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Published in:
IEEE Transactions on Automatic Control

DOI:
10.1109/TAC.2015.2414828

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2015

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Download date: 28-10-2023
Towards a Generic Constructive Nonlinear Control Design Tool Using Relaxed Control

Bayu Jayawardhana

Abstract—In this technical note, we revisit a control design approach for general (non-affine) nonlinear systems using relaxed control. Using the notion of relaxed input, where the ordinary real-valued control input is replaced by a measure-valued control input, we are able to manipulate the original system such that the resulting relaxed system is amenable to control. We show the applicability of using relaxed input for transforming a non-affine nonlinear system into an affine one such that the relaxed system has desirable control properties. Finally, we present a practical implementation of relaxed input via ordinary control input which is based on the sampled-data of the relaxed input.

Index Terms—Constructive nonlinear control design, relaxed control.

I. INTRODUCTION

The notion of relaxed control was firstly introduced by Warga in [24] for relaxing the optimal control problem of general nonlinear systems. It involves the replacement of the ordinary real-valued control input by a measure-valued control input. This approach is also related to the concept of generalized control as proposed by Gamkrelidze in [3]. Subsequently, Artstein in [1] applies the relaxed control notion for solving the stabilization problem of general nonlinear systems. In particular, it is shown in [1] that a necessary and sufficient condition for such systems to be stabilizable by relaxed control can be given by a control Lyapunov-type condition.

In this technical note, we revisit the application of relaxed control as part of a constructive control design tool for general nonlinear systems. It enables the transformation of a non-affine system (which can be difficult to control) to an affine relaxed system with a number of desirable control properties. Thus, it opens the possibility of applying well-known control design tools for affine systems, such as, the geometric control methods [4], [13], backstepping control design [9], [19], forwarding control design [15], [19] or passivity-based control design tools [14]. We extend the result in [6], [7] by discussing a generic constructive control design method via relaxed input and by presenting a practical implementation of relaxed input.

As an illustrative example, consider a nonlinear system $P$ described by

$$
\begin{align*}
\dot{x}_1 &= \sin(u) \\
\dot{x}_2 &= \cos(u)
\end{align*}
$$

(1)

where $x_1(t), x_2(t) \in \mathbb{R}$ and $u(t) \in [-\pi, \pi)$. This example is motivated by the dynamics of a unicycle where $(x_1, x_2)$ is its planar position, $u$ is its heading angle and its linear velocity is taken to be unity. The system $P$ can not be stabilized at any point using continuous state feedback since there is no equilibrium point associated with any constant input $u$ but it can be practically stabilized, for example, via an application of pulse-width modulation (PWM) signal as follows. Suppose that a multi-level PWM signal is used for $u$ and its duty cycle is controlled by two exogenous signals $v_1(t)$ and $v_2(t)$ designed to practically stabilize $P$. (b) Generated PWM signal $u$ on each duty cycle which is based on $v_1$ and $v_2$ and the period of the PWM is given by $\Delta$.

![Diagram](image)

Fig. 1. (a) Application of a multi-level PWM to the stabilization of (1). The system $P$ as in (1) and the block PWM is a PWM signal generator driven by two exogenous signals $v_1$ and $v_2$. The state feedback laws $v_1 = \alpha_1(x_1, x_2)$ and $v_2 = \alpha_2(x_1, x_2)$ are designed to practically stabilize $P$.
description, the state $x_1$ and $x_2$ can be controlled by the new control inputs $v_1$ and $v_2$, independently, and the stabilization problem can now be solved via a simple static continuous state-feedback law.

Notice that the averaging characteristic of the PWM signal which depends on $v_1$ and $v_2$ at each duty cycle can be formulated by a probability measure-valued function $\mu_{v_1,v_2}$ whose domain is defined on the input space $[-\pi, \pi]$ and it is defined by $\mu_{v_1,v_2}(E) = \int_E r_{v_1,v_2}(\tau) d\tau$ for all $E \subset [-\pi, \pi]$, where

$$r_{v_1,v_2}(\tau) = v_1 \delta_{-\pi/2} + v_2 \delta_0 + (0.5 - v_1) \delta_\pi + (0.5 - v_2) \delta_\pi$$

for all $v_1, v_2 \in [0,0.5]$ and $\delta_\epsilon$ is the Dirac measure at $\epsilon \in \mathbb{R}$. When $\mu_{v_1,v_2}$ is applied to the input of (1), computing the average of the RHS of (1), we get

$$\begin{align*}
\dot{x}_1 &= \int_{-\pi}^{\pi} \sin(\tau) r_{v_1,v_2}(\tau) d\tau = -2v_1 + 0.5 \\
\dot{x}_2 &= \int_{-\pi}^{\pi} \cos(\tau) r_{v_1,v_2}(\tau) d\tau = 2v_2 - 0.5
\end{align*}$$

which is identical to (2). The introduction of a probability measure-valued function to the ordinary control input $u$ is the basis of the relaxed control method.

For the general nonlinear systems described by

$$\dot{x} = f(x,u)$$

where $x \in X \subset \mathbb{R}^n$ and $u$ is defined in a compact set $U \subset \mathbb{R}^m$, the relaxed control method in [1], [24] replaces the ordinary control input $u$ by a Radon probability measure-valued function $\mu$ with $U$ as its support. In this case, the dynamics of (4) is given by

$$\dot{x} = \int_{U} f(x,\tau) d\mu(\tau).$$

If $f$ is locally Lipschitz then it is possible to have a practical implementation of $\mu$ using the ordinary input $u$ where the resulting state trajectory $x$ of (4) approximates that of the relaxed system (5). One such approximating input signal is detailed in [3, Theorem 3.2] where a sequence of piecewise-constant signals which resembles the multi-level PWM signal in the aforementioned example or in [20].

In Section II, we introduce the notion of a relaxed input $\mu_\psi$ to (4) which gives rise to a relaxed system with a new input $v$. The relaxed input $\mu_\psi$ enables the manipulation of the original system such that the relaxed system is amenable to control. For instance, in Section III, we discuss an application of relaxed input for transforming non-affine nonlinear systems into affine relaxed systems. Finally, in Section IV, we discuss a practical implementation of relaxed input $\mu_\psi$ by constructing an approximating ordinary control input $u_\psi$ based on the sampled-data of $\mu_\psi$.

II. RELAXED SYSTEMS

Throughout this technical note, we consider nonlinear systems described by

$$\begin{align*}
\dot{x} &= f(x,u), \quad x(0) = x_0 \\
y &= h(x)
\end{align*}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^p$. The functions $f$ and $h$ are assumed to be locally Lipschitz and $U$ is compact.

Let $rpm(U)$ be the set of all Radon probability measure defined on $U$ (see also, [16] and [24]). For a compact metric space $V \subset \mathbb{R}^n$, the space $R_f(V, rpm(U))$ is the space of all functions $\mu : V \to rpm(U)$ such that the function

$$(x,v) \mapsto \int_{U} f(x,\tau) d\mu_v(\tau)$$

is locally Lipschitz on $X \times V$. The subscript $f$ in $R_f$ describes its dependence on the vector field $f$. The space $R_f(V, rpm(U))$ is trivially non-empty. Throughout this technical note, $f_f : X \times V \to \mathbb{R}^n$ is defined by

$$f_f(x,v) := \int_{U} f(x,\tau) d\mu_v(\tau).$$

Using $R_f(V, rpm(U))$, the ordinary input $u$ in (6) can be replaced by $\mu_v$ such that the relaxed system is given by

$$\dot{x} = f_f(x,v), \quad x(0) = x_0$$

where $v \in V$. The variable $v$ becomes the new input variable in the RHS of (8). The system with the relaxed input $\mu \in R_f(V, rpm(U))$ as given in (8) is called relaxed system. The function $f_f$ is locally Lipschitz by the definition of $R_f(V, rpm(U))$.

For approximating a relaxed input $\mu_v$ using an ordinary input signal $u$, an appropriate concept of convergence of a sequence of measurable functions $(u_j)$ to $\mu_v$ is required. For this purpose, one can observe that for every $x \in C([0,T], X)$, for every measurable functions $v : [0,T] \to V$ and $(u_j) : [0,T] \to U$, the functions $F : t \mapsto \int_{0}^{t} f_f(x(\lambda), u_j(\lambda))d\lambda$ and $F_1 : t \mapsto \int_{0}^{t} f_f(x(\lambda), u(\lambda))d\lambda$ are both continuous functions on $[0,T]$. Hence, it is natural to use metric convergence for the space of continuous functions which is equipped with the sup-norm.

**Definition 2.1—[24]:** For a given vector field $f$ and relaxed input $\mu_v$, a sequence of measurable functions $(u_j) : [0,T] \to U$ converges to $\mu_v$ with a measurable function $v : [0,T] \to V$ if, for every continuous function $x \in C([0,T], X)$

$$\lim_{j \to \infty} \sup_{t \in [0,T]} \left\| \int_{0}^{t} f_f(x(\lambda), u_j(\lambda)) d\lambda - \int_{0}^{t} f_f(x(\lambda), \tau) d\mu_v(\lambda) d\lambda \right\| = 0. \quad (9)$$

The solution $x$ of (6) using the relaxed input $\mu_v$ with $v$ be a measurable function, is the Carathéodory solution $x$ of the relaxed system (8).

The following lemma which describes the approximation of the solution using an ordinary input signal is due to Warga [24].

**Lemma 2.2:** Let $X$ be open and $T > 0$. Suppose that the relaxed input $\mu_v$ with measurable $v : [0,T] \to V$ such that the solution $x$ of (8) is defined on $[0,T]$. Then for every sequence of measurable functions $(u_j) : [0,T] \to U$ that converges to $\mu_v$, there exist $j_0 \in \mathbb{N}$ and a sequence $(x_j)_{j \geq j_0}$ such that $(x_j, u_j)$ is the solution of (6) on $[0,T]$ and

$$\lim_{j \to \infty} \sup_{t \in [0,T]} \| x_j(t) - x(t) \| = 0. \quad (10)$$

**Proof:** The proof follows immediately from Lemma VI.1.4 in [24] by taking there $\psi_W : I_d, T := [0,T], y := x, \sigma := \mu_v$ and the fact that $f$ is a locally Lipschitz function.

We remark that the notion of relaxed control is a state space coordinate-free notion. Hence the design of a relaxed input (which is relevant for our control design purposes later) is independent of a particular choice of coordinate.

1. For a given set $X \subset \mathbb{R}$, the Dirac measure $\delta_\epsilon : X \to \mathbb{R}$ is defined by $\delta_\epsilon(X) = 1$ if $\epsilon \in X$ or, otherwise, $\delta_\epsilon(X) = 0$.

2. A Radon probability measure is a probability measure defined on Borel sets that is inner regular and locally finite. See also, [16], [24].
As mentioned before in the Introduction, a relaxed input can be used as part of a constructive control design tool for general nonlinear systems when it is combined with other control design methods. Notice that if \( k : X \to V \) is locally Lipschitz and \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \), then trivially \( \mu \circ k \in \mathcal{R}_f(X, \text{rpm}(U)) \). This shows that after designing a relaxed input, one can proceed with the design of a stabilizing state-feedback via the relaxed system equation

\[
\dot{x} = f_R(x, k(x))
\]

where \( f_R \) is locally Lipschitz. Artstein in [1] describes the stabilization of (4) by designing state-feedback relaxed control \( \mu \circ k \in \mathcal{R}_f(X, \text{rpm}(U)) \) such that \( f_R(0, k(0)) = 0 \) and the resulting relaxed system

\[
\dot{x} = f_R(x, k(x))
\]

is (globally) asymptotically stable in the origin. The following theorem is the main result of [1].

**Theorem 2.3:** The system (4) with locally Lipschitz \( f \) is locally asymptotically stabilizable by a state-feedback relaxed control if and only if \( x = 0 \) is (globally) asymptotically stable in the origin. The following theorem provides flexibility in designing a smooth control input, Lemma 2.2 shows that we can only achieve practical stabilization.

Suppose now that for the nonlinear system (6), there exists a Lyapunov function \( H \in C^1(X, \mathbb{R}_+) \), a function \( s : X \times U \to \mathbb{R} \) and positive definite functions \( \alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\alpha_1(||x||) \leq H(x) \leq \alpha_2(||x||)
\]

\[
\frac{\partial H(x)}{\partial x} f(x, u) \leq s(x, u) \quad \forall (x, u) \in X \times U.
\]

It follows then that for any \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \), the derivative of \( H \) along the vector field of relaxed system (8) satisfies

\[
\frac{\partial H(x)}{\partial x} f_R(x, v) \leq \int_U (x, \tau) d\mu_X(\tau) =: s_R(x, v) v(x, v) \in X \times V.
\]

As a consequence, we can immediately obtain the following result on the passivity-preserving property [5], [18] of some relaxed inputs.

**Proposition 2.5:** Suppose that the nonlinear system (6) is passive, i.e., there exist \( H \in C^1(X, \mathbb{R}_+) \) and positive definite functions \( \alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that (10) and (11) hold with \( s(x, u) = y^T U \) where \( y = h(x) \). Then for any \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \) with a compact and connected \( V \) containing the origin such that \( \int_U (x, \tau) d\mu_X(\tau) =: v \), the relaxed system (8) is also passive, i.e., (12) holds with \( s_R(x, v) = y^T v \).

The proof of this proposition is obtained directly from the computation of \( s_R(x, v) \) in (12).

### III. Transformation to Affine Relaxed Systems

For an affine nonlinear system, a relaxed input \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \) does not yield an advantage over the ordinary control input. Indeed, let an affine nonlinear system be described by (6) with

\[
f(x, u) = f_1(x) + f_2(x) u
\]

where \( f_1 \) and \( f_2 \) are locally Lipschitz. For every \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \), the computation of (8) yields

\[
\dot{x} = f_1(x) + f_2(x) \int_U \tau d\mu_X(\tau).
\]

It is evident from the above equation that the resulting relaxed system is also an affine nonlinear system with the same \( f_1 \) and \( f_2 \).

On the other hand, for a non-affine nonlinear system, relaxed input can be designed such that the resulting relaxed system has a number of useful control properties. For instance, it is possible to transform a non-affine nonlinear system to an affine one.

**Proposition 3.1:** Consider a non-affine nonlinear system described by (6) with \( m = 1 \). Let \( V = [v_1, v_2] \) where \( v_1 \) and \( v_2 \) are constants in \( \mathbb{R} \). Then there exists \( \mu \in \mathcal{R}_f(V, \text{rpm}(U)) \) such that the relaxed system in (8) is an affine nonlinear system.

**Proof:** For every \( v \in V \), define \( \mu_v(E) \) by

\[
\mu_v(E) = \int_E \frac{v - v_1}{v_2 - v_1} \delta_{v_2} + \frac{v_2 - v}{v_2 - v_1} \delta_{v_1} \quad \forall E \subset U.
\]

Using the above \( \mu \), it can be shown that

\[
\dot{x}(t) = \int_U f(x(t), \tau) d\mu_{x(t)}(\tau) = \int_U v(t) f(x(t), v_2) + \frac{v_2 - v(t)}{v_2 - v_1} f(x(t), v_1) = f_1(x(t)) + f_2(x(t)) v(t)
\]

where \( f_1 \) and \( f_2 \) are defined by

\[
\begin{align*}
f_1(x) &= \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1}, \\
f_2(x) &= \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1}.
\end{align*}
\]
In the proof of Proposition 3.1, the resulting affine system is constructed based on the convex hull of the vector fields \( f(x, v_1) \) and \( f(x, v_2) \), and the relaxed input is based on Dirac measures. However, the construction is not unique and Example 4.4 in Section IV shows a possible variation to the relaxed input which is based on a uniform probability measure. It is straightforward to extend Proposition 3.1 to the case of \( m > 1 \) where the vector field of the relaxed systems lies in the convex hull of the vectors \( \{f(x, v_1), \ldots, f(x, v_q)\} \), \( q > 1 \) with arbitrary \( \{v_i\} \) and \( V \) is the convex hull of \( \{v_i\} \).

Proposition 3.2: Consider a non-affine nonlinear system described by (6). Let \( V \) be a convex hull of \( \{v_i \in U | i = 1, \ldots, q\} \). Then there exists \( \mu \in R_f(V, \text{rpm}(U)) \) such that the relaxed system in (8) is an affine nonlinear system.

Related to the passivity property as given in Proposition 2.5, the following proposition shows that for any lossless (or conservative) system, there exists a relaxed input such that the corresponding relaxed system is affine, lossless and satisfies the well-known Hill-Moylan equations for lossless systems [14], [18].

Proposition 3.3: Consider a non-affine nonlinear system described by (6) with \( m = 1 \) and output \( y = h(x) \). Suppose that there exist \( H \in C^1(X, \mathbb{R}^+ \) and positive definite functions \( \alpha_1, \alpha_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that (10) and

\[
\frac{\partial H(x)}{\partial x} f_1(x) = 0 \quad \frac{\partial H(x)}{\partial x} f_2(x) = h(x)
\]  
(15)

hold for all \( (x, u) \in X \times U \). Then for every \( V = [v_1, v_2] \subset U \) containing the origin, the relaxed system (8) with

\[
\mu_{\ast}(E) = \int_E \frac{v - v_1}{v_2 - v_1} \delta_{v_2}(\tau) + \frac{v_2 - v}{v_2 - v_1} \delta_{v_1}(\tau) \, d\tau \quad \forall E \subset U
\]

is affine, lossless, and satisfies

\[
\frac{\partial H(x)}{\partial x} f_1(x) = 0 \quad \frac{\partial H(x)}{\partial x} f_2(x) = h(x)
\]

where \( f_1 \) and \( f_2 \) are the vector fields of the relaxed system \( \dot{x} = f_1(x) + f_2(x)u \).

Proof: The proof for transforming (6) to an affine system is the same as the proof of Proposition 3.1 where we have used the same relaxed input \( \mu \). For proving the second part, it remains to show that the Hill-Moylan equations for lossless system hold. Note that \( f_1 \) and \( f_2 \) are as given in (14). Thus,

\[
\frac{\partial H(x)}{\partial x} f_1(x) = \frac{\partial H(x)}{\partial x} \left( \frac{f(x, v_1)v_2 - f(x, v_2)v_1}{v_2 - v_1} \right) = \frac{yv_1v_2 - yv_2v_1}{v_2 - v_1} = 0
\]

where the second equality is due to (15). Similarly

\[
\frac{\partial H(x)}{\partial x} f_2(x) = \frac{\partial H(x)}{\partial x} \left( \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1} \right) = \frac{yv_2 - yv_1}{v_2 - v_1} = y
\]

where we have also used (15) in obtaining the second equality.

An extended system to the nonlinear system (6) has also been considered in the literature for constructing an affine system from a non-affine one, by taking the input as an extended state and its time derivative is assigned as a new affine input (see, for example, [13, Chapter 6]). However, designing a stabilizing state-feedback controller for such an extended system can be restrictive. For example, we cannot design a controller for the extended system if the original system can only stabilized by discontinuous state-feedback laws.

IV. IMPLEMENTATION OF THE RELAXED INPUT

In practice, a relaxed input must be implemented using an ordinary control input \( u \) as in (6). In this section, we discuss the approximation of a relaxed input \( \mu_k \) with \( m = 1 \) and \( \mu = [a, b] \) and let \( \eta \in C(R_+, V) \). Our approximating input is based on the sampled-data of \( \eta \) that is projected to the continuous-time approximating input \( u_j \). Roughly speaking, if the sequence of sampled-time is given by \( (t_k) \), the approximating input \( u_j \) is obtained by concatenating sequences of \( (u_{j,k}) \) where each \( u_{j,k} : [t_k, t_{k+1}] \rightarrow U \) is defined such that the time proportion of \( u_{j,k} \) spent on any subset \( A \subset U \) is equal to \( \mu_{\eta}(t_k)(A) \).

Let \( \Delta > 0 \) be the sampling period, i.e., \( \Delta = t_{k+1} - t_k \) for all \( k \), and denote \( j := \lfloor \Delta \rfloor \). Let us construct \( u_{j,k} : [t_k, t_{k+1}] \rightarrow U \), which is based on \( \mu_{\eta}(t_k) \), as follows. For every \( w \in [a, b] \), define \( \gamma_k : \eta \mapsto \mu_{\eta}(t_k)([a, w]) \) which is a non-decreasing function. Let \( \Gamma_k \subset [0, 1] \) be defined by \( \Gamma_k := \{\gamma_k(w) | w \in [a, b]\} \) which can be a union of disjoint sets due to Dirac measures in \( \mu(\eta(t_k)) \).

For every \( \nu \in [0, 1] \), let

\[
\eta(\nu) := \min \{\xi \in \Gamma_k | \xi - \nu \geq 0\}
\]

(17)

In other words, \( \eta(\nu) \) is the closest point in \( \Gamma_k \) to \( \nu \) from above and \( \eta(\xi) = \xi \) for all \( \xi \in \Gamma_k \). For every \( \xi \in \Gamma_k \), we define \( \gamma_k^{-1} : \xi \mapsto \min \{w \in [a, b] | \gamma_k(w) = \xi\} \). Using these functions, we can define \( u_{j,k}(t) := \gamma_k(\eta(t)) \) for all \( t \in [0, \Delta] \). The signal \( u_{j,k} \) is nondecreasing.

For every \( A \subset [a, b] \) and by defining \( T_{j,k}(A) := \{t \in [t_k, t_{k+1}] | u_{j,k}(t) \in A\} \), we have that

\[
\mu_{j,k}(A) := \frac{\int_{T_{j,k}(A)} dt}{\Delta} = \mu_{\eta}(t_k)(A)
\]

(18)

Based on the above construction, the concatenation of \( u_{j,k} \) gives the sequence of approximating input signals \( (u_j) \) which converge to \( \mu \) as \( j \rightarrow \infty \) (or, equivalently, as \( \Delta \rightarrow 0 \)) as shown in the following proposition. Let the sequence of approximating input signals \( (u_j) \) be defined by

\[
u_j(t) = \begin{cases} u_{j,1}(t) & t \in (0, 1] \\ u_{j,2}(t) & t \in (1, 2] \\ \vdots & \vdots \\ u_{j,k}(t) & t \in (k-1, k] \Delta \end{cases}
\]

(19)

where \( k \in \mathbb{N} \). Note that (18) holds for every \( k \) where \( t_k = (k-1) \Delta \) (or, equivalently, \( t_k = (k-1)/j \)).

Proposition 4.1: For any locally Lipschitz vector field \( f \) and for any continuous signal \( v \), the sequence of approximating input signals \( (u_j) \) as in (19) converges to \( \mu \).

See the Appendix for the proof.

Example 4.2: Let \( U = [a, b], a, b \in \mathbb{R} \) and consider \( \mu \), given by

\[
\mu_{\eta}(E) = \int_E v(t) \delta_{c_1}(E) + (1 - v(t)) \delta_{c_2}(E)
\]

for all \( E \subset [a, b] \), \( v(t) \in [0, 1] \), where \( a < c_1 < c_2 < b \). Following the construction of \( u_{j,k} \) as before, we first compute \( \gamma_k \) by:

\[
\gamma_k(w) = \begin{cases} 0 & w \in [a, c_1] \\ v(t_k) & w \in [c_1, c_2] \\ 1 & w \in [c_2, b] \end{cases}
\]

The set \( \Gamma_k = \{0, v(t_k), 1\} \) and it is straightforward to check that

\[
u_{j,k}(t_k + \xi) = \begin{cases} c_1 & 0 \leq \xi < v(t_k)/j \\ c_2 & v(t_k)/j \leq \xi < 1/j \end{cases}
\]
It is worth to note that the implementation of a relaxed input in the Example 4.2 has been used widely for implementing control signal using PWM signal where the width of the pulse is modified according to \( v \) and the range of the PWM signal is \( \{c_1, c_2\} \). The following example describes the approximation of relaxed input \( \mu_v \), that is based on a uniform probability measure-valued function.

**Example 4.3:** Let \( U = [a, b] \). Let \( \mu_v \) be a uniform probability measure-valued function defined on \( U \) and defined by

\[
\mu_v(t) = \frac{|E \cap [v(t), b]|}{b - v(t)} \quad \forall E \subset [a, b]
\]

where \(| \cdot |\) is the Lebesgue measure. In this case, \( v(t) \in [a, b] \) defines the lower interval of the uniform probability measure valued function.

Using the same construction of \( u_{j,k} \) as before

\[
\gamma_k(w) = \begin{cases} 
0 & w \in [a, v(t_k)] \\
\frac{w - v(t_k)}{b - v(t_k)} & w \in [v(t_k), b].
\end{cases}
\]

The set \( \Gamma_k = [0, 1] \) and \( u_{j,k} \) is given by

\[
u_{j,k}(t_k + \xi) = v(t_k) + j\xi (b - v(t_k)) \quad \forall \xi \in [0, 1/j).
\]

It can be observed that the approximating input signal \( u_j \) as in (19) is a discontinuous signal. Hence for the unicycle example as given in the Introduction where the input is the heading angle, we can not implement such a discontinuous approximating input signal. However, if \( \gamma_k \) is continuous for every \( v(t_k) \) (such as, the one in Example 4.3), then we can design an approximating \( u_j \) which is continuous, and thus, can be practically implemented in the unicycle example. One of such continuous approximating input signals is given by

\[
u_j(t) = \begin{cases} 
u_{j,1}(t) & t \in (0, 1]\Delta \\
u_{j,2}(2\Delta - t) & t \in (1, 2]\Delta \\
\vdots & \\
u_{j,k}(t) & t \in (k\Delta, k+1] \Delta \\
u_{j,(k+1)}((k+2\Delta - t) & t \in (k+1, k+2] \Delta \\
\vdots & \\
u_{j,(k+2)}(\Delta - t) & t \in (0, 1] \Delta 
\end{cases}
\]

**Example 4.4:** Let us consider again the example in the Introduction and we will implement the relaxed input \( \mu_v \) which is based on a uniform probability measure-valued function. Recall again the system equations

\[
\begin{align*}
\dot{x}_1 &= \sin(x) \\
\dot{x}_2 &= \cos(x)
\end{align*}
\]

where \( x_1(t), x_2(t) \in \mathbb{R} \) and \( u(t) \in [-\pi/2, 3\pi/2] \). Consider the following relaxed input \( \mu_v \) with \( v = \frac{v_1}{v_2} \):

\[
\mu_v(t) = \int_{E} v_1(t) \chi_{[0, \pi]}(\tau) + v_2(t) \chi_{[-\pi/2, \pi/2]}(\tau) + (0.5 - v_1(t)) \chi_{[\pi, 3\pi/2]}(\tau) + (0.5 - v_2(t)) \chi_{[-\pi/2, -\pi/2]}(\tau) \pi d\tau
\]

for all \( E \subset \mathbb{R} \) where \( \chi_{[a, b]} \) is the indicator function on the interval \([a, b]\) and \( v_1(t), v_2(t) \in [0, 0.5] \). Using \( \mu_v \), the relaxed system of (21) is given by

\[
\begin{align*}
\dot{x}_1 &= 4v_1 - 1 \\
\dot{x}_2 &= v_2 - 1
\end{align*}
\]

By setting \( v_1 = 0.25 - 0.25\text{sat}(x_1) \) and \( v_2 = 0.25 - 0.25\text{sat}(x_2) \), the closed-loop relaxed system is given by

\[
\begin{align*}
\dot{x}_1 &= -\text{sat}(x_1) \\
\dot{x}_2 &= -\text{sat}(x_2)
\end{align*}
\]

\[\text{PROOF OF PROPOSITION 4.1}\]

In this technical note, we have discussed the applicability of relaxed control where we are able to manipulate a nonlinear system using a relaxed input such that the resulting relaxed system is amenable to control. It has allowed us to transform a non-affine nonlinear system into an affine one such that the relaxed system has desirable control properties.

It can potentially be combined with other well-known control design techniques as a generic constructive control design tool for general nonlinear systems. The implementation of relaxed input that is based on Dirac measures leads to the multi-level PWM signals. By considering different type of relaxed inputs, we can obtain approximating input signals which are more general than the PWM signals and it is possible to construct continuous approximating input signals which is relevant for some applications, such as, unicyles.
Using the Lipschitz constants of $f$, the continuity modulus of $x$ and $v$ and the Lipschitz continuity of $f_R$, it follows from the above inequality that for all $t \in \mathcal{T}$

$$
\left\| \int_0^t f(x(\tau), u(\lambda)) \, d\lambda - \int_0^t f(x(\tau), \lambda) \, d\mu_v(\tau) \right\| \leq \frac{1}{T} \int_0^T \int f(t_k, u(\lambda)) \, d\lambda
$$

$$+ \frac{1}{T} \int_0^T \int f(t_k, \lambda) \, d\mu_v(\tau) + 2tT \Omega_k \left( \frac{1}{T} \right)
$$

where $\ell_R$ is the Lipschitz constant of $f_R$ on $X \times V$. Since we have

$$
\int_{[t_k, t_k+1]} f(x(t_k), u(\lambda)) \, d\lambda = \Delta \int_{[t_k, t_k+1]} f(x(t_k), \lambda) \, d\mu_{v,k}(\lambda)
$$

$$= \frac{1}{T} \int_0^T \int f(t_k, \lambda) \, d\mu_v(\tau) + 2tT \Omega_k \left( \frac{1}{T} \right)
$$

This inequality implies that

$$
\lim_{j \to \infty} \sup_{t \in \mathcal{T}} \left\| \int_0^t f(x(\lambda), u(\lambda)) \, d\lambda - \int_0^t f(x(\lambda), \lambda) \, d\mu_v(\lambda) \right\| = 0.
$$

**ACKNOWLEDGMENT**

The author would like to thank the anonymous reviewers for their careful reading of the manuscript that led to a substantial improvement of the technical note.

**REFERENCES**


