0, k \in [n]$, such that $x_{i_0}(t) = \sum_{k=1}^n \hat{\lambda}_k(t) v_k(\delta_0)$, and $\sum_{k=1}^n \hat{\lambda}_k(t) = 1$. Hence, condition (29) implies

$$\begin{aligned} &h_0^\top \dot{x}_{i_0}(t) \leq h_0^\top [A x_{i_0}(t) + B u(x_{i_0}(t)) + a] \\ &\quad + \|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_0^\top \Gamma(z - x_{i_0}(t)) \\ &= \sum_{k=1}^n \hat{\lambda}_k(t) [h_0^\top [A v_k(\delta_0) + B u(v_k(\delta_0)) + a] \\ &\quad + \|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_0^\top \Gamma(z - v_k(\delta_0))] \end{aligned} \quad (39)$$

Since $v_k(\delta_0) = (1 - \delta_0)v_0 + \delta_0 v_k$, for all $k \in [n]$, by using a similar argument in (36), we have

$$\begin{aligned} & h_0^\top [Av_k(\delta_0) + Bu(v_k(\delta_0)) + a] \\ & \quad + \|G\|_\infty \max_{z \in S_{v_0}(\delta_0)} h_0^\top \Gamma(z - v_k(\delta_0)) \\ & = (1 - \delta_0)h_0^\top \eta_0 + \delta_0(h_0^\top \eta_k + \mu_0(v_k)) \\ & = \frac{1}{1 - \xi} [-\xi h_0^\top \eta_0 + (h_0^\top \eta_k + \mu_0(v_k))], \end{aligned} \quad (40)$$

where $\delta_0 - 1 = \frac{\xi}{1 - \xi}$ with $\delta_0 = \frac{1}{1 - \xi}$ is used. Furthermore, when $k \in [n], l = 0$, the strong invariance conditions (15) become $-\xi h_0^\top \eta_0 + (h_0^\top \eta_k + \mu_0(v_k)) < 0$. Hence, combining (39) and (40), we obtain that $h_0^\top \dot{x}_{i_0}(t) < 0$. ■

The next lemma asserts that when the strong invariance conditions (15) hold, the states of all the agents belong to the convex extension $S_{v_0}(\delta_0)$ of S for all time $t \geq 0$.

Lemma 3.2 *Assume that the strong invariance conditions (15) hold for the multi-agent system (11). Then*

$$x_i(t) \in S_{v_0}(\delta_0), \quad \forall t \geq 0, i \in [N]. \quad (41)$$

Proof: Suppose on the contrary that there exists a $T \geq 0$, such that $x_i(T) \notin S_{v_0}(\delta_0)$, for some i . In this case, let $s \in [0, T]$ be the first leaving time of $\{x_i(\cdot)\}_{i=1}^N$ from $S_{v_0}(\delta_0)$, which implies that there exists $i_0 \in [N]$, and $\varepsilon > 0$, such that

$$x_i(\tau) \in S_{v_0}(\delta_0), \forall \tau \in [0, s], i \in [N]; \quad (42)$$

$$x_{i_0}(s) \in \partial S_{v_0}(\delta_0); \quad (43)$$

$$x_{i_0}(\tau) \notin S_{v_0}(\delta_0), \forall \tau \in (s, s + \varepsilon]. \quad (44)$$

Furthermore, according to the geometric property (22) of $S_{v_0}(\delta)$, (43) and (44) imply that exactly one of the following two cases must hold:

Case (I): there exists $l \in \mathcal{I}$, and $\varepsilon_0 \in (0, \varepsilon]$ such that

$$x_{i_0}(s) \in H(h_l, c_l), \quad (45)$$

$$x_{i_0}(\tau) \notin L(h_l, c_l), \forall \tau \in (s, s + \varepsilon_0]; \quad (46)$$

Case (II): there exists $\varepsilon_0 \in (0, \varepsilon]$ such that

$$x_{i_0}(s) \in H(h_0, c'_0), \quad (47)$$

$$x_{i_0}(\tau) \notin L(h_0, c'_0), \forall \tau \in (s, s + \varepsilon_0], \quad (48)$$

where c'_0 is given in Lemma 3.1.

For Case (I), noticing that from Definition (2) of hyperplane $H(h, c)$, (45), and (46) are equivalent to

$$h_l^\top x_{i_0}(s) = c_l, \quad (49)$$

$$h_l^\top x_{i_0}(\tau) > c_l, \forall \tau \in (s, s + \varepsilon_0]. \quad (50)$$

On the other hand, from (42) and (45), we deduce that $h_l^\top \dot{x}_{i_0}(s) < 0$ using (31) in Lemma 3.1. Then, by continuity of $\dot{x}_{i_0}(\cdot)$, there exists $\varepsilon_1 > 0$ such that

$$h_l^\top \dot{x}_{i_0}(t) < 0, \quad \forall t \in [s - \varepsilon_1, s + \varepsilon_1]. \quad (51)$$

Let $\varepsilon_2 = \min\{\varepsilon_0, \varepsilon_1\}$. Then, from (51) and (49), we get

$$\begin{aligned} h_l^\top x_{i_0}(s + \varepsilon_2) &= h_l^\top x_{i_0}(s) + \int_s^{s + \varepsilon_2} h_l^\top \dot{x}_{i_0}(\tau) d\tau \\ &< h_l^\top x_{i_0}(s) = c_l, \end{aligned} \quad (52)$$

which contradicts (50).

For Case (II), using (33) in Lemma 3.1, a similar argument used in the above case (I) can deduce a contradiction with (48). Hence the proof is completed. ■

Remark 3.1 *In the statement of Problem 2, the time when each agent leaves the simplex S through an admissible facet can be different. Lemma 3.2 guarantees that all the states of the agents belong to the convex extension $S_{v_0}(\delta_0)$, so it would not happen that the agents that have exited S reenter S through a restricted facet even under the influence of those neighbors still in S .*

In view of the result of Lemma 3.2, the additional condition (16) will guarantee that the tangent vector of each agent's trajectory always has an acute angle with the direction $\hat{h}_0$ and finally all the agents exit the simplex through $\mathcal{F}_0$.

Lemma 3.3 *Assume that the strong invariance conditions (15) and the strong flow conditions (16) hold for the multi-agent system (11). Then, there exists $t_f > 0$, such that $x_i(t_f) \notin S, \forall i \in [N]$.*

Proof: Suppose on the contrary that for any $t > 0$ there exists $i_t \in [N]$ such that $x_{i_t}(t) \in S$. Then by $S \in L(h_0, c_0)$ in (4), one has

$$h_0^\top x_{i_t}(t) \leq c_0, \quad \text{for all } t > 0. \quad (53)$$

Now, define a function $y(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ as follows

$$y(t) = x_{\sigma_N(t)}(t) \quad (54)$$

with

$$\sigma_N(t) = \max \left\{ k \in [N] : h_0^\top x_k(t) = \min_{i \in [N]} h_0^\top x_i(t) \right\}.$$

By the definition of $y(t)$, we can deduce that the function $y(\cdot) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is a piecewise continuously differentiable, more specifically, according to the finite subcover property of a compact set, for any time interval $[0, T]$, $y(\cdot)$ has the following property :

- There exists a finite subdivision $\{t_0, t_1, \dots, t_{M_T}\}$ of $[0, T]$ with $t_0 = 0, t_{M_T} = T$, and $i_m \in \{1, \dots, N\}$, for all $m = 1, \dots, M_T$ such that

$$y(t) = x_{i_m}(t), \text{ on } I_m = [t_{m-1}, t_m]. \quad (55)$$

Then, from (53) we have, for all $t \geq 0$,

$$h_0^\top y(t) = \min\{h_0^\top x_i(t) : i \in [N]\} \leq h_0^\top x_{i_t}(t) \leq c_0, \quad (56)$$

which implies that $y(t) \in L(h_0, c_0)$ for all $t \geq 0$. Furthermore, one has

$$y(t) \in L(h_0, c_0) \cap C_{S, v_0} = S, \forall t \geq 0, \quad (57)$$

by combining Lemma 3.2. Let

$$P(t) \triangleq \hat{h}_0^\top y(t). \quad (58)$$

Then, it is obvious that for all $t \geq 0$,

$$P(t) \leq D_0 \triangleq \max\{\hat{h}_0^\top z : z \in S\}, \quad (59)$$

where $\hat{h}_0$ is given by (18). Take

$$t_f = \frac{D_0 - P(0)}{\min_{i \in [\bar{n}]} \{\hat{h}_0^\top \eta_i\}} + 1. \quad (60)$$

Assume that $\{t_0, t_1, \dots, t_{M_T}\}$ is the finite subdivision of $[0, t_f]$ such that on each $I_m = [t_{m-1}, t_m]$, $m = 1, \dots, M_T$, the equation (55) holds. Then, for each $m = 1, \dots, M_T$,

$$h_0^\top x_{i_m}(t) = \min_{1 \leq i \leq N} h_0^\top x_i(t) \text{ on } I_m = [t_{m-1}, t_m]. \quad (61)$$

Noticing $g_{ij} \geq 0$, it follows from the definition of $y(t)$ in (54) that

$$\begin{aligned} & \sum_{j \in [N]} g_{ij} \hat{h}_0^\top \Gamma(x_j(t) - x_{i_m}(t)) \\ &= \sum_{j \in [N]} g_{ij} h_0^\top (x_j(t) - x_{i_m}(t)) \geq 0. \end{aligned}$$

Hence, under the feedback controller $u(x_i) = Kx_i + b$ determined by $u_j, j \in [\bar{n}]$, for $t \in I_m$,

$$\begin{aligned} \dot{P}(t) &= \hat{h}_0^\top \dot{x}_{i_m}(t) \\ &= \hat{h}_0^\top (A + BK)x_{i_m}(t) + \hat{h}_0^\top (Bb + a) \\ &\quad + \sum_{j \in [N]} g_{ij} \hat{h}_0^\top \Gamma(x_j(t) - x_{i_m}(t)) \\ &\geq \hat{h}_0^\top (A + BK)x_{i_m}(t) + \hat{h}_0^\top (Bb + a). \end{aligned} \quad (62)$$

Furthermore, we have $x_{i_m}(t) \in S$ for $t \in I_m$ by (57), and therefore there exist $\lambda_i(t) \geq 0, i = 1, \dots, n$, with $\sum_{i=0}^n \lambda^i(t) = 1$ such that

$$x_{i_m}(t) = \sum_{i=0}^n \lambda_i(t) v_i. \quad (63)$$

Thus, for all $t \in I_m, m = 1, \dots, M_T$,

$$\hat{h}_0^\top [(A + BK)x_{i_m}(t) + (Bb + a)] \geq \min_{i \in [\bar{n}]} \{\hat{h}_0^\top \eta_i\}, \quad (64)$$

by noticing that $Bu_k = BKv_k + Bb$ for all $k = 0, 1, \dots, n$.

Since $P(\cdot)$ is piecewise continuously differentiable, we have

$$\begin{aligned} P(t_f) &= P(0) + \int_0^{t_f} \dot{P}(t) dt \\ &= P(0) + \sum_{1 \leq m \leq M_T} \int_{t_{m-1}}^{t_m} \hat{h}_0^\top \dot{x}_{i_m}(t) dt \\ &\geq P(0) + \sum_{1 \leq m \leq M_T} \int_{t_{m-1}}^{t_m} \min_{i \in [\bar{n}]} \{\hat{h}_0^\top \eta_i\} dt \\ &= P(0) + t_f \min_{i \in [\bar{n}]} \{\hat{h}_0^\top \eta_i\} = D_0 + \min_{i \in [\bar{n}]} \{\hat{h}_0^\top \eta_i\} > D_0, \end{aligned}$$

which contradicts (59). Hence the proof is completed. ■

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: For each $i \in [N]$, define

$$t_i \triangleq \inf \{t \geq 0 : x_i(t) \notin S\}. \quad (65)$$

Lemma 3.3 guarantees that the well posedness of definition (65) and $t_i \leq t_f$ for all $i \in [N]$. Furthermore, by noticing that $x_i(\cdot)$ is continuous and the simplex S is compact, for each $i \in N$, there exist $\varepsilon_i > 0$ such that

$$\begin{aligned} \text{(C1)} \quad & x_i(t) \in S, \forall t \in [0, t_i]; \\ \text{(C2)} \quad & x_i(t_i) \in \partial S; \\ \text{(C3)} \quad & x_i(t) \notin S, \forall t \in (t_i, t_i + \varepsilon_i). \end{aligned}$$

(C1) and (C3) imply that the multi-agent systems (11) satisfies the conditions (i) and (iii) of Problem 2 with $\varepsilon = \min_{i \in [N]} \{\varepsilon_i\}$. In addition, combining with the conditions (C2), (C3), and Lemma 3.2, we have that $x_i(t_i) \in F_0$ for all $i \in [N]$. The proof is completed. ■

Remark 3.2 For the reach control problem of a single affine system, necessary and sufficient conditions (Theorem 2.1) can be derived, while for the coupled multi-agent system (11), only a sufficient condition is derived in Theorem 3.1 and a necessary condition is still missing. This is due to the difficulty in precise characterization on the effect of the coupling term.

4 Numerical example

In this section we revisit the example considered in Section 2 and verify the derived results in the previous section.

To solve the reach control problem of the multi-agent system (11) with the parameters given in (13), it suffices to find a set of $u_0 = [u_0^1, u_0^2]^\top, u_1 = [u_1^1, u_1^2]^\top, u_2 = [u_2^1, u_2^2]^\top$ satisfying the conditions (15) and (16).

First, calculate $\max_{v \in V} h_1^\top \Gamma v = \max_{i=0,1,2} \{h_1^\top \Gamma v_i\} = h_1^\top \Gamma v_2 = 3.8/\sqrt{5}$. Hence, by noticing that $\|G\|_\infty = 4.25$, the condition $h_1^\top \eta_0 + \mu_1(v_0) \leq \xi h_1^\top \eta_0$ is equivalent to $-u_0^1 + 2u_0^2 + \frac{1}{1-\xi}16.15 < 0$. Repeating the above process, we have:

(1) *Strong invariance conditions*

$$\begin{cases} -u_0^1 + 2u_0^2 + \frac{1}{1-\xi}16.15 < 0, \\ -u_0^2 < 0, \\ 13.075 + 2u_1^1 + u_1^2 < \xi(2u_0^1 + u_0^2) \\ -u_1^2 \leq -\xi u_0^2, \\ 5 + 2u_2^1 + u_2^2 < \xi(2u_0^1 + u_0^2) \\ -u_2^1 + 2u_2^2 \leq \xi(-u_0^1 + 2u_0^2), \end{cases}$$

(2) *Strong flow conditions*

$$\begin{cases} 10u_0^1 + 0.25u_0^2 > 0, \\ 100 + 40u_1^1 + u_1^2 > 0, \\ 81 + 40u_2^1 + u_2^2 > 0. \end{cases}$$

Choose $u_0 = [53, 10]^\top, u_1 = [11, 11]^\top, u_3 = [50, -50]^\top$, with $\xi = 0.5$ that satisfy the above conditions. Then, the parameters in the affine feedback control $u = Kx + b$ can be calculated as

$$K = \begin{bmatrix} -16.8 & 30.6 \\ 0.4 & -60.8 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 53 \\ 10 \end{bmatrix}. \quad (66)$$

The trajectories of all the agents are shown in Fig. 4, which illustrates that all the agents exist the simplex S_2 through the exit facet $\mathcal{F}_0$ in a finite time, but always remain within $S_{v_0}(2)$, which is consistent with the result of Lemma 3.2.

The choice of the inputs $u_i, i = 0, 1, 2$, is not unique. One can choose another set of inputs as $u_0 = [43, 2]^\top, u_1 = [-1, 9]^\top, u_2 = [40, -5]^\top$, with $\xi = 0.6$ that also satisfies the above conditions. The corresponding parameters of

Fig. 4. Trajectories of the six agents under the control (12) with (66).

Fig. 5. Trajectories of the six agents under the control (12) with (67).

the affine feedback control are

$$K = \begin{bmatrix} -17.6 & 32.2 \\ 2.8 & -12.6 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 43 \\ 2 \end{bmatrix}. \quad (67)$$

The reach control problem is solved under this affine feedback controller which is confirmed by Fig. 5 where $\delta_0 = \frac{1}{1-\xi} = 2.5$.

5 Conclusion

In this paper, the reach control problem for an affine multi-agent system has been studied. It has been shown by an example that the affine feedback control proposed for the single affine system in the literature does not work for the multi-agent system in general. A sufficient condition consisting of strong invariance conditions and strong flow conditions has been proposed to solve this

problem. Our result has been verified by a numerical example. Our future research is to look into finding weaker sufficient conditions of the reach control problem for affine multi-agent systems. The necessary conditions are of interest to investigate as well so that the gap between the sufficient and necessary ones can be identified. Moreover, the reach control problem for discontinuous multi-agent systems is also a subject for future research.

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