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Formation control of a multi-agent system subject to Coulomb friction[☆]

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ABSTRACT

This paper considers the formation control problem for a network of point masses which are subject to Coulomb friction. A dynamical model including the planar discontinuous friction force is presented in the port-Hamiltonian framework. Moreover, continuous and discontinuous controllers are designed in order to achieve a desired prescribed formation. The main results are derived using tools from nonsmooth Lyapunov analysis. It is shown that the continuous static feedback controller fails to achieve the exact formation, while the discontinuous controller achieves the desired task exactly. Numerical simulations are provided to illustrate the effectiveness of the approach.

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1. Introduction

Cooperative motion coordination of mobile agents has attracted increasing attention in recent years owing to its wide range of applications from biology and social networks to sensor/robotic networks. Distributed formation control is a cooperative control problem which aims at achieving a desired collective behavior, mainly forming a desired geometrical shape, for a group of agents using local feedback laws rather than a centralized controller (Arcak, 2007).

The dynamics of agents play an important role in the problem of formation control. In this regard, different classes of dynamic agents have been considered e.g. Bai, Arcak, and Wen (2011), Nuño, Ortega, Basañez, and Hill (2011), Ren (2009) and van der Schaft and

Maschke (2013). For dynamic agents, the dissipation due to friction forces plays a key role in the stability analysis of the whole network e.g. Bai et al. (2011), Jafarian and De Persis (2015) and Vos, Scherpen, and van der Schaft (2012). In the literature, to the authors best knowledge, only continuous friction forces have been considered for the formation control problem of networks. This paper considers formation control of a group of agents in the presence of Coulomb friction which is a discontinuous friction law (van de Wouw & Leine, 2004). Coulomb friction is a quantification of the friction force that exists between two (dry) surfaces in contact with each other. Coulomb friction renders the networked system nonsmooth, thereby requiring tools from nonsmooth systems for the analysis. Nonsmooth stability theory has been already considered in formation keeping control, e.g. in finite-time consensus algorithms (Cortés, 2006), and in quantized coordination (Ceragioli, De Persis, & Frasca, 2011; De Persis & Jayawardhana, 2012; Jafarian & De Persis, 2013, 2015).

In this paper, the model of the network and the formation controller design are defined in the port-Hamiltonian framework (Duindam, Macchelli, & Stramigioli, 2009; van der Schaft & Jeltsema, 2014). Being an energy-based modeling framework (Ortega, van der Schaft, Mareels, & Maschke, 2002), the port-Hamiltonian framework enables the modeling of a physical phenomenon like Coulomb friction in a natural way (i.e., the model provides a clear physical interpretation). The port-Hamiltonian framework interconnects the various sub-systems in a power preserving manner using the so-called *power ports* (Duindam et al., 2009). Moreover,

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the Hamiltonian equals the total energy stored in the system and can be used as a Lyapunov function. In addition, one can design controllers using energy-related elements, e.g. springs, in order to shape the total energy of the system into a desired one. The term *virtual spring* used throughout this paper refers to the application of this concept in the control design. We build upon the recently introduced theory of *port-Hamiltonian systems on graphs* (van der Schaft & Maschke, 2013) where graph theory is used to model the interconnection amongst agents in the network, and upon nonsmooth formation control (Jafarian & De Persis, 2013, 2015).

We consider a network of point masses moving in $\mathbb{R}^2$ and assume that each of the agents is subject to Coulomb friction. To achieve the desired formation, we consider assigning two types of virtual springs between the agents: continuous and discontinuous springs. The way in which agents are interconnected by virtual springs is modeled by an undirected connected acyclic graph. The physical interconnection structure modeled by the graph (van der Schaft & Maschke, 2013) is assumed to be equal to the information exchange amongst agents (i.e., two agents exchange (local) information if there is a virtual spring assigned in between the agents). The way in which we define the springs fits well within the position-based formation control setting described in Arcak (2007) and Bai et al. (2011). Compared with existing literature (Vos et al., 2012; Vos, Scherpen, & van der Schaft, 2014) the major difference is the presence of discontinuity in the agents dynamics due to Coulomb friction. This discontinuity prevents continuous springs to achieve the formation control objectives which is the motivation behind proposing discontinuous springs for the control. For other results about discontinuous position-based formation control, we refer the reader to De Persis and Jayawardhana (2012) and Jafarian and De Persis (2015).

Main contribution. We present modeling and analysis of a network of planar heterogeneous dynamic point masses subject to Coulomb friction in the port-Hamiltonian framework. We show that continuous virtual springs fail to achieve a desired formation exactly. Moreover, we present a discontinuous distributed design to achieve the desired goals of the formation control problem. Both the network and the controller are modeled within the port-Hamiltonian framework which provides a clear physical interpretation of the results. It is worth noting that the use of nonsmooth analytical tools for formation keeping control in the port-Hamiltonian framework has not been studied before. Preliminary results of this work were presented in Jafarian, Vos, De Persis, van der Schaft, and Scherpen (2014) which considers the formation control of a group of homogeneous agents in $\mathbb{R}$ communicating over a star graph. In this paper, we generalize the results to the class of *tree graphs*. Furthermore, we provide a rigorous nonsmooth stability analysis of the closed-loop system.

The outline of this paper is as follows. First we recall some preliminaries on port-Hamiltonian systems, graph theory and analytical tools for nonsmooth systems. Section 3 presents a port-Hamiltonian model for the agents which are subject to Coulomb friction in $\mathbb{R}$ and $\mathbb{R}^2$. Section 4 continues with the control design and the closed-loop analysis for both continuous and discontinuous springs. Section 5 illustrates the effectiveness of the approach by simulation results. Finally, Section 6 concludes the paper.

Notation. For a square matrix $A \in \mathbb{R}^{n \times n}$, $\text{tr}(A)$ denotes the trace of A , which is defined as $\text{tr}(A) = \sum_{i=1}^n a_{ii}$. The symbol $\times_{k=1}^m S_k$ denotes the Cartesian product $S_1 \times S_2 \times \dots \times S_m$. The symbol $\bigcup_{r=1}^n S_r$ denotes the union $S_1 \cup S_2 \cup \dots \cup S_n$. An empty set is denoted by $\emptyset$. Given a matrix M of real numbers, we denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and the null space, respectively. The symbols $\mathbf{1}$, $\mathbf{0}$ denote appropriately sized vectors or matrices of all 1 and 0 respectively. Sometimes the size of the matrix is explicitly given (i.e., $\mathbf{1}_n$ is the n -dimensional vector of all 1). I_p is the $p \times p$

identity matrix. Given two matrices A, B , the symbol $A \otimes B$ denotes the Kronecker product. $\text{block.diag}(A_1, \dots, A_n)$ denotes the block diagonal matrix such that A_i is its i th diagonal element. For a scalar function $H : \mathbb{R}^n \mapsto \mathbb{R}$, $\frac{\partial H}{\partial x}$ denotes the column vector of partial derivatives of the function $H(x)$ with respect to $x = (x_1, \dots, x_n)^T$. For vectors $a, b \in \mathbb{R}^n$, $a \cdot b$ denotes their inner product. Finally, the 2-norm and 1-norm of $a \in \mathbb{R}^n$ are denoted as $\|a\|$ and $\|a\|_1$ respectively.

2. Preliminaries

This section presents some preliminaries on the theory of port-Hamiltonian systems, graph theory and analytical tools for nonsmooth systems which are used in the remainder of this paper.

Port-Hamiltonian systems

The port-Hamiltonian framework is energy-based modeling framework which describes a large class of (nonlinear) multi-domain systems (Duindam et al., 2009; van der Schaft & Jeltsema, 2014). A port-Hamiltonian system consists of energy storing elements, energy dissipating elements and a Dirac structure which describes how the elements are interconnected in a power-preserving way. Furthermore, external ports are used to describe the interaction with the external systems like the control system and the environment.

There are several representations for port-Hamiltonian systems (van der Schaft & Jeltsema, 2014). In this work we are dealing with input-state-output port-Hamiltonian systems. Consider state $x \in \mathbb{R}^n$, skew-symmetric structure matrix $J(x) = -J^T(x) \in \mathbb{R}^{n \times n}$, positive semi-definite dissipation matrix $R(x) = R^T(x) \geq 0 \in \mathbb{R}^{n \times n}$, and let $H(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the Hamiltonian of the system. The Hamiltonian $H(x)$ is the sum of the kinetic and potential energy stored in the system. The systems considered in this work have two interaction ports (see Duindam et al., 2009): a control port (u, y) and a resistive port (u^r, y^r) . Each port has two port-variables, being the inputs $u \in \mathbb{R}^{m_1}$, $u^r \in \mathbb{R}^{m_2}$ and outputs $y \in \mathbb{R}^{m_1}$, $y^r \in \mathbb{R}^{m_2}$. The product of the port-variables has the dimension of power and equals the energy flow through the port. The port-Hamiltonian dynamics (Duindam et al., 2009) are given by

$$\begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u + g^r(x)u^r \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \\ y^r &= g^{rT}(x) \frac{\partial H}{\partial x}(x), \end{aligned} \quad (1)$$

with input matrices $g(x) \in \mathbb{R}^{n \times m_1}$, $g^r(x) \in \mathbb{R}^{n \times m_2}$ corresponding to the control port and the resistive port respectively. The resistive port enables us to model the discontinuous Coulomb friction (see Section 3). A resistive element dissipates energy and hence the resistive port-variables satisfy $y^{rT}u^r \leq 0$.

Graph theory

The information exchange between agents is modeled by a connected and undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where node-set $\mathcal{V}$ corresponds to n agents and the edge-set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ corresponds to m virtual springs. In order to use the tools of graph theory, we assign a positive/negative label to each of the nodes connected by a link. The labeling of the nodes can be done in an arbitrary manner, and it does not affect the final results. Label one end of each edge in $\mathcal{E}$ with a positive sign and the other end with a negative sign. The incidence matrix B associated to $\mathcal{G}(\mathcal{V}, \mathcal{E})$ describes which nodes are coupled by an edge, and is defined as

$$b_{i\ell} = \begin{cases} +1 & \text{if node } i \text{ is at the positive side of edge } \ell \\ -1 & \text{if node } i \text{ is at the negative side of edge } \ell \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we define the relative position z_k between agents i and j as follows

$$z_k = \begin{cases} q_i - q_j & \text{if node } i \text{ is the positive end of edge } k \\ q_j - q_i & \text{if node } j \text{ is the positive end of edge } k \end{cases}$$

where $q_i = (q_{xi}, q_{yi})^T \in \mathbb{R}^2$ is the position of agent i expressed in an inertial reference frame. By definition of B , we can represent the relative position variable z , with $z = (z_1^T, \dots, z_m^T)^T$, $z \in \mathbb{R}^{2m}$, as

$$z = (B^T \otimes I_2)q, \quad (2)$$

which implies that z belongs to the range space $\mathcal{R}(B^T \otimes I_2)$.

We say node i is a *neighbor* of node j if the link (i, j) exists in the graph $\mathcal{G}$. Note that for undirected graphs, if node i is a neighbor of node j , then node j is also a neighbor of node i . For an undirected graph, the degree of node i , $\deg i$, is defined equal to the number of its neighboring agents.

In this paper, we consider the subclass of *tree graphs*. A graph $\mathcal{T}(\mathcal{V}, \mathcal{E})$ is called a tree graph if $\mathcal{T}$ is an undirected graph in which any two nodes are connected by exactly one unique path. This is equivalent to $\mathcal{T}$ being connected and acyclic (Godsil & Royle, 2001). For any tree graph $\mathcal{T}$ the nodes of degree 1 are called *terminal nodes*. In other words, a terminal node is a node with exactly one edge incident to it. Furthermore, we define *terminal edges* as those edges which are incident to terminal nodes.

Consider a tree graph $\mathcal{T}$. Since it is acyclic and connected, $\mathcal{T}$ has at least two terminal nodes. Hence, the incidence matrix B associated to $\mathcal{T}$ has at least two rows with exactly one nonzero entry. Removing a terminal node and its corresponding edge from $\mathcal{T}$ results in a new tree graph $\mathcal{T}'$, since $\mathcal{T}'$ remains connected and acyclic. More details on the properties discussed in this section may be found in Bollobás (1998) and Godsil and Royle (2001).

Nonsmooth analysis

We now recall some basic notations from the theory of nonsmooth control (Bacciotti & Ceragioli, 1999) which will be used throughout the paper. Consider the system $\dot{x} = f(x)$ where $f(x)$ is a discontinuous function. In this paper, we adopt the Krasovskii notion of solution. A function $x(\cdot)$ defined on an interval $I \subset \mathbb{R}$ is a Krasovskii solution to the system on I if it is an absolutely continuous function and satisfies the differential inclusion

$$\dot{x} \in \mathcal{K}(f(x)) := \bigcap_{\delta > 0} \overline{\text{co}}(f(B(x, \delta))),$$

for almost every $t \in I$. The operator $\overline{\text{co}}$ denotes the closed convex hull of a set and $B(x, \delta)$ denotes a ball centered at x with radius δ . Local existence of Krasovskii solutions is guaranteed if $f(x)$ is locally bounded.

Denote the set-valued map $\mathcal{K}(f(x))$ by $F(x)$. A set $\mathcal{S}$ is weakly (respectively, strongly) invariant for the differential inclusion $\dot{x} \in F(x(t))$ if for any initial condition $\bar{x} \in \mathcal{S}$ at least one (all the) Krasovskii solution(s) $x(t)$ starting from $\bar{x}$ belong(s) to $\mathcal{S}$ for all t in the domain of the definition of $x(t)$. Let V be a locally Lipschitz continuous function. Recall that by Rademacher's theorem, the gradient ∇V of V exists almost everywhere. Let $\mathcal{R}$ be the set of measure zero where $\nabla V(x)$ does not exist. Then the Clarke generalized gradient of V at x is the set $\partial V(x) = \text{co}\{\lim_{i \rightarrow +\infty} \nabla V(x_i) : x_i \rightarrow x, x_i \notin \mathcal{S}, x_i \notin \mathcal{R}\}$ where $\mathcal{S}$ is any set of measure zero in $\mathbb{R}^n$. We define the set-valued derivative of V at x with respect to $\dot{x} \in F(x(t))$ as the set $\dot{V}(x) = \{a \in \mathbb{R} : \exists v \in F(x) \text{ s.t. } a = p \cdot v, \forall p \in \partial V(x)\}$. In the case V is differentiable at x , one has $\dot{V}(x) = \{\nabla V \cdot v, v \in F(x)\}$. A nonsmooth version of LaSalle's invariance principle (Bacciotti & Ceragioli, 1999) is used in this paper to prove stability.

Fig. 1. System of the motivational example: mass m is subject to a Coulomb friction force F_d and is controlled by a virtual spring force F_s .

3. Problem formulation and dynamical model of agents subject to Coulomb friction

The goal of this paper is to design a distributed control law for a network of n agents in order to achieve a desired formation at zero momentum (velocity). Let $z_\ell \in \mathbb{R}^2$ denote the relative position between two agents which are interconnected by virtual spring ℓ and let $z_\ell^* \in \mathbb{R}^2$ denote the desired relative position. The formation control objective is to make the relative position $z = (z_1^T, \dots, z_m^T)^T$ converge to a desired prescribed relative position $z^* = (z_1^{*T}, \dots, z_m^{*T})^T$ (i.e., achieve a formation). Furthermore, let $p = (p_1^T, \dots, p_n^T)^T \in \mathbb{R}^{2n}$ denote the momentum vector of the agents (see the agent dynamics defined in (6)). We formulate the control objective as follows

$$\begin{cases} p \rightarrow \mathbf{0}, \\ z \rightarrow z^*, \end{cases} \quad \text{as } t \rightarrow \infty. \quad (3)$$

To achieve (3), we present and compare two types of controllers (virtual springs): a continuous and a discontinuous controller. We show that only the discontinuous controller achieves (3) exactly. In the remainder of this section we continue with deriving the dynamical model of the agents. But first, we present a motivational example (in $\mathbb{R}$) to provide some intuition on the differences between continuous and discontinuous virtual springs.

Example (Motivation). Consider a single mass m moving in $\mathbb{R}$ with position $q \in \mathbb{R}$ (see Fig. 1). The mass is subject to a Coulomb friction force $F_d = -c \text{sign}(\dot{q})$ (van de Wouw & Leine, 2012), with friction coefficient c_i and velocity $\dot{q} \in \mathbb{R}$. Assume that the control objective is to move m to a prescribed position q^* . We achieve this by assigning a virtual spring to m with corresponding spring force F_s . We consider two types of springs: a continuous spring where $F_s(q) = k(q - q^*)$ and a discontinuous spring where $F_s(q) = k \text{sign}(q - q^*)$.

In order to get m moving, the virtual spring needs to overcome a *friction threshold* of $\pm c$. Intuitively, as long as $|q - q^*| > \frac{c}{k}$ the continuous spring gets m moving, but once $|q - q^*| \leq \frac{c}{k}$ mass m_i comes to a hold. Hence the control objective might not be achieved. On the other hand, the discontinuous spring provides a spring force of $\pm k$ as long as $|q - q^*| \neq 0$. Hence, if $k > c$ the control objective is achieved.

The example above provides some intuition why the continuous virtual springs might not achieve (3), while the discontinuous counterpart achieves the desired goals. Before presenting the two controller designs, we first derive the agents' dynamics.

3.1. Dynamical model of agents subject to Coulomb friction

In this section, we present a model for the Coulomb friction force in $\mathbb{R}^2$, followed by the dynamical model of the agents.

In general, the friction force between two bodies in contact is composed of several components including the Coulomb friction

Fig. 2. Illustration of the Coulomb friction force in $\mathbb{R}^2$.

force (De Wit, Olsson, Astrom, & Lischinsky, 1995). This paper considers dry contacts where the Coulomb friction force is the major friction force between the rigid bodies in contact. The Coulomb friction force is composed of two components: a static and a kinetic (sliding) component. The static force presents an upper bound for the friction force of two bodies with zero relative motion. The kinetic force on the other hand presents a constant friction force for the bodies in motion. In this paper, we assume that the friction coefficients of these two forces are equal. However, this assumption will not affect the main result of this paper (see Remark 5).

Consider a point mass m_i and let q_i denote its position. Assume that the mass is subject to the Coulomb friction force $F_i(v_i)$, with $v_i = \dot{q}_i$ the velocity of the mass. The Coulomb friction force for mass m_i moving in $\mathbb{R}$ (Jafarian et al., 2014; van de Wouw & Leine, 2004, 2012) is given by

$$F_i(v_i) := \begin{cases} \{c_i \text{sign}(v_i)\} & \text{if } v_i \neq 0 \\ [-c_i, c_i] & \text{if } v_i = 0, \end{cases} \quad (4)$$

where the function $\text{sign} : \mathbb{R} \rightarrow \{-1, +1\}$ is defined as $\text{sign}(v_i) = +1$ if $v_i \geq 0$ and $\text{sign}(v_i) = -1$ if $v_i < 0$. Now consider a planar mass m_i in $\mathbb{R}^2$. The model of Coulomb friction in $\mathbb{R}^2$ is more involved. Instead of having two decoupled friction forces along the x and y directions, here we consider a more natural model for the friction force that is defined along the direction of motion. We present the model for the Coulomb friction force acting on mass m_i as a set-valued map $F_i : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where $F_i(v_i)$ is given by

$$F_i(v_i) := \begin{cases} c_i \frac{v_i}{\|v_i\|} & \text{if } v_i \neq 0 \\ B(0, c_i) & \text{if } v_i = 0, \end{cases} \quad (5)$$

with $c_i \in \mathbb{R}^+$ the friction coefficient and $B(0, c_i)$ a disc with radius c_i centered at the origin (see Fig. 2). Both (4) and (5) are set-valued maps and their definitions are in accordance with the definition of the Krasovskii map for $\text{sign}(v)$ in $\mathbb{R}$ and $\mathbb{R}^2$ respectively (see Preliminaries on nonsmooth analysis).

Remark 1. The disc $B(0, c_i)$ has a clear physical interpretation. When agent i stands still, the controller needs to overcome a threshold of $\pm c_i$ before the agent starts moving. The disc $B(0, c_i)$ represents the physical fact that there is a friction force even though the agent might not be moving. This corresponds to the situation where the magnitude of the control input is too small to overcome the threshold value.

Now consider agent i with position $q_i = (q_{xi}, q_{yi})^T$ and momentum $p_i = (p_{xi}, p_{yi})^T = M_i \dot{q}_i$, with mass $M_i = m_i I_2$. Let $u_i = (u_{xi}, u_{yi})^T$, $y_i = (y_{xi}, y_{yi})^T$ and $u_i^r = (u_{xi}^r, u_{yi}^r)^T$, $y_i^r = (y_{xi}^r, y_{yi}^r)^T$ denote the port-variables of the control port and the resistive port respectively (see Section 2). Here, the inputs u_i , u_i^r are forces, while the corresponding outputs y_i , y_i^r are the velocities of agent i . The

dynamics of agent i may now be written in the form (1) (Jafarian et al., 2014) as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H_i^a}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H_i^a}{\partial q_i} + u_i + u_i^r, \\ y_i &= y_i^r = \frac{\partial H_i^a}{\partial p_i}, \end{aligned} \quad (6)$$

where the Hamiltonian $H_i^a(p_i) = \frac{1}{2m_i} p_i^2$ equals the kinetic energy of agent i . The resistive port relation is set as $u_i^r = -F_i(y_i^r)$. From (5) it immediately follows that

$$\begin{cases} y_i^{rT} u_i^r = -c_i \frac{\|y_i^r\|^2}{\|y_i^r\|} < 0 & \text{for } y_i^r \neq 0 \\ y_i^{rT} u_i^r = 0 & \text{for } y_i^r = 0. \end{cases} \quad (7)$$

The above implies $y_i^{rT} u_i^r \leq 0$ (i.e., the resistive element dissipates power) which means that each agent i is output passive with respect to its velocity.

Now consider a network of n agents of the form (6). To derive the dynamics of the network in a compact form, define the stacked vectors $q = (q_1^T, \dots, q_n^T)^T$, $p = (p_1^T, \dots, p_n^T)^T$, $u = (u_1^T, \dots, u_n^T)^T$, $u^r = (u_1^{rT}, \dots, u_n^{rT})^T$, $y = (y_1^T, \dots, y_n^T)^T$, $y^r = (y_1^{rT}, \dots, y_n^{rT})^T$, and mass matrix $M = \text{block.diag}(M_1, \dots, M_n)$. The dynamics of the network follow directly from (6) and are given by

$$\begin{aligned} \dot{q} &= \frac{\partial H^a}{\partial p}, \\ \dot{p} &= -\frac{\partial H^a}{\partial q} + u + u^r, \\ y &= y^r = \frac{\partial H^a}{\partial p}, \end{aligned} \quad (8)$$

with Hamiltonian $H^a(p) = \sum_{i=1}^n H_i^a(p_i) = \frac{1}{2} p^T M^{-1} p$ and $u^r = -F(y^r)$ where $u_i^r = -F_i(y_i^r)$ is defined in (5).

4. Control design and analysis

In this section we present two types of controllers for agents of the form (8) to achieve the formation control objectives (3). Each of the controllers acts like a virtual spring which is assigned in between the agents. The prescribed relative position z_ℓ^* for $\ell \in \{1, \dots, m\}$ corresponds to the desired relative position amongst the two agents which are interconnected by spring ℓ . We make a distinction between *continuous* and *discontinuous* virtual springs (Jafarian et al., 2014) in order to achieve the so-called *position-based* formation control in terms of Arcak (2007) and Bai et al. (2011).

Before continuing with the analysis of the two types of springs, we present the formal analysis of the motivational example in Section 3. The nonsmooth tools used in this example are illustrative for the remainder of this paper.

4.1. Formal analysis of the motivational example

Recall the motivational example in Section 3, where mass m is subject to a Coulomb friction force F_d and is controlled by a virtual spring force F_s (see Fig. 1). For simplicity we assume that the control objective is to reach a zero distance from the wall (i.e., $q^* = 0$).

The port-Hamiltonian dynamics are obtained from (6) by taking $H^a = \frac{1}{2m}p^2$. We obtain

$$\begin{aligned}\dot{q} &= \frac{p}{m}, \\ \dot{p} &= u + u^r, \\ y &= y^r = \frac{p}{m},\end{aligned}\quad (9)$$

where $u, u^r \in \mathbb{R}$ are respectively the control law (spring force) and Coulomb friction force defined in (4) and $y \in \mathbb{R}$ is the velocity of mass m . Hence, $u^r = F(\frac{p}{m})$ is

$$F\left(\frac{p}{m}\right) = \begin{cases} c \left\{ \text{sign}\left(\frac{p}{m}\right) \right\} & \text{if } \frac{p}{m} \neq 0, \\ [-c, c] & \text{if } \frac{p}{m} = 0. \end{cases}$$

Considering the above set-valued map, the dynamics (9) can be rewritten as

$$\begin{aligned}\dot{q} &= \frac{p}{m}, \\ \dot{p} &\in -F\left(\frac{p}{m}\right) + u, \\ y &= y^r = \frac{p}{m}.\end{aligned}\quad (10)$$

As mentioned in the Preliminaries, we adopt a Krasovskii notion of solution to analyze the solutions of the above differential inclusion.

Now, we consider two types of controllers. First, consider a continuous virtual spring of the form

$$u = -\frac{\partial H^c}{\partial q} = -kq,$$

with the corresponding potential energy $H^c = \frac{1}{2}kq^2$. The closed-loop Hamiltonian is $H(p, q) = H^a(p) + H^c(q)$. To analyze the stability and convergence of the solutions of (10), take $H(p, q)$ as the candidate Lyapunov function and calculate its set-valued derivative (for brevity we skip the calculation of set-valued derivative here). We obtain $\dot{H} \in -\frac{p}{m}F(\frac{p}{m}) \subseteq (-\infty, 0]$. Now, applying the nonsmooth LaSalle's invariance principle (Bacciotti & Ceragioli, 1999), the system converges to the largest weakly invariant set where $0 \in \dot{H}$. The latter implies that on the invariant set $p = 0$. Substituting $p = 0$ in (10), we conclude that $\dot{q} = 0$ and therefore $q = q^{eq}$ where q^{eq} is a constant. Moreover, we obtain

$$0 \in [-c, c] - kq^{eq}$$

which implies $q^{eq} \in [-\frac{c}{k}, \frac{c}{k}]$. In other words, the position q converges to a value in an interval containing the origin. Therefore, the convergence of the position to zero cannot be guaranteed.

Second, consider a discontinuous spring where the potential energy is $H^d = k|q|$. The control law u is equal to the Clarke generalized gradient (Bacciotti & Ceragioli, 1999) of H^d , that is

$$u = \begin{cases} -k\{\text{sign}(q)\} & \text{if } q \neq 0 \\ [-k, k] & \text{if } q = 0. \end{cases}$$

The closed-loop Hamiltonian is now $H(p, q) = H^a(p) + H^d(q)$, which is locally Lipschitz and regular (Bacciotti & Ceragioli, 1999). Same as the previous case, we take $H(p, q)$ as the candidate Lyapunov function and obtain $\dot{H} \in -\frac{p}{m}F(\frac{p}{m}) \subseteq (-\infty, 0]$. Now, applying the nonsmooth LaSalle's invariance principle, the system converges to the largest weakly invariant set where $p = 0$. Substituting $p = 0$ in (10), we conclude that $\dot{q} = 0$ and therefore $q = q^{eq}$ where q^{eq} is a constant. If $q^{eq} > 0$, we have

$$0 \in [-c, c] - k.$$

If $q^{eq} < 0$, we obtain

$$0 \in [-c, c] + k.$$

Now, assume that $k > c$. Therefore, for $q^{eq} > 0$ both the lower and upper bounds of the interval $[-(c+k), c-k]$ are negative, while for $q^{eq} < 0$ both of the bounds of $[-c+k, c+k]$ are positive. Since zero cannot belong to these intervals, we conclude (by contradiction) that q^{eq} is necessarily equal to zero for $k > c$.

The above example provides the motivation for the design and analysis of continuous and discontinuous virtual springs for formation control in the presence of Coulomb friction. We now continue with the dynamics, interconnection structure, control design and closed-loop analysis for a network of point masses in $\mathbb{R}^2$ controlled by continuous and discontinuous virtual springs.

4.2. Continuous virtual springs

In this paper, we pursue the position-based formation keeping control which aims at a desired shape and a desired orientation for a group of agents (Bai et al., 2011). For both types of virtual springs, we assign one spring along the x-direction and one spring along the y-direction. For spring ℓ consider the relative position $z_\ell = (z_{x\ell}, z_{y\ell})^T$, desired relative position $z_\ell^* = (z_{x\ell}^*, z_{y\ell}^*)^T$, spring constants $K_\ell = \text{diag}(k_{x\ell}, k_{y\ell})$, and let $\tilde{z}_\ell = (\tilde{z}_{x\ell}, \tilde{z}_{y\ell})^T$ denote the error variable defined as $\tilde{z}_\ell = z_\ell - z_\ell^*$. The input to each spring ℓ is a relative velocity $w_\ell = (w_{x\ell}, w_{y\ell})^T$, while the output is the corresponding spring force $\tau_\ell = (\tau_{x\ell}, \tau_{y\ell})^T$.

For m springs let $z = (z_1^T, \dots, z_m^T)^T$, $\tilde{z} = (\tilde{z}_1^T, \dots, \tilde{z}_m^T)^T$, $z^* = (z_1^{*T}, \dots, z_m^{*T})^T$, $w = (w_1^T, \dots, w_m^T)^T$, $\tau = (\tau_1^T, \dots, \tau_m^T)^T$, and $K = \text{block.diag}(K_1, \dots, K_m)$. We assume that the desired relative position z^* for the agents of the network is the prescribed vector $z^* \in (B^T \otimes I_2)x^*$, where B denotes the incidence matrix of the underlying tree graph which describes the interaction topology amongst the agents. The dynamics of the continuous virtual springs are well known (Duindam et al., 2009; Vos et al., 2012) and given by

$$\begin{aligned}\dot{\tilde{z}} &= w, \\ \tau &= \frac{\partial H^c}{\partial \tilde{z}},\end{aligned}\quad (11)$$

with Hamiltonian $H^c(\tilde{z}) = \frac{1}{2}\tilde{z}^T K \tilde{z}$. Note that the corresponding partial derivatives are given by

$$\frac{\partial H^c}{\partial \tilde{z}_{x\ell}} = k_{x\ell}\tilde{z}_{x\ell}, \quad \frac{\partial H^c}{\partial \tilde{z}_{y\ell}} = k_{y\ell}\tilde{z}_{y\ell}.$$

The coupling law to assign the virtual springs in between the agents (Jafarian et al., 2014) is given by

$$\begin{cases} u = -(B \otimes I_2)\tau, \\ w = (B^T \otimes I_2)y. \end{cases}\quad (12)$$

For the continuous springs, the closed-loop dynamics follow (8), (11), (12) and are given by

$$\begin{aligned}\dot{p} &= -(B \otimes I_2) \frac{\partial H}{\partial \tilde{z}} + u^r, \\ \dot{\tilde{z}} &= (B^T \otimes I_2) \frac{\partial H}{\partial p}.\end{aligned}\quad (13)$$

where $H(p, \tilde{z})$ is the closed-loop Hamiltonian given by

$$\begin{aligned}H(p, \tilde{z}) &= H^a(p) + H^c(\tilde{z}) \\ &= \frac{1}{2}p^T M^{-1}p + \frac{1}{2}\tilde{z}^T K \tilde{z}.\end{aligned}\quad (14)$$

Since all agents in the network are subject to Coulomb friction, the term u^r in (13) represents the friction and is equal to a set-valued map $u^r = -F(M^{-1}p)$, where $u_i^r = -F_i(M_i^{-1}p_i)$ is defined in (5). Hence, the closed loop system is a differential inclusion given by

$$\begin{aligned}\dot{p} &\in -(B \otimes I_2) \frac{\partial H}{\partial \tilde{z}} - F(M^{-1}p), \\ \dot{\tilde{z}} &= (B^T \otimes I_2) \frac{\partial H}{\partial p}.\end{aligned}\quad (15)$$

Hence, the closed-loop dynamics can be written in a compact form $(\dot{p}, \dot{\tilde{z}}) \in \mathcal{K}_1(p, \tilde{z})$ with

$$\mathcal{K}_1(p, \tilde{z}) = \left(\begin{array}{c} -(B \otimes I_2)K\tilde{z} - F(M^{-1}p) \\ (B^T \otimes I_2)M^{-1}p \end{array} \right),$$

where $F(M^{-1}p) = \times_{i=1}^n F_i(M_i^{-1}p_i)$. The map $F_i(M_i^{-1}p_i)$ follows (5) and it is given by

$$F_i(M_i^{-1}p_i) = \begin{cases} c_i \frac{M_i^{-1}p_i}{\|M_i^{-1}p_i\|} & \text{if } p_i \neq \mathbf{0} \\ B(\mathbf{0}, c_i) & \text{if } p_i = \mathbf{0}. \end{cases} \quad (16)$$

Note that $M_i^{-1}p_i$ is equal to the velocity of agent i . Before presenting the analysis of the closed-loop system controlled with continuous springs, we present some definitions on the terminal node and edge sets of the graph corresponding to the steps in the proofs.

Definition 2 (Terminal Node and Edge Set). Let $\mathcal{V}^s$ denote the set of nodes for step s ($s \geq 1$), which is defined as $\mathcal{V}^s = \mathcal{V} - \bigcup_{r=0}^{s-1} \mathcal{V}^r$. Here, $\mathcal{V}^r$ denotes the set of terminal nodes, which is defined as

$$\mathcal{V}^r := \{v_i \in \mathcal{V}^r \mid \deg v_i = 1\},$$

with $\mathcal{V}^0 = \emptyset$. In a similar way, let $\mathcal{E}^s$ denote the set of edges for step s , which is defined as $\mathcal{E}^s = \mathcal{E} - \bigcup_{r=0}^{s-1} \mathcal{E}^r$. Here, $\mathcal{E}^r$ denotes the set of terminal edges, which is defined as

$$\mathcal{E}^r := \{e_k \in \mathcal{E}^r \mid e_k = (v_i, v_j), v_i \in \mathcal{V}^r \text{ or } v_j \in \mathcal{V}^r\},$$

with $\mathcal{E}^0 = \emptyset$.

We are now ready to present the result on the continuous springs.

Theorem 3 (Continuous Virtual Springs). The solutions to the closed-loop dynamics (15) converge to the largest weakly invariant set where $p = \mathbf{0}$ and $\tilde{z}_\ell \in B(\mathbf{0}, \alpha_\ell)$ for all $\ell \in \mathcal{E}$, where α_ℓ is a positive constant depending on the spring constants $K_1, \dots, K_m$ and Coulomb friction coefficients $c_1, \dots, c_n$.

Proof. Take the Hamiltonian $H(p, \tilde{z})$ in (14) as the Lyapunov function candidate. Since $H(p, \tilde{z})$ is continuously differentiable, the set-valued derivative $\dot{H}(p, \tilde{z})$ along (15) is

$$\dot{H}(p, \tilde{z}) = \{\nabla H(p, \tilde{z}) \cdot w, w \in \mathcal{K}_1(p, \tilde{z})\}.$$

By definition of $F(M^{-1}p)$ in (16), for any $w \in \mathcal{K}_1(p, \tilde{z})$ there exists $w^p \in F(M^{-1}p)$ such that

$$w = \begin{pmatrix} -(B \otimes I_2)K\tilde{z} - w^p \\ (B^T \otimes I_2)M^{-1}p \end{pmatrix}.$$

Hence, $\dot{H}(p, \tilde{z}) = \{a \in \mathbb{R} : a = -p^T M^{-T} w^p\} \subset (-\infty, 0]$. Therefore, applying the nonsmooth LaSalle's invariance principle (Bacciotti & Ceragioli, 1999), the solutions to the closed-loop system converge to the largest weakly invariant set of points $(p, \tilde{z})$ where $p = \mathbf{0}$ and

$$\left\{ (\mathbf{0}, \tilde{z}) \mid \mathbf{0} = \begin{pmatrix} -w^p - (B \otimes I_2)K\tilde{z} \\ \mathbf{0} \end{pmatrix} \right\}. \quad (17)$$

Consider the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and define the terminal node-set $\bar{\mathcal{V}}^1$ and edge-set $\bar{\mathcal{E}}^1$ accordingly. From (17), for all $v_i \in \bar{\mathcal{V}}^1$ we have

$$\mathbf{0} = -w_i^p + \sum_{\ell \in \bar{\mathcal{E}}^1} b_{i\ell} K_\ell \tilde{z}_\ell,$$

where $w_i^p \in F_i(M_i^{-1}p_i)$ and $b_{i\ell}$ is the element of the incidence matrix corresponding to node i and edge ℓ . Note that, similar to z_ℓ , w_i^p belongs to $\mathbb{R}^2$ and it has a vector representation as $w_i^p = (w_{xi}^p, w_{yi}^p)^T$. Since v_i is a terminal node, the above equation simplifies to

$$\mathbf{0} = -w_i^p + b_{i\ell} K_\ell \tilde{z}_\ell, \quad (18)$$

since there is only one $b_{i\ell} \neq 0$ for each $v_i \in \bar{\mathcal{V}}^1$. From (18) it immediately follows that

$$\tilde{z}_\ell = b_{i\ell} K_\ell^{-1} w_i^p, \quad (19)$$

for all $\ell \in \bar{\mathcal{E}}^1$. Since $w_i^p \in B(\mathbf{0}, c_i)$, the equality (19) results in

$$\tilde{z}_\ell \in \{e \mid e = b_{i\ell} K_\ell^{-1} w_i^p, w_i^p \in B(\mathbf{0}, c_i)\},$$

and thus we can obtain a bound on the size (2-norm) of $\tilde{z}_\ell$ given by

$$\|\tilde{z}_\ell\| \in \left[0, c_i \sqrt{\text{tr}(K_\ell^{-1} (K_\ell^{-1})^T)} \right].$$

Recall that $K_\ell = \text{diag}(k_{x\ell}, k_{y\ell})$. Then, the above equation implies

$$\|\tilde{z}_\ell\| \in \left[0, \frac{c_i}{k_{x\ell} k_{y\ell}} \sqrt{k_{x\ell}^2 + k_{y\ell}^2} \right].$$

In this way we find a bound for all springs corresponding to an edge in the terminal edge set $\bar{\mathcal{E}}^1$.

Now continue with the node set $\mathcal{V}^2$ and edge set $\mathcal{E}^2$ (see Definition 2) and the corresponding terminal sets $\bar{\mathcal{V}}^2$ and $\bar{\mathcal{E}}^2$. Similar to the previous step, for all $v_i \in \bar{\mathcal{V}}^2$, we have

$$\mathbf{0} = -w_i^p + \sum_{\ell \in \bar{\mathcal{E}}} b_{i\ell} K_\ell \tilde{z}_\ell. \quad (20)$$

We rewrite (20) as

$$\mathbf{0} = -w_i^p + \sum_{k \in \bar{\mathcal{E}}^1} b_{ik} K_k \tilde{z}_k + \sum_{\ell \in \bar{\mathcal{E}}^2} b_{i\ell} K_\ell \tilde{z}_\ell. \quad (21)$$

Similar to the first step, there is exactly one $b_{i\ell} \neq 0$ for all $v_i \in \bar{\mathcal{V}}^2$, $e_\ell \in \bar{\mathcal{E}}^2$. In addition, each $\tilde{z}_k \in \bar{\mathcal{E}}^1$ follows (19). Therefore, we obtain

$$b_{i\ell} K_\ell \tilde{z}_\ell \in \left\{ e \mid e = w_i^p - \sum_{j \in \bar{\mathcal{V}}^1} w_j^p, w_j^p \in B(\mathbf{0}, c_j), w_i^p \in B(\mathbf{0}, c_i) \right\}.$$

Hence, we find a bound for all springs corresponding to an edge in the terminal edge set $\bar{\mathcal{E}}^2$. Repeating the steps above until $\bigcup_i \bar{\mathcal{E}}^i = \mathcal{E}$, we find a bound for all springs in the graph, depending on the friction coefficients $c_1, \dots, c_n$ and spring constants $K_1, \dots, K_m$, which completes the proof. $\square$

The above proof shows that the convergence of z to z^* cannot be guaranteed by means of continuous virtual springs. Hence, the formation control objective might not be achieved. In the next section, we design a discontinuous controller which achieves the exact desired formation.

4.3. Discontinuous virtual springs

The previous section showed that continuous virtual springs cannot achieve the formation objectives (3) exactly. Motivated by the example in Section 3 we replace the continuous virtual springs

by its discontinuous counterpart. Using the same variables as in (11), the dynamics of m discontinuous springs are given by

$$\begin{aligned}\dot{\tilde{z}} &= w \\ \tau &= \partial H^d,\end{aligned}\quad (22)$$

with a locally Lipschitz Hamiltonian $H^d(\tilde{z}) = \|K\tilde{z}\|_1$, where $\|K\tilde{z}\|_1$ denotes the one-norm that is $\|K\tilde{z}\|_1 = \sum_{\ell=1}^m (k_{x\ell}|\tilde{z}_{x\ell}| + k_{y\ell}|\tilde{z}_{y\ell}|)$. Since H^d is a nonsmooth Hamiltonian function (Fujimoto, Sakai, & Sugie, 2012), it is not differentiable everywhere. Hence, we calculate ∂H^d based on the Clarke generalized gradient (see Bacciotti & Ceragioli, 1999 and Eq. (27) later on).

Using the same coupling law (12) for the continuous springs, the closed-loop dynamics for the discontinuous springs follow directly from (8), (12), (22) and are given by

$$\begin{aligned}\dot{p} &\in -(B \otimes I_2) \partial_{\tilde{z}} H - F(M^{-1}p) \\ \dot{\tilde{z}} &= (B^T \otimes I_2) \frac{\partial H}{\partial p},\end{aligned}\quad (23)$$

where $\partial_{\tilde{z}} H(\tilde{z}) = \partial H^d(\tilde{z})$ and $H(p, \tilde{z})$ is the closed-loop Hamiltonian given by

$$\begin{aligned}H(p, \tilde{z}) &= H^a(p) + H^d(\tilde{z}) \\ &= \frac{1}{2} p^T M^{-1} p + \|K\tilde{z}\|_1.\end{aligned}\quad (24)$$

Note that (23) is a differential inclusion due to the set-valued model of Coulomb friction and the discontinuous spring force ∂H^d . Here, $\partial H^d = K \mathcal{K} \text{sign}(\tilde{z})$ where $\mathcal{K} \text{sign}(\tilde{z}) = \times_{\ell=1}^m \mathcal{K} \text{sign}(\tilde{z}_\ell)$ with

$$\mathcal{K} \text{sign}(\tilde{z}_\ell) = \begin{cases} \left\{ \frac{\tilde{z}_{x\ell}}{|\tilde{z}_{x\ell}|} \right\} \times \left\{ \frac{\tilde{z}_{y\ell}}{|\tilde{z}_{y\ell}|} \right\} & \text{if } \tilde{z}_{x\ell} \neq 0, \tilde{z}_{y\ell} \neq 0 \\ [-1, +1] \times \left\{ \frac{\tilde{z}_{y\ell}}{|\tilde{z}_{y\ell}|} \right\} & \text{if } \tilde{z}_{x\ell} = 0, \tilde{z}_{y\ell} \neq 0 \\ \left\{ \frac{\tilde{z}_{x\ell}}{|\tilde{z}_{x\ell}|} \right\} \times [-1, +1] & \text{if } \tilde{z}_{x\ell} \neq 0, \tilde{z}_{y\ell} = 0 \\ [-1, +1] \times [-1, +1] & \text{if } \tilde{z}_{x\ell} = 0, \tilde{z}_{y\ell} = 0. \end{cases} \quad (25)$$

Note that (25) considers discontinuous springs along x and y separately. This is a design choice that complies with the position-based control design as stated in Section 4.2.

Writing the closed-loop dynamics in a compact form $(\dot{p}, \dot{\tilde{z}}) \in \mathcal{K}_2(p, \tilde{z})$, we obtain

$$\mathcal{K}_2(p, \tilde{z}) = \left(\begin{array}{c} -(B \otimes I_2) K \mathcal{K} \text{sign}(\tilde{z}) - F(M^{-1}p) \\ (B^T \otimes I_2) M^{-1}p \end{array} \right), \quad (26)$$

where $F(M^{-1}p)$ is defined in (16). Local existence of Krasovskii solutions is guaranteed by the fact that the right hand side of (23) is measurable and locally bounded (i.e., bounded on each bounded subset of its domain Hájek, 1979). Moreover, completeness of the solutions to the system (23) can be deduced by their boundedness (Nemytskii & Stepanov, 1960) which is proven below (see De Persis & Jayawardhana, 2012, Lemma 3.1, Theorem 3.2 for more details).

Now we are ready to formulate our main result:

Theorem 4 (Discontinuous Springs). Assume that $\min\{k_{x\ell}, k_{y\ell}\} > \max\{c_i, c_j\}$ for $e_\ell = (n_i, n_j) \in \mathcal{E}$. Then the solutions of the closed-loop dynamics (23) converge to the origin $(p, \tilde{z}) = (\mathbf{0}, \mathbf{0})$, thereby achieving the control objectives (3).

Proof. Take the Hamiltonian in (24) as the candidate Lyapunov function which is a regular and locally Lipschitz function. Since the Hamiltonian is not differentiable everywhere, first we calculate its

corresponding Clarke generalized gradient (Bacciotti & Ceragioli, 1999) $\partial H(p, \tilde{z})$ as

$$\partial H(p, \tilde{z}) = \left\{ v : v = \begin{pmatrix} M^{-1}p \\ K v^{\tilde{z}} \end{pmatrix} \text{ s.t. } v_k^{\tilde{z}} \in \mathcal{K} \text{sign}(\tilde{z}_k) \right\}. \quad (27)$$

Calculating the set-valued derivative $\dot{H}(p, \tilde{z})$ along (23), we obtain

$$\begin{aligned}\dot{H}(p, \tilde{z}) &= \{a \in \mathbb{R} : \exists w \in \mathcal{K}_2(p, \tilde{z}) \text{ s.t.} \\ &\quad a = w \cdot v, \text{ for all } v \in \partial H(p, \tilde{z})\}.\end{aligned}$$

By definition of $\mathcal{K}_2(p, \tilde{z})$ in (25)–(26), for any $w \in \mathcal{K}_2(p, \tilde{z})$ there exist $w^p \in F(M^{-1}p)$ and $w^{\tilde{z}} \in \mathcal{K} \text{sign}(\tilde{z})$ such that

$$w = \begin{pmatrix} -(B \otimes I_2) K w^{\tilde{z}} - w^p \\ (B^T \otimes I_2) M^{-1}p \end{pmatrix}.$$

For each $v \in \partial H(p, \tilde{z})$, choose $w^{\tilde{z}} = v^{\tilde{z}}$. Hence, we obtain $\dot{H}(p, \tilde{z}) = \{a \in \mathbb{R} : a = -p^T M^{-T} w^p\} \subset (-\infty, 0]$. Assume that $\dot{H} \neq \emptyset$. Therefore, applying the nonsmooth version of LaSalle's invariance principle (Bacciotti & Ceragioli, 1999), the solutions of the closed-loop system converge to the largest weakly invariant set of points $(p, \tilde{z})$ such that $p = \mathbf{0}$ and

$$\left\{ (\mathbf{0}, \tilde{z}) \mid \mathbf{0} = \begin{pmatrix} -w^p - (B \otimes I_2) K w^{\tilde{z}} \\ \mathbf{0} \end{pmatrix} \right\}, \quad (28)$$

where $w^p \in \mathbb{R}^2$ and $w^{\tilde{z}} \in \mathbb{R}^2$. Now, consider the tree graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and define the terminal node-set $\tilde{\mathcal{V}}^1$ and edge-set $\tilde{\mathcal{E}}^1$ accordingly. Based on (28), all nodes $v_i \in \tilde{\mathcal{V}}^1$ obey

$$\mathbf{0} = w_i^p + \sum_{\ell \in \tilde{\mathcal{E}}^1} b_{i\ell} K_\ell w^{\tilde{z}_\ell}.$$

Noting that v_i is a terminal node, the above equation simplifies to

$$\mathbf{0} = w_i^p + b_{i\ell} K_\ell w^{\tilde{z}_\ell}, \quad (29)$$

where $w_i^p = (w_{xi}^p, w_{yi}^p)^T$, $w_i^p \in B(\mathbf{0}, c_i)$ and $w^{\tilde{z}_\ell} = (w_{x\ell}^{\tilde{z}_\ell}, w_{y\ell}^{\tilde{z}_\ell})^T$. Writing (29) for each of the components of w_i^p , we obtain

$$0 = w_{xi}^p + b_{i\ell} k_{x\ell} w_{x\ell}^{\tilde{z}_\ell} \quad (30)$$

$$0 = w_{yi}^p + b_{i\ell} k_{y\ell} w_{y\ell}^{\tilde{z}_\ell}. \quad (31)$$

We will now prove Theorem 4 by contradiction. Consider (30) and assume that $\tilde{z}_{x\ell} \neq 0$, then $w_{x\ell}^{\tilde{z}_\ell}$ is equal to either $+1$ or -1 . Moreover, since $w_i^p \in B(\mathbf{0}, c_i)$, we have $w_{xi}^p \in [-c_i, c_i]$. Take $w_{x\ell}^{\tilde{z}_\ell} = +1$, then from (30) we obtain

$$w_{xi}^p + b_{i\ell} k_{x\ell} w_{x\ell}^{\tilde{z}_\ell} \in [-c_i + k_{x\ell}, c_i + k_{x\ell}].$$

By assumption, we have $k_{x\ell} > c_i$. Hence, both of the upper and lower bounds of the above interval are positive and zero cannot belong to this interval. Similarly, if $w_{x\ell}^{\tilde{z}_\ell} = -1$, we obtain the interval $[-c_i - k_{x\ell}, c_i - k_{x\ell}]$ where both of its bounds are negative. This result contradicts (30). As a result, $\tilde{z}_{x\ell}$ should necessarily be zero. A similar argument holds for y -direction (31) which results in $\tilde{z}_{y\ell} = 0$. In this way we show that for all springs corresponding to an edge in the terminal edge set $\tilde{\mathcal{E}}^1$, the error position $\tilde{z}_\ell$ is equal to zero on the invariant set (28).

Now continue with the node set $\mathcal{V}^2$ and edge set $\mathcal{E}^2$ and the corresponding terminal sets $\tilde{\mathcal{V}}^2$ and $\tilde{\mathcal{E}}^2$. Using similar arguments as above, we can show that the relative error position is equal to zero for all springs corresponding to an edge in the terminal edge set $\tilde{\mathcal{E}}^2$.

Repeating the steps above until $\bigcup_i \bar{\mathcal{E}}^i = \mathcal{E}$, we can conclude that for each of edges in the graph the corresponding relative position error converges to zero, thereby completing the proof. $\square$

Remark 5. Considering the two components of Coulomb friction (see Section 3.1), this paper assumes that the coefficients for static friction s_i (for $v_i = \mathbf{0}$) and kinetic friction force c_i (for $v_i \neq \mathbf{0}$) are equal. Although, the latter assumption is not necessary. In fact, if $s_i > c_i$, there will only be an update in the condition of Theorem 4. That is the results of Theorem 4 hold if $\min\{k_{x\ell}, k_{y\ell}\} > \max\{s_i, s_j\}$.

5. Simulations

In this section, we present simulation results illustrating Theorems 3 and 4. Consider a network of $n = 5$ agents interconnected using $m = 4$ virtual couplings using the following incidence matrix:

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

Each agent has the unit mass ($m_i = 1$). The Coulomb friction coefficient is set equal to $c_i = 2$ for $i = 1, \dots, 5$. For the virtual couplings we set $k_{x\ell} = k_{y\ell} = 2.5$ for $\ell = 1, \dots, 4$ (i.e., the condition $\min\{k_{x\ell}, k_{y\ell}\} > \max\{c_i, c_j\}$ in Theorem 4 is satisfied).

The desired formation has a pentagon shape with edge length equal to two and is defined by the following inter-agent position vectors: $z_1^* = (0, -2)^T$, $z_2^* = (-1, -1 - \sqrt{3})^T$, $z_3^* = (-1, 1 + \sqrt{3})^T$, $z_4^* = (0, 2)^T$. The initial conditions for the agents are set as $q_{xi}(0) = (0.5, 0.2, -1, 2, 0.8)^T$ and $q_{yi}(0) = (1, -0.2, 0.9, 1.5, 0)^T$. The simulations are done using MATLAB with a fixed-step solver (ode4, step size = 0.001). Figs. 3 and 4 show the plots of the error relative position $\tilde{z}_x = z_x - z_x^*$, momentum p_x , and control input u_x along the x -direction for the continuous springs (Theorem 3) and the discontinuous springs (Theorem 4) respectively. The plots of the relative position $\tilde{z}_y$, momentum p_y and control input u_y along the y -direction follow similar trends.

The top plot in Fig. 3 shows that the relative position z does not converge to the desired one, z^* , using continuous springs. However, using discontinuous springs the relative position z does converge to the desired relative position z^* (see Fig. 4, top). The middle plots in Figs. 3 and 4 show that p_x converges to zero for both types of springs.

Comparing the control action u_x in (Fig. 3, bottom) and (Fig. 4, bottom), the control actions related to the discontinuous springs show fast switchings while $\tilde{z}_x$ converges to zero. This switching behavior is a known undesired phenomenon due to the use of a sign-based controller (Jafarian & De Persis, 2015).

Treating the fast switchings of the control action is among the future extensions (Ceragioli et al., 2011; De Persis & Frasca, 2013). However, as an example and motivated by hysteretic quantizers in Ceragioli et al. (2011), here, we include the simulation result of the time evolution of the system using a hysteretic-quantizer-based sign function. Fig. 5 shows the time evolution of $\tilde{z}_x$, p_x and u_x with a hysteretic sign controller. The model and the analysis of such a controller are beyond the scope of this paper. For the sake of clarity, only the control action of one of the agents is shown. As shown, the oscillations of the control action in Fig. 5 are highly reduced compared with Fig. 4. The design of such a controller is currently under investigation to improve the overall performance of the system.

Fig. 3. Time evolution of the relative position $\tilde{z}_x$, momentum p_x , and control input u_x using continuous springs.

Fig. 4. Time evolution of the relative position $\tilde{z}_x$, momentum p_x , and control input u_x using discontinuous springs.

6. Conclusions

This paper analyzed the problem of formation control for a group of agents communicating over a tree graph in the presence of Coulomb friction. Two types of controllers (virtual springs) were presented to achieve the formation objective. Using tools from nonsmooth analysis, it was shown that the continuous virtual springs cannot achieve exact formations. Discontinuous virtual springs on the other hand can achieve a desired formation exactly, under the condition that the virtual spring constant is strictly larger than the Coulomb friction coefficient. The results were illustrated using numerical simulations.

Directions for future research include generalization of the types of graphs and mitigating the undesired fast switching of the control action related to the discontinuous springs (e.g. by the adoption of the hysteretic quantizers or self-triggered control

Fig. 5. Time evolution of the relative position $\bar{z}_x$ and momentum p_x of all agents together with the control input u_x of agent 1 using hysteretic discontinuous springs.

algorithms Ceragioli et al., 2011; De Persis & Frasca, 2013). Other future avenues include time-varying communication topologies, experimental implementation and more general friction models.

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